Algorithms
Chapter 1 Preliminaries

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Outline

- **Mathematical Notions and Terminology** 數學符號與術語
- Definitions, Theorems, and Proofs
- Types of Proof 建構法,矛盾法,歸納法

Sets$_{1/3}$

- A **set** is a group of objects represented as a unit.
  將一群相同性質的事或物用一個群組表示，稱此群組為集合
- Sets may contain any type of object, including numbers, symbols, and even other sets. 集合中可以有集合
- The objects in a set are called its **elements** or **members**.
- One way to describe sets formally is by listing its elements inside braces. 用大括號將集合中的元素括起
  (表示集合的方式 ⇒ 列舉)
- Thus the set {7, 21, 57} contains the elements 7, 21, and 57.

Sets$_{2/3}$

屬於　不屬於

- The symbols $\in$ and $\notin$ denote set membership and non-membership, respectively.

- We write $7 \in \{7, 21, 57\}$ and $8 \notin \{7, 21, 57\}$.

子集

- For two sets A and B, we say that A is a **subset** of B, written $A \subseteq B$, if every member of A is also a member of B.

A是B的子集⇒每一個在A的element也一定在B裡 ($A \subseteq B$)

- We say that A is a **proper subset** of B, written $A \subsetneq B$, if A is a subset of B and not equal to B. 真子集

A是B的子集，但 $A \neq B$

- Let $A = \{7, 21\}$ and $B = \{7, 21, 57\}$. Then, we can write $A \subseteq B$ and $A \subsetneq B$.

- The set of **natural numbers** N is $\{1, 2, 3, \ldots\}$. 自然數的集合

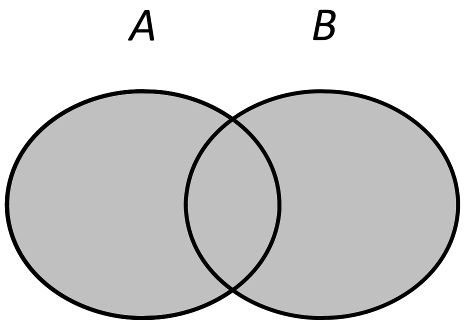
Sets$_{3/3}$

- The set with 0 members is called the **empty set** and is written ϕ.
  集合中沒有任何元素即為空集合（{ } = ϕ）且空集合為任何集合的子集
- A set containing elements according some rule is denoted by $\{n \mid \text{rule about } n\}$. （表示集合的方法 II）{n | 關於 n 的規則}
  - $\{n \mid n = m^2 \text{ for some } m \in N\}$ ➔ the set of perfect squares. 完全平方數
  Ex: $\{c \mid c \in N,\ 1 \le c \le 5\} = \{1, 2, 3, 4, 5\}$
- The **union** of A and B, written $A \cup B$, is the set we get by combining all the elements in A and B into a single set.
  聯集：將 A 和 B 合併形成一個新集合
- The **intersection** of A and B, written $A \cap B$, is the set of elements that are in both A and B.
  交集：取共同的元素
- The **complement** of A, written $\bar{A}$, is the set of all elements under consideration that are **not** in A.
  補集：所有不在 A 之中的元素所形成的集合

Venn diagram

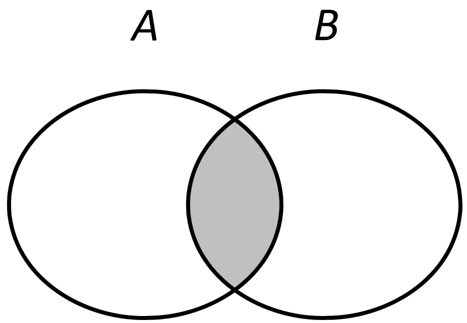
文氏圖

- The next two **Venn diagrams** depict the union and intersection of sets *A* and *B*.



(a) $A \cup B$

聯集



(b) $A \cap B$

交集

Sequences

序列：依某種順序將組成元素表列出來

- A **sequence** of objects is a list of these objects in some order.
  - We usually designate a sequence by writing the list within parentheses. 表列在小括號中
  - In a set the order doesn't matter, but in a sequence it does.
    集合中：順序不重要 序列：順序是最重要的
- For example:
  - The sequence 7, 21, 57 would be written (7, 21, 57).
  - Hence (7,21,57) is not the same as (57, 7, 21).
    (7,21,57)與(57,7,21)不相等
- Repetition does matter in a sequence, but it doesn't matter in a set. 重覆在序列中是有意義的，但在集合中無意義
  - Thus (7,7,21,57) is different from (7,21,57) and (57, 7, 21).
    不相同
  - The set {7, 21, 57} is identical to the set {7, 7, 21, 57}.
    相同

Tuples and power sets

元組：有限長度的序列通常稱為"元組"

- Finite sequences often are called **tuples**.
  - A sequence with *k* elements is a *k*-tuple. k-"元組"
  - Thus (7,21,57) is a 3-tuple.
  - A 2-tuple is also called a **pair**. 2-"元組"又稱為"對"
- The **power set** of *A* is the set of all subsets of *A*.
- For example: 冪集合：A的子集所形成的集合
  - If *A* is the set {O, I}, the power set of *A* is the set {ϕ, {O}, {I}, {O, I}}. 0個元素 1個元素 2個元素
  - The set of all pairs whose elements are Os and 1s is {(0, 0), (0, 1), (1, 0), (1, 1)}.

0和1所形成的pair (,)

可放0或1

Cartesian product

- The **Cartesian product** of A and B, written $A \times B$, is the set of all pairs wherein the first element is a member of A and the second element is a member of B.

  pair所形成的集合(放A的元素, 放B的元素) 一般指A×B

- For example:
  - If $A = \{1, 2\}$ and $B = \{x, y, z\}$, then $A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z)\}$.

- We can also take the Cartesian product of k sets, $A_1, A_2, \ldots, A_k$, written $A_1 \times A_2 \times \ldots \times A_k$. 2個以上為特殊情形

- For example:
  - If $A = \{1, 2\}$ and $B = \{x, y, z\}$, then $A \times B \times A = \{(1, x, 1), (1, x, 2), (1, y, 1), (1, y, 2), (1, z, 1), (1, z, 2), (2, x, 1), (2, x, 2), (2, y, 1), (2, y, 2), (2, z, 1), (2, z, 2)\}$.

Outline

- Mathematical Notions and Terminology
- **Definitions, Theorems, and Proofs**
- Types of Proof

Definitions, theorems, and proofs

- Theorems and proofs are the heart and soul of mathematics, and definitions are its spirits.
  定理和證明是數學的靈魂，定義是數學的精神
- These three entities are central to every mathematical subjects, including algorithms.
  對於任何數學科目都很重要，包括演算法
- In this class, you are expected to pick up the ability of reading and writing a concrete proof.

  目標：(1) 讀證明
  (2) 寫證明

Definitions 定義是用來描述事或物，以及要使用的符號

- **Definitions** describe the objects and notions that we use.
  - Precision is essential to any mathematical definition. 精確是最重要的
  - When defining some object we must make clear what constitutes that object and what does not.

  描述事或物時，要清楚說明包含與不包含的東西
- For example:
  - A **set** is a group of objects represented as a unit.
  - The objects in a set are called its **elements** or **members**.
  - A **tree** graph is a connected graph without cycles.

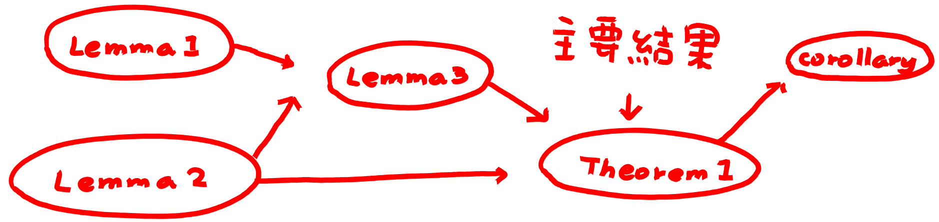
Proofs 在定義事或物以及符號後，我們會以"數學敘述"來描述一些性質

- After we have defined various objects and notions, we usually make **mathematical statements** about them.
  - A statement expresses that some object has a certain property.
  - The statement mayor may not be true. 可能錯也可能正確
  - The statements must be precise without any ambiguity.
    一定要精確，不可模稜兩可
- A **proof** is a convincing logical argument that a statement is true. 有說服力的邏輯論證，來說明 statement 是對的
  - In mathematics an argument must be airtight, that is, convincing in an absolute sense. 論證是無懈可擊的
  - A mathematician demands proof beyond any doubt.
    禁得起任何質疑

Theorems, lemmas, and corollaries

- A **theorem** is a mathematical statement proved true. Generally we reserve the word for statements of special interest. 定理：是證明真實的一個數學敘述
- **Lemmas** are the proved statements that are interesting only for their assistance in the proof of another statement.
  引理：用來輔助定理
- Occasionally a theorem or its proof may allow us to conclude easily that other, related statements are true. These statements are called **corollaries** of the theorem.

corollary 推論：由主要性質容易推導的性質
(結果)



Finding Proofs 證明是唯一可以決定statement是否正確的方法

- The only way to determine the truth or falsity of a mathematical statement is with a mathematical proof.
- Unfortunately, finding proofs isn't always easy.
  - It can't be reduced to a simple set of rules or processes.
- However, some helpful general strategies are available.
  - Carefully read the statement you want to prove. 敘述要了解
  - Make sure you understand all the notation. 符號要懂得
  - Rewrite the statement in your own words. 以自己的話重寫敘述
  - Break it down and consider each part separately. 將步驟分步去思考
  - Experimenting with examples. 舉出例子
  - Try to find an object that fails to have the property, called a **counterexample**. 去尋找反例，若無反例則可確定為是

Multipart statements$_{1/2}$

多方敘述 Ex：$A \Leftrightarrow B$

- One frequently occurring type of **multipart statement** has the form "*P* if and only if *Q*", often written "*P* iff *Q*".
  - Both *P* and *Q* are mathematical statements.
  - The first part is "*P* only if *Q*," which means: If *P* is true, then *Q* is true, written $P \Rightarrow Q$. 若P則Q，若P為真則Q成立
  - The second is "*P* if *Q*," which means: If *Q* is true, then *P* is true, written $P \Leftarrow Q$. 若Q則P，若Q為真則P成立
  - We write "*P* if and only if Q" as $P \Leftrightarrow Q$.
    若且唯若
  - To prove a statement of this form you must prove each of the two directions.

Multipart statements$_{2/2}$

- Another type of multipart statement states that two sets *A* and *B* are equal. 若A.B集合相等
  - The first part states that *A* is a subset of *B*. 先證A為B的子集 ($A \subseteq B$)
  - The second part states that *B* is a subset of *A*. 再證B為A的子集 ($B \subseteq A$)
- To show that *A* = *B*, we must prove the following two statements.
  - Every member of *A* also is a member of *B*.
  - Every member of *B* also is a member of *A*.

  每一個在A的element也一定在B裡 ($A \subseteq B$)

  每一個在B的element也一定在A裡 ($B \subseteq A$)

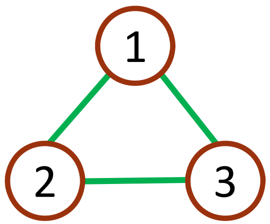
An example for finding Proofs$_{1/2}$

- Suppose that you want to prove the statement

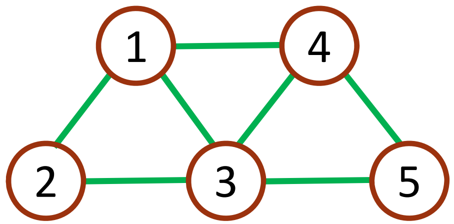
  "**For every graph *G*, the sum of the degrees of all the nodes in *G* is an even number**".

  證明degree和為偶數(degree為與此點相鄰的點個數)

- **First**, pick a few graphs and observe this statement.



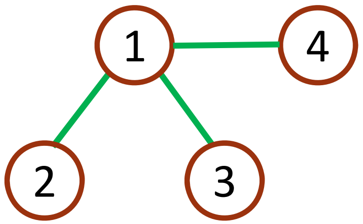
sum = 2 + 2 + 2
= 6



sum = 3 + 2 + 4 + 3 + 2
= 14

An example for finding Proofs$_{2/2}$

- **Next**, try to find a counterexample, that is, a graph in which the sum is an odd number. 尋找反例



Every time an edge is added,
the sum increases by 2.
每增加一條邊，degree總和多2

  - Can you now begin to see why the statement is true and how to prove it?

- If you are still stuck trying to prove the statement, try to prove a **special case** of the statement. 若仍無法證明，則找尋特例
  - First try for $k = 1$, as well as $k = 2$, $k = 3$, $k = 4$ and so on.

Writing proofs

- A well-written proof is a sequence of statements, wherein each one follows by simple reasoning from previous statements in the sequence.

一個好的證明：為一連串的敘述，且下一個敘述可由上一個敘述簡單接得

$$\text{statement1} \xrightarrow{\text{simple reasoning}} \text{statement2} \xrightarrow{\text{simple reasoning}} \text{statement3}$$

- Carefully writing a proof is important. 寫出證明的好處
  - Enable a reader to understand it. 使讀者易於理解
  - Make sure that it is free from errors. 避免出錯

Tips for producing a proof

- Be patient. 有耐心
  - Researchers sometimes work for weeks or even years to find a single proof.
- Come back to it.
  - Look over the statement you want to prove, think about it a bit, leave it, and then return a few minutes or hours later.
- Be concise. 小心且精確，清楚簡要的表達出來
  - Good mathematical notation is useful for expressing ideas concisely.
  - But be sure to include enough of your reasoning when writing up a proof so that the reader can easily understand what you are trying to say.

證明degree和為偶數（degree為與此點相鄰的點個數）

An example for producing a proof

- Theorem : **For every graph *G*, the sum of the degrees of all the nodes in *G* is an even number.**

- Proof.

(I) - Every edge in *G* is connected to two nodes. 每個edge需連接2個點

(II) - Each edge contributes 1 to the degree of each node to which it is connected. 連接到這條邊的點，degree增加1

(III) - Therefore each edge contributes 2 to the sum of the degrees of all the nodes. 一個edge會使degree增加2（由I、II得知）

(IV) - Hence, if *G* contains *e* edges, then the sum of the degrees of all the nodes of *G* is 2*e*, which is an even number. 若有e條edge ⇒ degree和為2e

Outline

- Mathematical Notions and Terminology
- Definitions, Theorems, and Proofs
- **Types of Proof**

Types of proof

- Several types of arguments arise frequently in mathematical proofs.
  - Proof by construction. 建構法
  - Proof by contradiction. 反證法
  - Proof by induction. 歸納法
- Note that a proof may contain more than one type of argument. 證明中可能會有好幾種方法,因為會包含許多子證明
  - Because the proof may contain within it several different subproofs.

Proof by construction$_{1/2}$

- A graph is said to be ***k*-regular** if every node in the graph has degree k. 每個 node 的 degree 都是 k

- Theorem : **For each even number n greater than 2, there exists a 3-regular graph $G(V, E)$ with n nodes.**

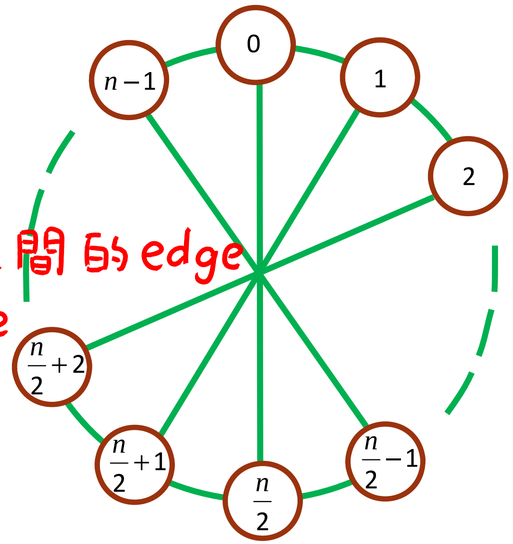
大於 2 個 node 的所有偶數個 node，必有一種畫法使得每個的 degree 為 3

- Proof.
  - Let n be an even number greater than 2.
  - The set $V = \{0, 1, \ldots, n-1\}$. 0 ~ (n-1) 個桌
  - The set $E = \{(i, i+1) \mid \text{for } 0 \le i \le n-2\}$ 0 ~ (n-1) 之間的 edge

    $\cup \{(n-1, 0)\}$ (n-1) ~ 0 之間的 edge

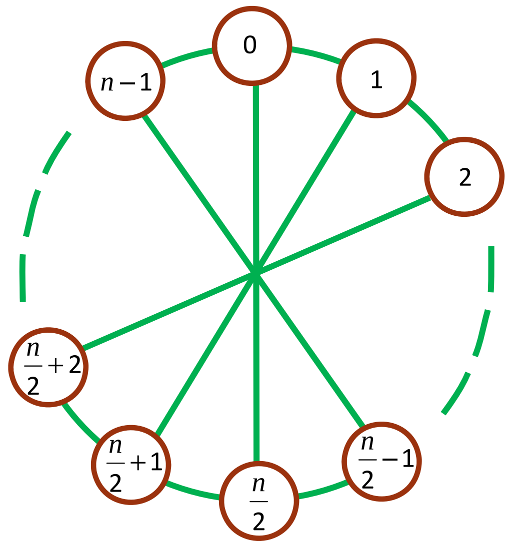
    $\cup \{(i, i+n/2) \mid \text{for } 0 \le i \le n/2-1\}$.

    i 與 (i + n/2) 之間的 edge



Proof by construction$_{2/2}$

- The set $E = \{(i, i + \mathrm{I}) \mid \text{for } 0 \le i \le n - 2\}$
  $\cup \{(n - 1, \mathrm{O})\}$
  $\cup \{(i, i + n/2) \mid \text{for } 0 \le i \le n/2 - \mathrm{I}\}$.
- The edges described in the top and middle lines of E go between adjacent pairs around the circle.
- The edges described in the bottom line of E go between nodes on opposite sides of the circle.
- This mental picture clearly shows that every node in G has degree 3.



Proof by contradiction 反證法

(I) - In this kind of arguments, we first **assume** that the theorem to prove is **false**. 首先假設是錯的

(II) - Then we show that this assumption leads to an obviously false consequence, called a **contradiction**. 最後發現一個矛盾的結果(與事實相反)

- We use this type of reasoning frequently in everyday life.
  - Jack just came in from outdoors and he is completely dry.
  - We want to prove that It's not raining.
  - If it were raining, Jack would be wet.

    (assume false) **(contradiction)**
  - Therefore, it must not be raining.

An example for proof by contradiction

- A number is **rational** if it is a fraction m/n where m and n are integers. 有理數可表示成 $\frac{m}{n}$
- Theorem: $\sqrt{2}$ is **irrational**. 證明 $\sqrt{2}$ 為無理數
- Proof.
  - Assume for a contradiction that $\sqrt{2} = \dfrac{m}{n}$ is rational, where m and n are relatively prime integers.
    互質
  - $\sqrt{2} = \dfrac{m}{n} \quad \Rightarrow \quad n\sqrt{2} = m \quad \Rightarrow \quad 2n^2 = m^2.$
  - Since m^2 is even, m must be even as well. m^2 為偶數 $\Rightarrow$ m 必為偶數
  - Let $m = 2k$. Then we have $2n^2 = (2k)^2 = 4k^2$, which implies $n^2 = 2k^2$.
  - Since n^2 is even, n is even.
  - Both m and n are even, a **contradiction** to our assumption.
    與假設矛盾

Proof by induction 歸納法

- An advanced method to show that **all elements of an infinite set** have a specified property.
  證明無限集合中的元素都有某種特殊性質的方法
- Suppose that our goal is to prove that $P(k)$ is true for each natural number $k \in \{1, 2, 3, \ldots\}$.
- The format for writing down a proof by induction is as follows.
  - **Basis:** Prove that $P(1)$ is true.
  - **Induction step:** For each $i \geq 1$, assume that $P(i)$ is true and use this assumption to show that $P(i+1)$ is true. 假設p(i)為是⇒p(i+1)也為是(i≥1)
- $P(1)$ is true: basis.
- $P(2)$ is true: $P(1)$ is true + induction step.
- $P(3)$ is true: $P(2)$ is true + induction step.

… P(k) is true

An example for proof by induction

- For each $n \geq 1$, we have

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n\,(n+1)\,(2n+1)}{6}.$$

- Proof.
  - **The basis**: For $n = 1$, $1^2 = 1 = \dfrac{1\,(1+1)\,(2\times 1+1)}{6}.$
  - **Induction step**: For each $k \geq 1$, assume that the formula is true for $n = k$ and show that it is true for $n = k + 1$.

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k\,(k+1)\,(2k+1)}{6} + (k+1)^2 = \frac{k\,(k+1)\,(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)\,(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)\,(2(k+1)+1)}{6}$$