3. Acoustic Wave Equation

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ERSE 210 Seismology

Wave equation(s)

- **Principle of Inertia** (aka **Equation of motion**)

  Links variations in time and space of deformations with stresses → generates movement in first place

- **Hooke's law** (aka **Deformation equation**)

  Links stresses and deformations → acts as contrasting force to initial movement, leading to oscillations

Acoustic Wave Equation

- Validity: isotropic fluid (no viscosity) → only compressional waves
- Quantities: pressure (scalar) and particle velocity (vector)

$$\underbrace{p_t}_{\text{total}} = \overbrace{p_0}^{\text{static}} + \underbrace{p}_{\text{variations due to acoustic field}}$$

$$\boldsymbol{v}_t = \boldsymbol{v}_0 + \boldsymbol{v}$$

=0 medium assumed fixed

$$c_t = c_0 + \cancel{c}$$

$$\rho_t = \rho_0 + \cancel{\rho}$$

Principle di Inertia

Links pressure gradients to temporal variations of velocity (i.e., acceleration)

Starting from Newton 2nd law:

Total derivative of $\mathbf{v}(\mathbf{r}(\mathrm{t}), \mathrm{t})$

Given a volume $\Delta\mathrm{V}$ with mass $\Delta\mathrm{m}$ onto which acts a force $\Delta\mathbf{F}$

$$\Delta\mathbf{F} = \frac{\mathrm{d}}{\mathrm{dt}}(\Delta\mathrm{m}\mathbf{v}) = \Delta\mathrm{m}\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right)$$

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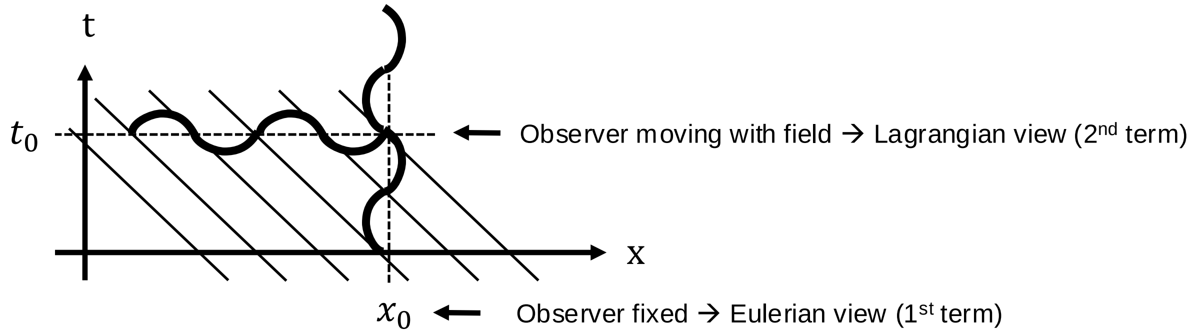

Principle di Inertia

Links pressure gradients to temporal variations of velocity (i.e., acceleration)

Assuming the fluid to be static (only field moves):

$$\Delta\mathbf{F} = \Delta\mathrm{m}\frac{\partial\mathbf{v}}{\partial t} \xrightarrow[\text{Use pressure force relation}]{} -\nabla p\Delta\mathrm{V} = \Delta\mathrm{m}\frac{\partial\mathbf{v}}{\partial t} \xrightarrow[\rho_0 = \frac{\Delta\mathrm{m}}{\Delta\mathrm{V}}]{} \boxed{-\nabla p = \rho_0\frac{\partial\mathbf{v}}{\partial t}}$$

Principle di Inertia

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→ Spatial variations of pressure are proportional to time variations of the particle velocity vector scaled by medium density

Hooke’s law

Links velocity divergence to temporal variations of pressure

Starting from the principle of conservation of mass to link changes in density to changes iun volume (i.e. perturb expression of mass):

$$m = \rho \Delta \mathrm{V} = (\rho + \mathrm{d}\rho)(\Delta \mathrm{V} + \mathrm{dV}) = \rho \Delta \mathrm{V} + \rho \mathrm{dV} + \mathrm{d}\rho \Delta \mathrm{V} + \mathrm{d}\rho \mathrm{dV}$$

Hooke’s law

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$$m = \rho\cancel{\Delta}\mathrm{V} = (\rho + \mathrm{d}\rho)(\Delta\mathrm{V} + \mathrm{dV}) = \rho\cancel{\Delta}\mathrm{V} + \rho\mathrm{dV} + \mathrm{d}\rho\Delta\mathrm{V} + \mathrm{d}\cancel{\rho}\mathrm{dV}$$

=0 discard higher order terms

$$0 = \rho\mathrm{dV} + \mathrm{d}\rho\Delta\mathrm{V} \rightarrow \frac{\mathrm{d}\rho}{\rho} = -\frac{\mathrm{d}V}{\Delta\mathrm{V}}$$

Hooke’s law

Links velocity divergence to temporal variations of pressure

Let’s now introduce also the adiabatic transformation to link changes in density to changes in pressure:

$$p\rho^{-\frac{C_p}{C_V}} = \text{const.}$$

C_p, C_v specific heat at constant pressure and velocity

$$\downarrow$$

$$p^k\rho = \text{const.} \xrightarrow[\text{perturb}]{} p^k\rho = (p+dp)^k(\rho+\mathrm{d}\rho)$$

$$= p^k(1+dp/p)^k\,\rho(1+\mathrm{d}\rho/\rho)$$

$$= p^k\rho(1+kdp/p+\cdots)\,(1+\mathrm{d}\rho/\rho)$$

$$\longrightarrow \frac{d\rho}{\rho} = -k\frac{dp}{p}$$

Hooke's law

Links velocity divergence to temporal variations of pressure

Substitute $\frac{\mathrm{d}\rho}{\rho} = -\frac{\mathrm{d}V}{\Delta \mathrm{V}}$ into $\frac{d\rho}{\rho} = -k\frac{dp}{p}$

$$\frac{\mathrm{d}V}{\Delta\mathrm{V}} = k\frac{\mathrm{d}p}{p} = -\kappa \mathrm{d}p \rightarrow \mathrm{d}p = -K\frac{\mathrm{d}V}{\Delta\mathrm{V}}$$

$\kappa = -\frac{k}{p}$ Compressibility (measure of relative volume change of fluid as response to pressure change

$K = \frac{1}{\kappa} = \rho_0 c^2$ Bulk modulus (how resistant is a fluid to compression)

Hooke’s law

Links velocity divergence to temporal variations of pressure

Finally, given the definition of flux of a vectorial field, we write

$$\frac{\mathrm{d}V}{\Delta \mathrm{V}} = (\nabla \cdot \boldsymbol{v})\mathrm{dt} \longrightarrow \qquad \frac{\mathrm{d}p}{dt} = -K\frac{\mathrm{d}V}{\Delta \mathrm{V}} = -K(\nabla \cdot \boldsymbol{v})$$

$\approx \frac{\partial p}{\partial t}$, assuming low-velocity approximation

$$\boxed{\frac{\partial p}{\partial t} = -K(\nabla \cdot \boldsymbol{v})}$$

→ Variations in volume of fluid lead to the contraction of the velocity field (by its divergence), which in turn modifies the pressure of the fluid (- sign: reduction in pressure for increase in volume and viceversa)

Acoustic wave equation constituents

- **Principle of Inertia** $$-\nabla p = \rho_0 \frac{\partial \boldsymbol{v}}{\partial t}$$

- **Hooke's law** $$-\nabla \cdot \boldsymbol{v} = \frac{1}{K} \frac{\partial p}{\partial t}$$

Acoustic wave equation derivation

Applying the divergence to the principle of inertia:

$$\nabla \cdot \left(-\frac{1}{\rho_0}\nabla p\right) = \nabla \cdot \left(\frac{\partial \boldsymbol{v}}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}(\nabla \cdot \mathbf{v})$$

Swap ∇ and $\frac{\partial}{\partial t}$

$$= \frac{\partial}{\partial t}\left(-\frac{1}{K}\frac{\partial p}{\partial t}\right)$$

Insert Hooke’s law

$$= -\frac{1}{\rho_0 c^2}\frac{\partial^2 p}{\partial t^2}$$

Use $K = \rho_0 c^2$ (and assume time invariance of the medium)

Acoustic wave equation derivation

Use $K = \rho_0 c^2$

$$\boxed{\rho_0 \nabla \cdot \left(\frac{1}{\rho_0} \nabla p \right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0}$$

→ Variable velocity, variable density acoustic wave equation

Assuming $\rho_0 = \text{const.}$

$$\boxed{\nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0}$$

→ Variable velocity, constant density acoustic wave equation

$\nabla^2 = \nabla \cdot (\nabla)$ Laplacian

Helmoltz equation

Time independent representation of the wave equation:

$$\nabla^2 p(t,\boldsymbol{r}) - \frac{1}{c^2}\frac{\partial^2 p(t,\boldsymbol{r})}{\partial t^2} = 0 \xrightarrow[\text{Separation of variables}]{} \nabla^2 P(\omega,\boldsymbol{r}) + \frac{\omega^2}{c^2} P(\omega,\boldsymbol{r}) = 0$$

$$p(t,\boldsymbol{r}) = P(\boldsymbol{r})T(t) = P(\boldsymbol{r})e^{j\omega t}$$

$$\frac{\partial^2 p(t,\boldsymbol{r})}{\partial t^2} = P(\boldsymbol{r})(j\omega)^2 e^{j2\pi f t}$$

$$= -\omega^2 P(\boldsymbol{r})T(t)$$

$\omega = 2\pi f$ angular frequency

Helmoltz equation

Time independent representation of the wave equation:

$$\nabla^2 P(\omega, \boldsymbol{r}) + \frac{\omega^2}{c^2} P(\omega, \boldsymbol{r}) = 0 \longrightarrow \boxed{\nabla^2 P(\omega, \boldsymbol{r}) + k^2 P(\omega, \boldsymbol{r}) = 0}$$

Wavenumber: $k = \frac{\omega}{c}\ [1/m]$

Eigenvalue problem for the Laplacian operator

→ Another simpler way to remember about the Helmoltz equation: $p(t, \boldsymbol{r}) \to p(f, k), \nabla \to jk, \partial_t \to j\omega$

Green's function

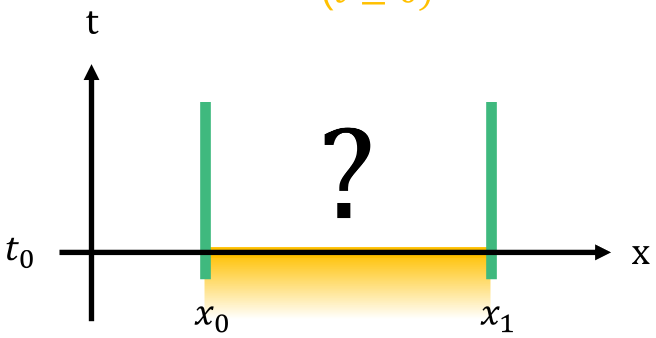
The acoustic wave equation is a 2nd order PDE → ICs/BCs to be fully defined

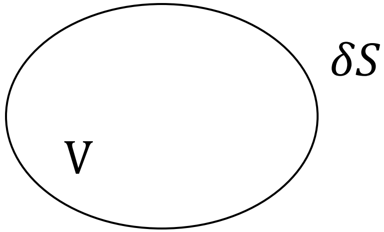
Source-free AWE

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \quad + \quad \begin{cases} p(x,t) = p_0 \\ \dfrac{\partial p}{\partial t} = \partial p_0 \end{cases} \quad + \quad \begin{cases} p(x \in \delta S, t) = p_{\delta S} & \text{Dirichelet BC} \\ \dfrac{\partial p}{\partial n}(x \in \delta S, t) = \partial p_{\delta S} & \text{Neumann BC} \end{cases}$$

Initial Conditions
$(t \leq 0)$

Boundary Conditions
(δS)





Green's function

The acoustic wave equation is a 2nd order PDE → ICs/BCs to be fully defined

AWE with sources

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -s(t,\boldsymbol{r}) \quad + \quad \begin{cases} p(x,t) = \mathbf{0} \\ \dfrac{\partial p}{\partial t} = \mathbf{0} \end{cases} \quad + \quad \begin{cases} p(x \in \delta S, t) = \mathbf{0} \\ \dfrac{\partial p}{\partial n}(x \in \delta S, t) = \mathbf{0} \end{cases}$$

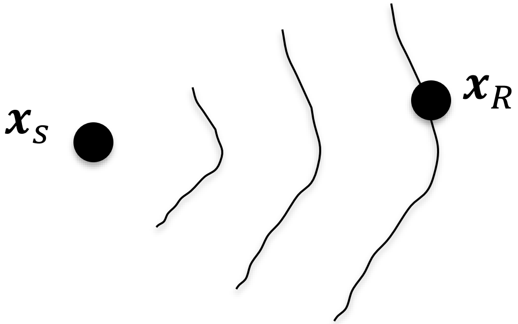
$$s(t,\boldsymbol{r}) = \rho \frac{\partial^2 i_v}{\partial t} - \nabla \cdot \boldsymbol{f}$$

Volume density of volume injection (monopole source) Volume density of external force (dipole source)

Green’s function

A Green’s function is the solution to the AWE for an **impulsive source**:

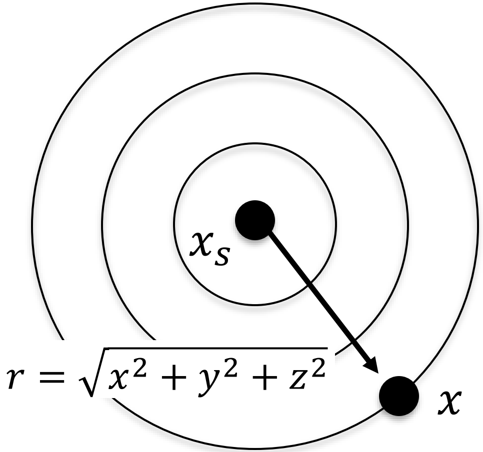
$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -\delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t')$$



$$p(t, \boldsymbol{x}_R) = s(t) * G(t, \boldsymbol{x}_R; t', \boldsymbol{x}_S)$$

3D Green's function

Spherical wave expanding from source



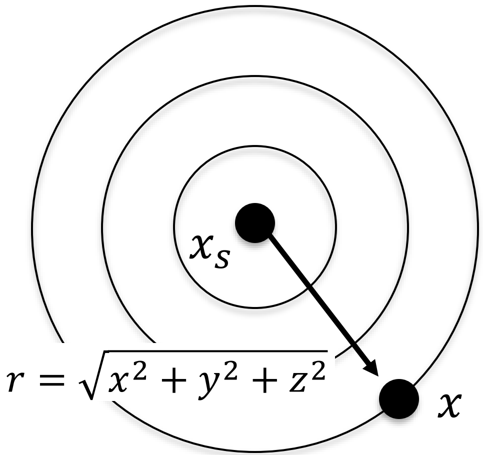
Amplitude decays with r

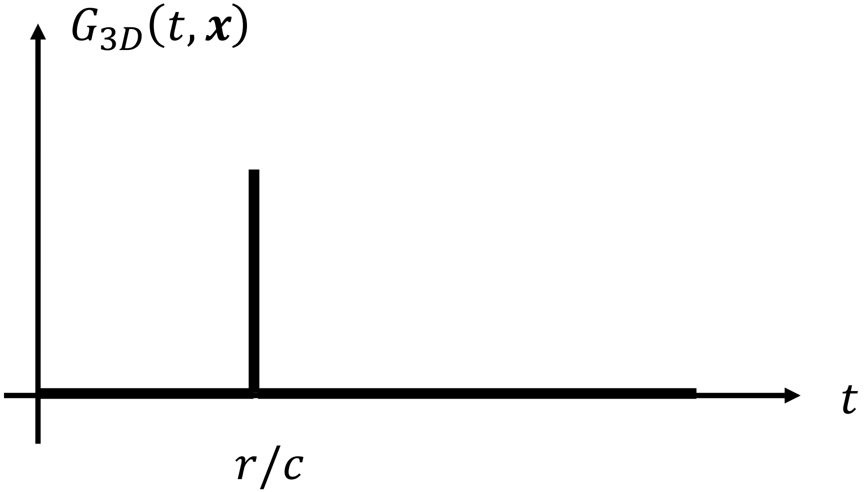
$$G_{3D}(t, \boldsymbol{x}) = \frac{1}{4\pi r} \delta\left(t - \frac{r}{c}\right)$$

$$G_{3D}(f, \boldsymbol{x}) = \frac{1}{4\pi r} e^{j2\pi f \frac{r}{c}}$$

3D Green's function

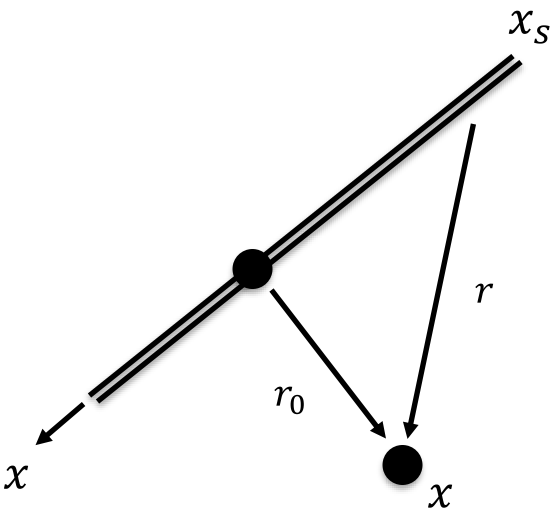
Spherical wave expanding from source



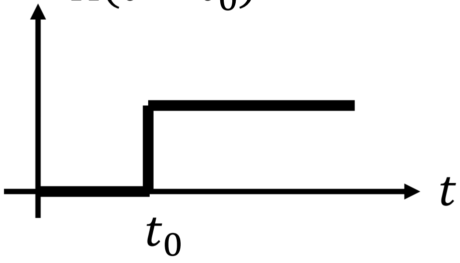


2D Green's function

Cilindrical wave with source along a line



$\mathrm{H}(t - t_0)$ Heaviside function



$$G_{2D}(t, \boldsymbol{x}) = \frac{1}{2\pi} \frac{\mathrm{H}\left(t - \frac{r_0}{c}\right)}{\sqrt{t^2 - \frac{r_0^2}{c^2}}}$$

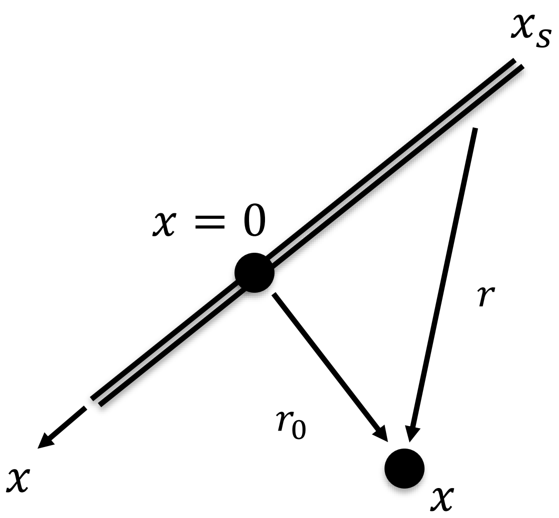
Hankel function

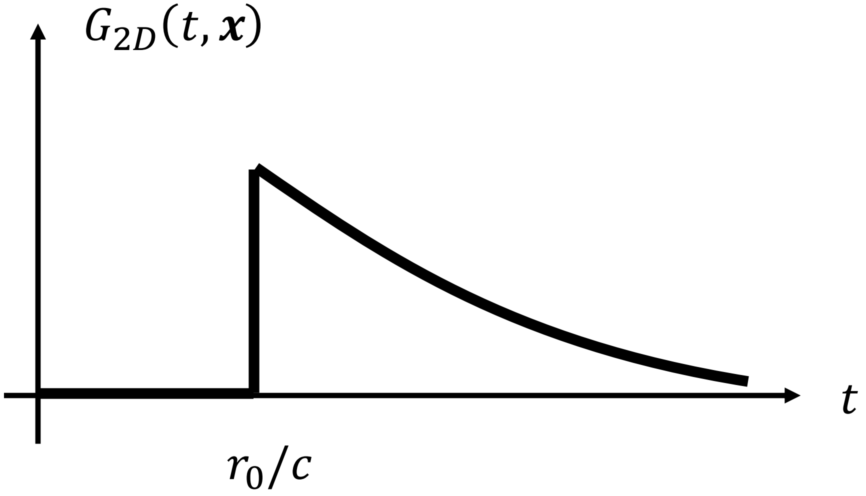
$$\mathrm{H}_0^1 = J_0 + jN_0$$

$$G_{2D}(f, \boldsymbol{x}) = \frac{j}{4\pi} \mathrm{H}_0^1\left(\omega \frac{r_0}{c}\right)$$

2D Green's function

Cilindrical wave with source along a line



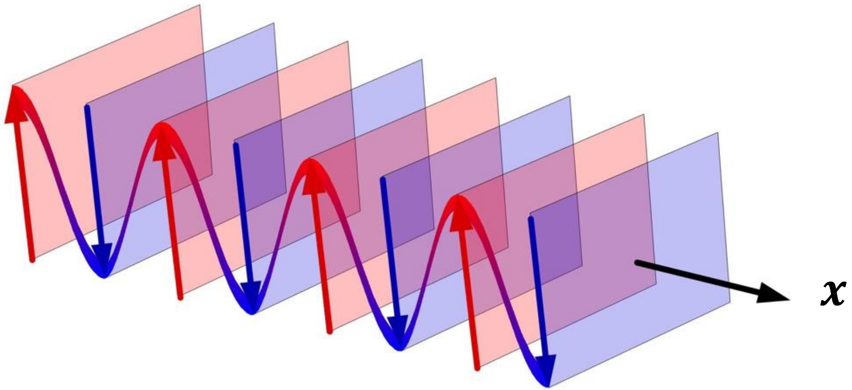


→ Long decay due to signals contributing from multiple locations along the source line

1D Green's function

Plane wave with source extending to infinity over a plane

yz-plane source, directed over x



$$G_{1D}(t,\boldsymbol{x}) = \frac{2}{c}\,\delta\left(t - \frac{\boldsymbol{x}}{c}\right)$$

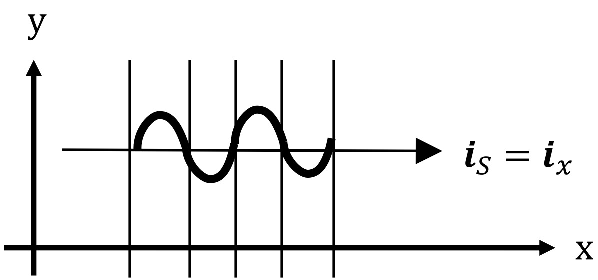
$$G_{1D}(f,\boldsymbol{x}) = \frac{1}{4\pi}\,e^{j\omega\frac{\boldsymbol{x}}{c}}$$

Amplitude does NOT decay

1D Green's function

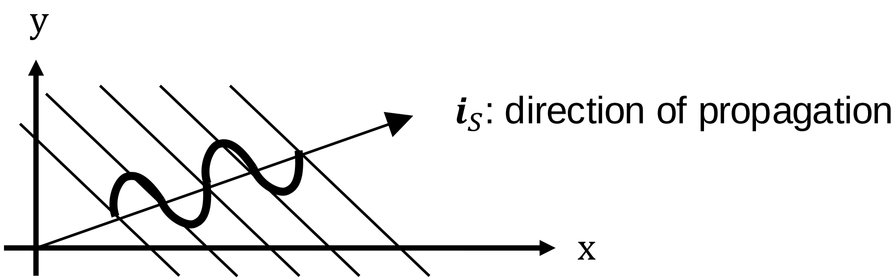
Plane wave with source extending to infinity over a plane

<u>Top view (x-direction)</u>



lines of constant displacement

<u>Top view (general direction)</u>



lines of constant displacement

1D Green's function

Plane wave with source extending to infinity over a plane

<u>Top view (x-direction)</u>

$$\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{f}\left(t \pm \frac{x}{c}\right)$$

- : positive direction of prop.
+ : negative direction of prop.

<u>Top view (general direction)</u>

$$\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{f}\left(t \pm \frac{\boldsymbol{i}_S \cdot \boldsymbol{x}}{c}\right)$$
$$= \boldsymbol{f}(t \pm \boldsymbol{s} \cdot \boldsymbol{x})$$

$$\boldsymbol{s} = \frac{\boldsymbol{i}_S}{c}$$ slowness vector

1D Green's function

- Plane waves are particular-type of waves whose displacement varies <u>only</u> in the direction of wave propagation and it is constant in orthogonal directions.

- Away from the source, every wave can be approximated by a plane wave

- Any wave can be decomposed as weighted sum of plane waves via the Fourier transform (i.e. FK_xK_y):

$$\boxed{\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{A}(\omega)e^{-j\omega(t\pm\boldsymbol{s}\cdot\boldsymbol{x})} = \boldsymbol{A}(\omega)e^{-j\,(\omega t\pm\boldsymbol{k}\cdot\boldsymbol{x})}}$$

Monocromatic plane wave

$\mathbf{k} = \frac{\omega}{c}\boldsymbol{i}_S$ Wavenumber vector