2023 MATH285 LAB Presentation

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CONTENTS

PART 01

Three Basic Methods

Euler Method

For the equation:

$$y' = f(t, y), y(t_0) = y_0$$

Its Euler Method is:

$$y(t+h) = y(t) + f\big(t, y(t)\big) * h$$

This method is the most intuitive method. However, its accuracy is not very high.

Improved Euler Method

For the equation:

$$y' = f(t, y), y(t_0) = y_0$$

Its Improved Euler Method is:

$$y_{predicted}(t+h) = y(t) + f\big(t, y(t)\big) * h$$

$$y(t+h) = y(t) + f\big(t, y(t)\big) * \frac{h}{2} + f(t+h, y_{predicted}(t+h))$$

This is based on the equation

$$y(t_1) = y(t_0) + \int_{t_0}^{t_1} f\big(t, y(t)\big)dt$$

$$\approx y(t_0) + \Big(f\big(t_0, y(t_0)\big) + f\big(t_1, y(t_1)\big)\Big) * \frac{h}{2}$$

Improved Euler Method

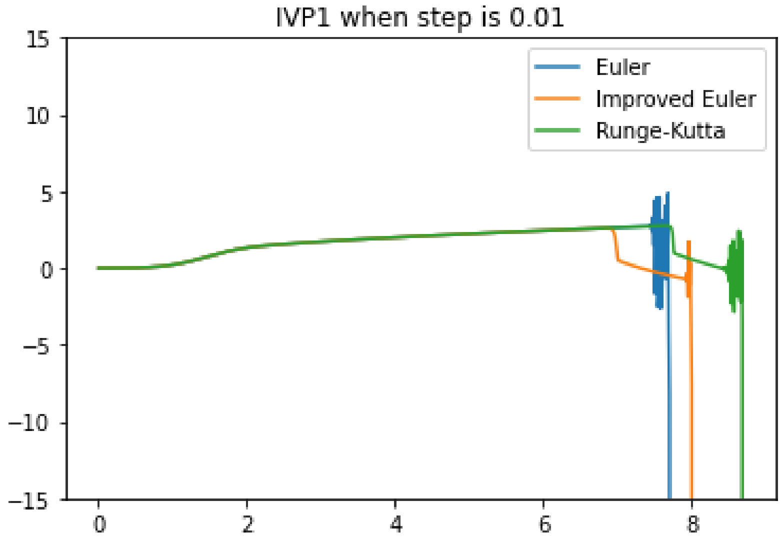
For the equation:

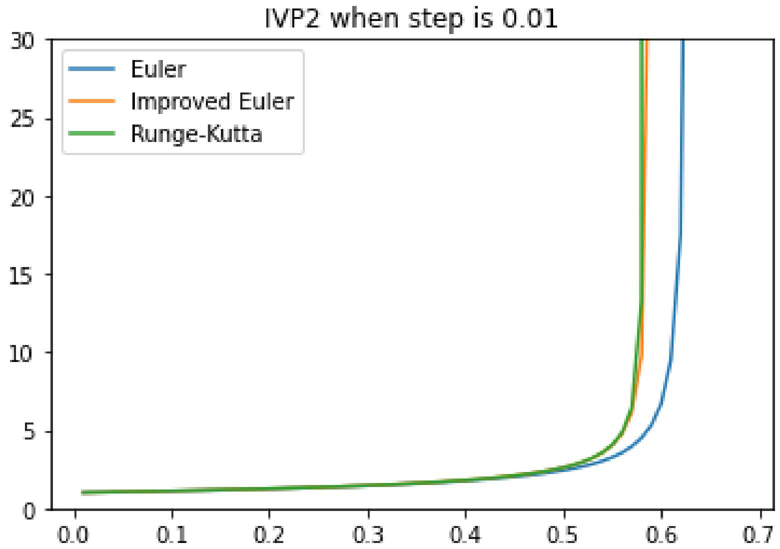
$$y' = f(t, y), y(t_0) = y_0$$

Its Runge-Kutta Method is:

$$k_1 = f(t_n + y_n)$$
$$k_2 = f\left(t_n + \frac{1}{3}h, y_n + \frac{1}{3}hk_1\right)$$
$$k_3 = f\left(t_n + \frac{2}{3}h, y_n - \frac{1}{3}hk_1 + hk_2\right)$$
$$k_4 = f(t_n + h, y_n + hk_1 - hk_2 + hk_3)$$
$$y_{n+1} = y_n + \frac{h}{8}(k_1 + 3k_2 + 3k_3 + k_4)$$

Comparison between different methods



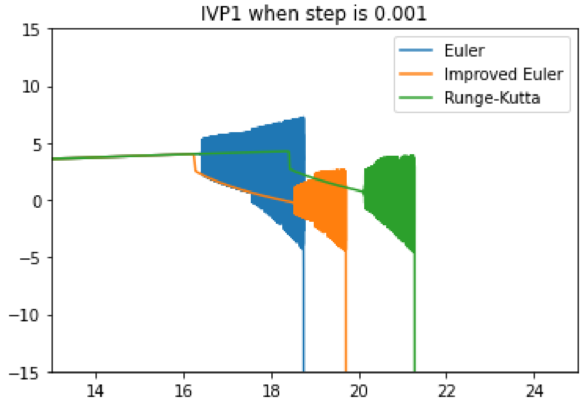


With the same step size, the Runge-Kutta Method is the most accurate. The other two have their pros and cons.

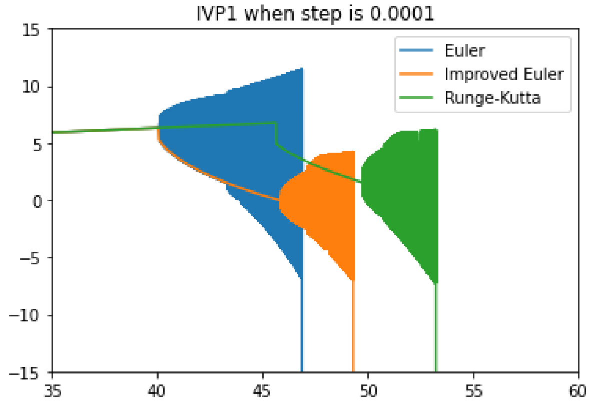
Three Basic Methods

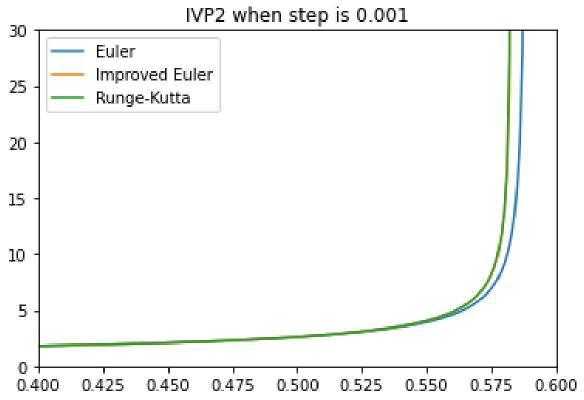
Comparison between different methods

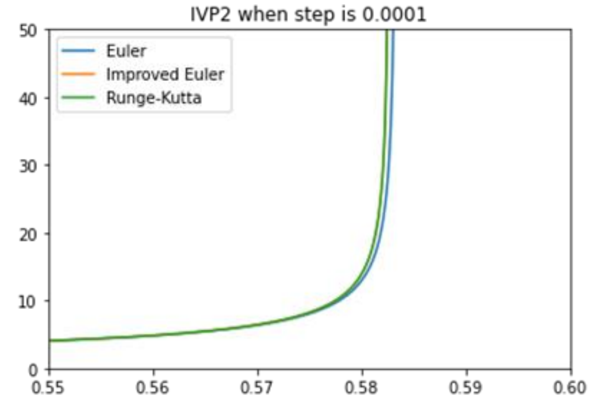
Step 0.001



Step 0.0001

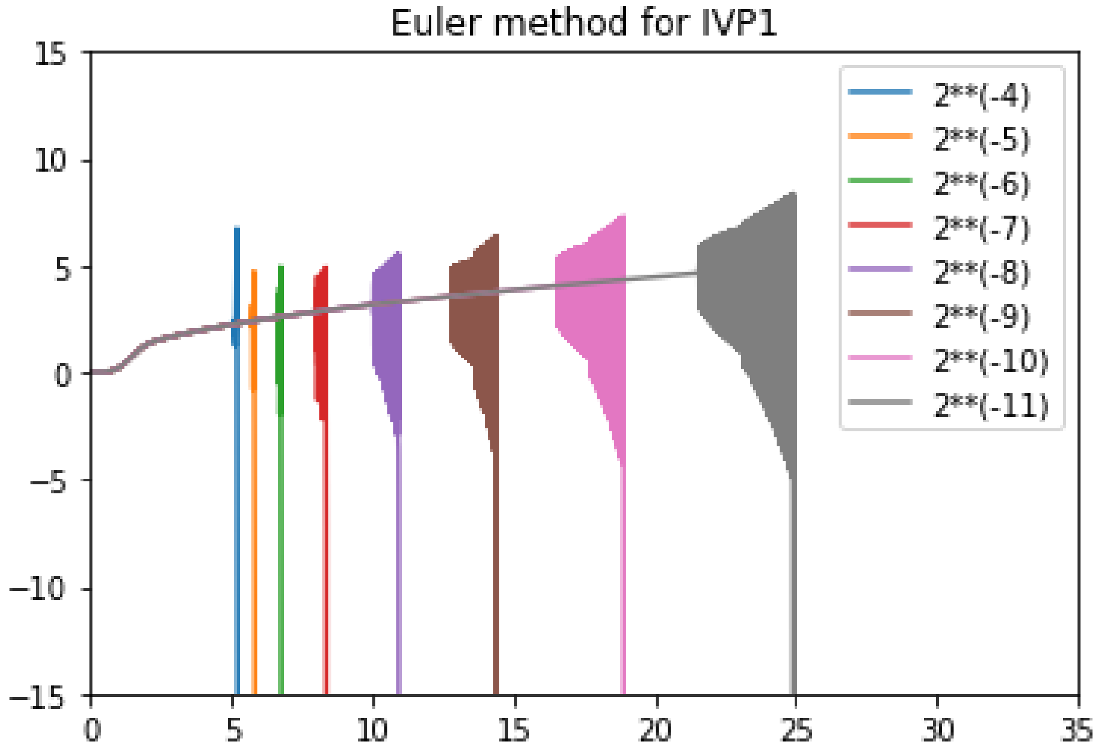




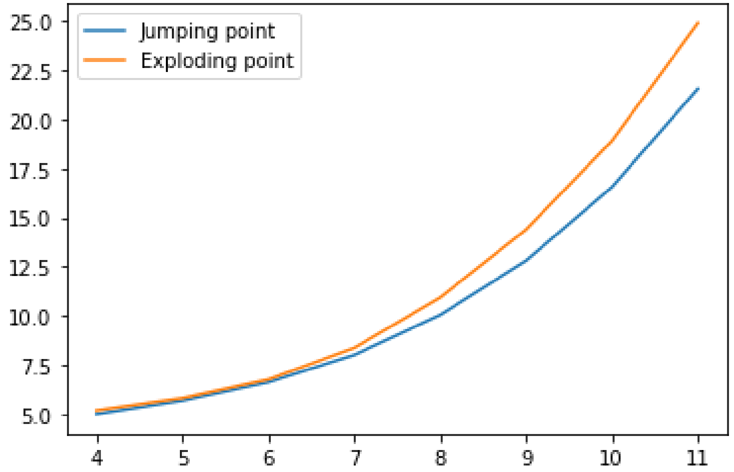


Three Basic Methods

Euler Method with different steps

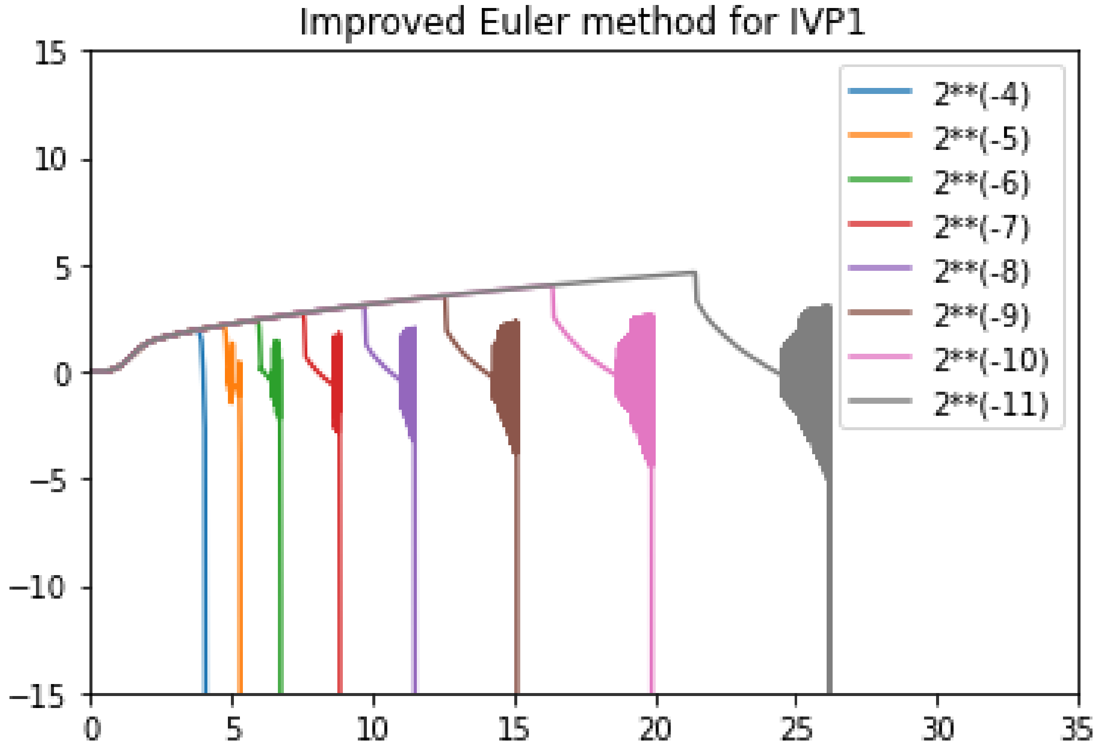


n	J(-n)	E(-n)
4	5.000000	5.187500
5	5.687500	5.812500
6	6.625000	6.765625
7	8.007812	8.367188
8	10.046875	10.933594
9	12.810547	14.392578
10	16.557617	18.912109
11	21.567871	24.912109

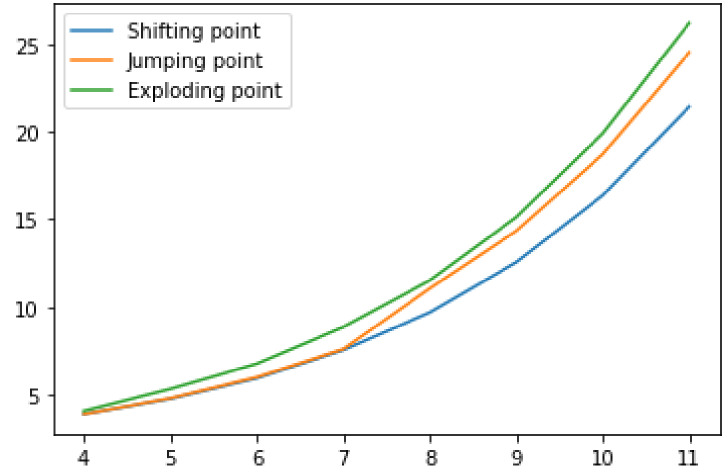


Three Basic Methods

Improved Euler Method with different steps

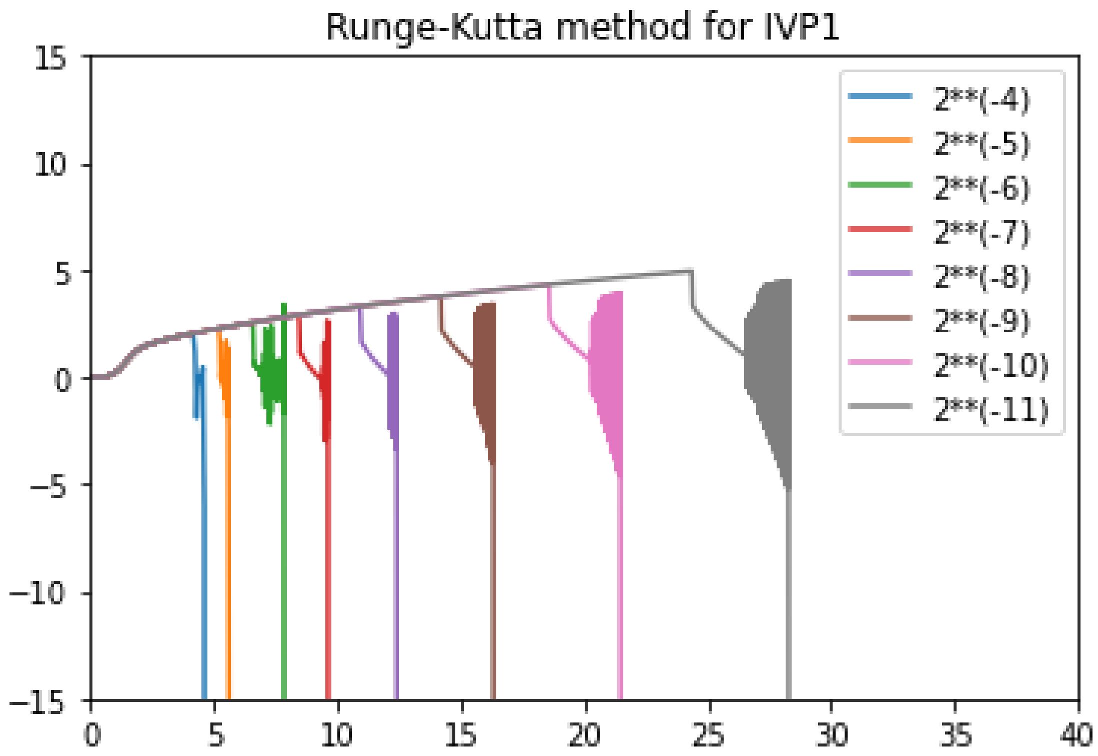


n	S(-n)	J(-n)	E(-n)
4	3.875000	3.875000	4.062500
5	4.750000	4.781250	5.312500
6	5.921875	5.984375	6.750000
7	7.531250	7.585938	8.843750
8	9.675781	11.050781	11.507812
9	12.544922	14.324219	15.113281
10	16.360352	18.705078	19.898438
11	21.420410	24.518555	26.196289

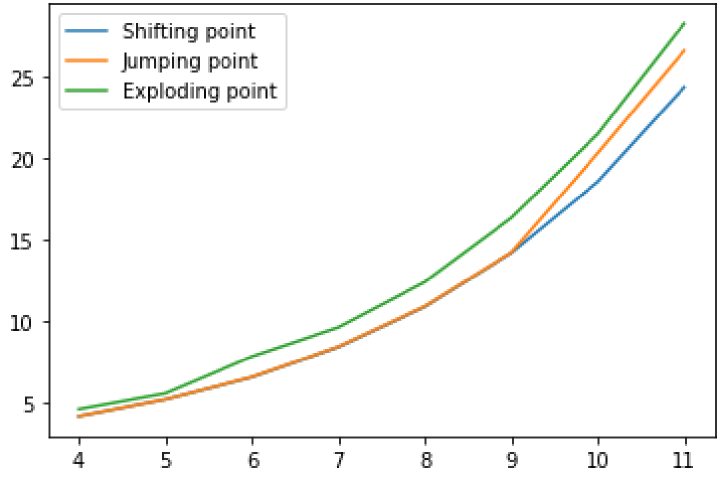


Three Basic Methods

Runge-Kutta Method
with different steps

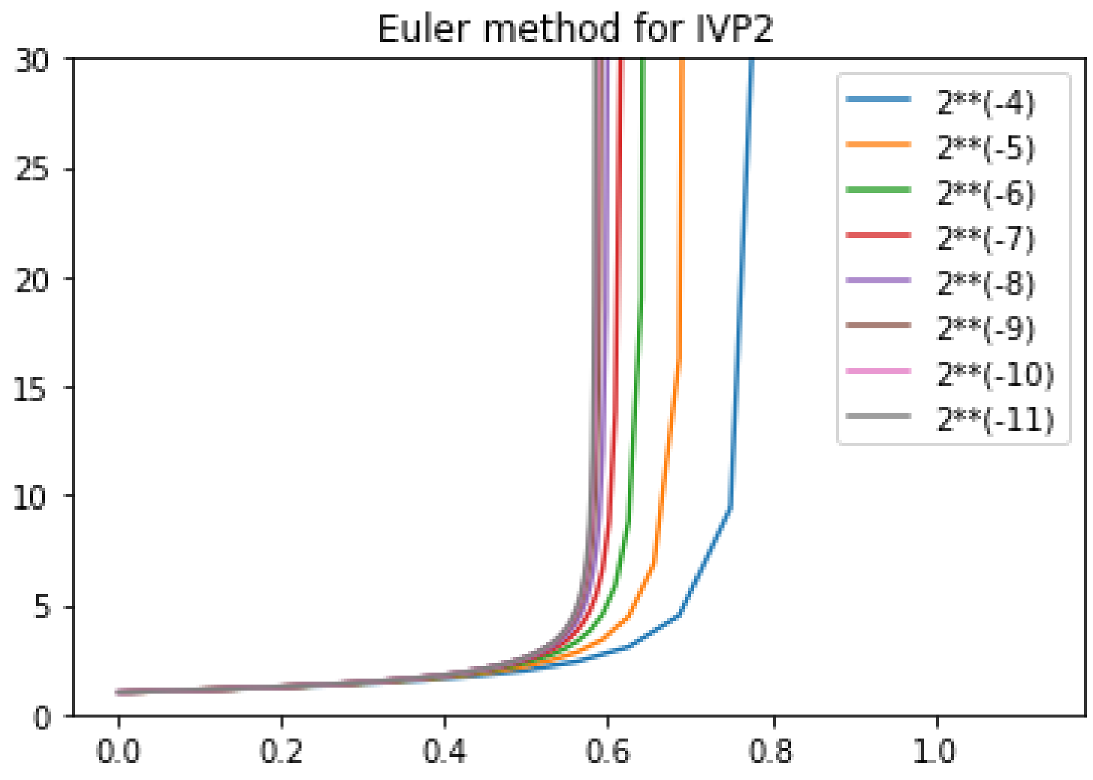


n	S(-n)	J(-n)	E(-n)
4	4.187500	4.187500	4.625000
5	5.218750	5.218750	5.593750
6	6.578125	6.593750	7.843750
7	8.421875	8.453125	9.632812
8	10.890625	10.925781	12.417969
9	14.187500	14.218750	16.347656
10	18.560547	20.310547	21.487305
11	24.354004	26.595215	28.273438

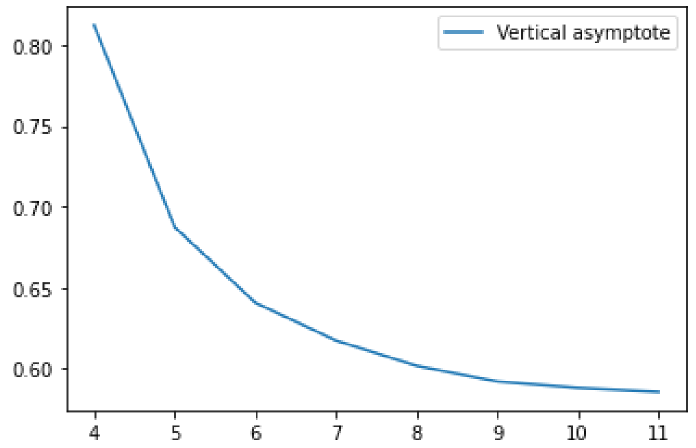


Three Basic Methods

Euler Method with
different steps



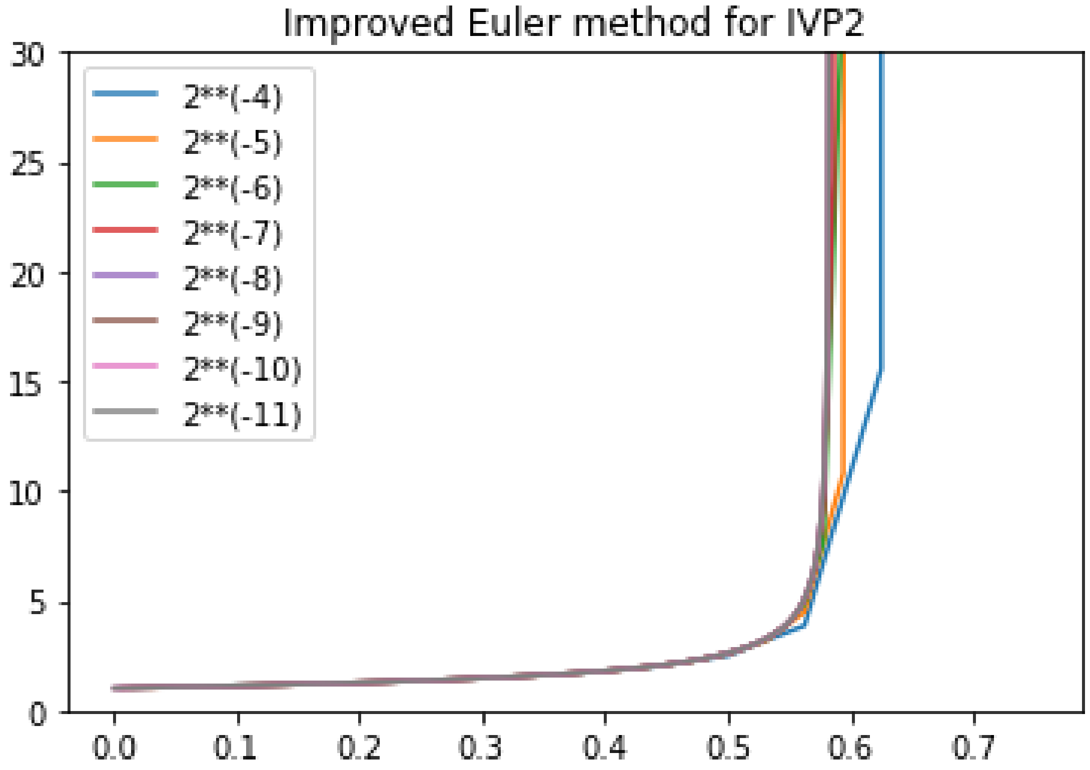
n	V(-n)
4	0.812500
5	0.687500
6	0.640625
7	0.617188
8	0.601562
9	0.591797
10	0.587891
11	0.585449

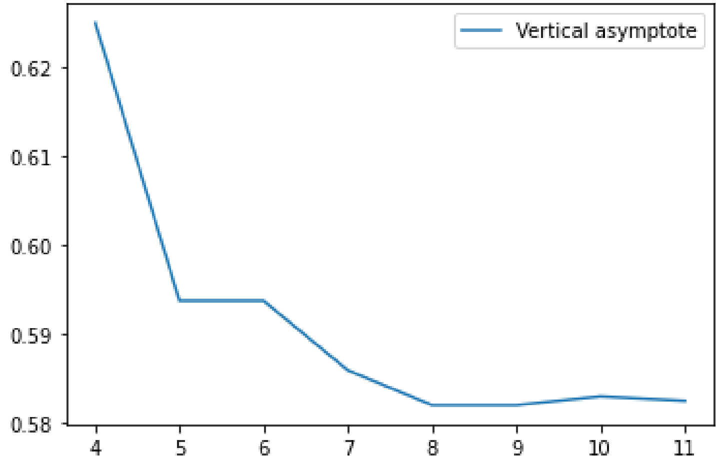


Three Basic Methods

Improved Euler Method
with different steps

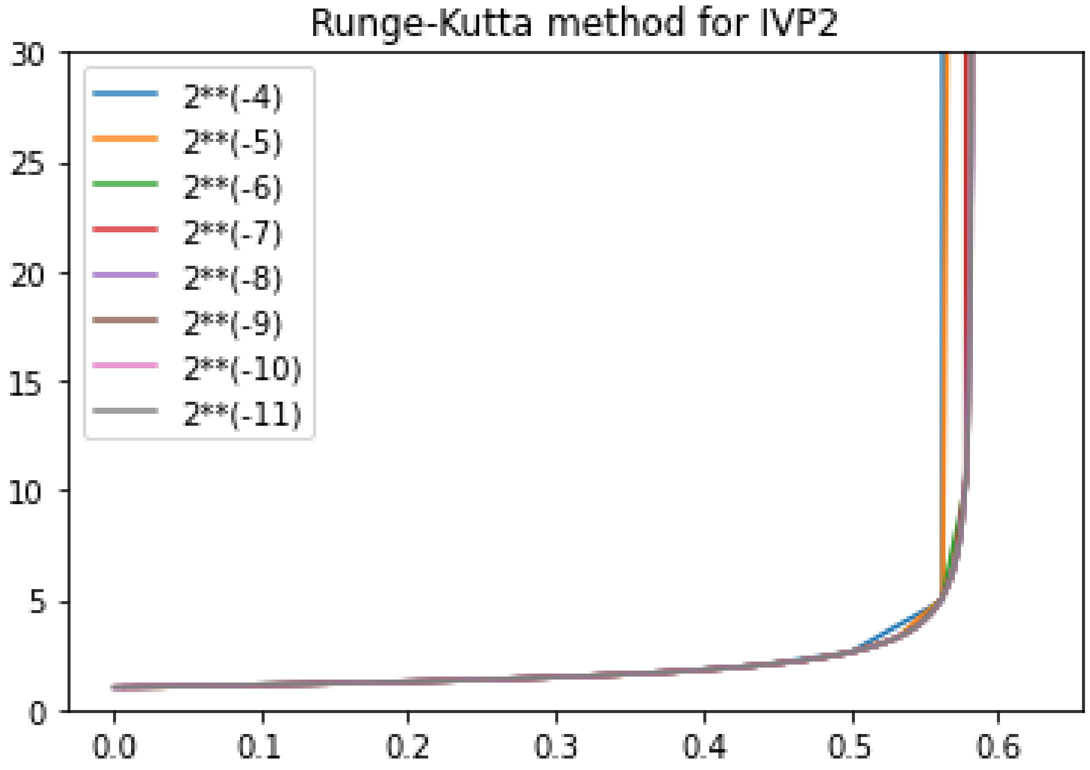
n	V(-n)
4	0.625000
5	0.593750
6	0.593750
7	0.585938
8	0.582031
9	0.582031
10	0.583008
11	0.582520



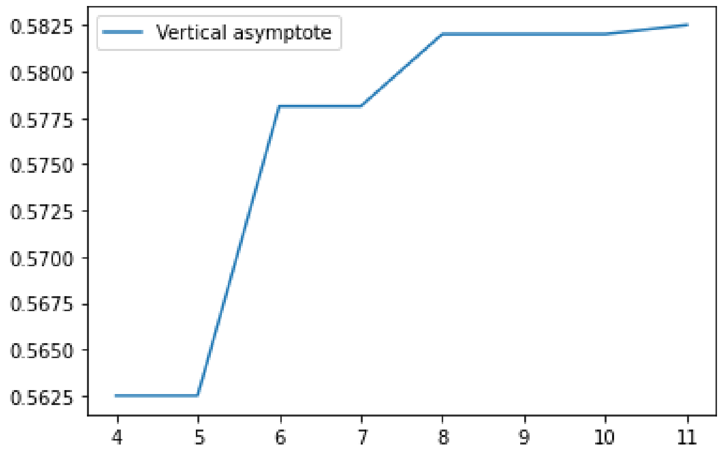


Three Basic Methods

Runge-Kutta Method
with different steps



n	V(-n)
4	0.562500
5	0.562500
6	0.578125
7	0.578125
8	0.582031
9	0.582031
10	0.582031
11	0.582520



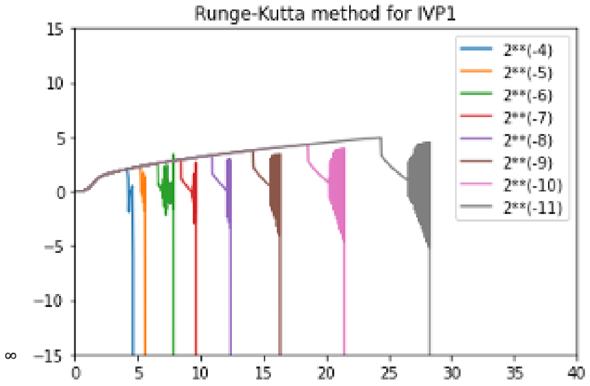
PART 02

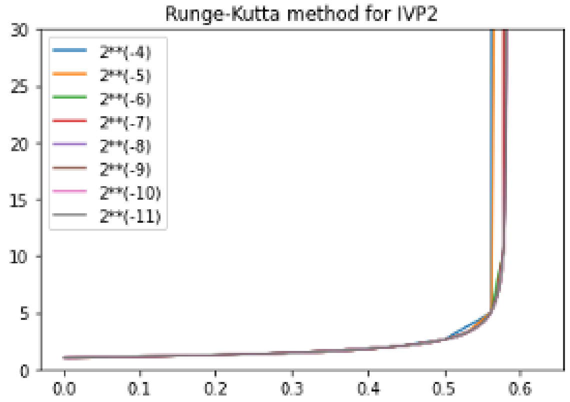
Domain of Definition/Asymptotes

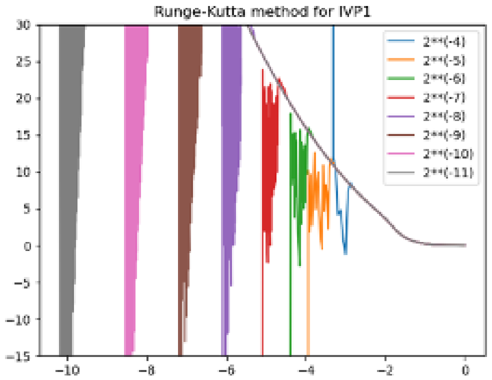
Domain of Definition/Asymptotes

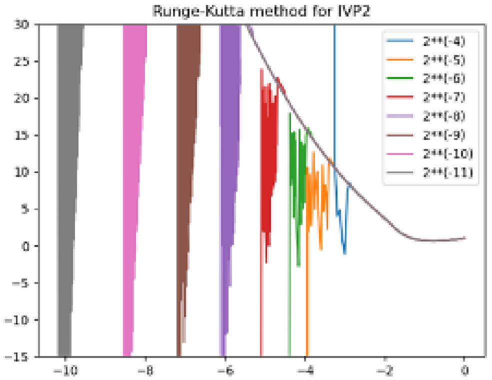
Method 1： Image method

1. Here IVP1 no vertical asymptote and IVP2 have only positive vertical asymptote. Then we could decide the accurate domain. IVP1’s domain is (-∞,∞), and IVP2’s domain is (-∞,0.583). According to our plots, we could easily find when step = 2**(-11), y(0.583008) = 3.195e+9， when step = 2**(-12), y(0.583008) = 1.497e+121,when step = 2**(-13), y(0.582764) = 3.858e+21, all the points with y approaches infinite. That we could find the negative and positive vertical asymptote, and the positive vertical asymptote is about t=0.583.









Domain of Definition/Asymptotes

Method 2: Numerical method

For IVP2, we could compute the date.

h	y(0.5)	y(0.6)
2^(-4)	2.606622	285.520
2^(-5)	2.607693	7.39e12
2^(-6)	2.607767	8.97e210
2^(-7)	2.607772	

The values at $t = 0.5$ are reasonable, and we might well believe that the solution has a value of about 2.61 at $t = 0.5$. However, it is not clear what is happening between $t = 0.5$ and $t = 0.6$. Thus, the solution of the problem has a vertical asymptote for some t in $0.5 \leq t \leq 0.6$. According to the precise point (0.583, 3194905643.584661), we find the asymptote is about x = 0.583.

PART 03

Three New Methods

Power Series

Adams-Bashforth-Moulton

Milne-Simpson

Power Series

Because $f(t,y) = (y - t^2)(y^2 - t)$ is analytic, its solution must be analytic. The solution can be represented as:

$$y(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n, t \in I$$

The original formula can be written as:

$$f(t,y) = y^3 - ty - t^2y^2 + t^3$$

We determine 0 to be t_0, therefore:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

And:

$$y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

$$y^2(t) = \sum_{n=0}^{\infty} b_n t^n$$

$$y^3(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (b_k a_{n-k}) t^n$$

$$(b_n = \sum_{k=0}^{n} a_k a_{n-k})$$

- The formula can be written as:

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(b_k a_{n-k})t^n - t\sum_{n=0}^{\infty} a_n t^n - t^2\sum_{n=0}^{\infty} b_n t^n + t^3$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+1)a_{n+1}t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(b_k a_{n-k})t^n - \sum_{n=1}^{\infty} a_{n-1}t^n - \sum_{n=2}^{\infty} b_{n-2}t^n + t^3$$

$$\Rightarrow a_1 + 2a_2 t + \sum_{n=2}^{\infty}(n+1)a_{n+1}t^n =$$

$$a_0 b_0 + (a_1 b_0 + a_0 b_1)t - a_0 t + \sum_{n=2}^{\infty}(\sum_{k=0}^{n} b_k a_{n-k})t^n - \sum_{n=2}^{\infty} a_{n-1}t^n - \sum_{n=2}^{\infty} b_{n-2}t^n + t^3$$

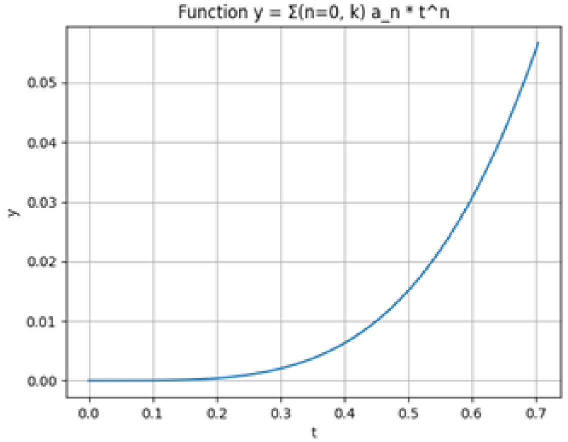
- **Based on the previous process, we can get the recursive relation of the series:**

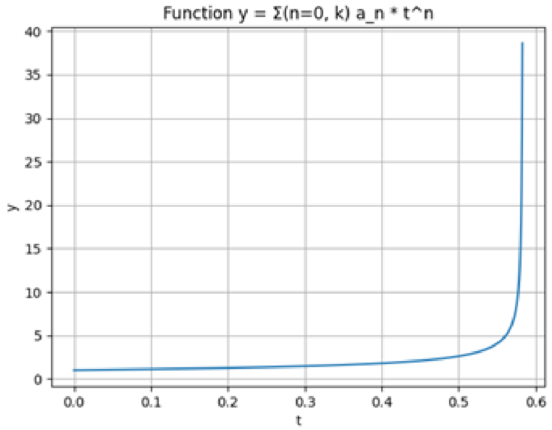
$$(n+1)a_{n+1} = \begin{cases} \left(\sum_{k=0}^{n} b_k a_{n-k}\right) - a_{n-1} - b_{n-2}, & n \neq 3 \\ \left(\sum_{k=0}^{n} b_k a_{n-k}\right) - a_{n-1} - b_{n-2} + 1, & n = 3 \end{cases}$$

$$\Downarrow$$

$$IVP1:\ y(t) = 0.25t^4 - 0.041667t^6 + 0.005208t^8 - 0.000521t^{10} - 0.005682t^{11} + \cdots$$

$$IVP2:\ y(t) = 1 + t + t^2 + 1.333333t^3 + 2.25t^4 + 3.283333t^5 + 5.238889t^6 + 8.364286t^7 + \cdots$$





Radius of Convergence:

$$\rho = \frac{1}{L} = \frac{1}{\lim\limits_{n\to\infty} sup \sqrt[n]{a_n}} = 1.4196 \qquad (IVP2)$$

$$\rho = \frac{1}{L} = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = 0.5829 \qquad (IVP2)$$

Adams-Bashforth-Moulton method

Introduction:
two main formula use first one to get the derivative and the Second one to get the value

$$y'_{k+1} = y_k + \int_{t_k}^{t_{k+1}} f(t,y) = y_k + \frac{h}{24}(-9f_{k-3} + 37f_{k-2} - 59f_{k-1} + 55f_k)$$

$$y_{k+1} = y_k + \int_{t_k}^{t_{k+1}} f(t,y) = y_k + \frac{h}{24}(f_{k-2} - 5f_{k-1} + 19f_k + 9f_{k+1})$$

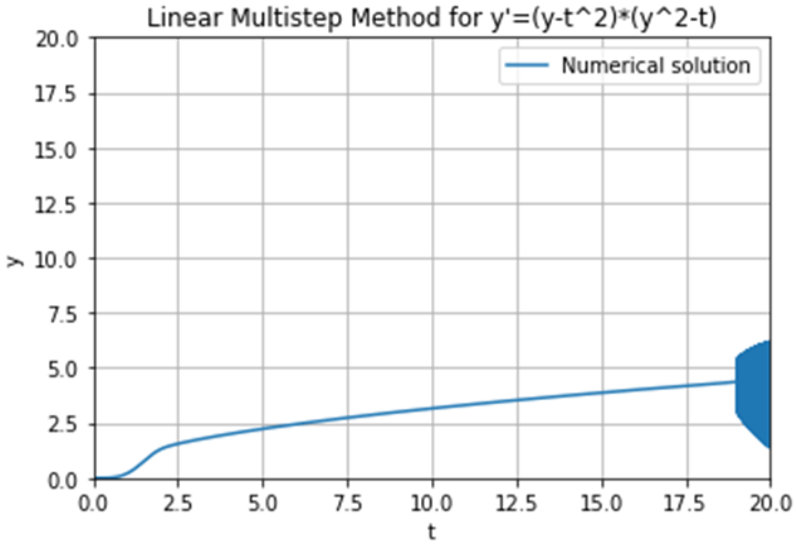
History solution: Four sample points
Use improved Euler

$$y\_predict(t+h) = y(t) + f(t, y(t)) * h$$

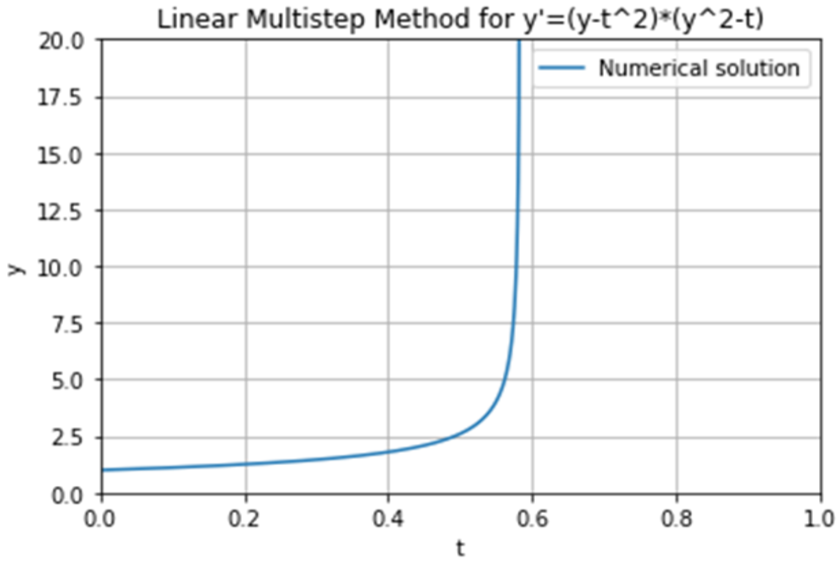
$$y(t+h) = y(t) + \frac{h}{2}(f(t, y(t)) + f(t+h, y\_predict(t+h))$$

Results

IVP1
Step:0.0001 times: 1000000



IVP2:
Step: 0.001 times:1000



Milne-Simpson Method

For Milne-Simpson method, the Milne formula is used as prediction formula and Simpson formula is used as correction formula. A fourth-order method as it is, it requires four initial values of y to start, while the main error term of both the predictor and the corrector contain h^5

Integrating $\frac{dy}{dt} = f(t, y)$ on (t_{k-3}, t_{k+1})

$y(t_{k+1}) - y(t_{k-3}) = \int_{t_{k-3}}^{t_{k+1}} f(t, y(t))dx$ ----> $\boldsymbol{y_{k+1} = y_{k-3} + \frac{4}{3}h(2f_k - f_{k-1} + 2f_{k-2})}$

<u>Milne formula</u>

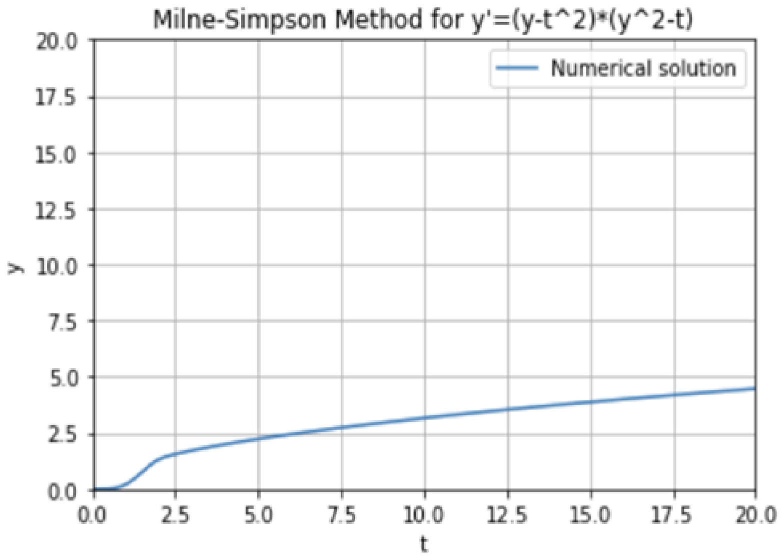
Integrating $\frac{dy}{dt} = f(t, y)$ on (t_{k-1}, t_{k+1})

$y(t_{k+1}) - y(t_{k-1}) = \int_{t_{k-1}}^{t_{k+1}} f(t, y(t))dx$ ----> $\boldsymbol{y_{k+1} = y_{k-1} + \frac{1}{3}h(f_{k+1} + 4f_k + f_{k-1})}$

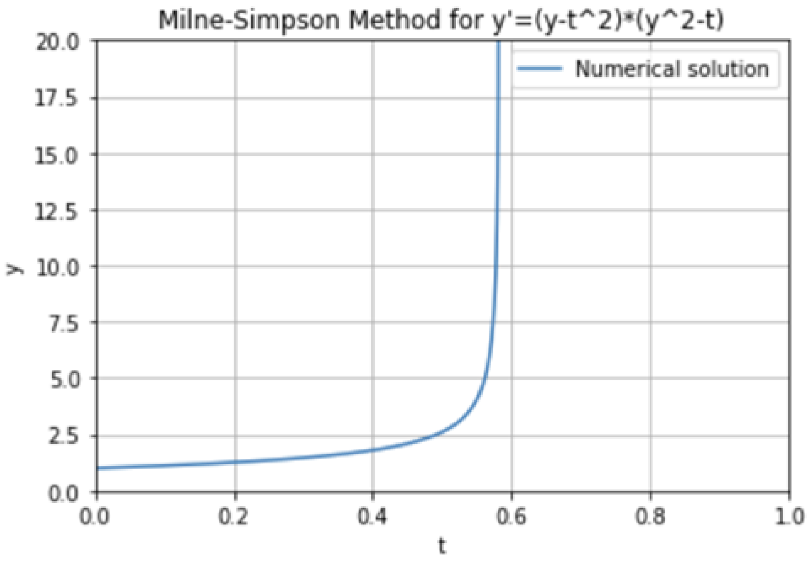
<u>Simpson formula</u>

Results

IVP1
Step:0.0001 times: 1000000



IVP2:
Step: 0.001 times:1000



第四部分

PART 04

Error Analysis

Error Analysis

(1) Euler Method

the Euler method comes from the Taylor approximation: $y(t+h) = y(t) + y'(t) * h \ + \frac{1}{2} * y''(c) * h^2$

where $c \in [t, t+h]$. For a small h, we can estimate $h^2 \approx h$, then the global error can be estimate

as $\frac{1}{2}\sum_{k=1}^{M} y^{(2)}(c_k)h^2 \approx \frac{1}{2}(b-a)y^{(2)}(c)h$, Thus the global error will be within **O(h).**

(2) Improved Euler Method

Since the Improved Euler Method is based on Euler method and estimated using the area of trapezoid, we also estimate the error by using the approximation of integral by area of trapezoid.

which is $-\sum_{k=1}^{M} y^{(2)} c_k \frac{h^3}{12} \approx \frac{b-a}{12} y^{(2)} c h^2$, Thus the global error will be within $\mathbf{O(h^2)}$.

(3) Runge-Kutta method

Due to the relation of order and highest accuracy by Butcher, it can be concluded that:

Order	2	3	4	5	6	7	$n \geq 8$
The highest accuracy	$0(h^2)$	$0(h^3)$	$0(h^4)$	$O(h^4)$	$0(h^5)$	$O(h^6)$	$O(h^{n-2})$

Thanks