SDSC 6015 Homework 1: Due on October 1 11:59pm

Submission Instructions
Please submit your solutions as **a single PDF file along with your code** via Canvas. The PDF should include your solutions to the theoretical problems as well as the outputs from the practical implementation.

Problem 1 (Jensen's inequality) [10 marks]
Let f be convex, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \mathbf{dom}(f), \lambda_1, \ldots, \lambda_m \in \mathbb{R}_+$ such that $\sum_{i=1}^m \lambda_i = 1$. Show that

$$f\Big(\sum_{i=1}^m \lambda_i \boldsymbol{x}_i\Big) \leqslant \sum_{i=1}^m \lambda_i f(\boldsymbol{x}_i)\,.$$

Problem 2 [20 marks]

(i) Let $f_1, f_2, \ldots, f_m$ be convex functions, $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}_+$. Show that $f := \sum_{i=1}^m \lambda_i f_i$ is convex on $\mathbf{dom}(f) := \bigcap_{i=1}^m \mathbf{dom}(f_i)$.

(ii) Let f be a convex function with $\mathbf{dom}(f) \subseteq \mathbb{R}^d$, $g : \mathbb{R}^m \to \mathbb{R}^d$ an affine function, meaning that $g(\boldsymbol{x}) = A\boldsymbol{x} + \mathbf{b}$, for some matrix $A \in \mathbb{R}^{d\times m}$ and some vector $\mathbf{b} \in \mathbb{R}^d$. Show that the function $f \circ g$ (that maps $\boldsymbol{x}$ to $f(A\boldsymbol{x} + \mathbf{b})$) is convex on $\mathbf{dom}(f \circ g) := \{\boldsymbol{x} \in \mathbb{R}^m : g(\boldsymbol{x}) \in \mathbf{dom}(f)\}$.

Problem 3 [10 marks]
Show that the quadratic function $f(\boldsymbol{x}) = \boldsymbol{x}^\top Q\boldsymbol{x} + \mathbf{b}^\top \boldsymbol{x} + c$, with symmetric matrix Q, is smooth with parameter $2||Q||$.

Problem 4 [10 marks]
Consider the projected gradient descent algorithm

$$\begin{aligned}\boldsymbol{y}_{t+1} &= \boldsymbol{x}_t - \eta\nabla f(\boldsymbol{x}_t)\\ \boldsymbol{x}_{t+1} &= \Pi_X(\boldsymbol{y}_{t+1})\end{aligned}$$

with a convex differentiable function $f : X \to \mathbb{R}$. Suppose that for some $t \geqslant 0$, $\boldsymbol{x}_{t+1} = \boldsymbol{x}_t$. Prove that in this case, $\boldsymbol{x}_t$ is a minimizer of f over the closed and convex set X.
(Hint: Use Fact 1 (i) from Lecture 3 and Lemma 4 from Lecture 2.)

Problem 5 Practical Implementation of Gradient Descent [50 marks]

- For implementation, it is recommended to use Google Colab.
- Please open the file "**HW1_Lab_GradientDescent.ipynb**" and insert your code following the provided instructions. All necessary datasets and helper functions can be found in the **HW1_Lab** folder.
- For submission, please submit your **completed "HW1_Lab_GradientDescent.ipynb"** file along with **a PDF containing the output results**.