4. Elastic Wave Equation

M. Ravasi

ERSE 210 Seismology

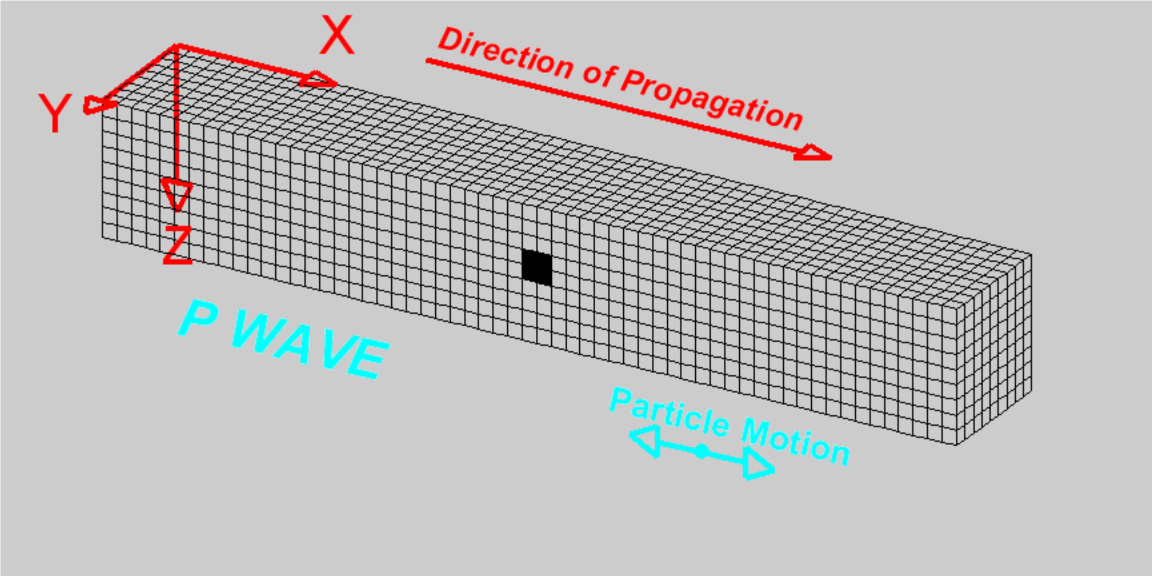
Elastic wave propagation

Two types of waves propagate in elastic media:

- **P-waves or compressional:** displacement longitudinal to propagation (like acoustic waves)

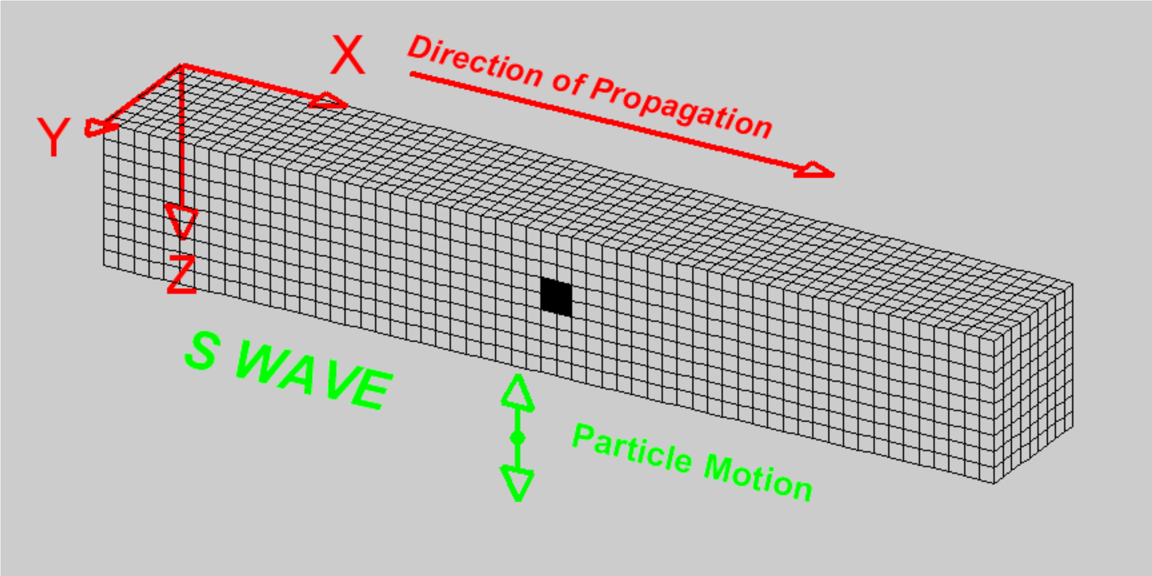
- **S-waves or shear:** displacement transverse to propagation (due to shearing of medium, not possible in acoustic media)

Compressional waves



Source: https://web.ics.purdue.edu/~braile/edumod/waves/Pwave.htm

Shear waves (transverse SV)



Source: https://web.ics.purdue.edu/~braile/edumod/waves/Swave.htm

Elastic wave propagation

To describe how elastic waves propagate, we need to describe **internal forces** and **deformation** in an elastic medium:

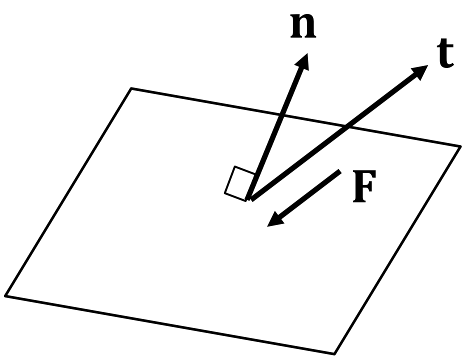
- **Strains (ϵ):** deformations in a 3D medium

- **Stresses (τ):** internal forces between different particles of the medium

→ Strains and stresses are linked via constitutive relations / Hooke's law

Stress tensor

Measure of the forces acting on an infinitesimal plan at each point in a solid medium (generalization of the concept of pressure force)



$\mathbf{n}$: normal vector (i.e., orientation of the plane)

$\mathbf{t}$: traction (force per unit area exerted on the side of the plane)

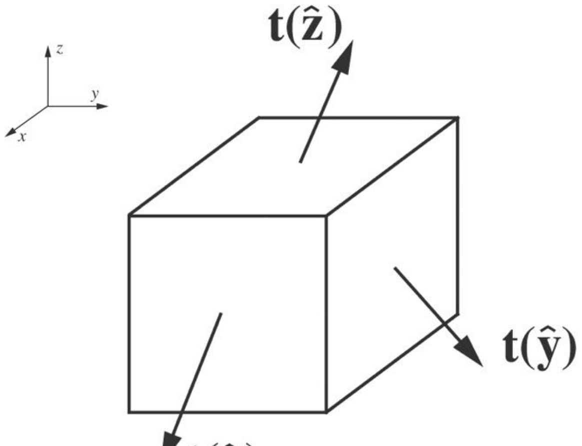
Convention: the traction force $\mathbf{F}$ is pulling in the opposite direction (towards the interface)

$$\mathbf{t} \; \alpha \; p \qquad \mathbf{F}_{\mathrm{p}} \; \alpha \; -\nabla \mathrm{p}$$

Compressional forces: -
Extensional forces: +

Stress tensor

In general $\mathbf{t}$ is a vector on each side of an infinitesimal volume, so it varies also as function of the orientation of the plane of interest → $\mathbf{t}(\mathbf{n})$ commonly represented as **tensor**

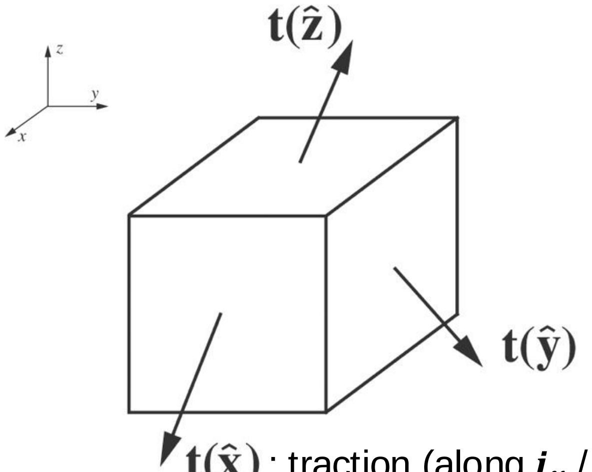


$\mathbf{t}(\hat{\mathbf{x}})$: traction (along $\boldsymbol{i}_x$ / across yz plane)

$$\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{t}(\boldsymbol{i}_x) \\ \boldsymbol{t}(\boldsymbol{i}_y) \\ \boldsymbol{t}(\boldsymbol{i}_z) \end{bmatrix} = \begin{bmatrix} t_x(\boldsymbol{i}_x) & t_y(\boldsymbol{i}_x) & t_z(\boldsymbol{i}_x) \\ t_x(\boldsymbol{i}_y) & t_y(\boldsymbol{i}_y) & t_z(\boldsymbol{i}_y) \\ t_x(\boldsymbol{i}_z) & t_y(\boldsymbol{i}_y) & t_z(\boldsymbol{i}_y) \end{bmatrix} \quad \left[\frac{N}{m^2} = Pa\right]$$

Stress tensor

In general $\mathbf{t}$ is a vector on each side of an infinitesimal volume, so it varies also as function of the orientation of the plane of interest → $\mathbf{t}(\mathbf{n})$ commonly represented as **tensor**



$\mathbf{t}(\hat{\mathbf{x}})$: traction (along $\boldsymbol{i}_x$ / across yz plane)

$$\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{t}(\boldsymbol{i}_x) \\ \boldsymbol{t}(\boldsymbol{i}_y) \\ \boldsymbol{t}(\boldsymbol{i}_z) \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

$$\left[\frac{N}{m^2} = Pa\right]$$

τ_{ij}

i: surface normal direction

j: component of the traction vector

Stress tensor

Because we consider a solid medium in static equilibrium:

$\tau_{ij} = \tau_{ji}$ → Symmetric tensor

$$\boldsymbol{\tau} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$ 6 indipendent params.

where:

$\sigma_i = \tau_{ii}$ Normal stresses (acoustic equivalent: $p = \frac{1}{3} tr(\tau)$)

τ_{ij} Shear stresses (acoustic equivalent: 0

Stress tensor

Properties:

- The traction across any plane with normal $\mathbf{n}$:

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\tau} \cdot \mathbf{n} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

Stress tensor

Properties:

- The traction across any plane with normal $\mathbf{n}$:

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\tau} \cdot \mathbf{n} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

- Principal axes: a direction n such that no shear stress occurs on the plane perpendicular to it

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\tau} \cdot \mathbf{n} = \lambda\mathbf{n} \rightarrow (\boldsymbol{\tau} - \lambda\mathbf{I})\mathbf{n} = 0$$

$$\downarrow$$

$$\boldsymbol{\tau}_{\mathrm{R}} = \mathbf{N}^{\mathrm{T}} \boldsymbol{\tau}\, \mathbf{N} = \mathbf{diag}(\tau_1, \tau_2, \tau_3)$$

Eigenvalue problem
λ eigenvalues
$\mathbf{N} = [\mathbf{n}^1, \mathbf{n}^2, \mathbf{n}^3]$ eigenvectors
(principal axes of stress)

If $\tau_1 = \tau_2 = \tau_3$, hydrostatic stress → no shear component

Stress tensor

Properties:

- Deviatoric stress: stresses in the Earth are dominated by large compressive components due to hydrostatic pressure → remove it to 'see' the remaining terms

$$\boldsymbol{\tau}_{\mathrm{D}} = \boldsymbol{\tau} - \mathbf{diag}(\tau_{\mathrm{m}}, \tau_{\mathrm{m}}, \tau_{\mathrm{m}})$$

$$\tau_{\mathrm{m}} = (\sigma_x + \sigma_y + \sigma_z)/3 = -\mathrm{p}$$

Mean normal stress=-pressure

We can always write

$$\boldsymbol{\tau} = \tau_{\mathrm{m}}\mathbf{I} + \boldsymbol{\tau}_{\mathrm{D}}$$

Strain tensor

We need to described changes in position of points within an elastic medium, and distinguish between:

- translations and rotations
- elastic strains

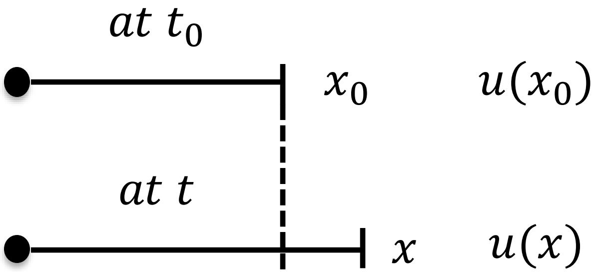
Displacement: absolute measure of position change

Strain: local measure of relative changes in displacement → spatial gradient of displacement

Strain tensor

<u>Extensional strain:</u>

1D



$$u(x) - u(x_0) = \frac{\partial u}{\partial x}\mathrm{d}x$$

$$\downarrow$$

$$\boxed{\epsilon_x = \frac{u(x) - u(x_0)}{\mathrm{d}x} = \frac{\partial u}{\partial x}}$$

Strain tensor

Strain and rotation tensors:

3D: $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x_0}) + \mathbf{J}\mathbf{d} + \cdots$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

$$\mathbf{d} = \mathbf{x} - \mathbf{x_0} = \begin{bmatrix} \mathrm{dx} \\ \mathrm{dy} \\ \mathrm{dz} \end{bmatrix}$$

Strain tensor $\boldsymbol{\epsilon}$ (symmetric)

$$\boldsymbol{\epsilon} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z\S} \end{bmatrix}$$

Rotation tensor Ω (antisymmetric)

$$\Omega = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) & 0 \end{bmatrix}$$

Strain tensor

Strain and rotation tensors:

3D: $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x_0}) + \mathbf{J}\mathbf{d} + \cdots$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial x} \end{bmatrix}$$

$$\mathbf{d} = \mathbf{x} - \mathbf{x_0} = \begin{bmatrix} \mathrm{dx} \\ \mathrm{dy} \\ \mathrm{dz} \end{bmatrix}$$

Strain tensor $\boldsymbol{\epsilon}$ (symmetric)

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}\right)$$

Rotation tensor Ω (antisymmetric)

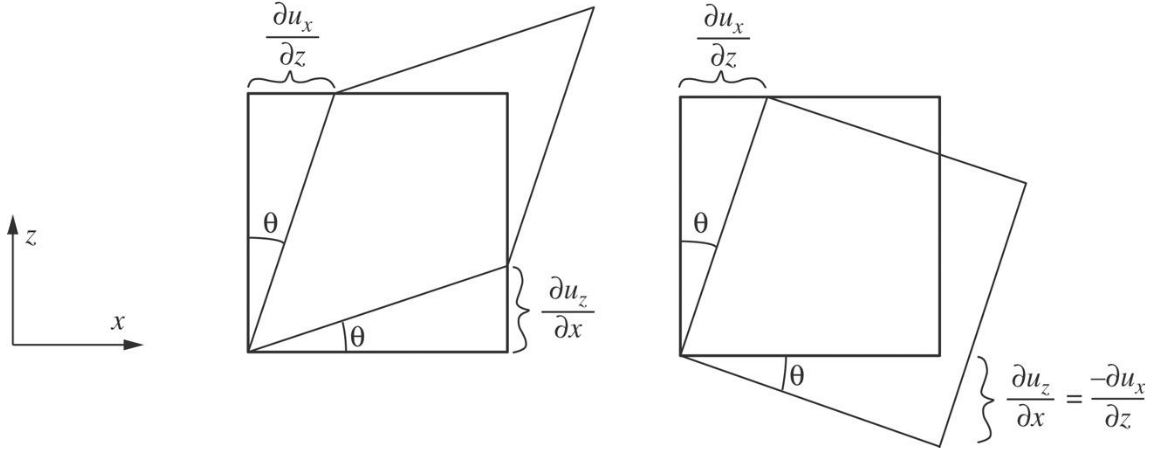
$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial j} - \frac{\partial u_j}{\partial i}\right)$$

Strain tensor

Ex: in 2D

- No volume change ($diag(\mathbf{J}) = 0$)

$$\boldsymbol{\epsilon} = \begin{bmatrix} 0 & \theta \\ \theta & 0 \end{bmatrix} \qquad \Omega = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$$



Strain tensor

Ex: in 2D

- No volume change ($diag(\mathbf{J}) = 0$)

$$\boldsymbol{\epsilon} = \begin{bmatrix} 0 & \theta \\ \theta & 0 \end{bmatrix} \qquad \Omega = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$$

- Volume change (or dilatation)

$$\Delta = \frac{V - V_0}{V_0} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = tr(\boldsymbol{\epsilon}) = \nabla \cdot \mathbf{u}$$

Hooke's law

A medium is defined **elastic** if after taking away the forces acting on it, it returns to its original position. As waves produce small perturbation, this is the case here.

In this scenario, there is a linear relationship between stresses and strains → **Constitutive relation or generalized Hooke's law**

$$\tau_{ij} = \underset{\uparrow}{C_{ijkl}}\, \varepsilon_{kl} = \frac{1}{2} C_{ijkl}\,(\partial_l u_k + \partial_k u_l)$$

Compliance or Elasticity tensor (4th order tensor, 3x3x3x3=81 elements!)

Einstein notation refresher

→ **Repeated indices are summed** $(i/j/k/l = 1,2,3)$

Inner product: $x_i y_i = \sum_{i=1,2,3} x_i y_i = x_1 y_1 + x_2 y_2 + x_3 y_3$

Matrix-matrix multiply: $x_{ij} y_{jk} = \sum_{j=1,2,3} x_{ij} y_{jk} = x_{i1} y_{1k} + x_{i2} y_{2k} + x_{i3} y_{3k}$

Tensor product: $C_{ijkl} y_{kl} = \sum_{k=1,2,3} \sum_{l=1,2,3} C_{ijkl} y_{kl} = \sum_{k=1,2,3} (C_{ijk1} y_{k1} + C_{ijk2} y_{k2} + C_{ijk3} y_{k3})$

$= \sum_{k=1,2,3} \left(C_{ijk1} y_{k1} + C_{ijk2} y_{k2} + C_{ijk3} y_{k3} \right) = C_{ij11} y_{11} + C_{ij21} y_{21} + C_{ij31} y_{31} + \cdots$

Elasticity tensor

- 4th order tensor, 3x3x3x3=81 elements!

- Due to symmetry of $\boldsymbol{\tau}$ and $\boldsymbol{\epsilon}$

$$\boldsymbol{\tau}' = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} \qquad \boldsymbol{\epsilon}' = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \epsilon_{xy} \\ \epsilon_{xz} \\ \epsilon_{yz} \end{bmatrix} \begin{array}{l} \longleftarrow 11 \\ \longleftarrow 22 \\ \longleftarrow 33 \\ \longleftarrow 12 \\ \longleftarrow 13 \\ \longleftarrow 23 \end{array}$$

$$\boldsymbol{\tau}' = \mathbf{C}_{6x6}\boldsymbol{\epsilon}'$$

- 21 terms: generic anisotropic medium (medium properties change with direction)

- 2 terms: isotropic medium (medium properties invariant with direction)

Isotropic elasticity tensor

Given the Lamè parameters (λ, μ [Pa])

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})$$

Kronecker delta: $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Ex:

$$C_{1111} = \lambda\delta_{11}\delta_{11} + \mu(\delta_{11}\delta_{11} + \delta_{11}\delta_{11}) = \lambda + 2\mu$$

$$C_{1212} = \lambda\delta_{12}\delta_{12} + \mu(\delta_{12}\delta_{21} + \delta_{11}\delta_{22}) = \mu$$

Isotropic Hooke's law

Plugging the definition of C_{ijkl} for isotropic media:

$$\tau_{ij} = \left[\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})\right]\epsilon_{kl}$$

$$= \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij}$$

$\delta_{kl}\epsilon_{kl} = \epsilon_{kk}$ (only terms with $k = l$)

$\delta_{il}\delta_{jk}\epsilon_{kl} = \epsilon_{ji}$ (only terms with $\mathrm{i} = l$ and $j = k$)

(only terms with $\mathrm{i} = k$ and $j = l$)

$\epsilon_{ij} = \epsilon_{ji}$

Isotropic Hooke's law

Plugging the definition of C_{ijkl} for isotropic media:

$$\boldsymbol{\tau} = \begin{bmatrix} \lambda tr\{\boldsymbol{\epsilon}\} + 2\mu\epsilon_{xx} & 2\mu\epsilon_{xy} & 2\mu\epsilon_{xz} \\ 2\mu\epsilon_{yx} & \lambda tr\{\boldsymbol{\epsilon}\} + 2\mu\epsilon_{yy} & 2\mu\epsilon_{yz} \\ 2\mu\epsilon_{zx} & 2\mu\epsilon_{zy} & \lambda tr\{\boldsymbol{\epsilon}\} + 2\mu\epsilon_{zz} \end{bmatrix}$$

Elastic properties

First Lamè parameter: λ [Pa]→ no simple physical explanation

Second Lamè parameter / shear modulus: $\mu = \tau_{xy}/2\epsilon_{xy}$ [Pa] → measure of resistance of material to shearing

Young's modulus : $\mathrm{E} = \frac{\tau_{xx}}{\epsilon_{xx}} = \frac{(3\lambda+2\mu)\mu}{\lambda+\mu}$ [Pa] → ratio of extensional stress to extensional strain

$\epsilon = \epsilon_{\mathrm{xx}} + \epsilon_{\mathrm{yy}} + \epsilon_{zz} = dV/\mathrm{V}$ volumetric dilatation

Bulk modulus : $\mathrm{K} = \frac{\tau_m}{\epsilon} = \lambda + \frac{2}{3}\mu$ [Pa] → ratio of hydrostatic pressure to volume change

Elastic properties

P-wave and S-wave velocities: $\alpha = c_P = \sqrt{\frac{\lambda+2\mu}{\rho}}$, $\beta = c_S = \sqrt{\frac{\mu}{\rho}}$

Poisson ratio: $\sigma = -\frac{\epsilon_{yy}}{\epsilon_{xx}} = -\frac{\epsilon_{zz}}{\epsilon_{xx}} = \frac{\lambda}{2(\lambda+\mu)}$ → ratio of lateral contraction of a cylinder to longitudinal extension

(written also as $\sigma = \frac{\alpha^2-2\beta^2}{2(\alpha^2-\beta^2)} = \frac{(\alpha/\beta)^2-2}{2(\alpha/\beta)^2-2}$) $0 < \sigma < 0.5$

$\sigma = 0.5 - \mu = \beta = 0$ → Fluid

$\sigma = 0 - \lambda = 0, \alpha/\beta = \sqrt{2}$ → Min. value for isotropic medium

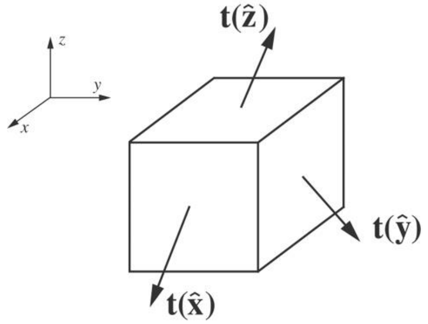
$\sigma = 0.25 - \lambda = \mu, \alpha/\beta = \sqrt{3}$ → Poisson solid (typical values for materials under ideal elastic conditions)

→ Any triplet (e.g., c_P, c_S, ρ) is sufficient to compute any other parameter!

Principle of inertia / momentum equation

Describe how stress-strain-displacement changes with time (seismic waves are time-dependent phenomena)

Starting from Newton 2nd law (forces on a surface of a cube are given by product of traction vector and surface area)



For one single side:

$$\mathbf{F}(\mathbf{i}_x) = \boldsymbol{t}(\boldsymbol{i}_x)\mathrm{dydz} = \boldsymbol{\tau} \cdot \boldsymbol{i}_x\mathrm{dydz}$$

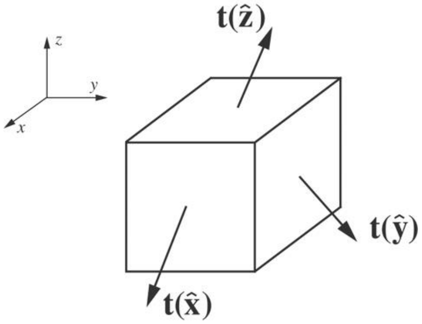
For variable stress field (gradient $\neq 0$), net forces

$$\mathbf{F}(\mathbf{i}_x) = \frac{\partial}{\partial x}\begin{bmatrix}\tau_{xx}\\ \tau_{xy}\\ \tau_{xz}\end{bmatrix}\mathrm{dxdydz}$$

Principle of inertia / momentum equation

Describe how stress-strain-displacement changes with time (seismic waves are time-dependent phenomena)

Starting from Newton 2nd law (forces on a surface of a cube are given by product of traction vector and surface area)



Given all sides of the cube:

$$\mathrm{F_i}(\mathbf{i}_x) = \textstyle\sum_i \partial\tau_{ij}/\partial x_j \ \mathrm{dxdydz} = \underbrace{\partial_j \tau_{\mathrm{ij}}}_{\text{Divergence of stress field}}\mathrm{dxdydz}$$

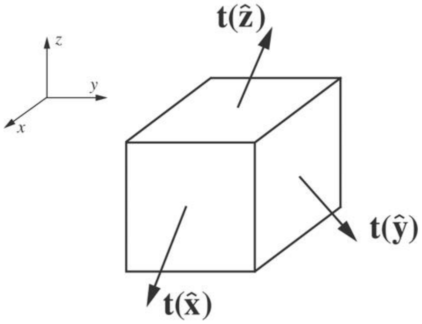
Defining:

$$\mathrm{m} = \rho\mathrm{dV} = \rho\mathrm{dxdydz} \qquad \mathbf{a} = \ddot{\mathbf{u}} = \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

Principle of inertia / momentum equation

Describe how stress-strain-displacement changes with time (seismic waves are time-dependent phenomena)

Starting from Newton 2nd law (forces on a surface of a cube are given by product of traction vector and surface area)



We obtain:

$$\rho \frac{\partial^2 u_i}{\partial t^2} \cancel{\mathrm{dxdydz}} = \partial_j \tau_{\mathrm{ij}} \cancel{\mathrm{dxdydz}}$$

In the presence of an external body force ($F_i^{body} = f_i \mathrm{dxdydz}$):

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j \tau_{ij} + f_i$$

Principle of inertia / momentum equation

Describe how stress-strain-displacement changes with time (seismic waves are time-dependent phenomena)

$$\boxed{\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j \tau_{ij} + f_i}$$

$$f_i = f_i^g + f_i^s$$

Gravity term (at very low freqs – e.g., normal modes)   Source term

→ In the absence of external forces, **homogenous equation of motion**

Elastic wave equation constituents

- **Principle of Inertia** $$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j \tau_{ij} + f_i$$

- **Hooke's law** $$\tau_{ij} = \frac{1}{2} C_{ijkl} \left(\partial_l u_k + \partial_k u_l \right)$$

$$(\tau_{ij} = \lambda \delta_{ij} \partial_k u_k + \mu(\partial_i u_j + \partial_j u_i))$$

First-order elastic wave equation

Written in terms of velocity ($\mathrm{v}_i = \partial u_i / \partial t$) and stresses ($\tau_{ij}$). In 2D:

$$\begin{cases} \rho \dfrac{\partial v_x}{\partial t} = \dfrac{\partial \tau_{xx}}{\partial x} + \dfrac{\partial \tau_{xz}}{\partial z} + f_x \\ \rho \dfrac{\partial v_z}{\partial t} = \dfrac{\partial \tau_{zx}}{\partial x} + \dfrac{\partial \tau_{zz}}{\partial z} + f_z \\ \dfrac{\partial \tau_{xx}}{\partial t} = \lambda \left(\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_z}{\partial z} \right) + 2\mu \dfrac{\partial v_x}{\partial x} \\ \dfrac{\partial \tau_{xz}}{\partial t} = \mu \left(\dfrac{\partial v_x}{\partial z} + \dfrac{\partial v_z}{\partial x} \right) \\ \dfrac{\partial \tau_{zz}}{\partial t} = \lambda \left(\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_z}{\partial z} \right) + 2\mu \dfrac{\partial v_z}{\partial z} \end{cases}$$

Elastic wave equation

For the isotropic case, inserting $\tau_{ij} = \lambda\delta_{ij}\partial_k u_k + \mu(\partial_i u_j + \partial_j u_i)$ from Hooke's law into the principle of inertia:

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \partial_j\left(\lambda\delta_{ij}\partial_k u_k + \mu(\partial_i u_j + \partial_j u_i)\right) + f_i$$

$$\equiv$$

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla\lambda(\nabla\cdot\mathbf{u}) + \nabla\mu\cdot[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}] + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \mu\nabla^{\mathbf{2}}\mathbf{u} + \mathbf{f}$$

Gradient of vector → Tensor

Laplacian: $\nabla\cdot\nabla\,\mathbf{u}$

Elastic wave equation

Using the following vector identity ($\nabla \times \nabla \times \mathbf{u} = \nabla\nabla \cdot \mathbf{u} - \nabla^2\mathbf{u}$) for the second last term:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \underbrace{\nabla\lambda(\nabla \cdot \mathbf{u}) + \nabla\mu \cdot [\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}]}_{\text{Only for heterogenous media } (\nabla\lambda,\ \nabla\mu \neq 0)} + (\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times \nabla \times \mathbf{u} + \mathbf{f}$$

Elastic (seismic) wave equation

Homogenous elastic wave equation

Two methods are commonly used in seismology with the homogenous elastic wave equation:

- **Layer cake model** (aka homogenous-layer methods): in each layer $\nabla\lambda = \nabla\mu = 0$, this leads to a simplified version of the wave equation and reflection&transmission coefficients are used to link layers. Useful for:
    - Surface waves
    - Low/medium frequency body waves

- **Ray-based methods**: these methods work in a high-frequency regime. Since one can show that $\nabla\lambda = \nabla\mu \; \alpha \; 1/\omega$, $\nabla\lambda = \nabla\mu = 0$ for $\omega \rightarrow \infty$.

Helmoltz decomposition theorem

→ We want to separate the homogenous elastic wave equation into its P- and S-wave components.

A displacement field $\mathbf{u}$ can be decomposed into its:

- **Irrotational or central component:** $\nabla \times \nabla \Phi = 0$
- **Solenoidal or rotational component:** $\nabla \cdot (\nabla \times \boldsymbol{\Psi}) = 0$

$$\mathbf{u} = \nabla \Phi + \nabla \times \boldsymbol{\Psi}$$

$$\nabla \cdot \boldsymbol{\Psi} = \mathbf{0}$$

Φ: scalar potential

$\boldsymbol{\Psi}$: vectorial potential

Helmoltz decomposition theorem

→ For elastic waves, Φ: P-wave, $\boldsymbol{\Psi}$: S-wave

Applying $\nabla \cdot$ to the displacement equation:

$$\nabla \cdot \mathbf{u} = \nabla \cdot \nabla\Phi + \cancel{\nabla \cdot \nabla \times \boldsymbol{\Psi}} = \nabla^2\Phi \longrightarrow \boxed{\nabla \cdot \mathbf{u} = \nabla^2\Phi}$$

Applying $\nabla \times$ to the displacement equation:

$$\nabla \times \mathbf{u} = \cancel{\nabla \times \nabla\Phi} + \nabla \times \nabla \times \boldsymbol{\Psi} = \nabla\cancel{\nabla \cdot \boldsymbol{\Psi}} - \nabla^2\boldsymbol{\Psi} = -\nabla^2\boldsymbol{\Psi} \longrightarrow \boxed{\nabla \times \mathbf{u} = -\nabla^2\boldsymbol{\Psi}}$$

P-wave equation

Taking the divergence of the homogenous, source-free elastic wave equation:

$$\nabla \cdot \left(\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \right) = \nabla \cdot \left((\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times \nabla \times \mathbf{u} \right)$$

$$\nabla \cdot \nabla = \nabla^2 \qquad \nabla \cdot \nabla \times = 0$$

$$\frac{\partial^2}{\partial t^2} \nabla \cdot \mathbf{u} = \left(\frac{\lambda + 2\mu}{\rho} \right) \nabla^2 (\nabla \cdot \mathbf{u})$$

$$\boxed{\nabla^2 \Phi_P - \frac{1}{\alpha^2} \frac{\partial^2}{\partial t^2} \Phi_P = 0}$$

$$\beta = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

$$\Phi_P \equiv \nabla \cdot \mathbf{u}$$

S-wave equation

Taking the curl of the homogenous, source-free elastic wave equation:

$$\nabla \times \left(\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \right) = \nabla \times \left((\lambda + 2\mu) \nabla \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u} \right)$$

$$\frac{\partial^2}{\partial t^2} \nabla \times \mathbf{u} = \frac{\mu}{\rho} \nabla^2 (\nabla \times \mathbf{u})$$

$$\boxed{\nabla^2 \mathbf{\Psi}_S - \frac{1}{\beta^2} \frac{\partial^2}{\partial t^2} \mathbf{\Psi}_S = 0}$$

$$\beta = \sqrt{\frac{\mu}{\rho}}$$

$$\mathbf{\Psi}_S \equiv \nabla \times \mathbf{u}$$

P- and S-wave equation

Putting all together:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times \nabla \times \mathbf{u}$$

$$\downarrow$$

$$\boxed{\frac{\partial^2 \mathbf{u}}{\partial t^2} = \alpha^2\nabla(\nabla \cdot \mathbf{u}) - \beta^2\nabla \times \nabla \times \mathbf{u}}$$

P-wave

S-wave

Polarization of waves

Using the plane wave formulation we can study how different waves behave.

- P-wave along x-dimension:

$$\alpha^2 \partial_{xx}\Phi = \partial_{tt}\Phi \rightarrow \Phi = \Phi_0(t \pm x/\alpha)$$

Since $\mathbf{u} = \nabla\Phi$, we have $\mathrm{u_x} = \partial_x\Phi$ and $\mathrm{u}_y = \mathrm{u}_z = 0$, meaning that the displacement happens only parallel to the direction of propagation with particle motion consisting of **Dilatation** and **Compression**.

$\downarrow$

Longitudinal waves

Polarization of waves

Using the plane wave formulation we can study how different waves behave.

- S-wave along x-dimension:

$$\beta^2(\partial_{xx}\Psi_x \boldsymbol{i}_x + \partial_{yy}\Psi_y \boldsymbol{i}_y + \partial_{zz}\Psi_z \boldsymbol{i}_z) = \partial_{tt}\boldsymbol{\Psi}$$

$$\rightarrow \boldsymbol{\Psi} = \Psi_x(t \pm x/\beta)\boldsymbol{i}_x + \Psi_y(t \pm y/\beta)\boldsymbol{i}_y + \Psi_z(t \pm z/\beta)\boldsymbol{i}_z$$

Since $\mathbf{u} = \nabla \times \boldsymbol{\Psi}$, and $\partial_z = \partial_y = 0$ (no changes of wavefield in the non-propagating directions), we have $\mathrm{u_x} = 0$, $\mathrm{u_y} = -\partial_x \Psi_z$, $\mathrm{u_z} = \partial_x \Psi_y$, displacement is perpendicular and motion consisting of 2 components:

→ **SV:** withing plane of propagation

→ **SH:** across plane of propagation