2. Grad-div-curl recap

M. Ravasi

ERSE 210 Seismology

Wave propagation

Key component of seismology, this is how we learn about the subsurface

Full wavefield
PDEs

High-freq approx.
Ray theories

Analytical

Wave propagation

Key component of seismology, this is how we learn about the subsurface

Full wavefield
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Analytical

Acoustic wave propagation
Elastic wave propagation

Hyperbolic
PDEs

Wave equation(s)

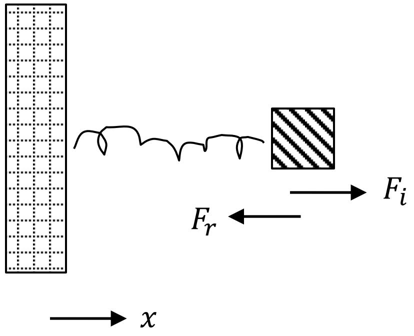
- **Principle of Inertia** (aka **Equation of motion**)

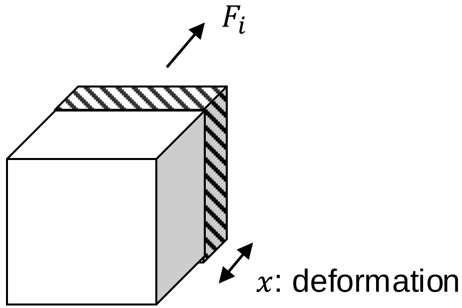
  Links variations in time and space of deformations with stresses → generates movement in first place

- **Hooke's law** (aka **Deformation equation**)

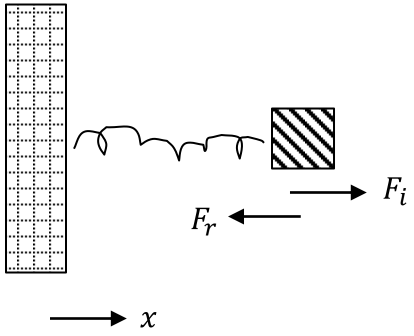
  Links stresses and deformations → acts as contrasting force to initial movement, leading to oscillations

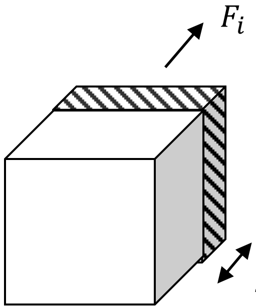
1D pendulum





1D pendulum

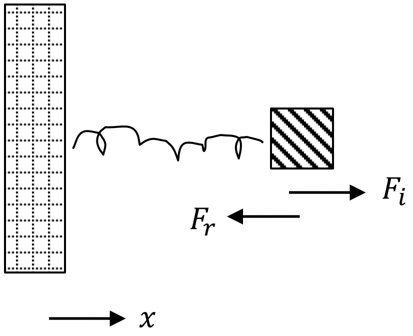


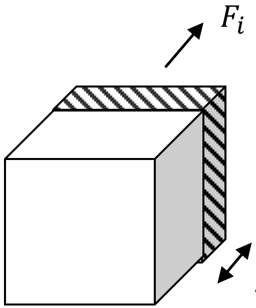


x: deformation

- **Principle of Inertia (2nd Newton's law):** $F_i = ma = m\ddot{x}$
- **Elastic reaction (Hooke's law):** $F_r = -kx$

1D pendulum





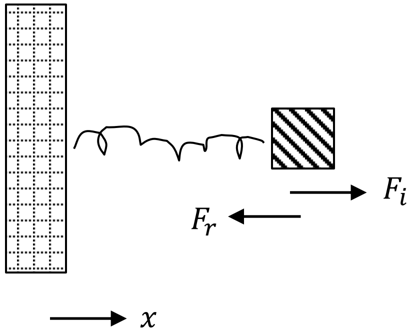
- **Principle of Inertia (2nd Newton's law):** $F_i = ma = m\ddot{x}$
- **Elastic reaction (Hooke's law):** $F_r = -kx$

At equilibrium ($F_r = F_r$):

$$m\ddot{x} = -kx \;\rightarrow\; \ddot{x} + \frac{k}{m}x = 0$$

Simple armonic motion (2nd order ODE)

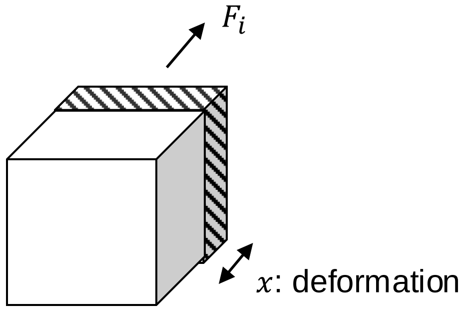
1D pendulum



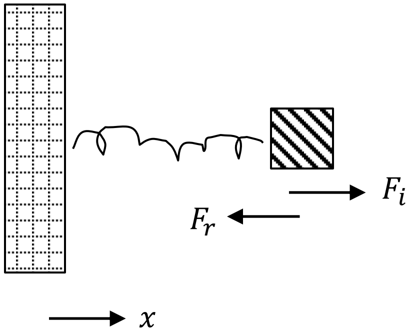
Simple armonic motion (2nd order ODE)

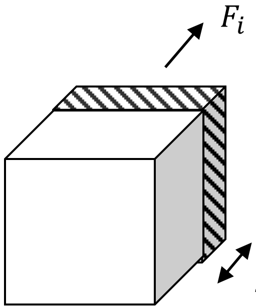
$$m\ddot{x} = -kx \;\rightarrow\; \ddot{x} + \frac{k}{m}x = 0$$

Analytical solution: $x(t) = Acos(\omega t - \phi)$



1D pendulum





Simple armonic motion (2nd order ODE)

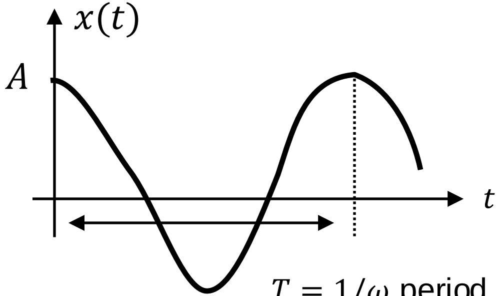
$$m\ddot{x} = -kx \;\rightarrow\; \ddot{x} + \frac{k}{m}x = 0$$

Analytical solution: $x(t) = Acos(\omega t - \phi)$

$\omega = \sqrt{\frac{k}{m}}$ Resonant frequency

A Amplitude

ϕ Initial phase



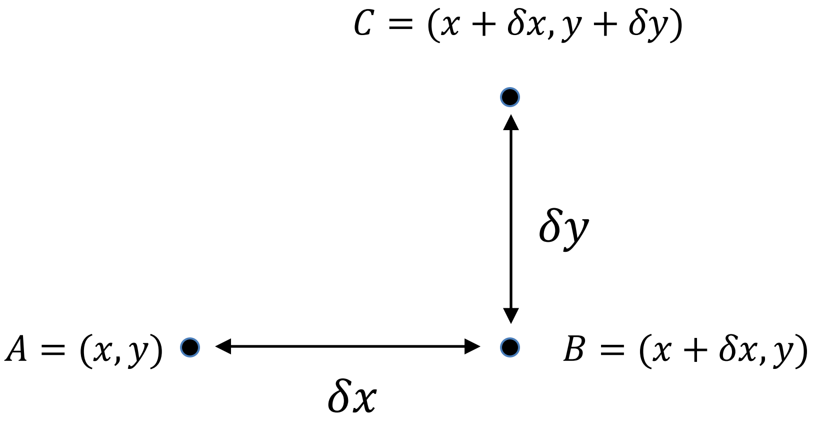
Gradient - definition

Given a generic scalar function f, we define the gradient as:

$$\mathbf{g} = \nabla f = \begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \\ \partial f / \partial z \end{bmatrix} \quad \mathbb{R} \rightarrow \mathbb{R}^3$$

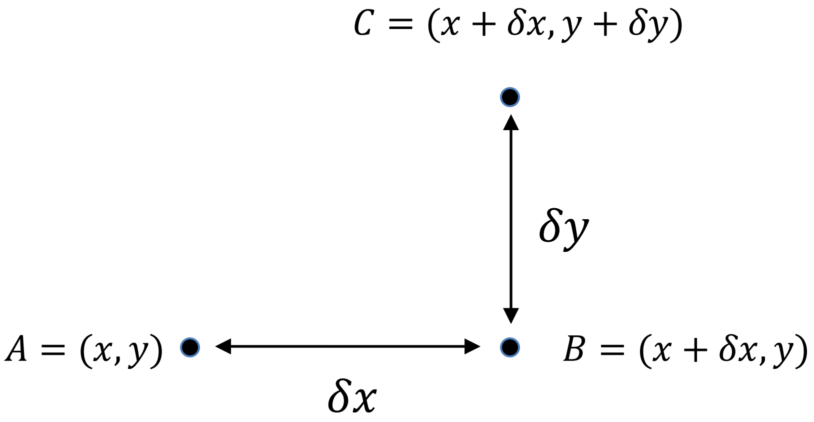
Fundamental in differentiation and integration of N-dimensional functions

Gradient - derivation



$$f_A = f(x, y)$$
$$f_B = f(x + \delta x, y)$$
$$f_B = f(x + \delta x, y + \delta y)$$

Gradient - derivation



$$f_A = f(x, y)$$
$$f_B = f(x + \delta x, y)$$
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Starting from:

$$\delta f = f_C - f_A = f_C - f_B + f_B - f_A$$

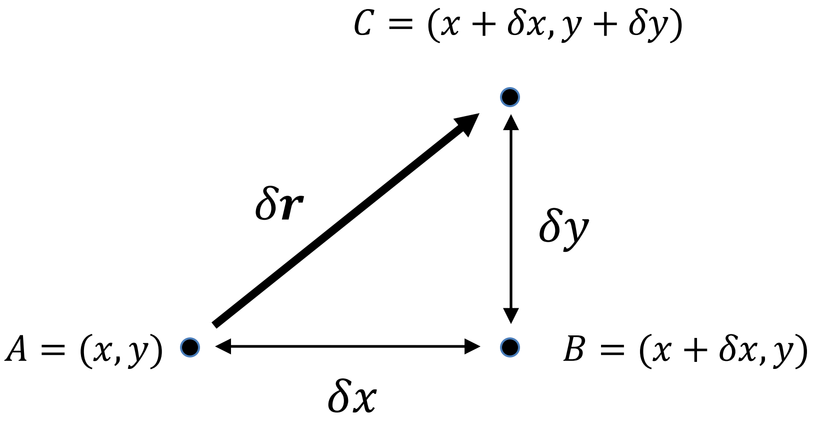
and using the definition of partial derivatives:

$$f_B - f_A = f(x + \delta x, y) - f(x, y)$$
$$= \delta x \frac{\partial f}{\partial x}(x, y)$$

$$f_C - f_B = f(x + \delta x, y + \delta y) - f(x + \delta x, y)$$
$$= \delta y \frac{\partial f}{\partial y}(x + \delta x, y) =$$ Taylor expansion
$$\delta y \frac{\partial f}{\partial y}(x, y) + \delta y \delta x \cancel{\frac{\partial^2 f}{\partial x \partial y}}(x, y)$$

=0 (higher order)

Gradient - derivation



$$f_A = f(x, y)$$
$$f_B = f(x + \delta x, y)$$
$$f_B = f(x + \delta x, y + \delta y)$$

Starting from:

$$\delta f = f_C - f_A = f_C - f_B + f_B - f_A$$

we get:

$$\delta f = \delta x \frac{\partial f}{\partial x}(x, y) + \delta y \frac{\partial f}{\partial y}(x, y)$$

$$\boxed{\delta f = \nabla f \cdot \delta \boldsymbol{r}}$$

Inner product between the gradient and the vector connecting A and C

Gradient - interpretation

The gradient is a vector with magnitude and direction:

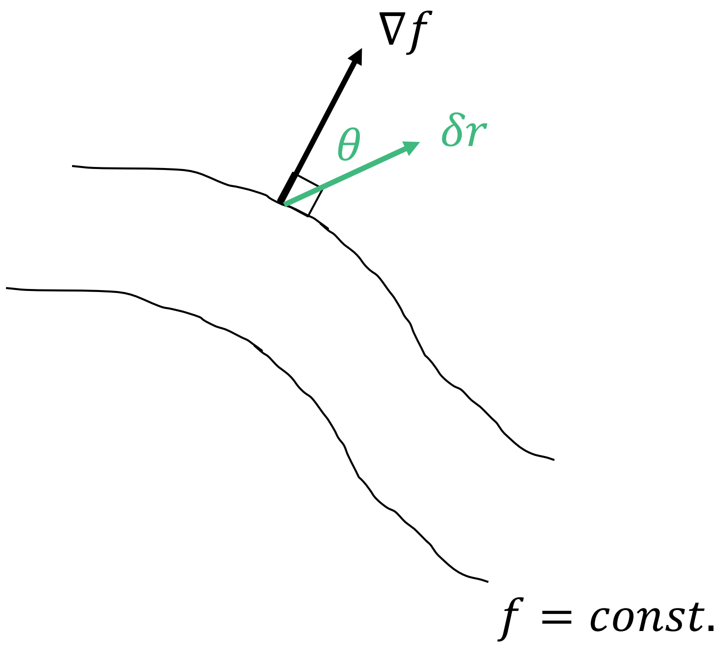
- **Magnitude**: $|\nabla f| = \delta f / |\delta \mathbf{r}|$ (change of f in the direction of largest increase divided by the distance in that direction)

- **Direction**: points in the direction of max change of function (perpendicular to contours):

$\delta\mathbf{r}$ random: $\delta f = |\nabla f||\delta\mathbf{r}|\cos\theta$

$\delta\mathbf{r}$ // f_{const} $(\theta = 90)$: $\delta\mathbf{r} \perp \nabla f \rightarrow \delta f = 0$

$\delta\mathbf{r} \perp f_{const}$ $(\theta = 0)$: $\delta f = |\nabla f||\delta\mathbf{r}|$

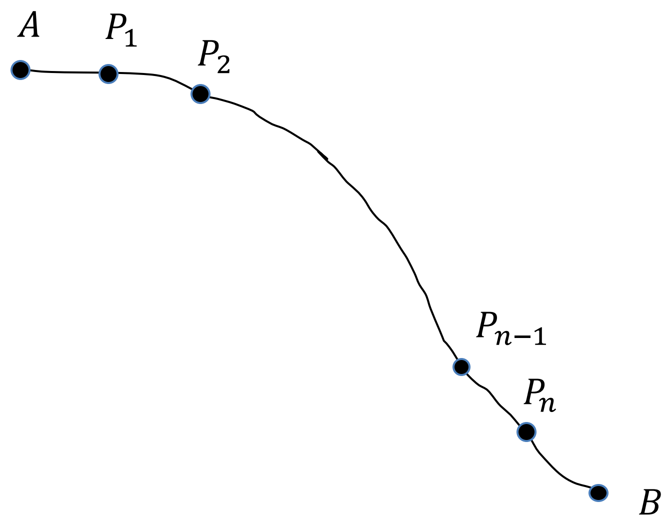


Gradient - interpretation

We can use the concept of gradient to **integrate lines**

$$\delta f = f_B - f_A = (f_B - f_{P_n}) + \cdots + (f_{P_1} - f_A) =$$

$$\underset{n \to \infty}{\sum \nabla f \cdot \delta \boldsymbol{r}} = \int_{\boldsymbol{A}}^{\boldsymbol{B}} \nabla f \cdot d\boldsymbol{r}$$



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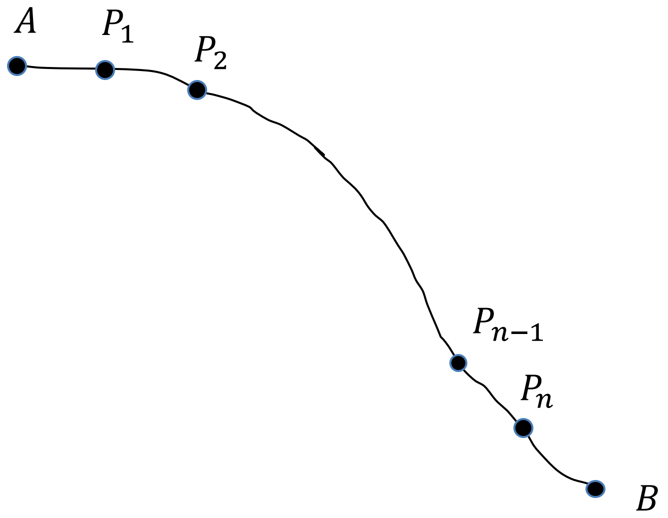


$$\delta f = f_B - f_A = (f_B - f_{P_n}) + \cdots + (f_{P_1} - f_A) =$$

$$\sum \nabla f \cdot \delta \boldsymbol{r} = \int_{\boldsymbol{A}}^{\boldsymbol{B}} \nabla f \cdot d\boldsymbol{r}$$

$$n \to \infty$$

! Can compute the change of a function between two points provided you know the gradient all along !

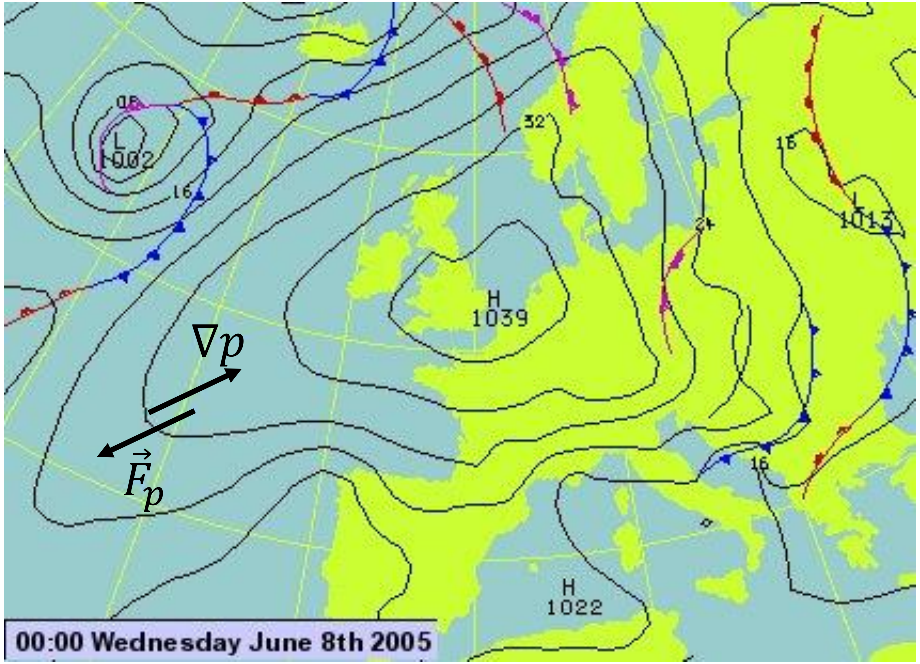
Generalization of the concept of integration in 1D: $\mathrm{d}\boldsymbol{r} = \mathrm{dx}\mathbf{i}_{\boldsymbol{x}} \to f_B - f_A = \int_A^B \frac{\partial f}{\partial x} dx$

Pressure force – definition

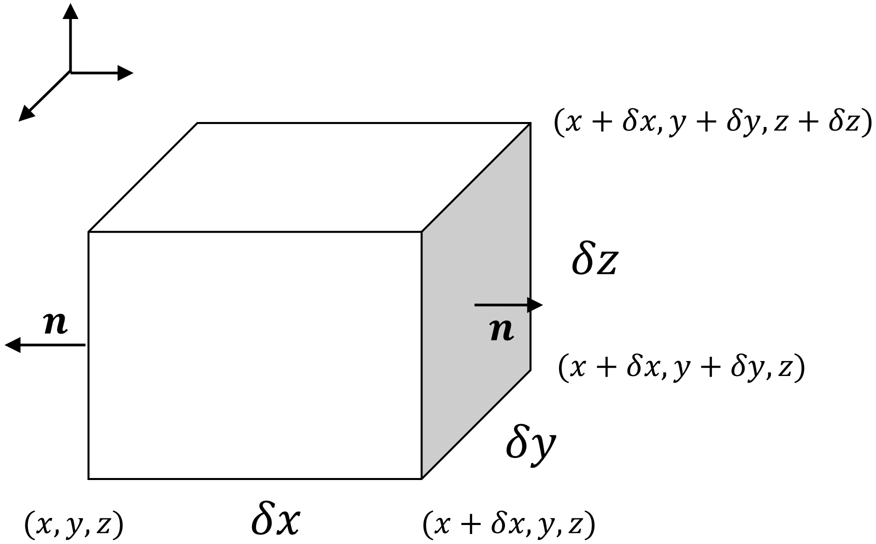
The gradient of the pressure field is:

$$\vec{F}_p = -\nabla p$$

This means that given a pressure field $p(x, y)$ that is non-constant, there is always a force that pushes in the direction of low pressure (i.e., high to low)



Pressure force – derivation



In the x-direction, the net force is:

$$\mathbf{F}_{\mathrm{x}} = \mathbf{F}_{\mathrm{left}} + \mathbf{F}_{\mathrm{right}}$$

$$\mathbf{F}_{\mathrm{left}} = \mathrm{p}(x, y, z)\delta y \delta z \mathbf{i}_x$$
$$\mathbf{F}_{\mathrm{right}} = -\mathrm{p}(x+\delta x, y, z)\delta y \delta z \mathbf{i}_x$$

So:

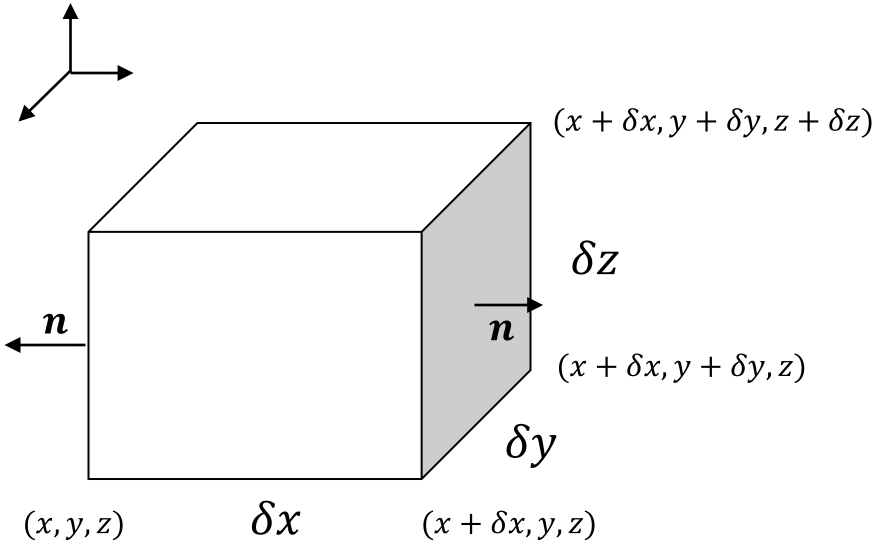
$$\mathbf{F}_{\mathrm{x}} = -[\mathrm{p}(x+\delta x, y, z) - \mathrm{p}(x, y, z)]\delta y \delta z \mathbf{i}_x$$

$$= -\frac{\partial p}{\partial x}\delta x \delta y \delta z \mathbf{i}_x = -\frac{\partial p}{\partial x}\delta V \mathbf{i}_x$$

Repeating for y and z:

$$\mathbf{F} = -\frac{\partial p}{\partial x}\delta V \mathbf{i}_x - \frac{\partial p}{\partial y}\delta V \mathbf{i}_y - \frac{\partial p}{\partial z}\delta V \mathbf{i}_z$$

Pressure force – derivation



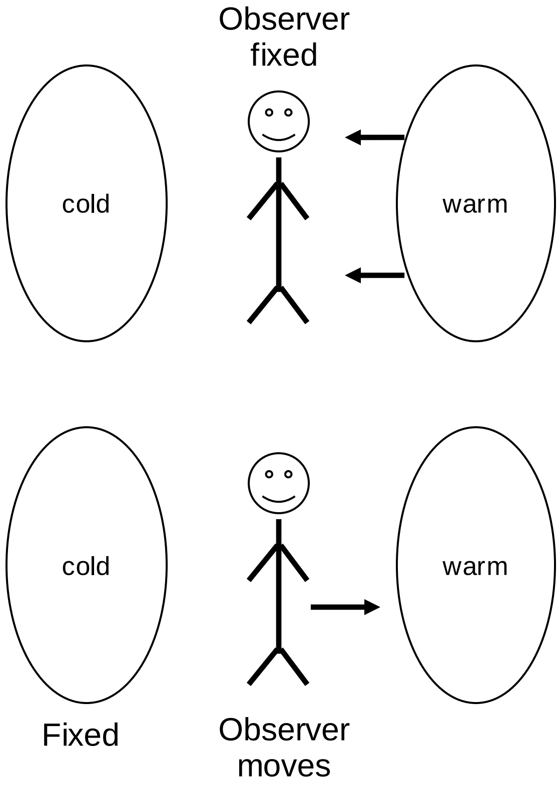
Repeating for y and z:

$$\mathbf{F} = -\frac{\partial p}{\partial x}\delta V \mathbf{i}_x - \frac{\partial p}{\partial y}\delta V \mathbf{i}_y - \frac{\partial p}{\partial z}\delta V \mathbf{i}_z$$

Which gives:

$$\mathbf{F}_\mathrm{p} = \frac{\mathbf{F}}{\delta V} = -\frac{\partial p}{\partial x}\mathbf{i}_x - \frac{\partial p}{\partial y}\mathbf{i}_y + \frac{\partial p}{\partial z}\mathbf{i}_z = -\nabla p$$

Total and partial derivatives - definition



Principle of of temporal changes caused by motion in a system

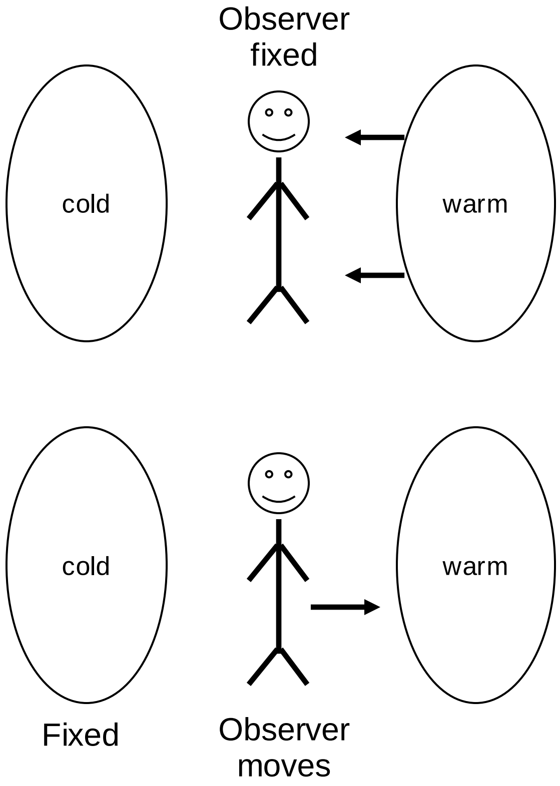
Given a temperature field $T(\mathbf{r}, \mathrm{t})$, we have 2 situations:

1. Temperature field moves right to left and observer is fixed; the observer perceives a change through time

$$\frac{\partial T(\mathbf{r}, \mathrm{t})}{\partial t} = \lim_{\partial t \to 0} \frac{T(\mathbf{r}, \mathrm{t} + \delta t) - T(\mathbf{r}, \mathrm{t})}{\delta t}$$

Partial derivative

Total and partial derivatives - definition



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Partial derivative

2.Temperature field is fixed ($\partial T/\partial t$=0), but observer moves and still experiences an increase in temperature over time

$$\frac{dT(\mathbf{r},\mathrm{t})}{dt} = \lim_{\partial t \to 0} \frac{T(\mathbf{r}(\mathrm{t}+\delta t), \mathrm{t} \cancel{+ \delta t}) - T(\mathbf{r}, \mathrm{t})}{\delta t}$$

Total derivative

$$= \lim_{\partial t \to 0} \frac{\nabla T \cdot \boldsymbol{\delta}\mathbf{r}}{\delta t} \qquad \delta\mathbf{r} = \mathbf{r}(\mathrm{t}+\delta t)\text{-}\mathbf{r}(\mathrm{t})$$

Total and partial derivatives - definition

In general, the two effects could be interwined:

$$\frac{dT(\mathbf{r},\mathrm{t})}{dt} = \lim_{\partial t \to 0} \frac{T(x(\mathrm{t}+\delta t), y(\mathrm{t}+\delta t), \mathrm{z}(\mathrm{t}+\delta t), \mathrm{t}+\delta t) - T(\mathbf{r},\mathrm{t})}{\delta t}$$

which becomes :

Lagrangian view

$$\delta x = x(\mathrm{t}+\delta t) - x(\mathrm{t}) \approx \frac{\partial x}{\partial t}\delta t = v_x \delta t$$

$$\delta y = y(\mathrm{t}+\delta t) - y(\mathrm{t}) \approx \frac{\partial y}{\partial t}\delta t = v_y \delta t \longrightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x} v_x + \frac{\partial T}{\partial y} v_y + \frac{\partial T}{\partial z} v_z + \frac{\partial T}{\partial t} = (\vec{v} \cdot \nabla f) + \frac{\partial T}{\partial t}$$

$$\delta z = \cdots$$

Eulerian view

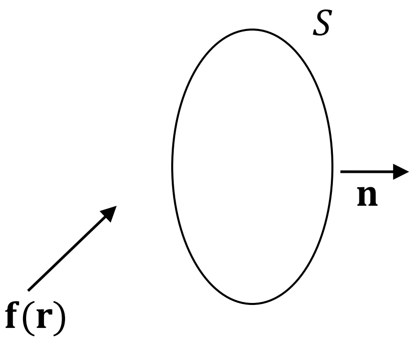
Divergence - definition

Given a generic vectorial function $\boldsymbol{f}$, we define the divergence as:

$$\nabla \cdot \boldsymbol{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \qquad \mathbb{R}^3 \rightarrow \mathbb{R}$$

Divergence - definition

To provide a physical interpretation of the divergence, we need to introduce the flux of a vectorial field:



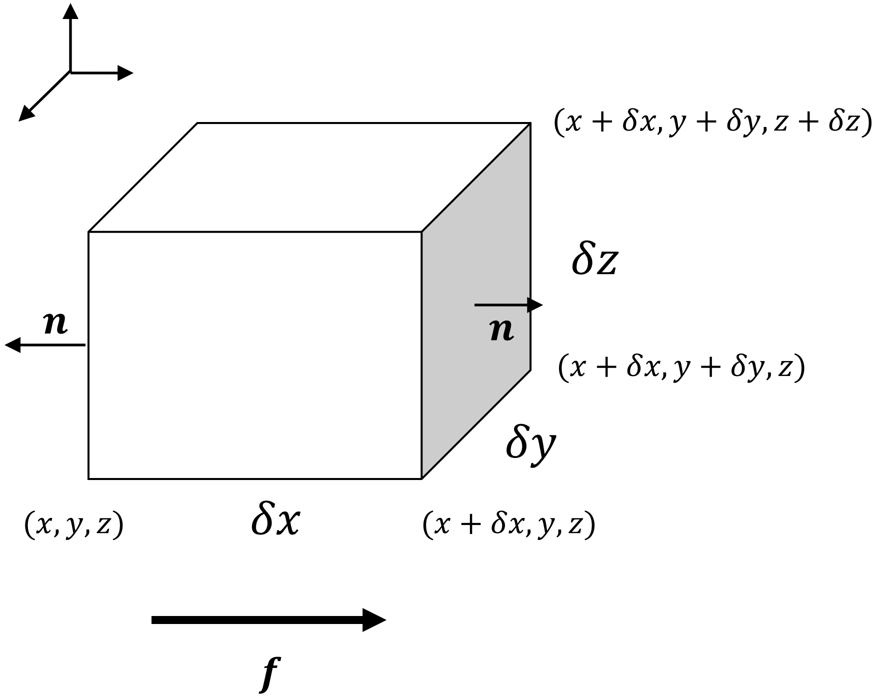
e.g., fluid with constant density

The volume of the fluid that flows in the surface S per unit of time is called flux Φ:

$$\Phi = \iint \mathbf{f} \cdot \mathbf{n}\, \mathrm{d}S = \iint \mathbf{f} \cdot \mathbf{dS}$$

where $\mathbf{dS} = \mathrm{d}S\mathbf{n}$

Divergence - derivation



Outward flux on the right surface

$$\mathrm{d}\Phi_{\mathrm{right}} = f_x(x+\delta x, y, z)\delta y \delta z$$

Outward flux on the left surface

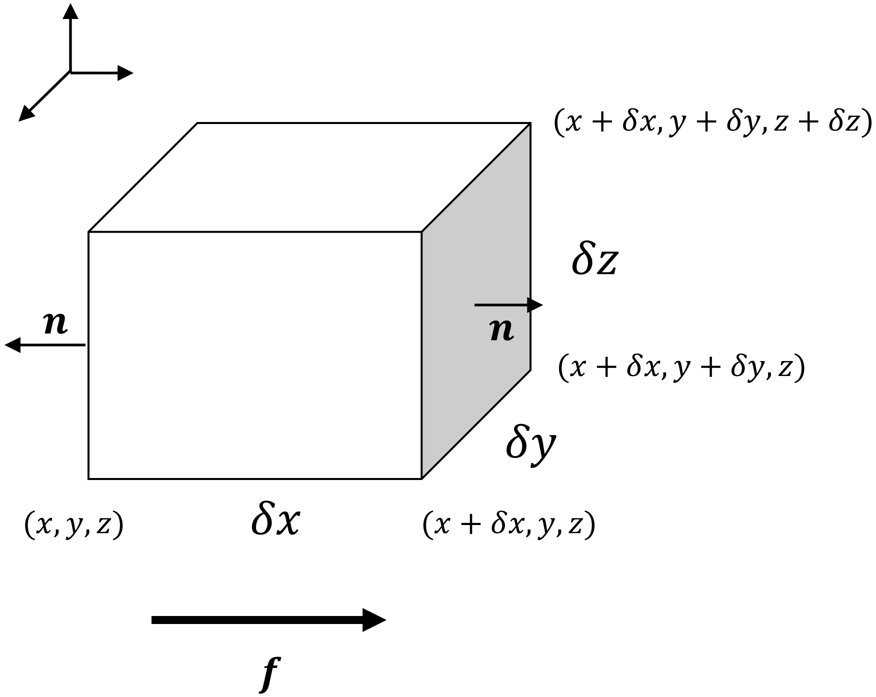
$$\mathrm{d}\Phi_{\mathrm{left}} = -f_x(x, y, z)\delta y \delta z$$

Total on the side surfaces

$$\mathrm{d}\Phi_{\mathrm{x}} = [f_x(x+\delta x, y, z) - f_x(x, y, z)]\delta y \delta z$$

$$= \frac{\partial f_x}{\partial x}\delta x \delta y \delta z = \frac{\partial f_x}{\partial x}\delta V$$

Divergence - derivation



Repeating for y and z:

$$\mathrm{d}\Phi_{\boldsymbol{f}} = \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right)\delta V = \nabla \cdot \boldsymbol{f}\,\delta V$$

$$\updownarrow$$

$$\boxed{\nabla \cdot \boldsymbol{f} = \frac{\mathrm{d}\Phi_{\boldsymbol{f}}}{\delta V}}$$

The divergence of a vector field is equivalent to the outward flux of the vector field per unit of volume

Curl - definition

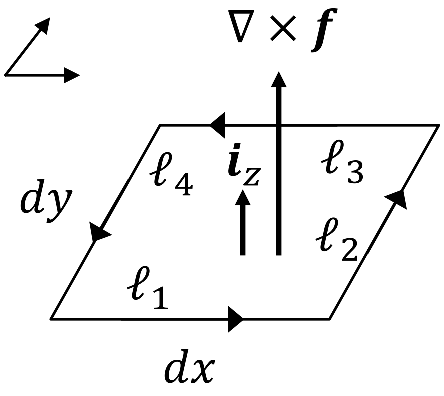
Given a generic vectorial function $\boldsymbol{f}$, we define the curl as:

$$\nabla \times \boldsymbol{f} = \begin{vmatrix} \boldsymbol{i}_x & \boldsymbol{i}_y & \boldsymbol{i}_z \\ \partial x & \partial y & \partial z \\ f_x & f_y & f_z \end{vmatrix} \qquad \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

Determinant

Curl - definition

To provide a physical interpretation of the curl, we need to introduce the infinitesimal circulation of a vectorial field:



Assuming that the curl is aligned with $\boldsymbol{i}_z$:

$$\nabla \times \boldsymbol{f} = (\partial_x f_y - \partial_y f_x)\boldsymbol{i}_z$$

Let's now consider the circulation defined as:

$$\oint_C \boldsymbol{f} \cdot d\boldsymbol{r} = \sum_{i=1}^{4} \int_{\ell_i} \boldsymbol{f} \cdot d\boldsymbol{r} =$$

Taylor expansion and higher order

$$= f_x\,(x,y)dx + f_y(x+\delta x, y)dy - f_x(x \cancel{+\delta x}, y+\delta y)dx - f_y(x, y \cancel{+\delta y})dy$$

$$= (-\partial_y f_x\,(x,y) - \partial_x f_y(x,y))dx\,dy$$

Curl - definition

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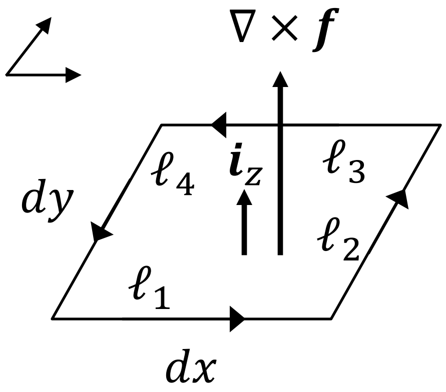


We get:

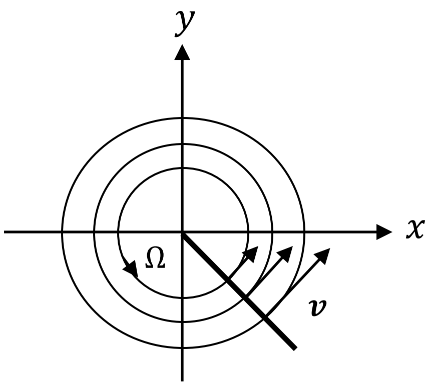
$$(\nabla \times \boldsymbol{f})_z = -\frac{\oint_C \boldsymbol{f} \cdot d\boldsymbol{r}}{dxdy}$$

Any component of the curl $\nabla \times \boldsymbol{f}$ in a given direction is equivalent to the closed line integral of the function $\boldsymbol{f}$ itself along a closed path perpendicular to such direction divided by the unit surface area.

Curl - interpretation

The curl has usually 2 components:

rigid rotation: fluid moves in (x,y)



Ω: angular velocity $\quad$ $\mathrm{T} = 2\pi/\Omega$: period

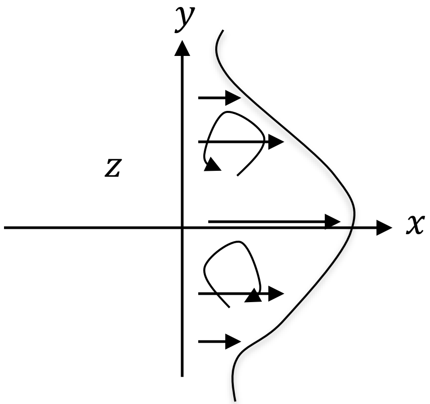
$\boldsymbol{v} = \Omega \mathbf{r}$ $\quad$ velocity increases linearly with distance

$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = 2\Omega \mathbf{i}_z$ $\quad$ vorticity is twice the rotational rate

Curl - interpretation

The curl has usually 2 components:

shearing: flow only in x but depends only by y



$$v_x = f(x),\ v_y = v_z = 0$$

$$\nabla \times \boldsymbol{v} = -\partial_y f_x \boldsymbol{i}_z$$