

## 4. Elastic Wave Equation

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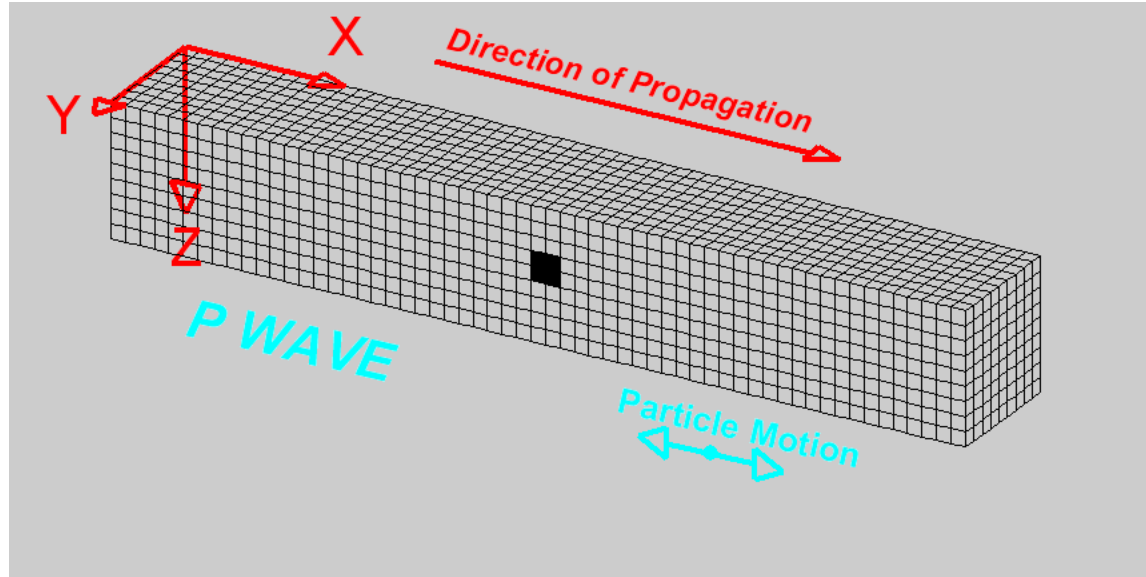
ERSE 210 Seismology

# Elastic wave propagation

Two types of waves propagate in elastic media:

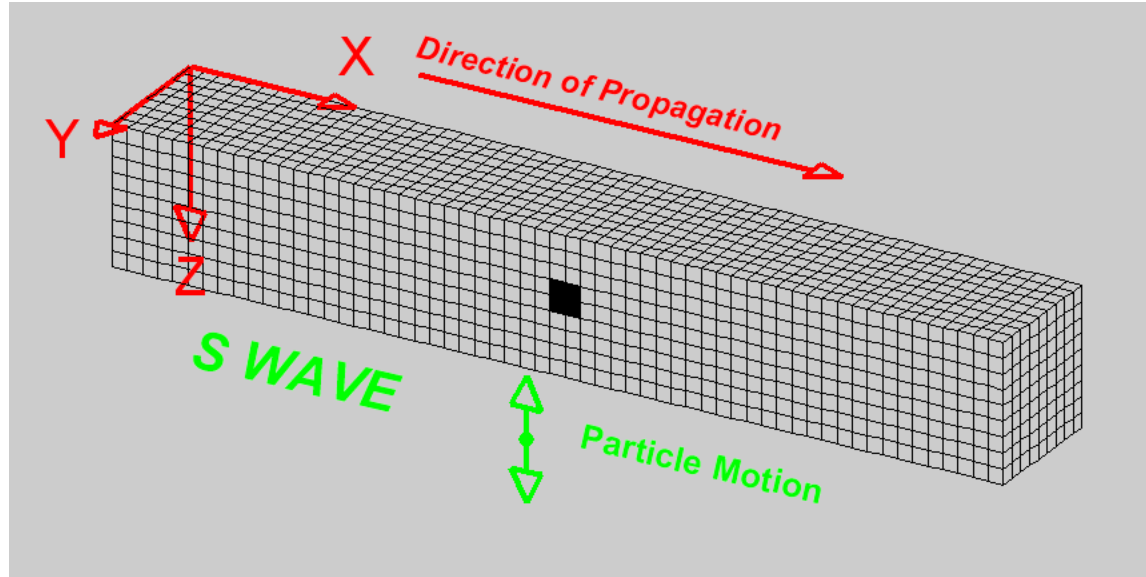
- **P-waves or compressional:** displacement longitudinal to propagation (like acoustic waves)
- **S-waves or shear:** displacement transverse to propagation (due to shearing of medium, not possible in acoustic media)

# Compressional waves



Source: <https://web.ics.purdue.edu/~braile/edumod/waves/Pwave.htm>

# Shear waves (transverse SV)



Source: <https://web.ics.purdue.edu/~braile/edumod/waves/Swave.htm>

# Elastic wave propagation

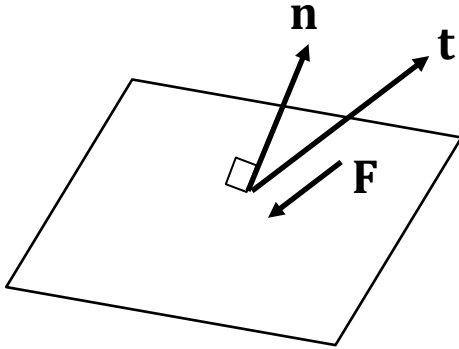
To describe how elastic waves propagate, we need to describe **internal forces** and **deformation** in an elastic medium:

- **Strains ( $\epsilon$ )**: deformations in a 3D medium
- **Stresses ( $\tau$ )**: internal forces between different particles of the medium

→ Strains and stresses are linked via constitutive relations / Hooke's law

# Stress tensor

Measure of the forces acting on an infinitesimal plan at each point in a solid medium  
(generalization of the concept of pressure force)



$\mathbf{n}$  : normal vector (i.e., orientation of the plane)

$\mathbf{t}$  : traction (force per unit area exerted on the side of the plane)

↳ **Convention:** the traction force  $\mathbf{F}$  is pulling in the opposite direction (towards the interface)

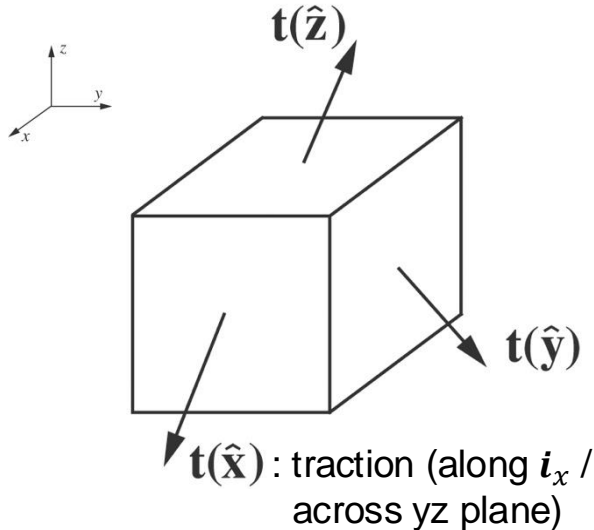
$$\mathbf{t} \propto p \quad \mathbf{F}_p \propto -\nabla p$$

Compressional forces: -

Extensional forces: +

# Stress tensor

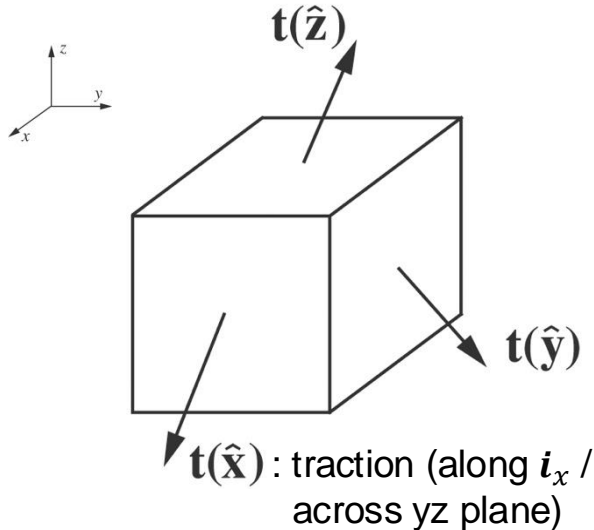
In general  $\mathbf{t}$  is a vector on each side of an infinitesimal volume, so it varies also as function of the orientation of the plane of interest  $\rightarrow \mathbf{t}(\mathbf{n})$  commonly represented as **tensor**



$$\boldsymbol{\tau} = \begin{bmatrix} \mathbf{t}(\mathbf{i}_x) \\ \mathbf{t}(\mathbf{i}_y) \\ \mathbf{t}(\mathbf{i}_z) \end{bmatrix} = \begin{bmatrix} t_x(\mathbf{i}_x) & t_y(\mathbf{i}_x) & t_z(\mathbf{i}_x) \\ t_x(\mathbf{i}_y) & t_y(\mathbf{i}_y) & t_z(\mathbf{i}_y) \\ t_x(\mathbf{i}_z) & t_y(\mathbf{i}_z) & t_z(\mathbf{i}_z) \end{bmatrix} \left[ \frac{N}{m^2} = Pa \right]$$

# Stress tensor

In general  $\mathbf{t}$  is a vector on each side of an infinitesimal volume, so it varies also as function of the orientation of the plane of interest  $\rightarrow \mathbf{t}(\mathbf{n})$  commonly represented as **tensor**



$$\boldsymbol{\tau} = \begin{bmatrix} \mathbf{t}(\mathbf{i}_x) \\ \mathbf{t}(\mathbf{i}_y) \\ \mathbf{t}(\mathbf{i}_z) \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \quad \left[ \frac{N}{m^2} = Pa \right]$$

$\tau_{ij}$      $i$ : surface normal direction  
           $j$ : component of the traction vector



# Stress tensor

Because we consider a solid medium in static equilibrium:

$$\tau_{ij} = \tau_{ji} \rightarrow \text{Symmetric tensor} \quad \boldsymbol{\tau} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \quad 6 \text{ independent params.}$$

where:

$$\sigma_i = \tau_{ii} \quad \text{Normal stresses (acoustic equivalent: } p = \frac{1}{3} \text{tr}(\boldsymbol{\tau}))$$

$$\tau_{ij} \quad \text{Shear stresses (acoustic equivalent: 0)}$$

# Stress tensor

Properties:

- The traction across any plane with normal  $\mathbf{n}$ :

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\tau} \cdot \mathbf{n} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

# Stress tensor

Properties:

- The traction across any plane with normal  $\mathbf{n}$ :

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\tau} \cdot \mathbf{n} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

- Principal axes: a direction  $\mathbf{n}$  such that no shear stress occurs on the plane perpendicular to it

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\tau} \cdot \mathbf{n} = \lambda \mathbf{n} \rightarrow (\boldsymbol{\tau} - \lambda \mathbf{I}) \mathbf{n} = 0$$



$$\boldsymbol{\tau}_R = \mathbf{N}^T \boldsymbol{\tau} \mathbf{N} = \mathbf{diag}(\tau_1, \tau_2, \tau_3)$$

Eigenvalue problem

$\lambda$  eigenvalues

$\mathbf{N} = [\mathbf{n}^1, \mathbf{n}^2, \mathbf{n}^3]$  eigenvectors  
(principal axes of stress)

If  $\tau_1 = \tau_2 = \tau_3$ , hydrostatic stress  $\rightarrow$  no shear component

# Stress tensor

Properties:

- Deviatoric stress: stresses in the Earth are dominated by large compressive components due to hydrostatic pressure → remove it to 'see' the remaining terms

$$\boldsymbol{\tau}_D = \boldsymbol{\tau} - \mathbf{diag}(\tau_m, \tau_m, \tau_m)$$

$$\tau_m = (\sigma_x + \sigma_y + \sigma_z)/3 = -p$$

Mean normal stress=-pressure

We can always write

$$\boldsymbol{\tau} = \tau_m \mathbf{I} + \boldsymbol{\tau}_D$$

# Strain tensor

We need to describe changes in position of points within an elastic medium, and distinguish between:

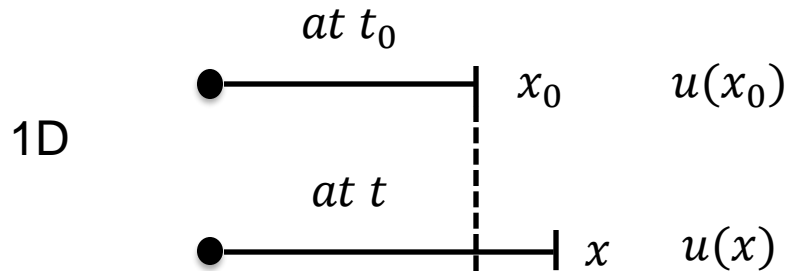
- translations and rotations
- elastic strains

**Displacement:** absolute measure of position change

**Strain:** local measure of relative changes in displacement  $\rightarrow$  spatial gradient of displacement

# Strain tensor

Extensional strain:



$$u(x) - u(x_0) = \frac{\partial u}{\partial x} dx$$




$$\epsilon_x = \frac{u(x) - u(x_0)}{dx} = \frac{\partial u}{\partial x}$$

# Strain tensor


Strain and rotation tensors:

$$3\text{D:} \quad \mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x}_0) + \mathbf{J}\mathbf{d} + \dots \quad \mathbf{J} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix} \quad \mathbf{d} = \mathbf{x} - \mathbf{x}_0 = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$


$$\begin{array}{ll} \text{Strain tensor } \boldsymbol{\epsilon} \text{ (symmetric)} & \text{Rotation tensor } \boldsymbol{\Omega} \text{ (antisymmetric)} \\ \boldsymbol{\epsilon} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix} & \boldsymbol{\Omega} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) & 0 \end{bmatrix} \end{array}$$

# Strain tensor

Strain and rotation tensors:

$$3\text{D:} \quad \mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x}_0) + \mathbf{J}\mathbf{d} + \dots$$
$$\mathbf{J} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix}$$
$$\mathbf{d} = \mathbf{x} - \mathbf{x}_0 = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$


Strain tensor  $\epsilon$  (symmetric)                      Rotation tensor  $\Omega$  (antisymmetric)

$$\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i} \right)$$

$$\Omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial j} - \frac{\partial u_j}{\partial i} \right)$$



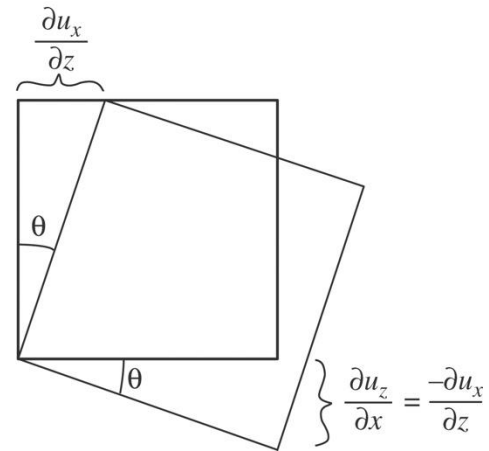
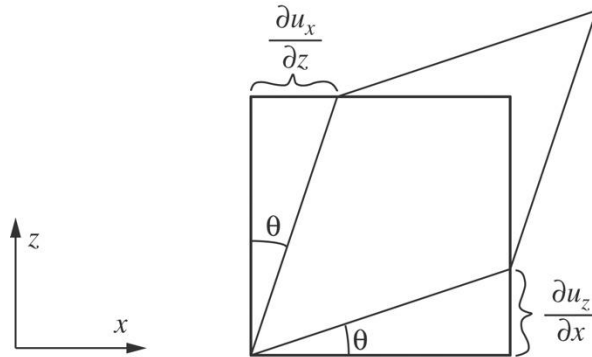
# Strain tensor

Ex: in 2D

- No volume change ( $\text{diag}(\mathbf{J}) = 0$ )

$$\boldsymbol{\epsilon} = \begin{bmatrix} 0 & \theta \\ \theta & 0 \end{bmatrix}$$

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$$



# Strain tensor

Ex: in 2D

- No volume change ( $diag(\mathbf{J}) = 0$ )

$$\boldsymbol{\epsilon} = \begin{bmatrix} 0 & \theta \\ \theta & 0 \end{bmatrix} \quad \boldsymbol{\Omega} = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$$

- Volume change (or dilatation)

$$\Delta = \frac{V - V_0}{V_0} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = tr(\boldsymbol{\epsilon}) = \nabla \cdot \mathbf{u}$$

# Hooke's law

A medium is defined **elastic** if after taking away the forces acting on it, it returns to its original position. As waves produce small perturbation, this is the case here.

In this scenario, there is a linear relationship between stresses and strains → **Constitutive relation or generalized Hooke's law**

$$\tau_{ij} = C_{ijkl} \varepsilon_{kl} = \frac{1}{2} C_{ijkl} (\partial_l u_k + \partial_k u_l)$$

↑

**Compliance or Elasticity tensor** (4th order tensor,  $3 \times 3 \times 3 \times 3 = 81$  elements!)

# Einstein notation refresher

→ Repeated indices are summed ( $i/j/k/l = 1,2,3$ )

Inner product:  $x_i y_i = \sum_{i=1,2,3} x_i y_i = x_1 y_1 + x_2 y_2 + x_3 y_3$

Matrix-matrix multiply:  $x_{ij} y_{jk} = \sum_{j=1,2,3} x_{ij} y_{jk} = x_{i1} y_{1k} + x_{i2} y_{2k} + x_{i3} y_{3k}$

Tensor product:  $c_{ijkl} y_{kl} = \sum_{k=1,2,3} \sum_{l=1,2,3} c_{ijkl} y_{kl} = \sum_{k=1,2,3} (c_{ijk1} y_{k1} + c_{ijk2} y_{k2} + c_{ijk3} y_{k3})$   
 $= \sum_{k=1,2,3} (c_{ijk1} y_{k1} + c_{ijk2} y_{k2} + c_{ijk3} y_{k3}) = c_{ij11} y_{11} + c_{ij21} y_{21} + c_{ij31} y_{31} + \dots$

# Elasticity tensor

- 4th order tensor,  $3 \times 3 \times 3 \times 3 = 81$  elements!
- Due to symmetry of  $\boldsymbol{\tau}$  and  $\boldsymbol{\epsilon}$

$$\boldsymbol{\tau}' = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

$$\boldsymbol{\epsilon}' = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \epsilon_{xy} \\ \epsilon_{xz} \\ \epsilon_{yz} \end{bmatrix} \begin{array}{l} \leftarrow 11 \\ \leftarrow 22 \\ \leftarrow 33 \\ \leftarrow 12 \\ \leftarrow 13 \\ \leftarrow 23 \end{array}$$

$$\boldsymbol{\tau}' = \mathbf{C}_{6 \times 6} \boldsymbol{\epsilon}'$$

- 21 terms: generic anisotropic medium (medium properties change with direction)
- 2 terms: isotropic medium (medium properties invariant with direction)

# Isotropic elasticity tensor

Given the Lamé parameters ( $\lambda, \mu$  [Pa])

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl})$$

$$\text{Kronecker delta: } \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Ex:

$$C_{1111} = \lambda \delta_{11} \delta_{11} + \mu (\delta_{11} \delta_{11} + \delta_{11} \delta_{11}) = \lambda + 2\mu$$

$$C_{1212} = \lambda \delta_{12} \delta_{12} + \mu (\delta_{12} \delta_{21} + \delta_{11} \delta_{22}) = \mu$$

# Isotropic Hooke's law

Plugging the definition of  $C_{ijkl}$  for isotropic media:

$$\tau_{ij} = \left[ \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}) \right] \epsilon_{kl}$$

$$\uparrow = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}$$

$$\delta_{kl} \epsilon_{kl} = \epsilon_{kk} \quad (\text{only terms with } k = l)$$

$$\delta_{il} \delta_{jk} \epsilon_{kl} = \epsilon_{ji} \quad (\text{only terms with } i = l \text{ and } j = k)$$

$$(\text{only terms with } i = k \text{ and } j = l)$$

$$\left. \vphantom{\begin{matrix} \delta_{il} \delta_{jk} \epsilon_{kl} = \epsilon_{ji} \\ (\text{only terms with } i = k \text{ and } j = l) \end{matrix}} \right\} \epsilon_{ij} = \epsilon_{ji}$$

# Isotropic Hooke's law

Plugging the definition of  $C_{ijkl}$  for isotropic media:

$$\boldsymbol{\tau} = \begin{bmatrix} \lambda \text{tr}\{\boldsymbol{\epsilon}\} + 2\mu\epsilon_{xx} & 2\mu\epsilon_{xy} & 2\mu\epsilon_{xz} \\ 2\mu\epsilon_{yx} & \lambda \text{tr}\{\boldsymbol{\epsilon}\} + 2\mu\epsilon_{yy} & 2\mu\epsilon_{yz} \\ 2\mu\epsilon_{zx} & 2\mu\epsilon_{zy} & \lambda \text{tr}\{\boldsymbol{\epsilon}\} + 2\mu\epsilon_{zz} \end{bmatrix}$$



# Elastic properties

First Lamè parameter:  $\lambda$  [Pa]  $\rightarrow$  no simple physical explanation

Second Lamè parameter / shear modulus:  $\mu = \tau_{xy}/2\epsilon_{xy}$  [Pa]  $\rightarrow$  measure of resistance of material to shearing

Young's modulus :  $E = \frac{\tau_{xx}}{\epsilon_{xx}} = \frac{(3\lambda+2\mu)\mu}{\lambda+\mu}$  [Pa]  $\rightarrow$  ratio of extensional stress to extensional strain

$$\epsilon = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = dV/V \text{ volumetric dilatation}$$

Bulk modulus :  $K = \frac{\tau_m}{\epsilon} = \lambda + \frac{2}{3}\mu$  [Pa]  $\rightarrow$  ratio of hydrostatic pressure to volume change

# Elastic properties

P-wave and S-wave velocities:  $\alpha = c_P = \sqrt{\frac{\lambda+2\mu}{\rho}}$ ,  $\beta = c_S = \sqrt{\frac{\mu}{\rho}}$

Poisson ratio:  $\sigma = -\frac{\epsilon_{yy}}{\epsilon_{xx}} = -\frac{\epsilon_{zz}}{\epsilon_{xx}} = \frac{\lambda}{2(\lambda+\mu)} \rightarrow$  ratio of lateral contraction of a cylinder to longitudinal extension

(written also as  $\sigma = \frac{\alpha^2-2\beta^2}{2(\alpha^2-\beta^2)} = \frac{(\alpha/\beta)^2-2}{2(\alpha/\beta)^2-2}$ )  $0 < \sigma < 0.5$

$$\sigma = 0.5 - \mu = \beta = 0 \rightarrow \text{Fluid}$$

$$\sigma = 0 - \lambda = 0, \alpha/\beta = \sqrt{2} \rightarrow \text{Min. value for isotropic medium}$$

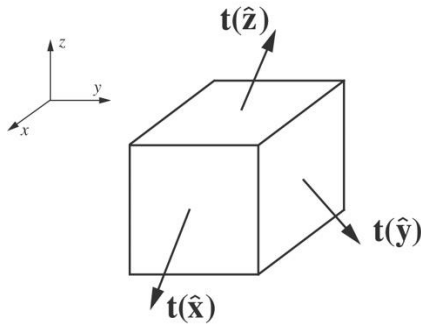
$$\sigma = 0.25 - \lambda = \mu, \alpha/\beta = \sqrt{3} \rightarrow \text{Poisson solid (typical values for materials under ideal elastic conditions)}$$

**$\rightarrow$  Any triplet (e.g.,  $c_P, c_S, \rho$ ) is sufficient to compute any other parameter!**

# Principle of inertia / momentum equation

Describe how stress-strain-displacement changes with time (seismic waves are time-dependent phenomena)

Starting from Newton 2<sup>nd</sup> law (forces on a surface of a cube are given by product of traction vector and surface area)



For one single side:

$$\mathbf{F}(\mathbf{i}_x) = \mathbf{t}(\mathbf{i}_x) dydz = \boldsymbol{\tau} \cdot \mathbf{i}_x dydz$$

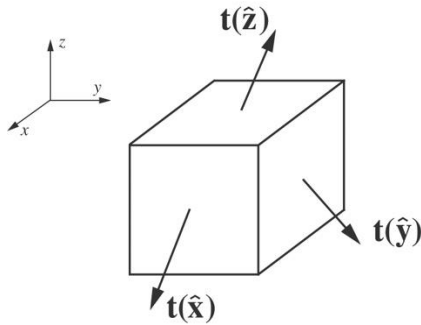
↓  
For variable stress field  
(gradient  $\neq 0$ ), net forces

$$\mathbf{F}(\mathbf{i}_x) = \frac{\partial}{\partial x} \begin{bmatrix} \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{bmatrix} dx dy dz$$

# Principle of inertia / momentum equation

Describe how stress-strain-displacement changes with time (seismic waves are time-dependent phenomena)

Starting from Newton 2<sup>nd</sup> law (forces on a surface of a cube are given by product of traction vector and surface area)



Given all sides of the cube:

$$F_i(\mathbf{i}_x) = \sum_j \partial_j \tau_{ij} / \partial x_j \, dx dy dz = \underbrace{\partial_j \tau_{ij}}_{\text{Divergence of stress field}} dx dy dz$$

Divergence of stress field

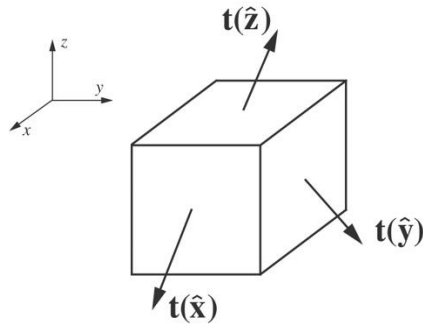
Defining:

$$m = \rho dV = \rho dx dy dz \quad \mathbf{a} = \ddot{\mathbf{u}} = \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

# Principle of inertia / momentum equation

Describe how stress-strain-displacement changes with time (seismic waves are time-dependent phenomena)

Starting from Newton 2<sup>nd</sup> law (forces on a surface of a cube are given by product of traction vector and surface area)



We obtain:

$$\rho \frac{\partial^2 u_i}{\partial t^2} dx dy dz = \partial_j \tau_{ij} dx dy dz$$

In the presence of an external body force ( $F_i^{body} = f_i dx dy dz$ ):

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j \tau_{ij} + f_i$$

# Principle of inertia / momentum equation

Describe how stress-strain-displacement changes with time (seismic waves are time-dependent phenomena)

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j \tau_{ij} + f_i$$

$$f_i = f_i^g + f_i^s$$

Gravity term (at very low freqs – e.g., normal modes)

Source term

→ In the absence of external forces, **homogenous equation of motion**

# Elastic wave equation constituents

- **Principle of Inertia**  $\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j \tau_{ij} + f_i$
- **Hooke's law**  $\tau_{ij} = \frac{1}{2} C_{ijkl} (\partial_l u_k + \partial_k u_l)$   
 $(\tau_{ij} = \lambda \delta_{ij} \partial_k u_k + \mu (\partial_i u_j + \partial_j u_i))$

# First-order elastic wave equation

Written in terms of velocity ( $v_i = \partial u_i / \partial t$ ) and stresses ( $\tau_{ij}$ ). In 2D:

$$\left\{ \begin{array}{l} \rho \frac{\partial v_x}{\partial t} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + f_x \\ \rho \frac{\partial v_z}{\partial t} = \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} + f_z \\ \frac{\partial \tau_{xx}}{\partial t} = \lambda \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} \right) + 2\mu \frac{\partial v_x}{\partial x} \\ \frac{\partial \tau_{xz}}{\partial t} = \mu \left( \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \\ \frac{\partial \tau_{zz}}{\partial t} = \lambda \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} \right) + 2\mu \frac{\partial v_z}{\partial z} \end{array} \right.$$



# Elastic wave equation

For the isotropic case, inserting  $\tau_{ij} = \lambda\delta_{ij}\partial_k u_k + \mu(\partial_i u_j + \partial_j u_i)$  from Hooke's law into the principle of inertia:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j (\lambda \delta_{ij} \partial_k u_k + \mu (\partial_i u_j + \partial_j u_i)) + f_i$$

$\equiv$

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \lambda (\nabla \cdot \mathbf{u}) + \nabla \mu \cdot [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} + \mathbf{f}$$



Gradient of vector  $\rightarrow$  Tensor




Laplacian:  $\nabla \cdot \nabla \mathbf{u}$

# Elastic wave equation

Using the following vector identity ( $\nabla \times \nabla \times \mathbf{u} = \nabla \nabla \cdot \mathbf{u} - \nabla^2 \mathbf{u}$ ) for the second last term:

Only for heterogenous media ( $\nabla \lambda, \nabla \mu \neq 0$ )


$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \lambda (\nabla \cdot \mathbf{u}) + \nabla \mu \cdot [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] + (\lambda + 2\mu) \nabla \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u} + \mathbf{f}$$

**Elastic (seismic) wave equation**

# Homogenous elastic wave equation

Two methods are commonly used in seismology with the homogenous elastic wave equation:

- **Layer cake model** (aka homogenous-layer methods): in each layer  $\nabla\lambda = \nabla\mu = 0$ , this leads to a simplified version of the wave equation and reflection&transmission coefficients are used to link layers. Useful for:
  - Surface waves
  - Low/medium frequency body waves
- **Ray-based methods**: these methods work in a high-frequency regime. Since one can show that  $\nabla\lambda = \nabla\mu \propto 1/\omega$ ,  $\nabla\lambda = \nabla\mu = 0$  for  $\omega \rightarrow \infty$ .

# Helmoltz decomposition theorem

→ We want to separate the homogenous elastic wave equation into its P- and S-wave components.

A displacement field  $\mathbf{u}$  can be decomposed into its:

- **Irrotational or central component:**  $\nabla \times \nabla \Phi = 0$
- **Solenoidal or rotational component:**  $\nabla \cdot (\nabla \times \mathbf{\Psi}) = 0$

$$\mathbf{u} = \nabla \Phi + \nabla \times \mathbf{\Psi} \quad \Phi: \text{scalar potential}$$

$$\nabla \cdot \mathbf{\Psi} = 0 \quad \mathbf{\Psi}: \text{vectorial potential}$$

# Helmoltz decomposition theorem

→ For elastic waves,  $\Phi$ : P-wave,  $\Psi$ : S-wave

Applying  $\nabla \cdot$  to the displacement equation:

$$\nabla \cdot \mathbf{u} = \nabla \cdot \nabla \Phi + \cancel{\nabla \cdot \nabla \times \Psi} = \nabla^2 \Phi \longrightarrow$$

$$\nabla \cdot \mathbf{u} = \nabla^2 \Phi$$

Applying  $\nabla \times$  to the displacement equation:

$$\nabla \times \mathbf{u} = \cancel{\nabla \times \nabla \Phi} + \nabla \times \nabla \times \Psi = \nabla \nabla \cdot \Psi - \nabla^2 \Psi = -\nabla^2 \Psi \longrightarrow$$

$$\nabla \times \mathbf{u} = -\nabla^2 \Psi$$

# P-wave equation

Taking the divergence of the homogenous, source-free elastic wave equation:

$$\nabla \cdot \left( \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \right) = \nabla \cdot \left( (\lambda + 2\mu) \nabla \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u} \right)$$

$\uparrow \qquad \qquad \uparrow$   
 $\nabla \cdot \nabla = \nabla^2 \quad \nabla \cdot \nabla \times = 0$

$$\frac{\partial^2}{\partial t^2} \nabla \cdot \mathbf{u} = \left( \frac{\lambda + 2\mu}{\rho} \right) \nabla^2 (\nabla \cdot \mathbf{u})$$

$$\boxed{\nabla^2 \Phi_P - \frac{1}{\alpha^2} \frac{\partial^2}{\partial t^2} \Phi_P = 0}$$

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad \Phi_P \equiv \nabla \cdot \mathbf{u}$$

# S-wave equation

Taking the curl of the homogenous, source-free elastic wave equation:

$$\nabla \times \left( \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \right) = \nabla \times \left( (\lambda + 2\mu) \nabla \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u} \right)$$

$$\frac{\partial^2}{\partial t^2} \nabla \times \mathbf{u} = \frac{\mu}{\rho} \nabla^2 (\nabla \times \mathbf{u})$$

$$\boxed{\nabla^2 \Psi_S - \frac{1}{\beta^2} \frac{\partial^2}{\partial t^2} \Psi_S = 0}$$

$$\beta = \sqrt{\frac{\mu}{\rho}}$$

$$\Psi_S \equiv \nabla \times \mathbf{u}$$

# P- and S-wave equation

Putting all together:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu) \nabla \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u}$$



$$\boxed{\frac{\partial^2 \mathbf{u}}{\partial t^2} = \alpha^2 \nabla (\nabla \cdot \mathbf{u}) - \beta^2 \nabla \times \nabla \times \mathbf{u}}$$



P-wave



S-wave



# Polarization of waves

Using the plane wave formulation we can study how different waves behave.

- P-wave along x-dimension:

$$\alpha^2 \partial_{xx} \Phi = \partial_{tt} \Phi \rightarrow \Phi = \Phi_0(t \pm x/\alpha)$$

Since  $\mathbf{u} = \nabla \Phi$ , we have  $u_x = \partial_x \Phi$  and  $u_y = u_z = 0$ , meaning that the displacement happens only parallel to the direction of propagation with particle motion consisting of **Dilatation** and **Compression**.



**Longitudinal waves**

# Polarization of waves

Using the plane wave formulation we can study how different waves behave.

- S-wave along x-dimension:

$$\beta^2(\partial_{xx}\Psi_x\mathbf{i}_x + \partial_{yy}\Psi_y\mathbf{i}_y + \partial_{zz}\Psi_z\mathbf{i}_z) = \partial_{tt}\Psi$$
$$\rightarrow \Psi = \Psi_x(t \pm x/\beta)\mathbf{i}_x + \Psi_y(t \pm x/\beta)\mathbf{i}_y + \Psi_z(t \pm x/\beta)\mathbf{i}_z$$

Since  $\mathbf{u} = \nabla \times \Psi$ , and  $\partial_z = \partial_y = 0$  (no changes of wavefield in the non-propagating directions), we have  $u_x = 0$ ,  $u_y = -\partial_x \Psi_z$ ,  $u_z = \partial_x \Psi_y$ , displacement is perpendicular and motion consisting of 2 components:

→ **SV:** withing plane of propagation

→ **SH:** across plane of propagation