$$\begin{array}{r} 393 \\ \times\ 252 \\ \hline \end{array}$$

3×2 出現4次

第一次就將3×2的答案記下來

Algorithms
Chapter 15
Dynamic Programming

動態規劃法 = 填表法 = 一種有技巧的暴力法

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Outline

- **Rod cutting**
- Matrix-chain multiplication
- Elements of dynamic programming 使用動態規劃法的要素
- Longest common subsequence
- Optimal binary search trees

台北到高雄的最短距是350公里

Dynamic Programming$_{1/2}$

路徑(一)：台北→台中→台南→高雄
路徑(二)：台北→花蓮→屏東→高雄

- Not a specific algorithm, but a technique, like divide-and-conquer. 像divide and conquer，非演算法，而是一種解決問題的技巧
- Dynamic programming is applicable when the subproblems are not independent. 適用於子問題重覆出現的時候
- A dynamic-programming algorithm solves every subsubproblem just once and then saves its answer in a table.
  对相同子問題，只解決一次，且將答案放入表格中
- "Programming" in this context refers to a tabular method, not to writing computer code. Programming指的是填表法
- Used for **optimization problems**: 用來解決最佳化問題
  - Find **a** solution with **the** optimal value.
  - Minimization or maximization. 可能是最大化(利益)或最小化(成本)

最佳解可能不止一個，但最佳解的值只有一個

Dynamic Programming$_{2/2}$

- **Four-step method**

1. Characterize the structure of an optimal solution.
   問題的最佳解也包含子問題的最佳解
2. Recursively define the value of an optimal solution.
   用子問題的答案定義最佳解
3. Compute the value of an optimal solution in a bottom-up fashion.
   先算子問題，再算原問題
4. Construct an optimal solution from computed information.
   用子問題的答案產生最佳解

Rod cutting$_{1/2}$

- How to cut steel rods into pieces in order to maximize the revenue you can get ?
  - Each cut is free.
  - Rod lengths are always an integral number of inches.長度為整數
- **The rod-cutting problem problem**
  - **Input:** A length n and table of prices p_i, for $i = 1, 2, \ldots, n$.
  - **Output:** The maximum revenue obtainable for rods whose lengths sum to n.

也可以都不切

- If p_n is large enough, an optimal solution might require no cuts.
- We can cut up a rod of length n in 2^{n-1} different ways.
  - can choose to cut or not cut after each of the first $n - 1$ inches.

n-1個切點:每一個都可以選擇切或不切

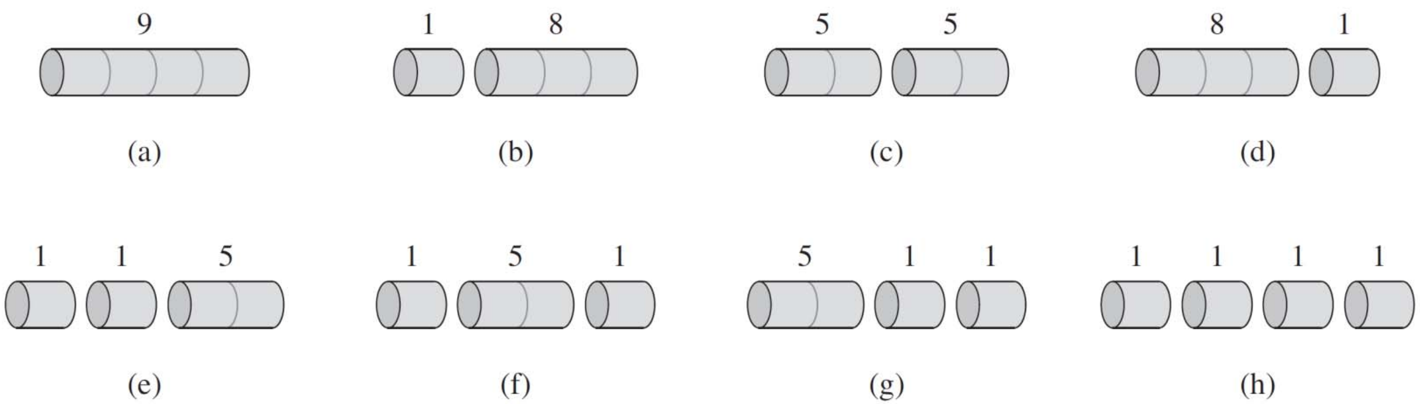
Rod cutting$_{2/2}$

- Consider the case when n = 4.

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

  - Here are all 8 ways to cut a rod of length 4.
  - The optimal strategy is part (c)—cutting the rod into two pieces of length 2—which has total value 10.

最好的切法



假設台北到高雄最短距離經过台中

台北到高雄最短距 = 台北到台中最短距 + 台中到高雄最短距

Structure of an optimal solution

- Let r_i be the maximum revenue for a rod of length i.

r_i：長度為 i 的最高價錢

- **Step 1:** Characterize the structure of an optimal solution.
  - Suppose a cut is made at distance j inches in an optimal solution of size n. 假設一個最好的切法在長度為 j 的地方切了一刀
  - The optimal revenue $r_n = r_j + r_{n-j}$.
  - An optimal solution to a problem contains within it an optimal solution to subproblems. 問題的最佳解包含子問題最佳解
  - This is **optimal substructure**.

$r_n = r_j + r_{n-j}$

長度為 n 的最高價錢 = 長度為 j 的最高價錢 + 長度為 $n-j$ 的最高價錢

將問題的size變小：用子問題的答案定義最佳解

Recursive solution

- **Step 2:** Recursively define the value of an optimal solution.

最好的切法可能是

- Can determine optimal revenue r_n by taking the maximum of
  - p_n: the price we get by not making a cut, 不用切
  - $r_1 + r_{n-1}$: the maximum revenue from a rod of 1 inch and a rod of n − 1 inches, 切在距離 1
  - $r_2 + r_{n-2}$: the maximum revenue from a rod of 2 inch and a rod of n − 2 inches,… 切在距離 2
  - $r_{n-1} + r_1$.
- More generally, $r_n = \max(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1)$.

所有可能性中取最大：共有n種

換一種看法

A simpler way to decompose the problem

- Every optimal solution has a leftmost cut. 每個最佳解都有最左的一刀
  - A first piece of length *i* cut off the left-hand end, and a remaining piece of length *n* − *i* on the right. 假設最左的那一刀切在距離 i
  - Need to divide only the remainder, not the first piece.
  - Leaves only one subproblem to solve, rather than two subproblems.
  - $r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$.

左邊不用再切，只需切右邊

```
CUT-ROD(p, n)
    if n == 0
        return 0
    q = −∞
    for i = 1 to n
        q = max(q, p[i] + CUT-ROD(p, n−i))
    return q
```

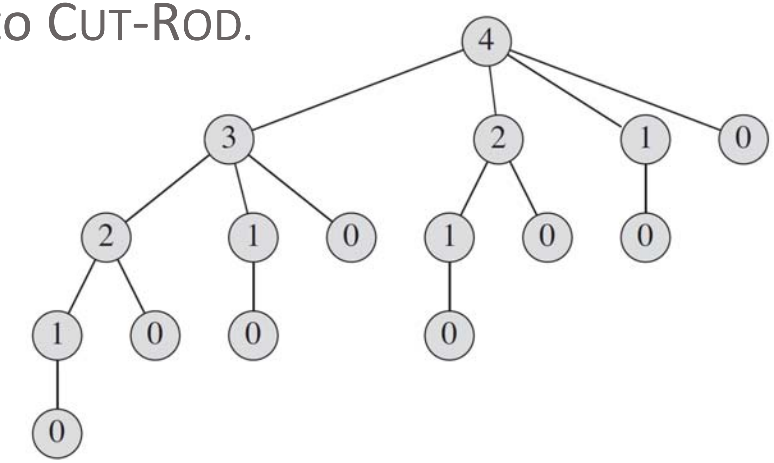
→左邊的長度由1~n

Recursive top-down implementation

由上往下遞迴

Running time of CUT-ROD(p, n)

- For n = 40, the program could take more than an hour.
  n=40時，电脑要跑至少1小時
- Each time you increase n by 1, the program's running time would approximately double. 每增加1，要多2倍
- Why is CUT-ROD so inefficient? 沒效率的原因：重覆算子問題
  - It solves the same subproblems repeatedly.
- Running time: n=4時，右边可能為 3，2，1，0
  - $T(n)$: total number of calls made to CUT-ROD.
  - $T(n) = \begin{cases} 1 & \text{if } n = 0, \\ \underbrace{1}_{\text{自己}} + \underbrace{\sum_{j=0}^{n-1} T(j)}_{\text{右边子問題}} & \text{if } n > 1. \end{cases}$
  - $T(n) = 2^n$. (exercise 15.1-1)



Dynamic programming 使用動態規劃法

- Using dynamic programming for optimal rod cutting
  - Instead of solving the same subproblems repeatedly, arrange to solve each sub-problem just once. 每個子問題只算一次
  - Save the solution to a subproblem in a table, and refer back to the table whenever we revisit the subproblem. 遇到相同子問題就查表
  - "Store, don't recompute" ➔ time-memory trade-off. 計算與記憶体空間不可兼得
  - Can turn an exponential-time solution into a polynomial-time solution.
- Two basic approaches: **top-down with memoization**, and **bottom-up method**.

2種方法 { top-down with memoization
bottom-up method

Top-down with memoization

將答案存在表中

- Save the result of each subproblem in an array or hash table.
- The procedure first checks whether it has previously solved this subproblem. 計算前先檢查是否有算过
  - Yes : return the saved value.
  - No : compute the value in the usual manner.

有：回傳答案
沒有：進行計算

- **Memoizing** is remembering what we have computed previously.
  - Storing the solution of length *i* in array entry *r*[*i*].

```
MEMOIZED-CUT-ROD-AUX(p,n, r)
    if r[n] ≥ 0
        return r[n]          有：回傳答案
    if n == 0
        q = 0                沒有：進行計算
    else q = −∞
        for i = 1 to n
            q = max(q, p[i] + MEMOIZED-CUT-ROD-AUX(p,n − i, r))
    r[n] = q
    return q
```

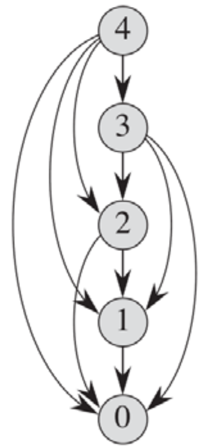
```
MEMOIZED-CUT-ROD(p,n)
    let r[0..n] be a new array
    for i = 0 to n
        r[n] = −∞
    return MEMOIZED-CUT-ROD-AUX(p,n, r)
```

Bottom-up method

- **Step 3:** Compute the value of an optimal solution in a bottom-up fashion. 先算子問題，再算原問題
- The procedure solves subproblems of sizes j = 0, 1,..., n, in that order. 先算 size = 1，再算 size = 2，…
- When solving a subproblem, have already solved the smaller subproblems we need. 先解決子問題：不需遞迴

```
BOTTOM-UP-CUT-ROD(p, n)
    let r[0..n] be a new array
    r[0] = 0
    for j = 1 to n
        q = −∞
        for i = 1 to j
            q = max(q, p[i] + r[j − i])
        r[j] = q
    return r[n]
```

j = size
i = 左邊的長度
j 種可能性



The subproblem graph.

Running time

- Both the top-down and bottom-up versions run in $\Theta(n^2)$ time.

size = 1 ⇒ 1 種可能性 size = 2 ⇒ 2 種可能性

- **Bottom-up**
  - Doubly nested loops 兩層 for loop $= \underbrace{1}_{size=0} + 1 + 2 + 3 + \dots + n = \Theta(n^2)$
  - Number of iterations of inner for loop forms an arithmetic series.

    等差級數
- **Top-down**
  - MEMOIZED-CUT-ROD solves each subproblem just once.
  - It solves subproblems for sizes 0, 1,…,n.
  - To solve a subproblem of size n, the for loop iterates n times.
  - Total number of iterations also forms an arithmetic series.

Memoized: ① 每一個子問題都只有算一次

② Size 為 n 的子問題有 n 種可能性

③ $\underbrace{1}_{size=0} + 1 + 2 + 3 + \dots + n = \Theta(n^2)$

Reconstructing a solution$_{1/2}$

- **Step 4:** Construct an optimal solution from computed information. 用子問題的答案產生最佳解
- Saves the first cut made in an optimal solution for a problem of size *i* in *s*[*i*]. 將size為i的最左那一刀記在S[i]

```
EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
    let r[0..n]  and s[0..n] be new arrays
    r[0] = 0
    for j = 1 to n
        q = −∞
        for i = 1 to j        j種可能性
            if q < p[i] + r[j − i]        如果價錢更高
                q = p[i] + r[j − i]           ①更新價錢
                s[j] = i                      ②記住最左那一刀
        r[j] = q        記住最好價錢
    return r and s
```

Reconstructing a solution$_{2/2}$

- To print out the cuts made in an optimal solution.

```
PRINT-CUT-ROD-SOLUTION (p, n)
    (r, s) = EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
    while n > 0
        print s[n]  →最左那一刀
        n = n − s[n]→右边子問題
```

- The call EXTENDED-BOTTOM-UP-CUT-ROD(p, n) return

i	0	1	2	3	4	5	6	7	8	9	10
$r[i]$	0	1	5	8	10	13	17	18	22	25	30
$s[i]$	0	1	2	3	2	2	6	1	2	3	10

- A call to PRINT-CUT-ROD-SOLUTION(p, 10) would print just 10.
- A call with $n = 7$ would print the cuts 1 and 6.

Outline

- Rod cutting
- **Matrix-chain multiplication**
- Elements of dynamic programming
- Longest common subsequence
- Optimal binary search trees

$[A]_{p\times q} \times [B]_{q\times r} = [C]_{p\times r}$

Matrix-chain multiplication

- When we multiply two matrices A and B, if A is a $p \times q$ matrix and B is a $q \times r$ matrix, the resulting matrix C is a $p \times r$ matrix.
  - The number of scalar multiplications is ***pqr***.

要算出C要算出pqr次，大小為pr

$[A_i]_{p_{i-1} \times p_i}$

- **Matrix-chain multiplication problem**

一連串的矩陣

$$\begin{array}{cc} & p_i \\ p_{i-1} & \boxed{A_i} \end{array}$$

  - **Input:** A chain $\langle A_1, A_2, \ldots, A_n \rangle$ of n matrices.
    (matrix A_i has dimension $p_{i-1} \times p_i$)
  - **Output:** A fully parenthesized product $A_1, A_2, \ldots, A_n$ that minimizes the number of scalar multiplications. 括號⇒乘法順序

將 $A_1 \ldots A_n$ 完全括弧，使得用到的純量乘法次數最少

- For example: The dimensions of the matrices A_1, A_2, and A_3 are 10×100, 100×5, and 5×50, respectively.
  - $((A_1A_2)A_3) = 10 \cdot 100 \cdot 5 + 10 \cdot 5 \cdot 50 = 7500.$
  - $(A_1(A_2A_3)) = 100 \cdot 5 \cdot 50 + 10 \cdot 100 \cdot 50 = 75000.$

$[A_2, A_3]_{100\times 50}$ $\qquad$ $[A_1]_{10\times 100} \cdot [A_2A_3]_{100\times 50} = [A_1A_2A_3]_{10\times 50}$

Counting the number of parenthesizations

- **Brute-force algorithm:** 暴力法：將所有可能性都試過
  - Checking all possible parenthesizations
- Time: $\Omega(2^n)$. (Exercise 15.2-3)

  $p(k)\ \boxed{A_1 \ldots A_k}\ \boxed{A_{k+1} \ldots A_n}\ p(n-k)$

  $\Rightarrow 1 \le k \le n-1$
  - Denote the number of alternative parenthesizations of a sequence of n matrices by $P(n)$.
  - A fully parenthesized matrix product is the product of two fully parenthesized matrix subproducts.
  - The split between the two subproducts may occur between the kth and $(k + 1)$st matrices.

    $p(n) = n$ 個矩陣完成括弧的可能數

    (純量乘法可能數)
  - Thus, we have $P(n) = \begin{cases} 1 & \text{if } n = 1, \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2. \end{cases}$

    前 k 個的可能數　　後 n-k 個的可能數

Step 1: The structure of an optimal solution

- An optimal solution to an instance contains optimal solutions to subproblem instances. 問題的最佳解包含子問題的最佳解
- For example:
  - If $((A_1A_2)A_3)(A_4(A_5A_6))$ is an optimal solution to $A_1, A_2, \ldots, A_6$.
  - Then, $((A_1A_2)A_3)$ is an optimal solution to A_1, A_2, A_3 and $(A_4(A_5A_6))$ is an optimal solution to A_4, A_5, A_6.

若最佳解為 $((A_1A_2)A_3)(A_4(A_5A_6))$

$\Rightarrow$ $((A_1A_2)A_3)$ 為 $A_1 \sim A_3$ 的最佳解

$(A_4(A_5A_6))$ 為 $A_4 \sim A_6$ 的最佳解

$$p_{i-1}\boxed{A_i}^{\,p_i}\ \boxed{A_{i+1}}^{\,p_{i+1}}\cdots\boxed{A_j}^{\,p_j} = p_{i-1}\boxed{A}^{\,p_j}$$

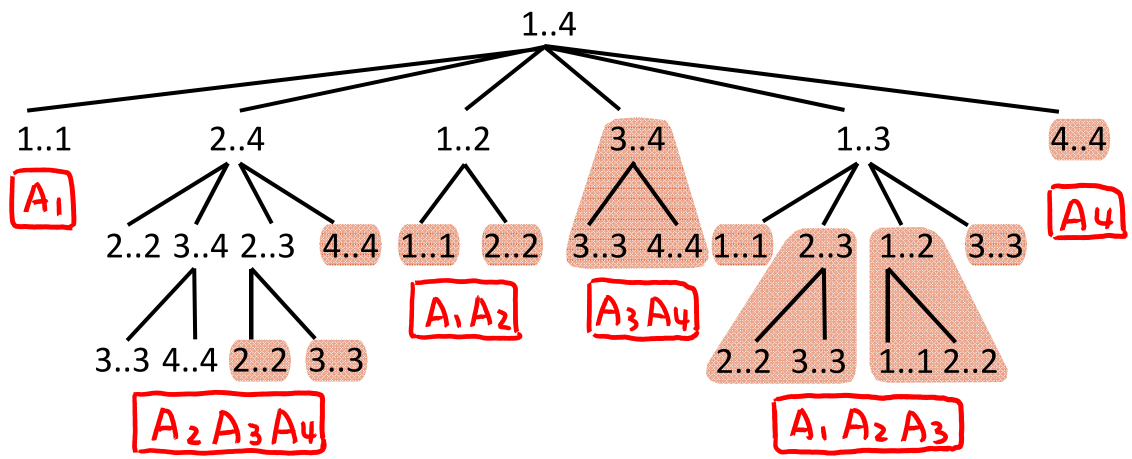
Step 2: A recursive solution

- Define $m[i, j]$ = the minimum number of scalar multiplications needed to compute $A_iA_{i+1}\ldots A_j$. m[i,j]= $A_i \sim A_j$ 最少的純量相乘數

$$m[i,j]=\begin{cases}0 & \text{if } i=j,\\ \min\limits_{i\le k<j}(m[i,k]+m[k+1,j]+p_{i-1}p_kp_j) & \text{if } i<j.\end{cases}$$

左邊最小乘法數+右邊最小乘法數+相乘的乘法數

- The recursion tree for the computation of $m[1,4]$.



*程式的流程是DFS(深度優先): 1..1→2..4→2..2→3..4→3..3→4..4→2..3→2..2→3..3→4..4

Step 3: Computing the optimal costs

- Based on the recursive formula, we could easily write an exponential-time recursive algorithm to compute the compute the minimum cost $m[1, n]$ for multiplying $A_1A_2 \ldots A_n$.

i,j不同 i,j相同

- There are only $\binom{n}{2} + n = \Theta(n^2)$ distinct subproblems, one problem for each choice of i and j satisfying $1 \leq i \leq j \leq n$.

- We can use dynamic programming to compute the solutions bottom up. 使用動態規劃法,由下往上,先算子問題,再算原問題

暴力法花指數時間
但子問題的個數只有 n^2 個

$\boxed{A_1}\boxed{A_2A_3} = 0 + 2625 + 30\times 35\times 5 = 7875,\ k=1$

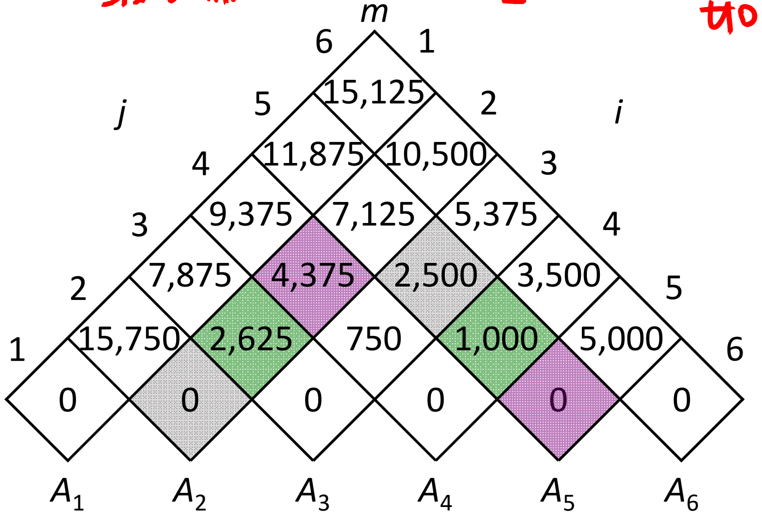
$\boxed{A_1A_2}\boxed{A_3} = 15750 + 0 + 30\times 15\times 5 = 18000,\ k=2$

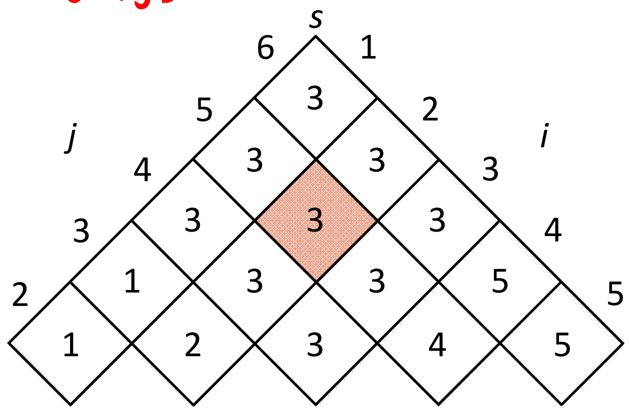
Dependencies between the subproblems

最少需要幾個純量乘法

如何算 m[i,j] 最好的 k (使純量乘法最少)

由下往上填表





matrix	A_1	A_2	A_3	A_4	A_5	A_6
dimension	30×35	35×15	15×5	5×10	10×20	20×25

$P_0 = 30,\ P_1 = 35,\ P_2 = 15,\ P_3 = 5,\ P_4 = 10,\ P_5 = 20,\ P_6 = 25$

- $s[i, j]$: index k achieved the optimal cost in computing $m[i, j]$.

$$m[2,5] = \min\begin{cases} m[2,2] + m[3,5] + p_1 p_2 p_5 = 0 + 2500 + 35\cdot 15\cdot 20 = 13000, & \boxed{A_2}\ \boxed{A_3A_4A_5} \\ m[2,3] + m[4,5] + p_1 p_3 p_5 = 2625 + 1000 + 35\cdot 5\cdot 20 = 7125, & \boxed{A_2A_3}\ \boxed{A_4A_5} \\ m[2,4] + m[5,5] + p_1 p_4 p_5 = 4375 + 0 + 35\cdot 10\cdot 20 = 11375. & \boxed{A_2A_3A_4}\ \boxed{A_5} \end{cases}$$

*先算長度=1，再算長度=2 … 再算長度=n

MATRIX-CHAIN-ORDER pseudocode

```
MATRIX-CHAIN-ORDER(p)
    n ← length[p] − 1
    for i ← 1 to n
        m[i, i] ← 0
    for ℓ ← 2 to n      /* ℓ is the chain length*/
        for i ← 1 to n − ℓ + 1
            j ← i + ℓ − 1
            m[i, j] ← ∞
            for k ← i to j − 1
                q ← m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
                if q < m[i, j]
                    m[i, j] ← q
                    s[i, j] ← k
    return m and s
```

長度為1

長度為ℓ的起頭和結尾

i：起頭

j：結尾

長度為2~n

算 m[i,j]

k有j-i種

- The loops are nested three deep, and each loop index (ℓ, i, and k) takes on at most $n - 1$ values.
- Time: $O(n^3)$.

表的大小 $O(n^2)$，算每一個要 $O(n)$，
因為每一格有k種且 $k < n$
$\Rightarrow O(n^2) \cdot O(n) = O(n^3)$

用已知資訊建構最佳解

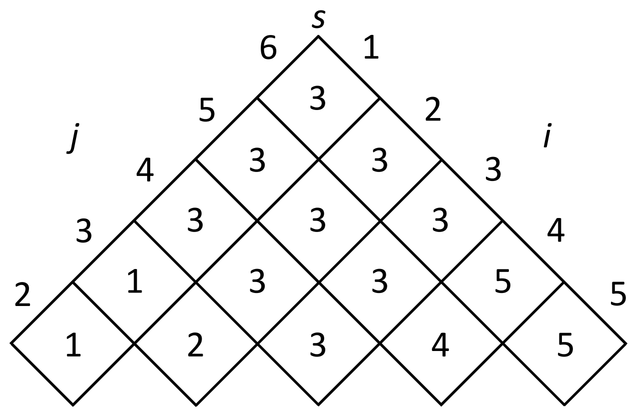
Step 4: Constructing an optimal solution

- Each entry $s[i, j]$ records the value of k such that the optimal parenthesization of $A_iA_{i+1}\cdots A_j$ splits the product between A_k and A_{k+1}.

(左 右)

```
PRINT-OPTIMAL-PARENS(s, i, j)
    if i = j
        print "A_i"
    else print "("
        PRINT-OPTIMAL-PARENS(s, i, s[i, j])
        PRINT-OPTIMAL-PARENS(s, s[i, j]+1, j)
        print ")"
```

若i=j時,直接印



- The call PRINT-OPTIMAL-PARENS(s, 1, n) prints the parenthesization $((A_1(A_2A_3))\,((A_4A_5)A_6))$.

(左 右)

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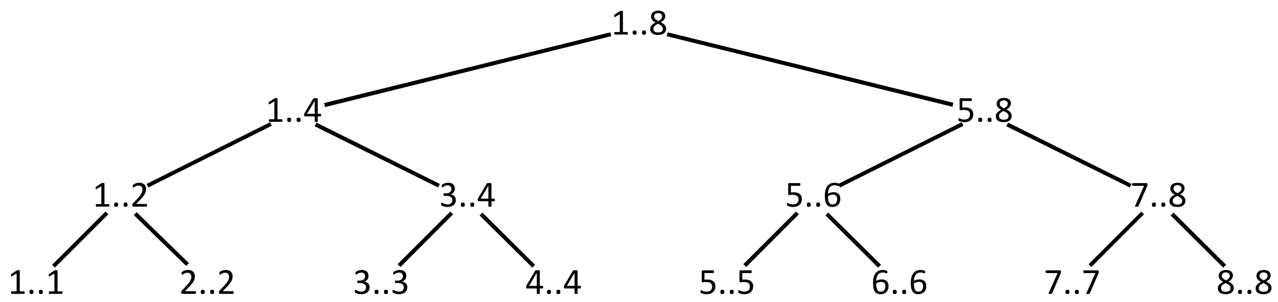
Elements of dynamic programming$_{1/2}$

- **Optimal substructure** 問題的最佳解也包含子問題的最佳解
  - An optimal solution to a problem contains an optimal solution to subproblems.
    - If $((A_1A_2)A_3)(A_4(A_5A_6))$ is an optimal solution to $A_1, A_2, \ldots, A_6$, then $((A_1A_2)A_3)$ is an optimal solution to A_1, A_2, A_3 and $(A_4(A_5A_6))$ is an optimal solution to A_4, A_5, A_6.
- **Overlapping subproblems** 重覆子問題
  - A recursive algorithm revisits the same problem over and over again. 遞迴演算法一再遇到相同問題
  - Typically, the total number of distinct subproblems is a polynomial in the input size. 但子問題的個數只有n的多項式個
  - In contrast, a problem for which a divide-and-conquer approach is suitable usually generates brand-new problems at each step of the recursion. matrix-chain 的子問題会重覆

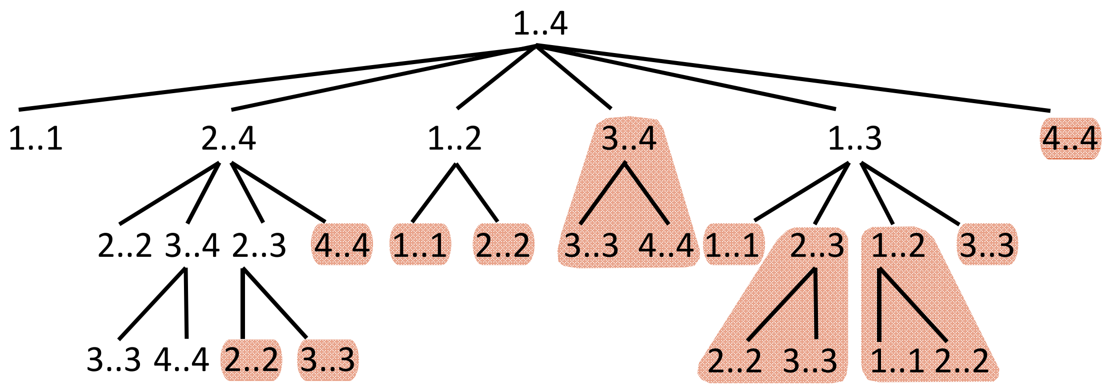
divide-and-conquer 会一直遇到新的子問題

Elements of dynamic programming$_{2/2}$

- Example: merge sort 問題不一樣



- Example: matrix-chain 重覆子問題



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- **Longest common subsequence**
- Optimal binary search trees

Longest-common-subsequence 最長相同子序列

- A **subsequence** is a sequence that can be derived from another sequence by deleting some elements.

子序列：將原本 sequence 的某些元素刪除

- For example:
  - $\langle K, C, B, A \rangle$ is a subsequence of $\langle \boldsymbol{K}, G, \boldsymbol{C}, E, \boldsymbol{B}, B, \boldsymbol{A} \rangle$.
  - $\langle B, C, D, G \rangle$ is a subsequence of $\langle A, C, \boldsymbol{B}, E, G, \boldsymbol{C}, E, \boldsymbol{D}, B, \boldsymbol{G} \rangle$.
- **Longest-common-subsequence problem**
  - **Input:** 2 sequences, $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$.
  - **Output:** A maximum-length common subsequence of X and Y.
- For example: $X = \langle A, B, C, B, D, A, B \rangle$ and $Y = \langle B, D, C, A, B, A \rangle$.
  - $\langle B, C, A \rangle$ is a common subsequence of both X and Y.
  - $\langle B, C, B, A \rangle$ is an longest common subsequence (**LCS**) of X and Y.

長度為4 最長相同子序列

Step 1: Characterizing an LCS

- **Brute-force algorithm:** 暴力法將所有可能性都看過
  - For every subsequence of X, check whether it is a subsequence of Y.
- Time: $\Theta(n2^m)$.
  - 2^m subsequences of X to check.
  - Each subsequence takes $\Theta(n)$ time to check: scan Y for first letter, from there scan for second, and so on.

x的子序列個數 2^m
檢查是不是y的子序列：$O(n)$
∴ 共花 $\Theta(n \cdot 2^m)$

- Given a sequence $X = \langle x_1, x_2,..., x_m \rangle$, we define the ith **prefix** of X, as $X = \langle x_1, x_2,..., x_i \rangle$.

x的第i個前置

- For example:
  - $X = \langle A, B, C, B, D, A, B \rangle$.
  - $X_4 = \langle A, B, C, B \rangle$ and X_0 is the empty sequence.

Optimal substructure of an LCS

若 $x_m \neq y_n$，z 有可能 $\begin{cases} x_{m-1} \text{ 和 } y_n \text{ 的 LCS} \\ x_m \text{ 和 } y_{n-1} \text{ 的 LCS} \end{cases}$

- **Theorem 15.1** 如果 $x_m = y_n$，可以 x_{m-1} 和 y_{n-1} 的 LCS 來形成 Z_{k-1}

  Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y.

  1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
  2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is an LCS of X_{m-1} and Y.
  3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is an LCS of X and Y_{n-1}.

- For example:
  - $X = \langle A, B, C, B, D, A, B \rangle$, $Y = \langle B, D, C, A, B \rangle$ and $Z = \langle B, C, A, B \rangle$ is an LCS of X and Y. If $x_7 = y_5$, then $z_4 = x_7 = y_5$ and $Z_3 = \langle B, C, A \rangle$ is an LCS of X_6 and Y_4.
  - $X = \langle A, B, C, B, D, A, D \rangle$, $Y = \langle B, D, C, B, A \rangle$ and $Z = \langle B, C, A \rangle$ is an LCS of X and Y. If $x_7 \neq y_5$, then $z_3 \neq x_7$ implies that $Z_3 = \langle B, C, A \rangle$ is an LCS of X_6 and Y_5.
  - $X = \langle A, B, C, B, D, A, A \rangle$, $Y = \langle B, D, C, A, B \rangle$ and $Z = \langle B, C, A \rangle$ is an LCS of X and Y. If $x_7 \neq y_5$, then $z_3 \neq y_5$ implies that $Z_3 = \langle B, C, A \rangle$ is an LCS of X_7 and Y_4.

在 $x_m \neq y_n$ 的情形下，$z_k \neq x_m$ 與 $z_k \neq y_n$ 至少有一個成立，
⇒ 因為兩個都是 common sequence，所以選較長者

Step 2: A recursive solution

- Define $c[i, j]$ = length of LCS of X_i and Y_j. We want $c[m, n]$.

x_i 和 y_j 的 LCS 長度

由定理而來

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0, \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j, \quad \text{case ①} \\ \max(c[i, j-1], c[i-1, j]) & \text{if } i, j > 0 \text{ and } x_i \neq y_j. \quad \text{case ② ③} \end{cases}$$

選較長者

Step 3: Computing the length of an LCS

- Based on the recursive formula, we could easily write an exponential-time recursive algorithm to compute the length of an LCS of two sequences.

  如果將遞迴公式直接寫成程式,則程式的時間複雜度將會是指數時間

- There are only $\Theta(mn)$ distinct subproblems.

  c[i,j],其中 $i=1\sim m$, $j=1\sim n$, ∴子問題個數 $\Theta(mn)$ 個

- We can use dynamic programming to compute the solutions bottom up. 使用動態規劃法,由下往上,先算子問題,再算原問題

LCS-LENGTH pseudocode

```
LCS-LENGTH(X, Y)
    m ← length[X]; n ← length[Y]
    for i ← 1 to m
        c[i, 0] ← 0
    for j ← 0 to n
        c[0, j] ← 0
    for i ← 1 to m
        for j ← 1 to n
            if x_i = y_j
                c[i, j] ← c[i – 1, j – 1] + 1
                b[i, j] ← "↖"
            else if c[i - 1, j] ≥ c[i, j - 1]
                c[i, j] ← c[i – 1, j]
                b[i, j] ← "↑"
            else  c[i, j] ← c[i, j – 1]
                b[i, j] ← "←"
    return c and b
```

初始化

先算 $i=1$

$i=2$

⋮

$i=m$

j		0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑0	↑0	↑0	↖1	←1	↖1
2	B	0	↖1	←1	←1	↑1	↖2	←2
3	C	0	↑1	↑1	↖2	←2	↑2	↑2
4	B	0	↖1	↑1	↑2	↑2	↖3	←3
5	D	0	↑1	↖2	↑2	↑2	↑3	↑3
6	A	0	↑1	↑2	↑2	↖3	↑3	↖4
7	B	0	↖1	↑2	↑2	↑3	↖4	↑4

x_i 與 y_j 相比

不同：上和左中取大的 ⇒ case ② ③

相同：左上的長度+1 ⇒ case ①

↑：記錄大的是從何處所取

- Time: $O(mn)$.

Step 4: Constructing an LCS

- Whenever we encounter a "↖" in entry $b[i, j]$, it implies that $x_i = y_j$ is an element of the LCS. 將回溯路徑印出

```
PRINT-LCS(b, X, i, j)
    if i = 0 or j = 0
        return
    if b[i, j] = "↖"
        PRINT-LCS(b, X, i – 1, j – 1) 往左上
        print x_i  直接印
    elseif b[i, j] = "↑"
        PRINT-LCS(b, X, i – 1, j) 往上
    else PRINT-LCS(b, X, i, j – 1) 往左
```

i \ j		0	1	2	3	4	5	6
	y_j		*B*	*D*	*C*	*A*	*B*	*A*
0	x_i	0	0	0	0	0	0	0
1	*A*	0	↑0	↑0	↑0	↖1	←1	↖1
2	*B*	0	↖1	←1	←1	↑1	↖2	←2
3	*C*	0	↑1	↑1	↖2	←2	↑2	↑2
4	*B*	0	↖1	↑1	↑2	↑2	↖3	←3
5	*D*	0	↑1	↖2	↑2	↑2	↑3	↑3
6	*A*	0	↑1	↑2	↑2	↖3	↑3	↖4
7	*B*	0	↖1	↑2	↑2	↑3	↖4	↑4

- This procedure prints "*BCBA*".

Outline

- Rod cutting
- Matrix-chain multiplication
- Elements of dynamic programming
- Longest common subsequence
- **Optimal binary search trees**

Optimal binary search trees

- **Input:** A sequence $K = \langle k_1, k_2, \ldots, k_n \rangle$ of n distinct keys in sorted order.
  A sequence $D = \langle d_0, d_1, \ldots, d_n \rangle$ of $n + 1$ dummy keys.
  - $k_1 < k_2 < \cdots < k_n$.
  - d_0 = all values < k_1. d_n = all values > k_n.
  - d_i = all values between k_i and k_{i+1}.
  - For each key k_i, a probability p_i that a search is for k_i.
  - For each key d_i, a probability q_i that a search is for d_i.
- **Output:** A BST with minimum expected search cost.
  - $$\begin{aligned} \text{E[search cost in } T] &= \sum_{i=1}^{n} (\text{depth}_T(k_i) + 1) \cdot p_i + \sum_{i=1}^{n} (\text{depth}_T(d_i) + 1) \cdot q_i \\ &= \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} q_i + \sum_{i=1}^{n} \text{depth}_T(k_i) \cdot p_i + \sum_{i=1}^{n} \text{depth}_T(d_i) \cdot q_i \\ &= 1 + \sum_{i=1}^{n} \text{depth}_T(k_i) \cdot p_i + \sum_{i=1}^{n} \text{depth}_T(d_i) \cdot q_i \end{aligned}$$

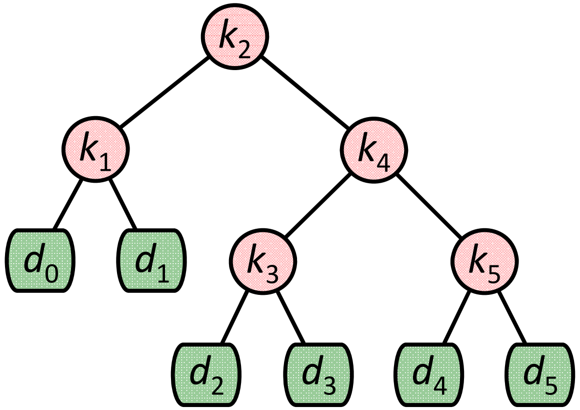
$$\boxed{\sum_{i=1}^{n} p_i + \sum_{i=1}^{n} q_i = 1}$$

$0.15 \times 2 + 0.1 \times 1 + 0.05 \times 3 + 0.10 \times 2 + 0.20 \times 3$
$+ 0.05 \times 3 + 0.10 \times 3 + 0.05 \times 4 + 0.05 \times 4 + 0.05 \times 4 + 0.10 \times 4 = 2.80$

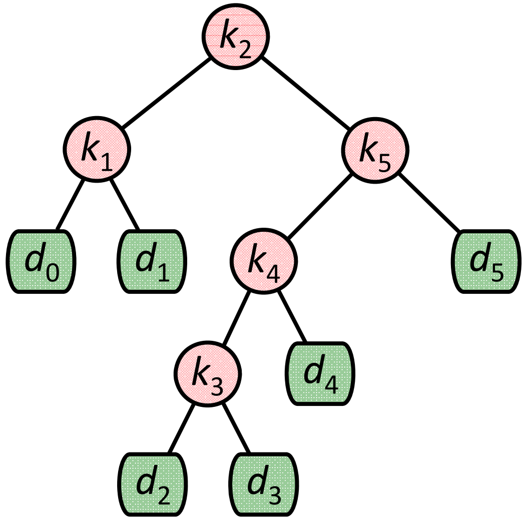
An example

k_i : 成功

d_i : 失敗



Expected search cost 2.80.



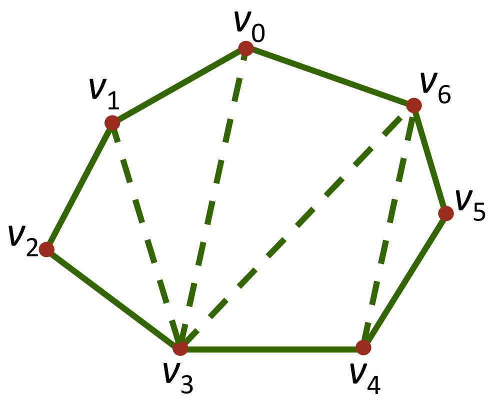
Expected search cost 2.75.
This tree is optimal.

- Two binary search trees for a set of n = 5 keys with the following probabilities:

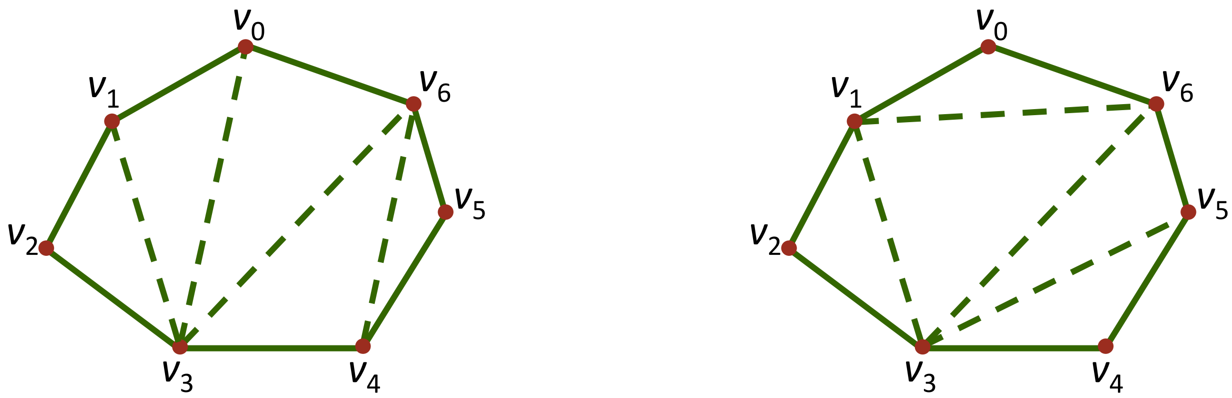
i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

Optimal polygon triangulation$_{1/2}$ 將多邊形三角化

- If $P = \langle v_0, v_1, \ldots, v_{n-1} \rangle$ is a **convex polygon**, it has n **sides**, $\overline{v_0 v_1}$, $\overline{v_1 v_2}$, …, $\overline{v_{n-1} v_0}$.
- Given two nonadjacent vertices v_i and v_j, the segment $v_i v_j$ is a **chord** of the polygon.
- A **triangulation** of a polygon is a set T of chords of the polygon that divide the polygon into disjoint triangles.



Optimal polygon triangulation$_{2/2}$



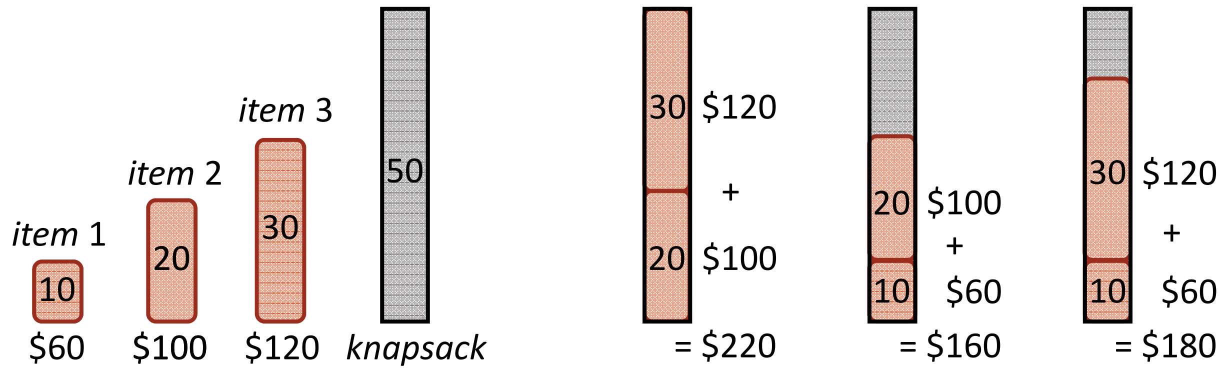
Two ways of triangulating a convex polygon.

- **Optimal polygon triangulation problem**
  - **Input:** A convex polygon $P = \langle v_0, v_1, \ldots, v_{n-1} \rangle$.
    - A weighting function w defined on triangles formed by sides and chords of P. 三角化後週的和為最小
  - **Output:** A triangulation that minimizes the sum of the weights of the triangles in the triangulation.

动態規劃法=填表法

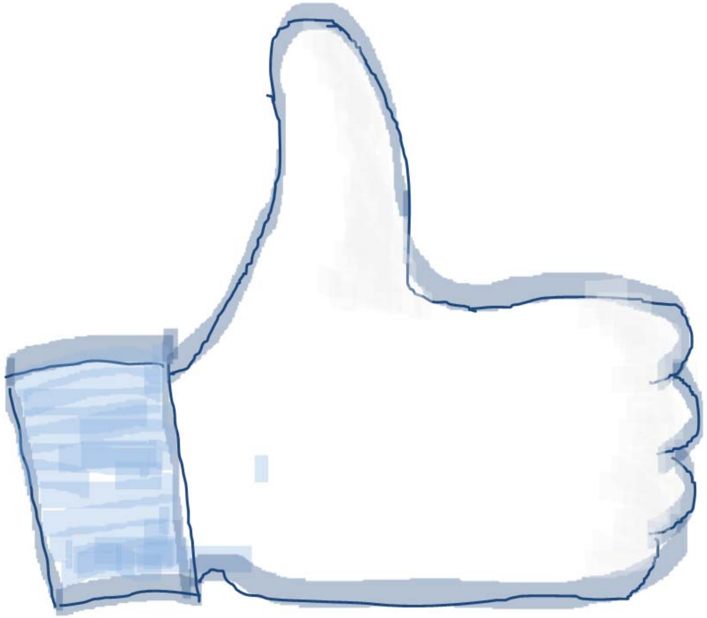
0-1 knapsack problem-- using DP 背包問題

- **Input:** A set $A = \{a_1, a_2, \ldots, a_n\}$ of n items and a knapsack of capacity C. 背包容量
  - Each item a_i is worth v_i dollars and weighs w_i pounds.
    各物品有其重量及價值
- **Output:** A subset of items whose total size is bounded by C and whose profit is maximized. 如何取有最大價值
  - **Each item must either be taken or left behind.**
    每個item只能選擇取或不取
- For example:



講義超精美

我沒蓋你



2014.01.06.

魏琳瑑.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大家要認真看講義喔!
2014. 12. 17