

# Underactuated Robots

## Lecture 1: Trajectory Optimization

Nicola Scianca

September 2024

- these slides were made by **Nicola Scianca**, researcher at Dipartimento di Ingegneria Informatica, Automatica e Gestionale, Sapienza University of Rome, Italy  
email: [scianca@diag.uniroma1.it](mailto:scianca@diag.uniroma1.it)
- the slides and the examples are available at this **repository**:  
<https://github.com/DIAG-Robotics-Lab/underactuated>



# trajectory optimization

- **trajectory optimization** is a mathematical approach to compute the best trajectory for a system to follow, while satisfying **constraints** and minimizing a **cost function**
- it involves optimizing the states and control inputs over time to achieve a specific goal, such as minimizing energy, time, effort, or following a desired path
- trajectory optimization can be used, in many forms, for tasks like motion planning, navigation, and control

- we consider a **discrete** sequence of input and states over  $N$  time-steps, covering the entire duration of the task

$$x_0, x_1, \dots, x_{N-1}, x_N$$

$$u_0, u_1, \dots, u_{N-1}$$

- these states and inputs are connected via the systems **dynamics**, which we write in discrete form

$$x_{k+1} = f(x_k, u_k)$$

- for physical systems, which have continuous dynamics, we assume to have discretized them in some way (e.g., Euler, Runge-Kutta, ...)

- to formulate a trajectory optimization problem, we choose a **cost function**, which is something that we want to minimize

$$\min \sum_{i=0}^{N-1} \underbrace{l_k(x_i, u_i)}_{\text{running cost}} + \underbrace{l_N(x_N)}_{\text{terminal cost}}$$

- for example, the cost function might be a measure of energy consumption, or the time necessary to complete the task, or the distance to some reference point or trajectory
- the cost function is usually constituted by a **running cost**, evaluating the cost of states and inputs along the trajectory, and a **terminal cost** evaluating the cost of the final state

- this cost function must be minimized while taking into account some necessary **constraints**

$$x_{i+1} = f(x_i, u_i) \quad \text{for } i = 0, \dots, N - 1$$
$$x_0 = \bar{x}_0$$

- the input and state sequences should satisfy the dynamics, which makes it so that the resulting input and state sequences constitute legitimate **trajectories** for the system
- the initial state should correspond to the initial state of the system  $\bar{x}_0$
- we could add additional constraints (e.g., max input, obstacle avoidance, ...)

# trajectory optimization

- a basic trajectory optimization problem has the following form

$$\begin{aligned} \min \quad & \sum_{i=0}^{N-1} l_k(x_i, u_i) + l_N(x_N) \\ \text{s. t.} \quad & x_{i+1} = f(x_i, u_i) \quad \text{for } i = 0, \dots, N-1 \\ & x_0 = \bar{x}_0 \end{aligned}$$

- the optimal input that we get by solving this problem depends on the current state  $\bar{x}_0$
- if we could solve this problem for **every** state, we would have found an **optimal control law**  $u(x)$

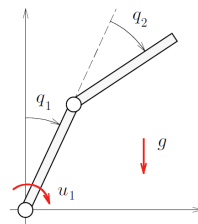
# trajectory optimization

- if we can't explicitly write out a feedback law  $u(x)$ , maybe we can still solve the optimization problem, starting from one initial state  $\bar{x}_0$ , and find a particular trajectory as solution
- by doing this we don't get an optimal control law, but rather a particular **optimal trajectory**, i.e., a sequence of inputs and states which satisfy the system dynamics
- we can perform this optimization **offline**, and then do trajectory tracking to keep the system as close as possible to this optimal trajectory



# example

- consider a **pendubot**: a double pendulum with one actuated joint
- the state variables are
  - ▶  $\theta_1$  angle of the first link
  - ▶  $\theta_2$  angle of the second link (relative)
  - ▶  $\dot{\theta}_1$  angular velocity of the first link
  - ▶  $\dot{\theta}_2$  angular velocity of the second link
- goal: swing up the pendubot to its upright configuration  $(0, 0, 0, 0)$
- initial condition: stable equilibrium  $(\pi, 0, 0, 0)$  in which both arms of the pendulum are pointing down



# example

- first, we discretize states and inputs

$$u_0, u_1, \dots, u_{N-1}$$

$$x_0, x_1, \dots, x_{N-1}, x_N$$

- then, we define a cost function, which in our case is just to minimize the sum of the squares of the input

$$\min \sum_{i=0}^N u_i^2$$

- and we define our initial state (given) and our final state (target) as constraints

$$x_0 = (\pi, 0, 0, 0)$$

$$x_N = (0, 0, 0, 0)$$

- the dynamics of the pendubot can be written as

$$M(q)\ddot{q} + c(q, \dot{q})\dot{q} + g(q) = \underbrace{(u, 0)^T}_{\text{zero component due to the underactuation}}$$

zero component due to the underactuation

- first we rewrite it as  $\dot{x} = f(x, u)$

$$\underbrace{\frac{d}{dt} \begin{pmatrix} q \\ \dot{q} \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} 0_{2 \times 2} & I_{2 \times 2} \\ 0_{2 \times 2} & -M^{-1}c \end{pmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} + \begin{pmatrix} 0_{2 \times 2} \\ -M^{-1}g \end{pmatrix} + \begin{pmatrix} 0_{2 \times 2} \\ M^{-1} \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix}}_{f(x, u)}$$

- we must discretize the dynamics, e.g. with Euler integration

$$x_{k+1} = x_k + \Delta \cdot f(x_k, u_k)$$

# example

```
import ... [numpy, casadi, model, etc ...]

f = model.get_pendubot_model()

# parameters
N = 200
delta = 0.01

# set up optimization problem
opt = cs.Opti()
opt.solver("ipopt", p_opts, s_opts)
X = opt.variable(4,N+1)
U = opt.variable(1,N+1)

for i in range(N):
    opt.subject_to( X[:,i+1] == X[:,i] + delta * f(X[:,i], U[0,i]) )

opt.subject_to( X[:,0] == (math.pi, 0, 0, 0) )
opt.subject_to( X[:,N] == (0, 0, 0, 0) )

cost_function = cs.sumsqr(U)
opt.minimize(cost_function)

# get solution
sol = opt.solve()
u = sol.value(U)
x = sol.value(X)
```

# how do we solve the optimization problem

- in the example we used `casADi` to set up a problem for `ipopt`, which is a very popular off-the-shelf optimization algorithm
- however, our problem is not a generic optimization problem, it is a problem with a particular structure
- sometimes it is possible to use techniques that exploit the particular structure of an optimal control problem (e.g., DDP)

# transcription methods

- **transcription** is the particular way in which we choose we write our optimization problem so that we can then solve it
- there are many transcription methods: shooting methods, collocation methods, ...
- we will discuss the two main paradigms that characterize shooting methods: **single shooting** and **multiple shooting**
- aside from very simple cases, the solution process will always involve **numerical iteration** that hopefully **converge** to an optimal solution

- the same problem can be **transcribed** in different ways: we will consider the two main possibilities (but there are others)
- **single shooting** methods: we perform substitutions to have less variables and constraints, but we increase the nonlinearity
- **multiple shooting** methods: we have a larger number of variables held together by constraints, so the problem is larger but less nonlinear

# transcription methods: single shooting

- in **single shooting** methods, the **inputs**  $u_0, \dots, u_{N-1}$  are decision variables: we eliminate states from the cost function by substituting in the dynamics

$$x_1 = f(\bar{x}_0, u_0)$$

$$x_2 = f(x_1, u_1) = f(f(\bar{x}_0, u_0), u_1)$$

...

$$x_k = f(f(\dots f(\bar{x}_0, u_0), u_1), u_2), \dots), u_{k-1})$$

- we also substitute  $x_0$  with  $\bar{x}_0$  to eliminate the initial state constraint
- we get a problem that is only dependent on the input sequence, and has **no constraints**



# transcription methods: multiple shooting

- in **multiple shooting** methods, the **inputs** and **states** are decision variables: they are linked together by constraints (the dynamic equation)

$$x_1 = f(x_0, u_0)$$

$$x_2 = f(x_1, u_1)$$

...

$$x_k = f(x_{k-1}, u_{k-1})$$

- now we have to solve a constrained problem, but the advantage is that we have a simpler cost function
- if we stop the solution before convergence, the dynamics might not be satisfied exactly

# transcription methods: example

consider this example with horizon length  $N = 2$

- discrete inputs and states

$$u_0, u_1,$$

$$x_0, x_1, x_2$$

- cost function

$$\sum_{i=0}^1 \left( u_i^2 + x_i^2 \right) + x_2^2$$

- dynamics

$$x_{k+1} = f(x_k, u_k)$$

- **single shooting** formulation

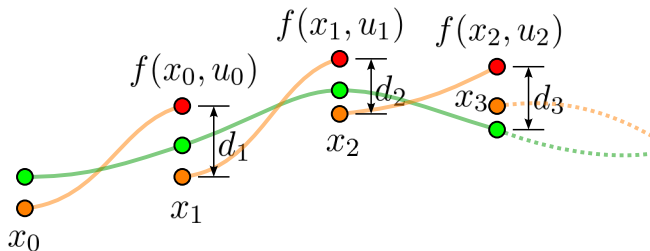
$$\min_{u_0, u_1} \left( u_0^2 + u_1^2 + \bar{x}_0^2 + f(\bar{x}_0, u_0)^2 + f(f(\bar{x}_0, u_0), u_1)^2 \right)$$

- **multiple shooting** formulation

$$\begin{aligned} \min_{\substack{u_0, u_1, \\ x_0, x_1, x_2}} & \left( u_0^2 + u_1^2 + x_0^2 + x_1^2 + x_2^2 \right) \\ \text{s. t.} \quad & x_0 = \bar{x}_0 \\ & x_1 = f(x_0, u_0) \\ & x_2 = f(x_1, u_1) \end{aligned}$$

# multiple shooting: defects

- in a multiple shooting formulation, the constraints are not necessarily satisfied until we reach convergence
- the difference between the forward integration and the next variable is called the **defect**  $d_{k+1} = x_{k+1} - f(x_k, u_k)$



# single shooting: numerical problems

- in single shooting, the entire trajectory is determined by forward integration from an initial guess for the control inputs
- this can lead to **numerical instability**, especially for long time horizons or highly nonlinear systems
- the initial trajectory might be too far away from the optimum, or in the worst case we might even have a software failure (numbers get too big)

# single shooting vs. multiple shooting

- let's recap the main differences between single and multiple shooting methods

single shooting	multiple shooting
less variables	more variables
cost function is very nonlinear	cost function is usually quadratic
less constraints	more constraints
initial guess could diverge	initial guess is stable
result is always a feasible trajectory	result could have defects

- if possible, it is usually better to do multiple shooting rather than single shooting