homework 1

question 1

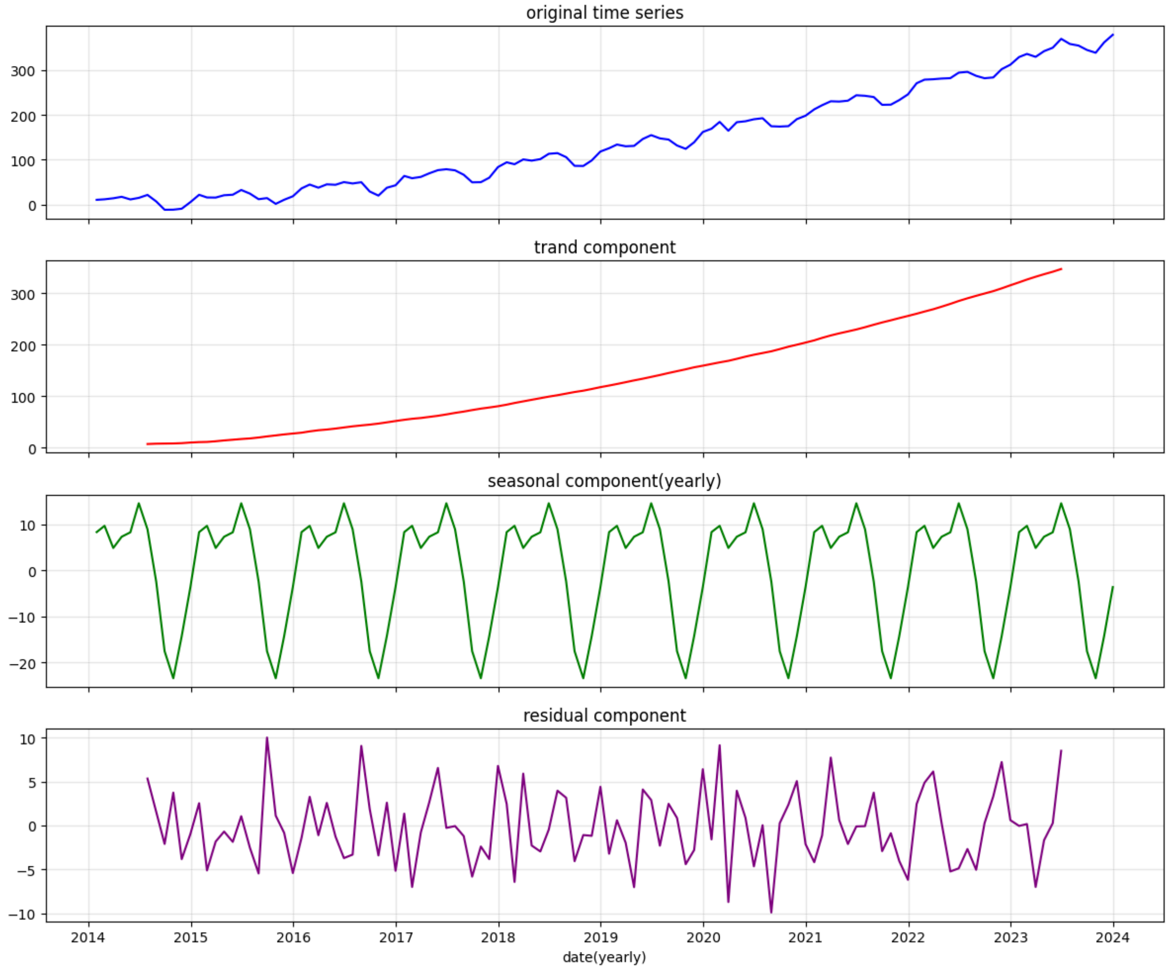
$X_t = trand_t + seasonal_t + noise_t$
$trand_t = 0.8t - 0.02t^2$
$seasonal_t = 15sin(\frac{2\pi t}{12}) + 8cos(\frac{2\pi t}{6})$
$noise_t \sim N(0, 5^2)$

In [30]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from statsmodels.tsa.seasonal import seasonal_decompose

np.random.seed(42)
n = 120  # 10*12=120
t = np.arange(n)

trend = 0.8 * t + 0.02 * t ** 2
seasonal = 15 * np.sin(2 * np.pi * t / 12) + 8 * np.cos(2 * np.pi * t / 6)
noise = np.random.normal(0, 5, n)

time_series = trend + seasonal + noise
dates = pd.date_range(start="2014-01-01", periods=n, freq="ME")
df = pd.DataFrame({"date": dates, "value": time_series})
df.set_index("date", inplace=True)
```

In [31]:
```
from statsmodels.tsa.seasonal import seasonal_decompose

# decompose the time series
decomposition = seasonal_decompose(df["value"], model="additive", period=12)

# extract the components
decomposed_trend = decomposition.trend
decomposed_seasonal = decomposition.seasonal
decomposed_resid = decomposition.resid
```

In [32]:
```
fig, axes = plt.subplots(4, 1, figsize=(12, 10), sharex=True)

# 1. original time series
axes[0].plot(df.index, df["value"], color="blue")
axes[0].set_title("original time series")
axes[0].grid(alpha=0.3)

# 2. trend component
axes[1].plot(decomposed_trend.index, decomposed_trend, color="red")
axes[1].set_title("trand component")
axes[1].grid(alpha=0.3)

# 3. seasonal component
axes[2].plot(decomposed_seasonal.index, decomposed_seasonal, color="green")
axes[2].set_title("seasonal component(yearly)")
axes[2].grid(alpha=0.3)
```

```
# 4. residual component
axes[3].plot(decomposed_resid.index, decomposed_resid, color="purple")
axes[3].set_title("residual component")
axes[3].set_xlabel("date(yearly)")
axes[3].grid(alpha=0.3)

plt.tight_layout()
plt.show()
```



question 2

consider the time series

$x_t = \beta_1 + \beta_2 t + \omega_t$

β_1 and β_2 are known constants and ω_t is a white noise process with variance σ_ω^2.

(a)determine whether x_t is stationary.

(b)show that the first-order difference of x_t ($y_t = x_t - x_{t-1}$) = is stationary.

(c)show the moving average

$v_t = \frac{1}{2q+1}\sum_{j=-q}^{q} x_{t-j}$ is $\beta_1 + \beta_2 t$, and give a simplified expression for the autocorrelation function.

(a) No, x_t is nonstationary

$E[x_t] = E[\beta_1 + \beta_2 t + \omega_t] = E[\beta_1 + \beta_2 t](because E[\omega_t] = 0)$ = $\beta_1 + \beta_2 t$

which is time-dependent, so x_t is nonstationary.

(b) $y_t = x_t - x_{t-1}$ is stationary

$y_t = x_t - x_{t-1} = \beta_1 + \beta_2 t + \omega_t - \beta_1 - \beta_2(t-1) - \omega_{t-1} = \beta_2 + \omega_t - \omega_{t-1}$

$E[y_t] = E[x_t - x_{t-1}] = E[\beta_1 + \beta_2 t + \omega_t - \beta_1 - \beta_2(t-1) - \omega_{t-1}] = E[\beta_2 t + \omega_t$

$Cov(y_t, y_{t+h}) = E[(y_t - E[y_t])(y_{t+h} - E[y_{t+h}])] = E[(\omega_t - \omega_{t-1})(\omega_{t+h} - \omega_{t+h-1}$

when h = 0 the $Cov(y_t, y_{t+h}) = E[\omega_t^2] + E[\omega_{t-1}^2] = 2\sigma_\omega^2$, which is constant, so y_t is stationary.

when $h = \pm 1, Cov(y_t, y_{t+h}) = -E[\omega_t^2] = -\sigma_\omega^2$, which is also constant, so y_t is stationary.

when $|h| > 1, Cov(y_t, y_{t+h}) = 0$, which means y_t is independent of y_{t+h}, so y_t is stationary. so y_t is stationary.

(c) $v_t = \frac{1}{2q+1} \sum_{j=-q}^{q} x_{t-j}$

$E[v_t] = E[\frac{1}{2q+1} \sum_{j=-q}^{q} x_{t-j}] = \frac{1}{2q+1} \sum_{j=-q}^{q} E[x_{t-j}] = \frac{1}{2q+1} \sum_{j=-q}^{q} (\beta_1 + \beta_2(t - j$

$Cov(v_t, v_{t+h}) = E[(v_t - E[v_t])(v_{t+h} - E[v_{t+h}])] = E[\frac{1}{2q+1} \sum_{j=-q}^{q} (x_{t-j} - E[x_{t-j}]$

= $E[(\frac{1}{2q+1} \sum_{j=-q}^{q} \omega_{t-j})(\frac{1}{2q+1} \sum_{j=-q}^{q} \omega_{t+h-k})]$ =

$\frac{1}{(2q+1)^2} \sum_{j=-q}^{q} \sum_{k=-q}^{q} E[\omega_{t-j}\omega_{t+h-k}]$

when k = j + h , $Cov(v_t, v_{t+h}) = \frac{1}{(2q+1)^2} \sum_{j=-q}^{q} \sigma_\omega^2 = \frac{2q+1-|h|}{(2q+1)^2} \sigma_\omega^2$

when k $\neq$ j + h , $Cov(v_t, v_{t+h}) = 0$

consider that j and k $\in$ [-q,q]

when |h| $\leq$ q , $Cov(v_t, v_{t+h}) = \frac{2q+1}{(2q+1)^2} \sigma_\omega^2$

when |h| > q , $Cov(v_t, v_{t+h}) = 0$

question 3

let ω_t for t = 0 , ± 1 ,± 2 ,± 3 ..... be a noise process, and consider the series

$x_t = \omega_t \omega_{t-1}$

determine the mean and autocovariance function of x_t for t = 0 , ± 1 ,± 2 ,± 3 ....., and state whether it is stationary or not.

$E[x_t] = E[\omega_t \omega_{t-1}] = E[\omega_t]E[\omega_{t-1}] = 0 (t \neq t-1)$

$\gamma_x(h) = E[(x_t - E[x_t])(x_{t+h} - E[x_{t+h}])] = E[x_t x_{t+h}]$

$= E[\omega_t \omega_{t-1} \omega_{t+h} \omega_{t+h-1}]$

when h = 0 , $\gamma_x(h) = E[\omega_t^2]E[\omega_{t-1}^2] = \sigma_\omega^4$

when h = ± 1 , $\gamma_x(h) = E[\omega_t \omega_{t-1}^2 \omega_{t-2}] = E[\omega_t]E[\omega_{t-1}^2]E[\omega_{t-2}] = 0$

when |h| > 1 ,

$\gamma_x(h) = E[\omega_t \omega_{t-1} \omega_{t+h} \omega_{t+h-1}] = E[\omega_t]E[\omega_{t-1}]E[\omega_{t+h}]E[\omega_{t+h-1}] = 0$

as E[x] is constant, $\gamma_x(h) = \sigma_\omega^4$ for h = 0 and h = ± 1 and $\gamma_x(h) = 0$ for $|h| \geq 1.$ so x_t is a stationary process.

question 3

for the AR(2) model given by:$x_t = 0.9x_{t-2} + \omega_t$, find the roots of the autoregressive polynomial and sketch the ACF , and $\rho(h)$ for the AR(2) model.

$x_t = 0.9x_{t-2} + \omega_t$
$\Phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3 - \ldots - \phi_n z^n$
$\Phi(z) = 1 - 0z - 0.9z^2 = 0$
$Z^2 = -\frac{10}{9}$
$Z = \sqrt{-\frac{10}{9}}$
$Z = \frac{\sqrt{10}}{3} i$
$|z| \geq 1$ so it is causal

for ACF($\rho(h)$) $\rho(h) = \phi_1 \rho(h-1) + \phi_2 \rho(h-2)(h \geq 1)$
as $\phi_1 = 0, \phi_2 = 0.9, \rho(0) = 1 \rho(-h) = h$ when h = 1 ,
$\rho(1) = 0 * \rho(0) + 0.9 * \rho(-1) = 0$
when h = 2 , $\rho(2) = 0 * \rho(1) + 0.9 * \rho(0) = -0.9$
when h = 3 , $\rho(3) = 0 * \rho(2) + 0.9 * \rho(1) = 0$
when h = 4 , $\rho(4) = 0 * \rho(3) + 0.9 * \rho(2) = 0.81$
when h = 5 , $\rho(5) = 0 * \rho(4) + 0.9 * \rho(3) = 0$
when h is odd , $\rho(h) = 0$
when h is even , $\rho(h) = (-0.9)^{h/2}$ so Even-only non-zero lag, alternating oscillation attenuation factor $\rho(h) = (-0.9)^{h/2}$

question 4

identify the following models as ARMA(p, q) models(watch out for parameter redundancy),
and determine whether they are causal and/ or invertible.
(a)$x_t = 0.8x_{t-1} - 0.15x_{t-2} + \omega_t - 0.3\omega_{t-1}$
(b)$x_t = x_{t-1} - 0.5x_{t-2} + \omega_t - \omega_{t-1}$

for(a)
$x_t - 0.8x_{t-1} + 0.15xt - 2 = \omega_t - 0.3\omega_{t-1}$
$\Phi(z) = 1 - 0.8z + 0.15z^2 = (1 - 0.3z)(1 - 0.5z)$
$z = \frac{10}{3}$ or z = 2
$\Theta(z) = 1 - 0.3z$
$z = \frac{10}{3}$
so it is ARMA(2, 1)
norm length of z root for $\Phi(z) = 0$ is larger than 1 so it is causal
norm length of z root for $\Theta(z) = 0$ is larger the 1 so it is vertible

for(b)
$x_t - x_{t-1} + 0.5x_{t-1} = \omega_t - \omega_{t-1}$ $\Phi(z) = 1 - z + 0.15z^2$
z = 1 $\pm$ i
$\Theta(z) = 1 - z$
z = 1

so it is ARMA(2, 1)
norm length of z root for $\Phi(z) = 0$ is larger than 1 so it is causal
norm length of z root for $\Theta(z) = 0$ is 1 so it is invertible