7. Raytracing

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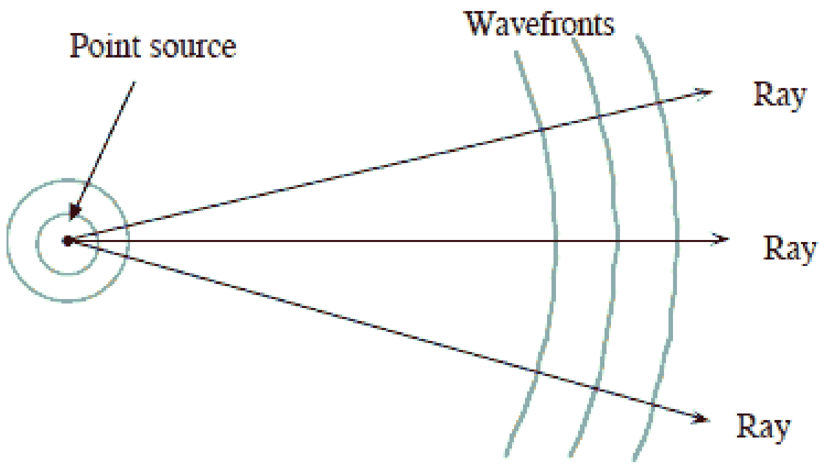
ERSE 210 Seismology

Rays and waves

Rays are a powerful concept to describe the propagation of any kind of wave; in fact, they originated from the field of **geometrical optics**.



Ray $\perp$ Wavefront

Rays and waves

Rays are a powerful concept to describe the propagation of any kind of wave; in fact, they originated from the field of **geometrical optics**.

→ Can be used to describe the propagation of waves across an interface (Snell's law) – previous lecture…

→ Can be used to describe the propagation of waves in heterogenous media (Eikonal equation law) – this lecture

Rays and waves

Rays are a powerful concept to describe the propagation of any kind of wave; in fact, they originated from the field of **geometrical optics**.

→ Can be used to describe the propagation of waves across an interface (Snell's law) – previous lecture…

→ Can be used to describe the propagation of waves in heterogenous media (Eikonal equation law) – this lecture

Pros: intuitive, efficient, relatively easy to generalize to 3D

Cons: high-frequency approximation (inaccurate at long wavelengths or steep velocity changes), only kinematic part of wavefield (at least for basic theories

Eikonal equation

Can be easily derived (next slides) from the acoustic wave equation (or elastic wave equation for P or S waves):

$$\underbrace{|\nabla T|^2}_{\text{Traveltime}} = \frac{1}{\alpha^2}$$

→ Shows (once again) that the traveltime of a wavefield is purely driven by the velocity of the medium (no density!)

Eikonal equation - derivation

Let us start from the wave equation for a P-wave:

$$\nabla^2\phi - \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2} = 0$$

Define the armonic solution to the w.e.: $\phi(\boldsymbol{x},t) = A(\boldsymbol{x})e^{-j\omega(t-T(\boldsymbol{x}))}$ A: amplitude, T:phase

Let's compute $\nabla^2\phi$ for this solution:

$$\nabla\,\phi = \nabla A e^{-j\omega(t-T)} + A(j\omega\nabla T)e^{-j\omega(t-T)}$$

$$\nabla^2\phi = \nabla^2 A e^{-j\omega(t-T)} + \nabla A j\omega\nabla T e^{-j\omega(t-T)} + j\omega\nabla A\nabla T e^{-j\omega(t-T)} + j\omega A\nabla^2 T e^{-j\omega(t-T)} + j\omega A\nabla T(j\omega\nabla T)e^{-j\omega(t-T)}$$

$$\nabla^2\phi = [\nabla^2 A - \omega^2 A|\nabla T|^2 + j[2\omega\nabla A\nabla T + \omega A\nabla^2 T]\,]e^{-j\omega(t-T)}$$

Eikonal equation - derivation

Let's now compute $\partial^2\phi/\partial t^2$ for this solution:

$$\partial\phi/\partial t = A(-j\omega)e^{-j\omega(t-T)}$$

$$\partial^2\phi/\partial t^2 = -Aj\omega(-j\omega)e^{-j\omega(t-T)} = -A\omega^2 e^{-j\omega(t-T)}$$

Putting them together:

$$\nabla^2 A - \omega^2 A|\nabla T|^2 + j[2\omega\nabla A\nabla T + \omega A\nabla^2 T] = -\frac{A\omega^2}{\alpha^2}$$

Real part ↙ ↘ Imag part

$$\nabla^2 A - \omega^2 A|\nabla T|^2 = -\frac{A\omega^2}{\alpha^2} \qquad 2\omega\nabla A\nabla T + \omega A\nabla^2 T = 0$$

Eikonal equation - derivation

From the real part:

$$\nabla^2 A - \omega^2 A|\nabla T|^2 = -\frac{A\omega^2}{\alpha^2} \xrightarrow[* \, 1/A\omega^2]{} |\nabla T|^2 = \frac{\nabla^2 A}{A\omega^2} + \frac{1}{\alpha^2}$$

Under a high-frequency assumption $(\omega \to \infty, 1/\omega \to 0)$

$$\boxed{|\nabla T|^2 = \left(\frac{\partial T}{\partial_x}\right)^2 + \left(\frac{\partial T}{\partial_y}\right)^2 + \left(\frac{\partial T}{\partial_z}\right)^2 = \frac{1}{\alpha^2} = p_\alpha^2}$$

where T is the wavefront, $\nabla T = \mathbf{p}$ is the ray direction

→ the same equation can be derived for the S-waves.

Eikonal equation - derivation

From the imaginary part:

$$2\omega\nabla A\nabla T + \omega A\nabla^2 T = 0 \xrightarrow{\nabla T = \mathbf{p} = \mathrm{p}_\alpha\, \mathbf{i}_{\mathrm{p}_\alpha}} 2\mathrm{p}_\alpha \mathbf{i}_{\mathrm{p}_\alpha}\nabla A = -A\nabla\cdot(\mathrm{p}_\alpha \mathbf{i}_{\mathrm{p}_\alpha}) \longrightarrow A = -\frac{2\mathrm{p}_\alpha\nabla A\, \mathbf{i}_{\mathrm{p}_\alpha}}{\nabla\cdot(\mathrm{p}_\alpha \mathbf{i}_{\mathrm{p}_\alpha})}$$

Integrating along the ray path:

$$\boxed{A = e^{-\frac{1}{2}\int_{path}\frac{\nabla\cdot(\mathrm{p}_\alpha \mathbf{i}_{\mathrm{p}_\alpha})}{\mathrm{p}_\alpha}d\ell} = \left(\frac{p_0}{p_\alpha}\right)^{1/2} e^{-\frac{1}{2}\int_{path}\nabla\cdot\mathbf{i}_{\mathrm{p}_\alpha}d\ell}}$$

Negative and real exponent: decay in amplitude along the ray

where p is the slowness at the start of the ray and p_α is the slowness at the end of the ray

Combining the results from the real and imaginary parts, we get:

$$\phi = \underbrace{\left(\frac{u_0}{u_\alpha}\right)^{1/2} e^{-\frac{1}{2}\int_{path}\nabla\cdot\mathbf{i}_{\mathrm{p}_\alpha}d\ell}}_{\text{Geometrical spreading}} e^{-j\omega \overbrace{\int_{path} p_\alpha d\ell}^{\text{Traveltime}}}$$

Eikonal equation in action

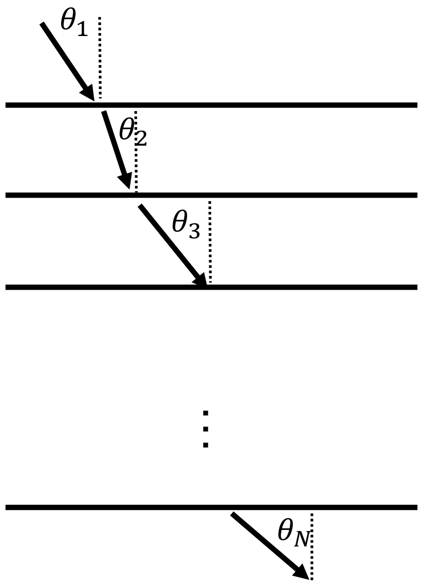
We are going to consider the following 3 scenarios:

→ Horizontally layered medium in Cartesian coordinates
- Near-surface and applied seismology (anything in the upper 30m)

→ Spherical medium (i.e., Earth)
- Global seismology (must take into account the Earth's sphericity.

→ Eikonal equation in general heterogenous medium

Raytracing in horizontally layered medium



Recall that the ray parameter (i.e. horizontal component of slowness vector) is constant

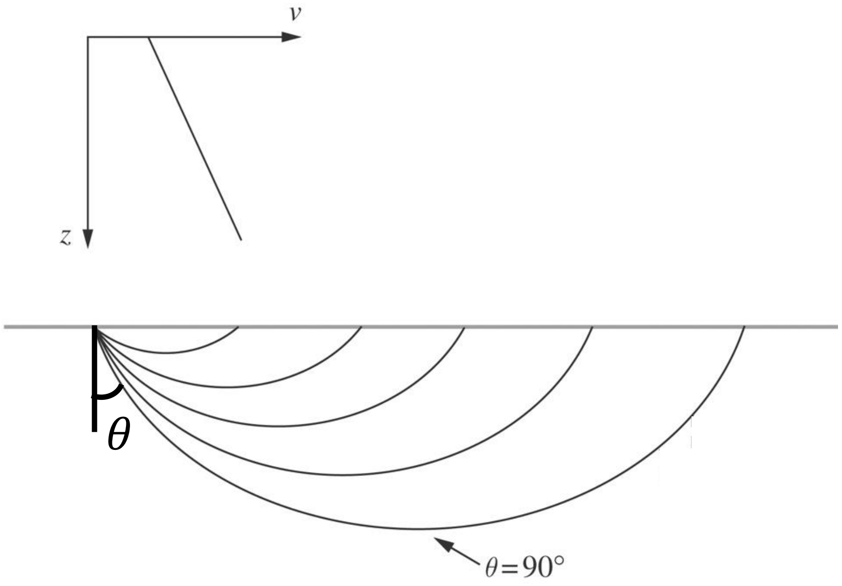
$$p_x = p_1 \sin\theta_1 = p_2 \sin\theta_2 = \ldots = p_N \sin\theta_N$$

If velocity increases with depth, we reach a point where:

$$\frac{p_{i-1}}{p_i} sin\theta_{i-1} = 1 \rightarrow \theta_i = 90$$

so the ray travels horizontally in the interface between layer i-1 and i: **turning point**

Raytracing in horizontally layered medium

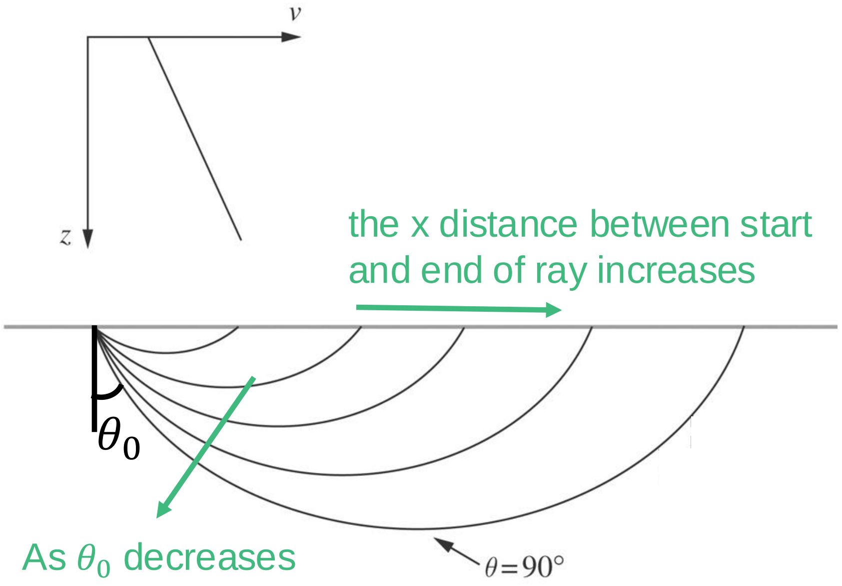


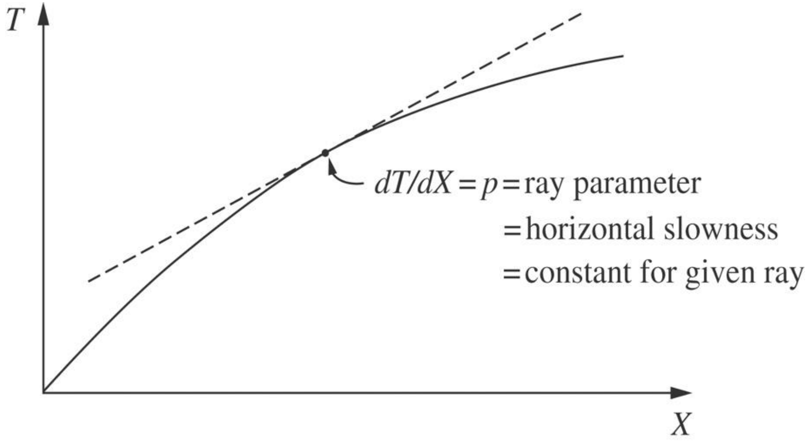
Assuming infinitesimally small layers with gradient as in figure, we know at which depth the ray turns based on the initial angle θ:

$$\frac{1}{v(z_p)} = p_{z_p} = p_0 sin\theta_0$$

so we just need to find at which depth the velocity equals $1/p_0 sin\theta_0$

Raytracing in horizontally layered medium

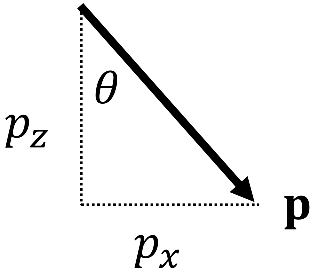




$$\frac{dx}{dp_x} < 0 \rightarrow prograde$$

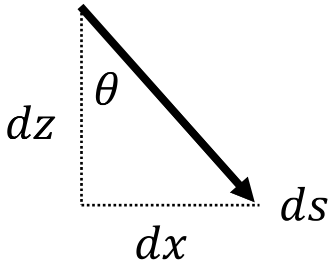
Raytracing in horizontally layered medium – deriv.

Let us recall the equations for the slowness vector



$$|\mathbf{p}| = p = 1/c \qquad p_x = sin\theta/c \qquad p_z = cos\theta/c = \sqrt{p^2 - p_x^2}$$

and write a similar expression for the infinitesimal distance and traveltime components of the ray



$$\frac{dx}{ds} = sin\theta = p_x/p \qquad \frac{dz}{ds} = cos\theta = \frac{1}{p}\sqrt{p^2 - \mathrm{p}_\mathrm{x}^2}$$

Raytracing in horizontally layered medium – deriv.

We can therefore write the infinitesimal horizontal change of the ray given the infinitesimal vertical change

$$\frac{dx}{dz} = \frac{dx}{ds}\frac{ds}{dz} = \frac{\frac{dx}{ds}}{\frac{dz}{ds}} = \frac{\frac{p_x}{p}}{\frac{1}{p}\sqrt{p^2 - \mathrm{p}_\mathrm{x}^2}} = \frac{p_x}{\sqrt{p^2 - \mathrm{p}_\mathrm{x}^2}}$$

If we integrate along two depths:

$$x(z_1, z_2, p_x) = p_x \int_{z_1}^{z_2} \frac{dz}{\sqrt{p(z)^2 - \mathrm{p}_\mathrm{x}^2}}$$

which gives the total horizontal distance travelled by the ray:

$$\boxed{X(p_x) = 2p_x \int_0^{z_p} \frac{dz}{\sqrt{p(z)^2 - \mathrm{p}_\mathrm{x}^2}}}$$

2 since ray is symmetrical

Raytracing in horizontally layered medium – deriv.

Similarly, given that $dt = pds$

$$\frac{dt}{dz} = \frac{dt}{ds}\frac{ds}{dz} = \frac{\frac{dt}{ds}}{\frac{dz}{ds}} = \frac{p}{\frac{1}{p}\sqrt{p^2 - \mathrm{p}_\mathrm{x}^2}} = \frac{p^2}{\sqrt{p^2 - \mathrm{p}_\mathrm{x}^2}}$$

If we integrate along two depths:

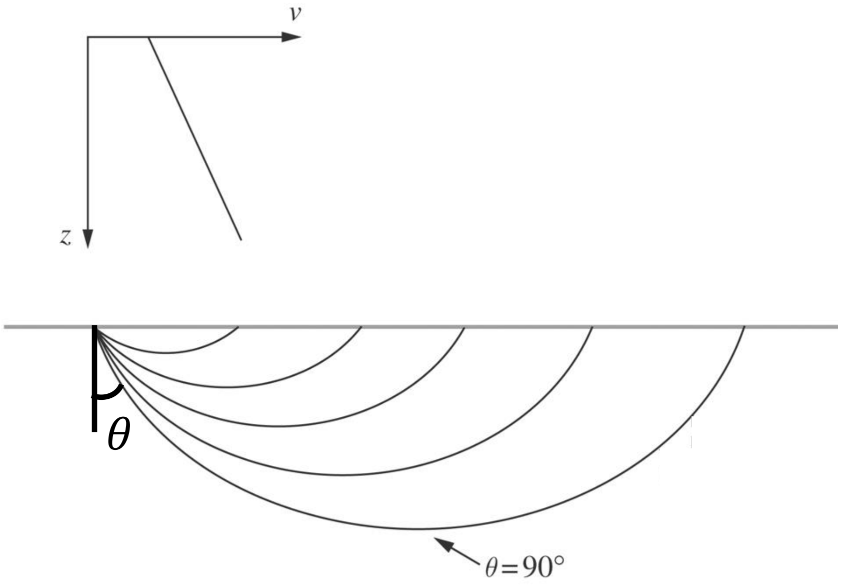
$$\mathrm{t}(z_1, z_2, p_x) = \int_{z_1}^{z_2} \frac{p(z)^2 dz}{\sqrt{p(z)^2 - \mathrm{p}_\mathrm{x}^2}}$$

which gives the total horizontal distance travelled by the ray:

$$\boxed{T(p_x) = 2\int_0^{z_p} \frac{p(z)^2 dz}{\sqrt{p(z)^2 - \mathrm{p}_\mathrm{x}^2}}}$$

2 since ray is symmetrical

Raytracing in horizontally layered medium



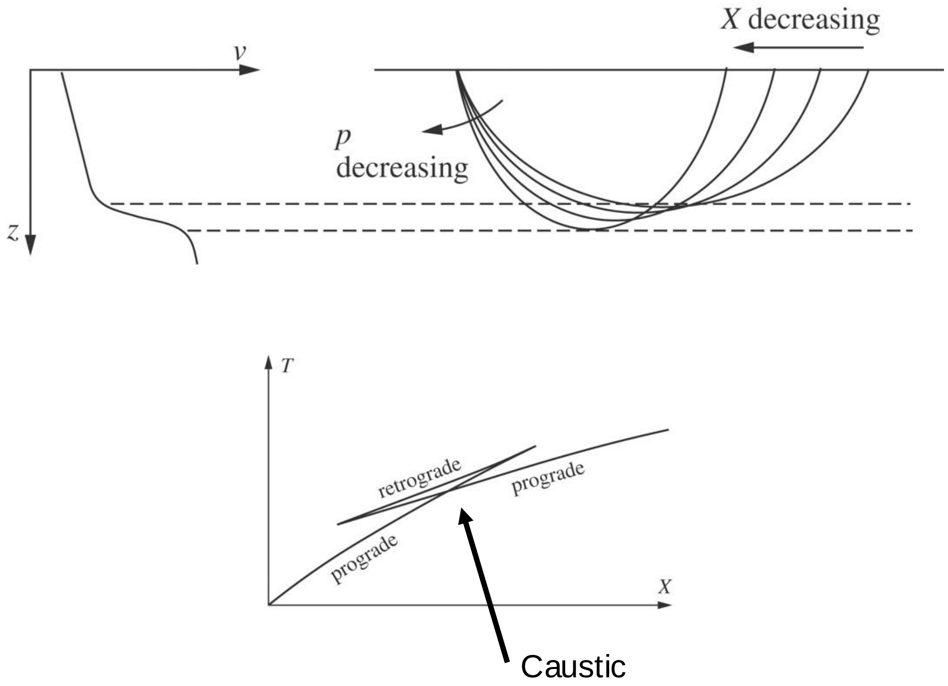
For general $v(z)$ medium

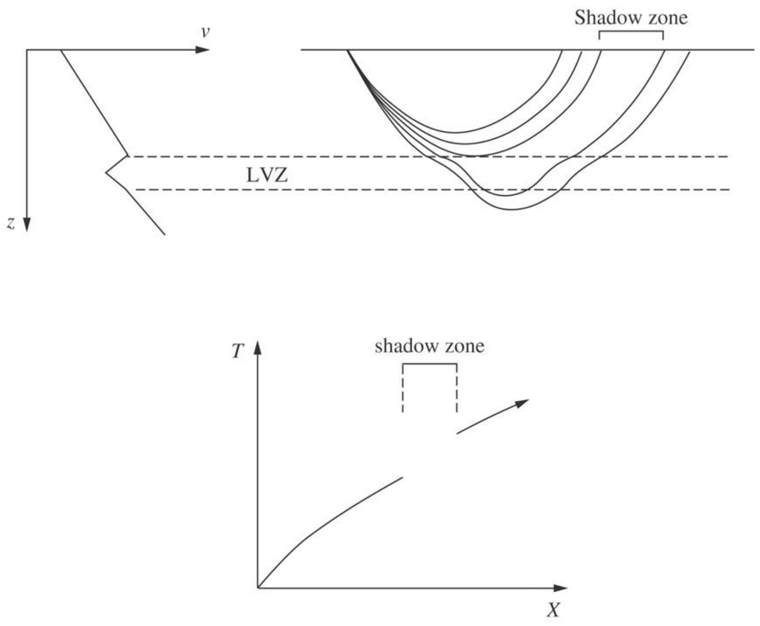
$$X(p_x) = 2p_x \int_0^{z_p} \frac{dz}{\sqrt{p^2(z) - p_x^2}}$$

$$T(p_x) = 2 \int_0^{z_p} \frac{p^2(z)dz}{\sqrt{p^2(z) - p_x^2}}$$

Raytracing in horizontally layered medium

For cases with the velocity gradient slowing down or becoming negative:





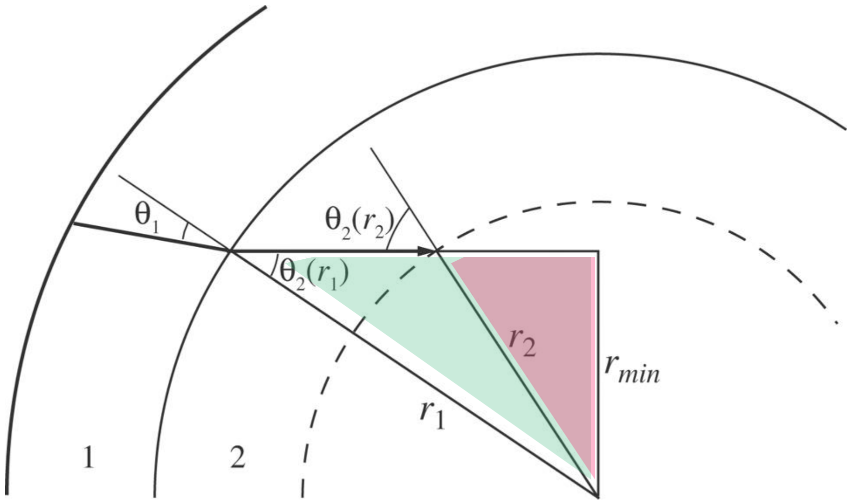
Raytracing in spherical medium

Raytracing in spherical medium (i.e., Earth) can be done via:

→ Use of **spherical coordinates**: change ray parameter $\mathbf{p}$ to account for spherical geometry

→ **Earth flattening transformation**: change velocity as function of depth to compensate for sphericity of the Earth

Raytracing in spherical coordinates



Snell's law needs to take into account that angle changes from entering to exiting a layer:

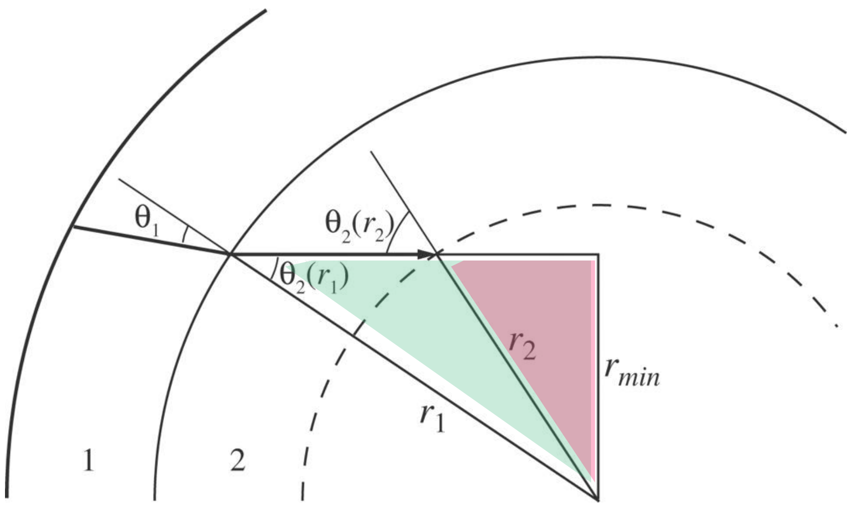
$$\theta_2(r_1) \neq \theta_2(r_2)$$

Indicates the radius with respect to the angle is taken

Starting from Snell's law at interface 1-2:

$$p_x = p_1 sin\theta_1(r_1) = p_2 sin\theta_2\,(r_1)$$

Raytracing in spherical coordinates



And noting we can compute r_{MIN} using both the green and red triangles:

$$r_{MIN} = r_1 \sin\theta_2\,(r_1) = r_2 \sin\theta_2\,(r_2)$$

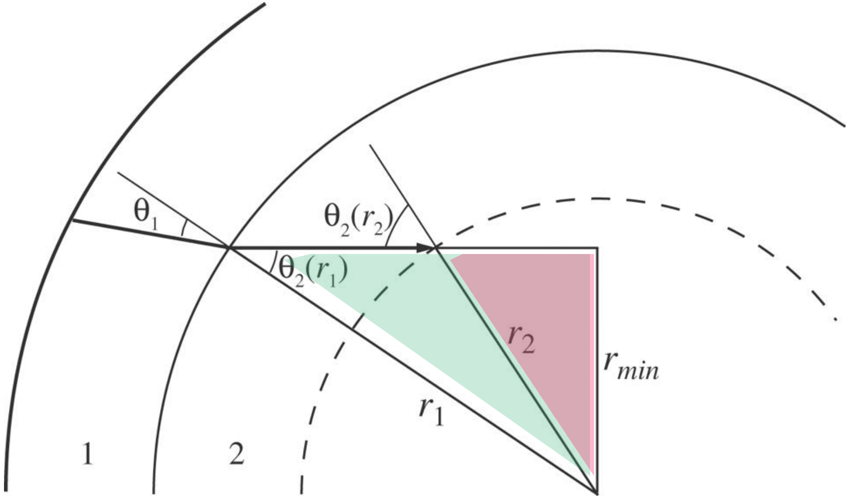
We get: $\sin\theta_2\,(r_1) = \dfrac{r_2}{r_1}\sin\theta_2\,(r_2)$

$$p_x = p_1 sin\theta_1(r_1) = p_2 sin\theta_2\,(r_1)$$

$$\downarrow$$

$$p_x = p_1 sin\theta_1(r_1) = p_2\frac{r_2}{r_1}\sin\theta_2\,(r_2)$$

Raytracing in spherical coordinates



We can therefore write Snell's law in spherical coordinates

$$\boxed{p_1 r_1 sin\theta_1(r_1) = p_2 r_2 \sin\theta_2\,(r_2)}$$

where on both sides of the equal we have incident angles and radius of the earth at the corresponding interface.

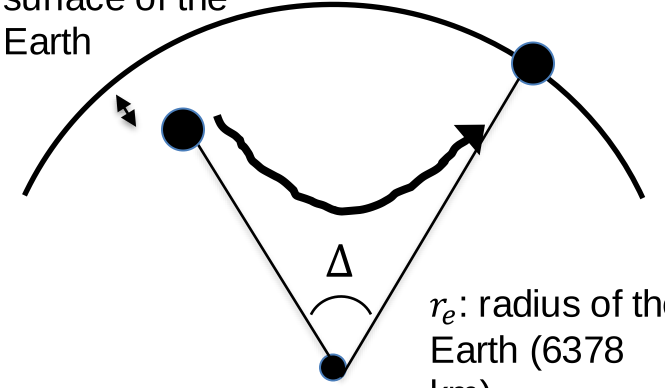
So:

Angle of radius with vertical in rads.

$$p_{sph} = rpsin\theta = dT/d\Delta$$

Raytracing in spherical coordinates

r_{tp}: part of radius from source to surface of the Earth



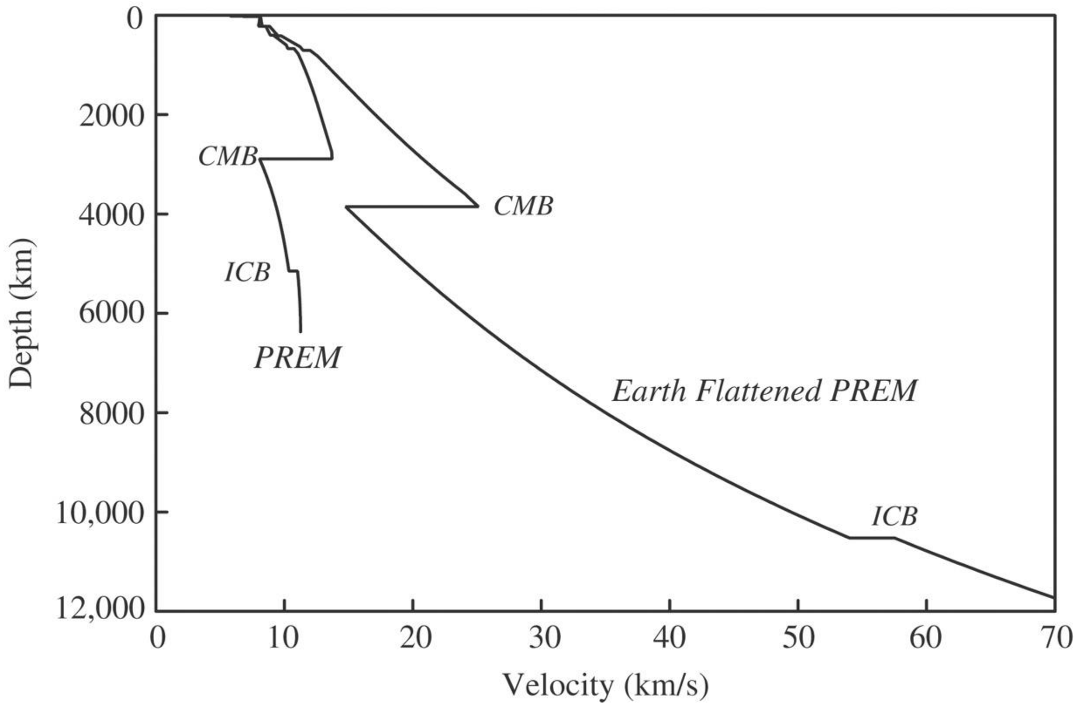
r_e: radius of the Earth (6378 km)

Under the assumption of source/rec at surface:

$$\Delta(p_{sph}) = 2p_{sph}\int_0^{r_e}\frac{1}{\sqrt{(ur)^2 - p_{sph}^2}}\frac{dr}{r}$$

$$T(p_{sph}) = 2\int_0^{r_e}\frac{(ur)^2}{\sqrt{(ur)^2 - p_{sph}^2}}\frac{dr}{r}$$

Earth-flattening transformation



Simply change the velocity in cartesian coordinate relations

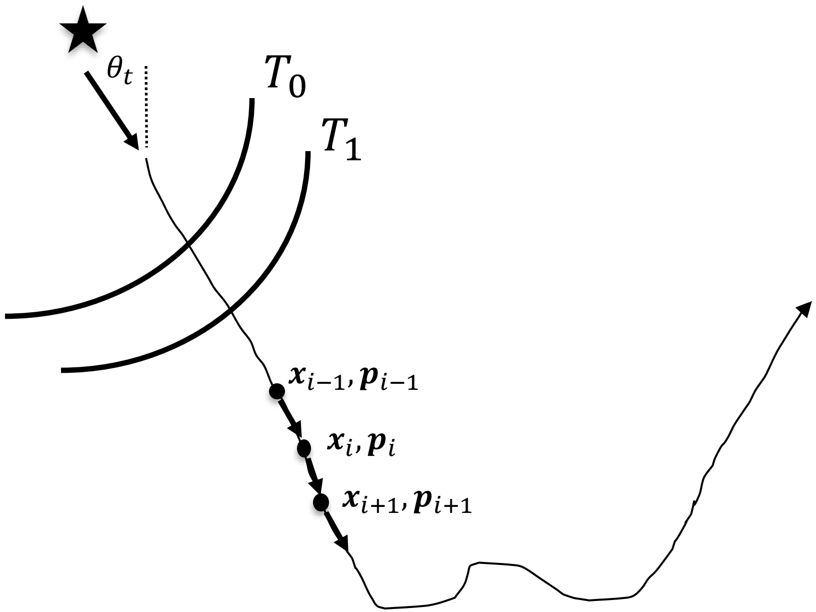
$$z_f = -r_e \ln\left(\frac{r_e - z_s}{r_e}\right)$$

$$v(z_f) = \frac{r_e}{r - z_s} v(r_e - z_s)$$

And then convert X into Δ :

$$\Delta_{deg} = \mathrm{X}_{km} 360/(2\pi r_e)$$

Raytracing in heterogeneous media



Given:

x_S, z_S, θ_t:take-off angle

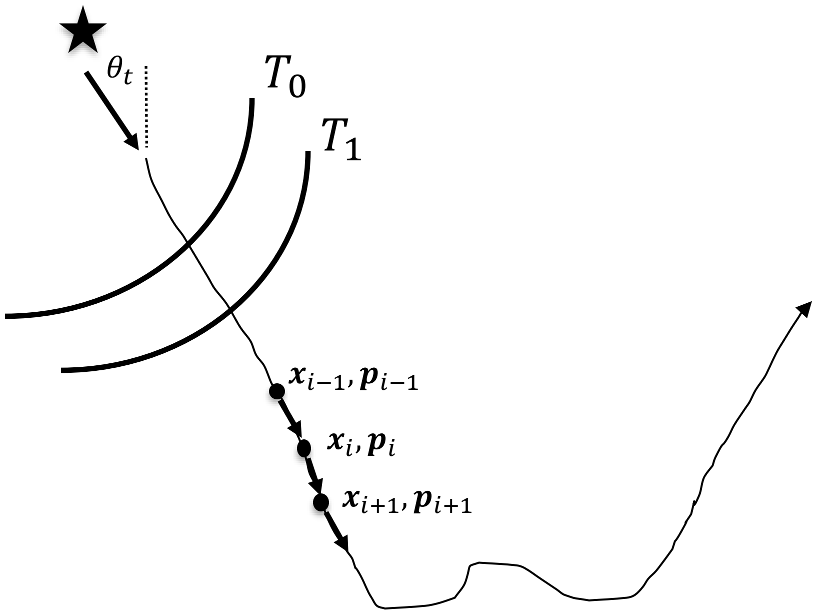
$c(x, z)$: velocity

Find:

$\boldsymbol{x}_1, \boldsymbol{x}_2 \ldots, \boldsymbol{x}_N$: locations

$\boldsymbol{p}_1, \boldsymbol{p}_2 \ldots, \boldsymbol{p}_N$: directions

Raytracing in heterogeneous media



Recall the Eikonal equation:

$$|\nabla T|^2 = \frac{1}{c^2}$$

$T(\boldsymbol{x}) = const$ represents a wavefront

$$\boldsymbol{p_i} = \nabla T(\boldsymbol{x_i})$$

Raytracing in heterogeneous media - derivation

Recall the eikonal equation:

$$|\nabla T|^2 = \frac{1}{c^2} \qquad T(\boldsymbol{x}) = const\text{: wavefront} \qquad \boldsymbol{p_i} = \nabla T(\boldsymbol{x_i})$$

In raytracing we want to track how T changes over the ray path, so we write

$$\boxed{\boldsymbol{p} = \mathrm{p}\frac{\mathrm{d}\mathbf{x}}{\mathrm{ds}} = \nabla T} \rightarrow \mathrm{p}\frac{\mathrm{d}x}{\mathrm{ds}} = \frac{\partial T}{\partial x}, \mathrm{p}\frac{\mathrm{d}y}{\mathrm{ds}} = \frac{\partial T}{\partial y}, \mathrm{p}\frac{\mathrm{dz}}{\mathrm{ds}} = \frac{\partial T}{\partial z}$$

And we evaluate

$$\frac{\mathrm{d}\nabla T}{\mathrm{ds}(\mathbf{x})} = \frac{\mathrm{d}}{\mathrm{ds}(\mathbf{x})}\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) \rightarrow \frac{\mathrm{d}}{\mathrm{ds}(\mathbf{x})}\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \mathrm{s}(\mathbf{x})\partial x} = \frac{\partial^2 T}{\partial x^2}\frac{\mathrm{dx}}{\mathrm{ds}(\mathbf{x})} + \frac{\partial^2 T}{\partial x \partial y}\frac{\mathrm{dy}}{\mathrm{ds}(\mathbf{x})} + \frac{\partial^2 T}{\partial x \partial z}\frac{\mathrm{dz}}{\mathrm{ds}(\mathbf{x})}$$

Raytracing in heterogeneous media - derivation

Recall the eikonal equation:

$$|\nabla T|^2 = \frac{1}{c^2} \qquad T(\boldsymbol{x}) = const\text{: wavefront} \qquad \boldsymbol{p_i} = \nabla T(\boldsymbol{x_i})$$

In raytracing we want to track how T changes over the ray path, so we write

$$\boldsymbol{p} = \mathrm{p}\frac{\mathrm{d}\mathbf{x}}{\mathrm{ds}} = \nabla T \rightarrow \mathrm{p}\frac{\mathrm{d}x}{\mathrm{ds}} = \frac{\partial T}{\partial x}, \mathrm{p}\frac{\mathrm{d}y}{\mathrm{ds}} = \frac{\partial T}{\partial y}, \mathrm{p}\frac{\mathrm{dz}}{\mathrm{ds}} = \frac{\partial T}{\partial z}$$

And we evaluate

$$\frac{\mathrm{d}\nabla T}{\mathrm{ds}(\mathbf{x})} = \frac{\mathrm{d}}{\mathrm{ds}(\mathbf{x})}\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) \rightarrow \frac{\mathrm{d}}{\mathrm{ds}(\mathbf{x})}\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \mathrm{s}(\mathbf{x})\partial x} = \frac{\partial^2 T}{\partial x^2}\frac{\mathrm{dx}}{\mathrm{ds}(\mathbf{x})} + \frac{\partial^2 T}{\partial x \partial y}\frac{\mathrm{dy}}{\mathrm{ds}(\mathbf{x})} + \frac{\partial^2 T}{\partial x \partial z}\frac{\mathrm{dz}}{\mathrm{ds}(\mathbf{x})}$$

Raytracing in heterogeneous media - derivation

We get

$$\frac{\partial T}{\partial s(\mathbf{x})\partial x} = \frac{1}{p}\left(\frac{\partial^2 T}{\partial x^2}\frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial y^2}\frac{\partial T}{\partial y} + \frac{\partial^2 T}{\partial z^2}\frac{\partial T}{\partial z}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial i}\right)^2 = 2\frac{\partial f}{\partial i}\frac{\partial^2 f}{\partial x \partial i} \longrightarrow \quad = \frac{1}{2p}\frac{\partial}{\partial x}\left(\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right)$$

Eikonal equation $\longrightarrow$ $$= \frac{1}{2p}\frac{\partial}{\partial x}(p^2) = \frac{1}{2p}2p\frac{\partial p}{\partial x} = \frac{\partial p}{\partial x}$$

Similarly: $$\frac{\partial T}{\partial s(\mathbf{x})\partial y} = \frac{\partial p}{\partial y} \qquad \frac{\partial T}{\partial s(\mathbf{x})\partial z} = \frac{\partial p}{\partial z}$$

$$\boxed{\frac{d}{\mathrm{ds}(\mathbf{x})}\nabla T = \nabla \mathrm{p}}$$

Raytracing in heterogeneous media - derivation

Using the two previously found relations

$$\mathbf{p} = \mathrm{p}\frac{\mathrm{d}}{\mathrm{ds}(\mathbf{x})}\mathbf{x} = \nabla T \qquad \frac{\mathrm{d}}{\mathrm{ds}(\mathbf{x})}\mathbf{p} = \nabla \mathrm{p}$$

We have now an ordinary ODE for $\mathbf{x}$ and $\mathbf{p}$:

$$\boxed{\begin{cases} \dfrac{\mathrm{d}\mathbf{x}}{\mathrm{ds}} = \dfrac{\mathbf{p}}{\mathrm{p}(\mathbf{x})} \\ \dfrac{\mathrm{d}\mathbf{p}(\mathbf{x})}{\mathrm{ds}} = \nabla \mathrm{p}(\mathbf{x}) \\ \mathbf{x}(s=0) = \mathbf{x}_0 \\ \mathbf{p}(s=0) = \mathbf{p}_0 \\ \dfrac{\mathrm{dT}}{\mathrm{ds}} = \mathrm{p}(\mathbf{x}) \end{cases}} \xrightarrow{\text{In 2D}} \begin{cases} \dfrac{\mathrm{dx}}{\mathrm{ds}} = \dfrac{\mathrm{p_x}}{\mathrm{p}(\mathbf{x})} \\ \dfrac{\mathrm{dz}}{\mathrm{ds}} = \dfrac{\mathrm{p_z}}{\mathrm{p}(\mathbf{x})} \\ \dfrac{\mathrm{dp_x}}{\mathrm{ds}} = \dfrac{\partial \mathrm{p}(\mathbf{x})}{\partial x} \\ \dfrac{\mathrm{dp_z}}{\mathrm{ds}} = \dfrac{\partial \mathrm{p}(\mathbf{x})}{\partial z} \end{cases} + \begin{matrix} x(s=0) = x_0 \\ z(s=0) = z_0 \\ p_x(s=0) = p_{x0} \\ p_z(s=0) = p_{z0} \end{matrix}$$