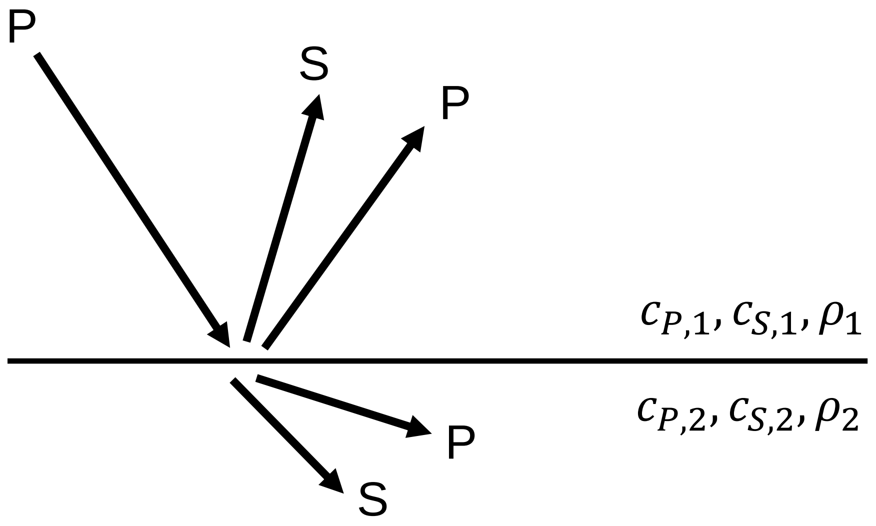
6. Waves across a planal interface

M. Ravasi

ERSE 210 Seismology

Seismic propagation across an interface

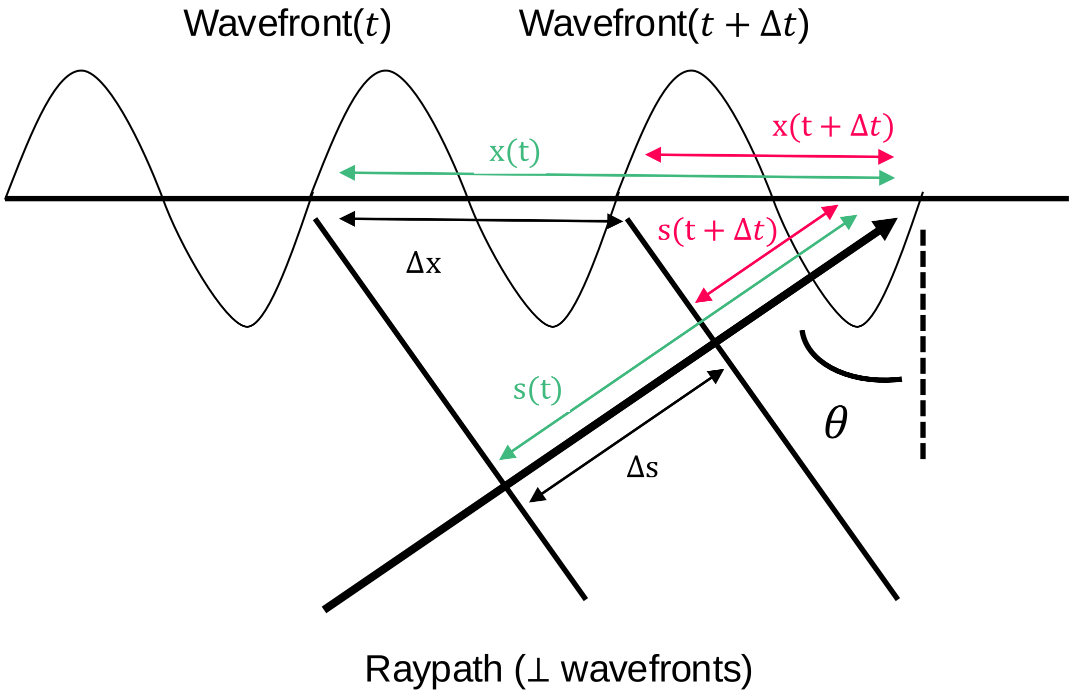


2 components:

- **kinematic:** Snell's Law
- **dynamic:** Lamb's problem

Snell's law

Given a plane wave propagating in a material with constant velocity c and incident angle θ



Snell's law

Given a plane wave propagating in a material with constant velocity c and incident angle θ

→ The wavefronts at t and $t + \Delta t$ are separated by a distance Δs along the ray path. Using Pytagoras theorem we can write:

$$\mathrm{s(t) = x(t)sin}\theta$$

$$\mathrm{s(t+\Delta t) = x(t+\Delta t)sin}\theta$$

$$\Delta\mathrm{s = s(t+\Delta t) - s(t) = (x(t+\Delta t) - x(t))sin}\theta$$

$$= \Delta\mathrm{xsin}\theta$$

→ We can also write a second relation for Δs: $\Delta s = \mathrm{c} \cdot \Delta\mathrm{t}$

Snell's law

→ Equating the two terms:

$$\boxed{\Delta x \sin\theta = c \cdot \Delta t} \quad \rightarrow \quad \frac{\Delta t}{\Delta x} = \frac{\sin\theta}{c} = p\,\sin\theta = p_x$$

Slowness [s/m]

Horizontal slowness or ray parameter

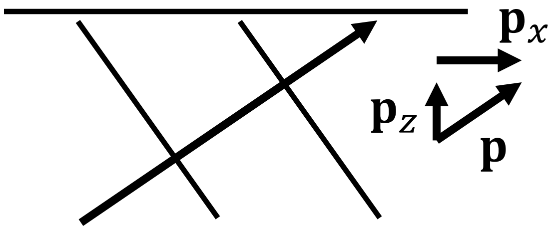
Snell's law

→ Equating the two terms:

$$\boxed{\Delta x \sin\theta = c \cdot \Delta t} \quad \rightarrow \quad \frac{\Delta t}{\Delta x} = \frac{\sin\theta}{c} = \mathrm{p}\sin\theta = p_x$$

Slowness [s/m]

Horizontal slowness or ray parameter



$$\mathrm{p}^2 = \mathrm{p}_\mathrm{x}^2 + \mathrm{p}_z^2 \rightarrow \mathrm{p}_z = \sqrt{\mathrm{p}^2 - \mathrm{p}_\mathrm{x}^2} = \mathrm{p}\sqrt{1 - sin\theta^2} = pcos\theta$$

Snell's law

1. Defines the equivalent of the Nyquist theorem for waves (i.e., spatial sampling)

Assuming $\Delta t = \mathrm{T}\ (\mathrm{period}) = 1/\mathrm{f}_{\mathrm{wave}}$ $\longrightarrow$ $\Delta\mathrm{x} = \dfrac{c\mathrm{T}}{\sin\theta}$ $\longrightarrow$ $\boxed{\mathrm{dx} \le \dfrac{\Delta\mathrm{x}}{2} = \dfrac{c\mathrm{T}}{2\sin\theta}}$

Distance between start and end of a period of the wave along the horizontal (i.e., recording) plane

Sampling along x-axis to avoid spatial aliasing for wave with incident angle θ

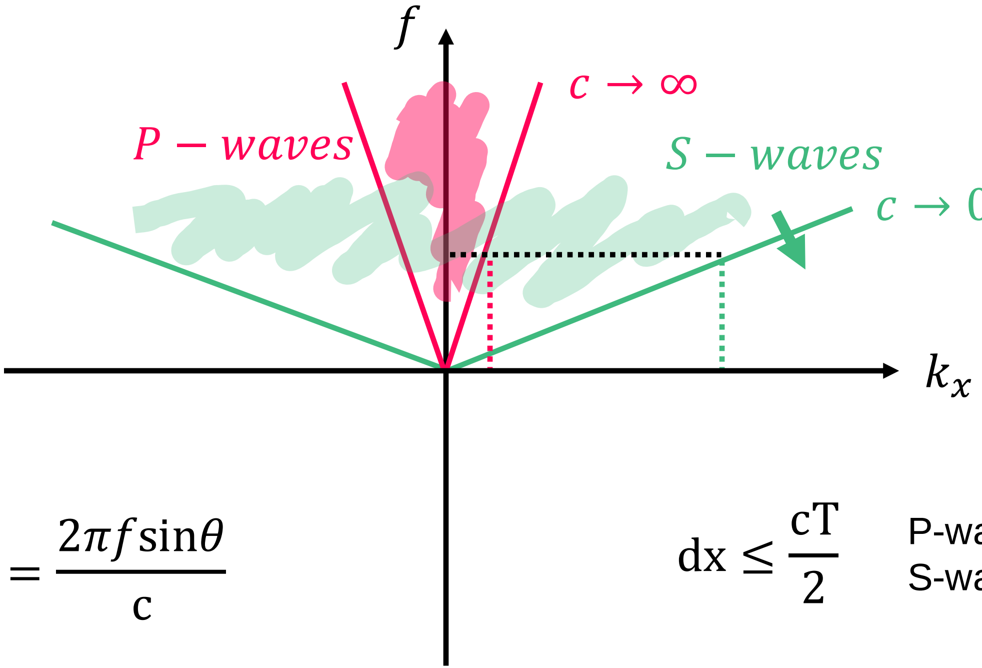
Expressing the Nyquist theorem in terms of horizontal wavenumber

$$\mathrm{k}_{\mathrm{x,c}} \ge 2\,\frac{2\pi f_{wave}\sin\theta}{\mathrm{c}} \longrightarrow \boxed{\mathrm{k}_{\mathrm{x,c}} \ge \frac{4\pi f_{wave}}{\mathrm{c}},\ \mathrm{dx} \le \frac{\mathrm{cT}}{2}}$$

Most stringent criterion for $\sin\theta = 1$
($\theta = 90$, wave propagating horizontally)

Snell's law

1. Defines the equivalent of the Nyquist theorem for waves (i.e., spatial sampling)



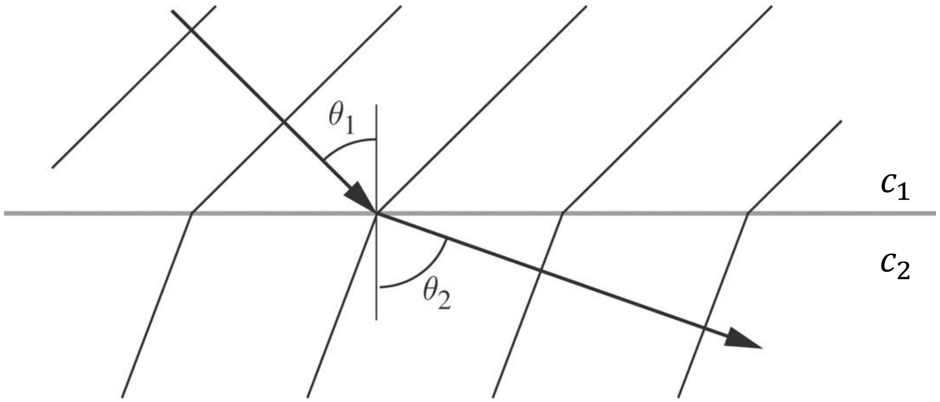
$$k_x = \frac{2\pi f \sin\theta}{c}$$

$$dx \leq \frac{cT}{2}$$

P-waves: larger dx
S-waves: smaller dx

Snell’s law

2. Explains how waves change direction when crossing an interface



Because we assume that a wave does not change frequency f_{wave} across an interface, also the period T does not change the wavefront are connected:

$$\sin\theta_1 \Delta\text{x} = c_1 \Delta\text{t} \rightarrow \frac{\Delta\text{t}}{\Delta\text{x}} = \frac{\sin\theta_1}{c_1}$$

$$\sin\theta_2 \Delta\text{x} = c_2 \Delta\text{t} \rightarrow \frac{\Delta\text{t}}{\Delta\text{x}} = \frac{\sin\theta_2}{c_2}$$

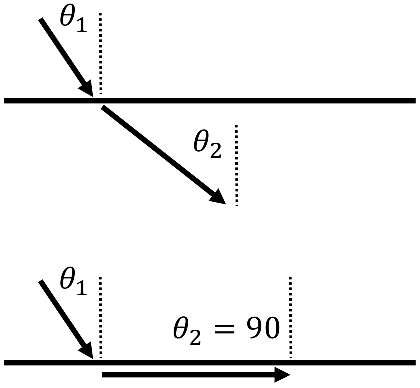
$$\boxed{\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}} \quad \boxed{p_{x1} = p_{x2}}$$

Snell’s law

2. Explains how waves change direction when crossing an interface

→ For layers of increasing velocity, the rays flatten

$$\sin\theta_2 = \frac{c_2}{c_1}\sin\theta_1$$



If $\frac{c_2}{c_1} > 1, \sin\theta_2 > \sin\theta_1, \theta_2 > \theta_1$

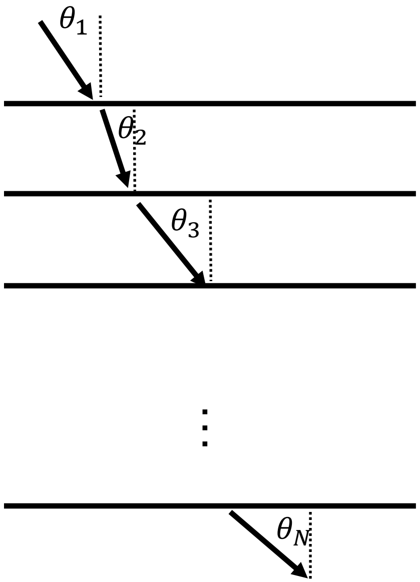
If $\frac{c_2}{c_1}\sin\theta_1 = 1, \theta_2 = 90$ (Refraction)

θ_1 is called critical angle as a flatter ray would lead to $\sin\theta_1 > 1$ → evanescent wave

Snell's law

2. Explains how waves change direction when crossing an interface

→ The horizontal slowness remains unchanged also for a stack of layers ($p_{x1} = p_{x2} = p_{x3} = \ldots = p_{xN}$)

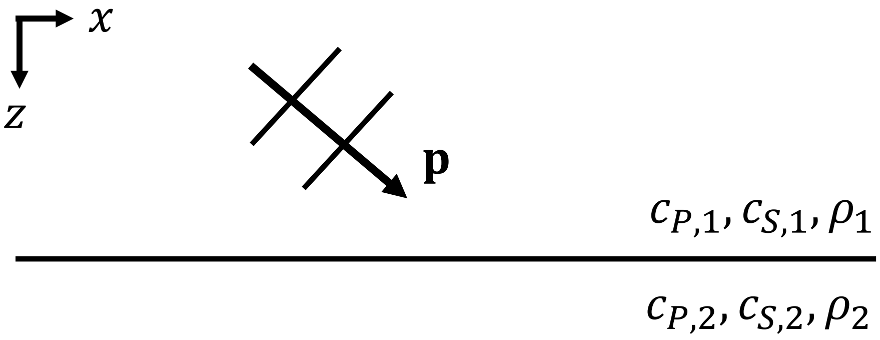


From layer 1 to 2: $\sin\theta_2 = \dfrac{c_2}{c_1}\sin\theta_1$

From layer 1 to N: $\sin\theta_N = \dfrac{c_N}{c_1}\sin\theta_1$

Lamb’s problem

Given a plane wave hitting an interface, we are interest to find out how the amplitude of the incident wave is split into **Reflection** and **Transmission** components.



Plane wave in x-z space:

$$\mathbf{u}(\mathbf{x}, \mathrm{t}) = \mathbf{f}(\mathrm{t} - \mathbf{p} \cdot \boldsymbol{x})$$

$$\mathbf{p} = \mathrm{p_x}\mathbf{i_x} + \mathrm{p_z}\mathbf{i_z}$$

Both up and downgoing

$$\mathbf{u}(\mathbf{x}, \mathrm{t}) = \mathbf{f}(\mathrm{t} - p_x x \pm p_z z)$$

Only going from left to right

Lamb’s problem

Given a plane wave hitting an interface, we are interest to find out how the amplitude of the incident wave is split into **Reflection** and **Transmission** components.

→ For vertically stratified media, 2 solutions:

- P-SV: P-waves and S-waves polarized along the plane of propagating (xz); since they are coupled, they cannot be treated independently.

- SH: S-waves polarixed in the horizontal direction outside of the propagation plane (y); never coupled with P-SV (not even at the discontinuity), therefore easier to analyse

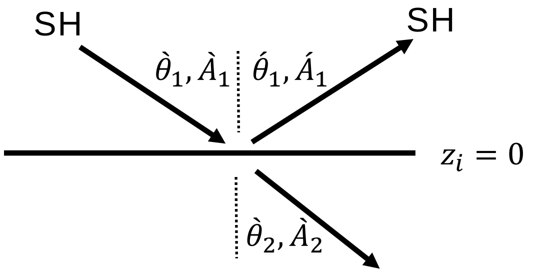
Lamb's problem

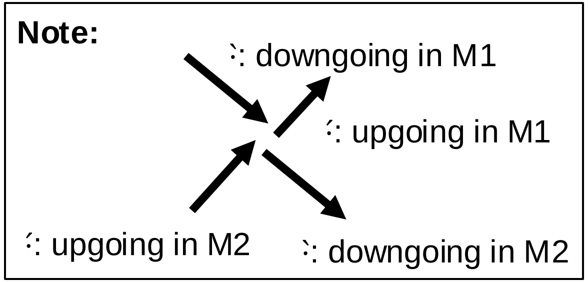
Given a plane wave hitting an interface, we are interest to find out how the amplitude of the incident wave is split into **Reflection** and **Transmission** components.

→ For vertically stratified media, 2 solutions:

- P-SV: P-waves and S-waves polarized along the plane of propagating (xz); since they are coupled, they cannot be treated independently.

- **SH: S-waves polarized in the horizontal direction outside of the propagation plane (y); never coupled with P-SV (not even at the discontinuity), therefore easier to analyse**





SH reflection and transmission

The displacement reduces to a single component:

$$\mathbf{u}(\mathbf{x},\mathrm{t};\omega)=\begin{bmatrix}u_x(\mathbf{x},\mathrm{t};\omega)\\u_y(\mathbf{x},\mathrm{t};\omega)\\u_z(\mathbf{x},\mathrm{t};\omega)\end{bmatrix}\begin{bmatrix}0\\\grave{A}f_y(t-p_xx-p_zz)+\acute{A}f_y(t-p_xx+p_zz)\\0\end{bmatrix}$$

Similarly the vertical traction (remember: $u_x=u_z=0$ and $\partial/\partial_y=0$ as displ. is constant over y)

$$\mathbf{t}(\mathbf{i}_z)(\mathbf{x},\mathrm{t};\omega)=\begin{bmatrix}\tau_{xz}(\mathbf{x},\mathrm{t};\omega)\\\tau_{yz}(\mathbf{x},\mathrm{t};\omega)\\\tau_{zz}(\mathbf{x},\mathrm{t};\omega)\end{bmatrix}\xrightarrow[\text{In isotropic media } \tau_{ij}=\lambda\delta_{ij}\epsilon_{kk}+2\mu\epsilon_{ij}]{}\mathbf{t}(\mathbf{i}_z)(\mathbf{x},\mathrm{t};\omega)=\begin{bmatrix}0\\\mu\frac{\partial u_y}{\partial_z}\\0\end{bmatrix}=\begin{bmatrix}0\\-p_z\mu[\grave{A}_1f_y'(t-p_xx-p_zz)-\acute{A}_1f_y(t-p_xx+p_zz)]\\0\end{bmatrix}$$

SH reflection and transmission

i) Continuity of displacement: $\mathbf{u}(\mathrm{x}, \mathrm{z_i} - \epsilon, \mathrm{t}; \omega) = \mathbf{u}(\mathrm{x}, \mathrm{z_i} + \epsilon, \mathrm{t}; \omega) \rightarrow \mathbf{u}^{+} = \mathbf{u}^{-}$

ii) Continuity of vertical traction: $\mathbf{t}(\mathbf{i}_z)(\mathrm{x}, \mathrm{z_i} - \epsilon, \mathrm{t}; \omega) = \mathbf{t}(\mathbf{i}_z)(\mathrm{x}, \mathrm{z_i} + \epsilon, \mathrm{t}; \omega) \rightarrow \mathbf{t}(\mathbf{i}_z)^{+} = \mathbf{t}(\mathbf{i}_z)^{-}$

SH reflection and transmission

i) Continuity of displacement: $\mathbf{u}(\mathrm{x}, \mathrm{z_i} - \epsilon, \mathrm{t}; \omega) = \mathbf{u}(\mathrm{x}, \mathrm{z_i} + \epsilon, \mathrm{t}; \omega) \rightarrow \mathbf{u}^{+} = \mathbf{u}^{-}$

ii) Continuity of vertical traction: $\mathbf{t}(\mathbf{i}_z)(\mathrm{x}, \mathrm{z_i} - \epsilon, \mathrm{t}; \omega) = \mathbf{t}(\mathbf{i}_z)(\mathrm{x}, \mathrm{z_i} + \epsilon, \mathrm{t}; \omega) \rightarrow \mathbf{t}(\mathbf{i}_z)^{+} = \mathbf{t}(\mathbf{i}_z)^{-}$

$$u_y^{+}(\mathrm{x}, \mathrm{z_i} = 0, \mathrm{t}; \omega) = \grave{A}_1 f_y(t - p_x x) + \acute{A}_1 f_y(t - p_x x)$$

$$u_y^{-}(\mathrm{x}, \mathrm{z_i} = 0, \mathrm{t}; \omega) = \grave{A}_2 f_y(t - p_x x)$$

$$\longrightarrow \boxed{\grave{A}_1 + \acute{A}_1 = \grave{A}_2}$$

SH reflection and transmission

i) Continuity of displacement: $\mathbf{u}(\mathrm{x}, \mathrm{z_i} - \epsilon, \mathrm{t}; \omega) = \mathbf{u}(\mathrm{x}, \mathrm{z_i} + \epsilon, \mathrm{t}; \omega) \rightarrow \mathbf{u}^{+} = \mathbf{u}^{-}$

ii) Continuity of vertical traction: $\mathbf{t}(\mathbf{i}_z)(\mathrm{x}, \mathrm{z_i} - \epsilon, \mathrm{t}; \omega) = \mathbf{t}(\mathbf{i}_z)(\mathrm{x}, \mathrm{z_i} + \epsilon, \mathrm{t}; \omega) \rightarrow \mathbf{t}(\mathbf{i}_z)^{+} = \mathbf{t}(\mathbf{i}_z)^{-}$

$$\tau_{yz}^{+}(\mathrm{x}, \mathrm{z_i} = 0, \mathrm{t}; \omega) = -(\grave{A}_1 - \acute{A}_1)\mu_1 p_{z,1} f_y{}'(t - p_x x)$$

$$\tau_{yz}^{-}(\mathrm{x}, \mathrm{z_i} = 0, \mathrm{t}; \omega) = -\grave{A}_2 \mu_2 p_{z,2} f_y{}'(t - p_x x)$$

$$\longrightarrow \boxed{(\grave{A}_1 - \acute{A}_1)\,\mu_1 p_{z,1} = \grave{A}_2 \mu_2 p_{z,2}}$$

SH reflection and transmission

2 equations in 2 unknowns (assuming $\grave{A}_1 = 1$):

$$\grave{A}_1 + \acute{A}_1 = \grave{A}_2 \qquad\qquad (\grave{A}_1 - \acute{A}_1)\,\mu_1 p_{z,1} = \grave{A}_2 \mu_2 p_{z,2}$$

$$\grave{S}\acute{S} = \acute{A}_1 = \frac{\mu_1 p_{z,1} - \mu_2 p_{z,2}}{\mu_1 p_{z,1} + \mu_2 p_{z,2}} = \frac{\rho_1 \beta_1 cos\theta_1 - \rho_2 \beta_2 cos\theta_2}{\rho_1 \beta_1 cos\theta_1 + \rho_2 \beta_2 cos\theta_2}$$

$$\grave{S}\grave{S} = \grave{A}_2 = \frac{2\mu_1 p_{z,1}}{\mu_1 p_{z,1} + \mu_2 p_{z,2}} = \frac{2\rho_1 \beta_1 cos\theta_1}{\rho_1 \beta_1 cos\theta_1 + \rho_2 \beta_2 cos\theta_2}$$

Using $\beta^2 = \mu/\rho$ and $p_z = cos\theta/\beta$

SH reflection and transmission

2 equations in 2 unknowns (assuming $\grave{A}_1 = 1$):

$$\grave{A}_1 + \acute{A}_1 = \grave{A}_2 \qquad\qquad (\grave{A}_1 - \acute{A}_1)\,\mu_1 p_{z,1} = \grave{A}_2 \mu_2 p_{z,2}$$

$$\grave{S}\acute{S} = \acute{A}_1 = \frac{\mu_1 p_{z,1} - \mu_2 p_{z,2}}{\mu_1 p_{z,1} + \mu_2 p_{z,2}} = \frac{\rho_1 \beta_1 cos\theta_1 - \rho_2 \beta_2 cos\theta_2}{\rho_1 \beta_1 cos\theta_1 + \rho_2 \beta_2 cos\theta_2}$$

$$\grave{S}\grave{S} = \grave{A}_2 = \frac{2\mu_1 p_{z,1}}{\mu_1 p_{z,1} + \mu_2 p_{z,2}} = \frac{2\rho_1 \beta_1 cos\theta_1}{\rho_1 \beta_1 cos\theta_1 + \rho_2 \beta_2 cos\theta_2}$$

Using $\beta^2 = \mu/\rho$ and $p_z = cos\theta/\beta$

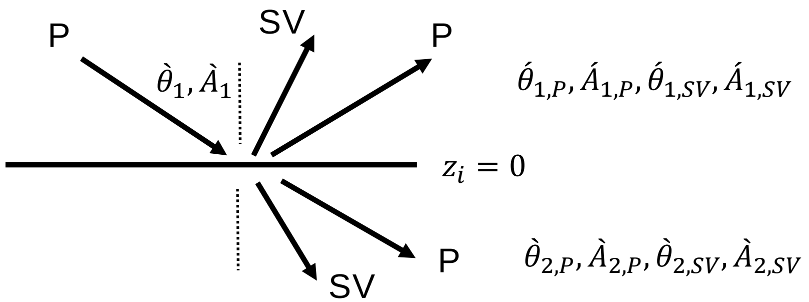
! Note that whilst the kinematic component of a seismic wavefield does not depend on density (ρ), the dynamic part does.

Lamb’s problem

Given a plane wave hitting an interface, we are interest to find out how the amplitude of the incident wave is split into **Reflection** and **Transmission** components.

→ For vertically stratified media, 2 solutions:

- SH: S-waves polarized in the horizontal direction outside of the propagation plane (y); never coupled with P-SV (not even at the discontinuity), therefore easier to analyse
- **P-SV: P-waves and S-waves polarized along the plane of propagating (xz); since they are coupled, they cannot be treated independently.**



Scattering matrices

For both the SH and the P-SV cases, the transmission and reflection coefficients can be 'summarized' into the so-called scattering matrices:

SH case

$$
\begin{array}{cc}
\multicolumn{2}{c}{\begin{pmatrix} \grave{S}\acute{S} & \acute{S}\acute{S} \\ \grave{S}\grave{S} & \acute{S}\grave{S} \end{pmatrix}} \\
\uparrow & \uparrow \\
\text{Incident from above} & \text{Incident from below}
\end{array}
$$

P-SV case

$$
\begin{pmatrix}
\grave{P}\acute{P} & \grave{S}\acute{P} & \acute{P}\acute{P} & \acute{S}\acute{P} \\
\grave{P}\acute{S} & \grave{S}\acute{S} & \acute{P}\acute{S} & \acute{S}\acute{S} \\
\grave{P}\grave{P} & \grave{S}\grave{P} & \acute{P}\grave{P} & \acute{S}\grave{P} \\
\grave{P}\grave{S} & \grave{S}\grave{S} & \acute{P}\grave{S} & \acute{S}\grave{S}
\end{pmatrix}
$$

$$
\underbrace{\qquad\qquad\qquad}_{\text{Incident from above}} \quad \underbrace{\qquad\qquad\qquad}_{\text{Incident from below}}
$$

Scattering matrices

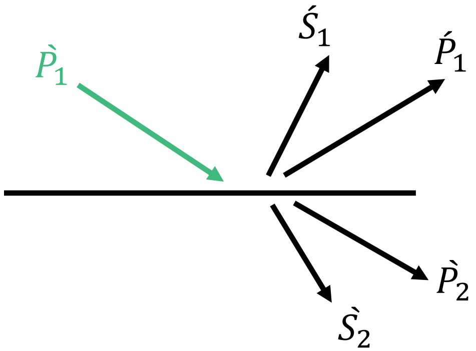
For both the SH and the P-SV cases, the transmission and reflection coefficients can be 'summarized' into the so-called scattering matrices:

P-SV case

$$\begin{pmatrix} \grave{P}\acute{P} & \grave{S}\acute{P} & \acute{P}\acute{P} & \acute{S}\acute{P} \\ \grave{P}\acute{S} & \grave{S}\acute{S} & \acute{P}\acute{S} & \acute{S}\acute{S} \\ \grave{P}\grave{P} & \grave{S}\grave{P} & \acute{P}\grave{P} & \acute{S}\grave{P} \\ \grave{P}\grave{S} & \grave{S}\grave{S} & \acute{P}\grave{S} & \acute{S}\grave{S} \end{pmatrix}$$

Incident from above

Incident from below



Scattering matrices

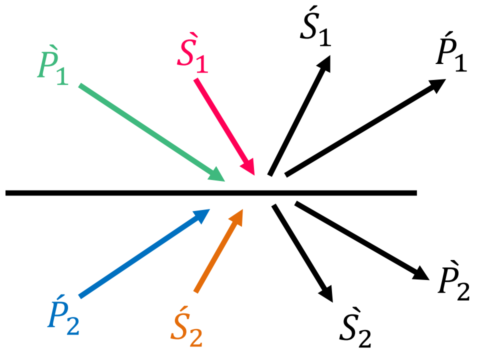
For both the SH and the P-SV cases, the transmission and reflection coefficients can be ‘summarized’ into the so-called scattering matrices:

P-SV case

$$\begin{pmatrix} \grave{P}\acute{P} & \grave{S}\acute{P} & \acute{P}\acute{P} & \acute{S}\acute{P} \\ \grave{P}\acute{S} & \grave{S}\acute{S} & \acute{P}\acute{S} & \acute{S}\acute{S} \\ \grave{P}\grave{P} & \grave{S}\grave{P} & \acute{P}\grave{P} & \acute{S}\grave{P} \\ \grave{P}\grave{S} & \grave{S}\grave{S} & \acute{P}\grave{S} & \acute{S}\grave{S} \end{pmatrix}$$

Incident from above

Incident from below



Scattering matrices

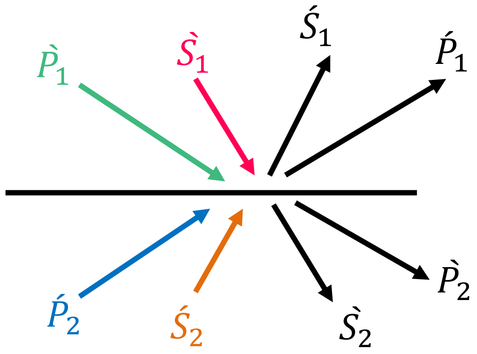
For both the SH and the P-SV cases, the transmission and reflection coefficients can be ‘summarized’ into the so-called scattering matrices:

P-SV case

$$\begin{pmatrix} \grave{P}\acute{P} & \grave{S}\acute{P} & \acute{P}\acute{P} & \acute{S}\acute{P} \\ \grave{P}\acute{S} & \grave{S}\acute{S} & \acute{P}\acute{S} & \acute{S}\acute{S} \\ \grave{P}\grave{P} & \grave{S}\grave{P} & \acute{P}\grave{P} & \acute{S}\grave{P} \\ \grave{P}\grave{S} & \grave{S}\grave{S} & \acute{P}\grave{S} & \acute{S}\grave{S} \end{pmatrix}$$

Incident from above

Incident from below

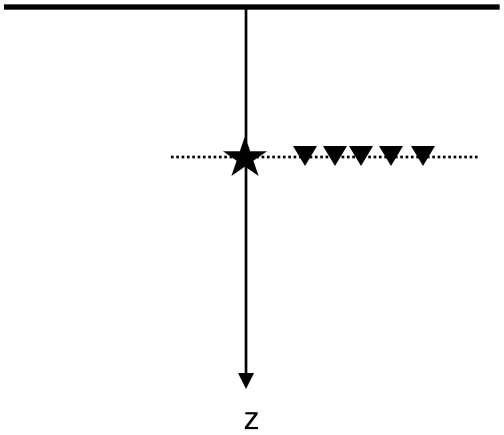


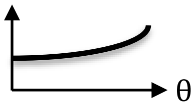
$$\mathbf{M}\begin{pmatrix} \grave{P}_1 \\ \grave{S}_1 \\ \acute{P}_2 \\ \acute{S}_2 \end{pmatrix} = \mathbf{N}\begin{pmatrix} \acute{P}_1 \\ \acute{S}_1 \\ \grave{P}_2 \\ \grave{S}_2 \end{pmatrix} \rightarrow \begin{pmatrix} \acute{P}_1 \\ \acute{S}_1 \\ \grave{P}_2 \\ \grave{S}_2 \end{pmatrix} = \mathbf{N}^{-1}\mathbf{M}\begin{pmatrix} \grave{P}_1 \\ \grave{S}_1 \\ \acute{P}_2 \\ \acute{S}_2 \end{pmatrix}$$

AVO modelling

Through reflection and transmission coefficients we can find a simple link between elastic properties and seismic responses – much easier than via the (elastic) wave equation

→ We can use these equations (also referred to as Zoeppritz equations) to model seismic data in 1D media





Amplitude variation versus offset (AVO) curve

- Modelling with α, β, ρ (what if scenarios, feasibility studies)

- Invert α, β, ρ from AVO responses

Zero-offset modelling

For $\theta_1 = 0$ (and therefore $\theta_2 = 0$), we get simplified relationships:

$$\grave{S}\acute{S} = \frac{\rho_1\beta_1 - \rho_2\beta_2}{\rho_1\beta_1 - \rho_2\beta_2}$$

$$\grave{P}\acute{S} = 0$$

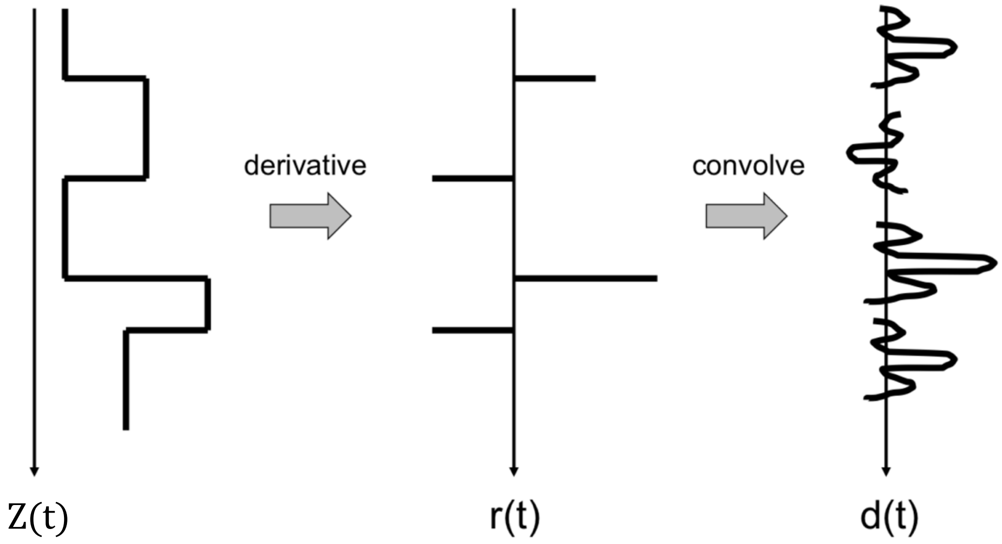
No conversion at zero incidence

$$\grave{P}\acute{P} = \frac{\rho_2\alpha_2 - \rho_1\alpha_1}{\rho_1\alpha_1 - \rho_2\alpha_2} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$Z = \rho\alpha$ Acoustic impedance

Note: the sign difference between $\grave{S}\acute{S}$ and $\grave{P}\acute{P}$ is given by the fact that the polarity of SH waves is indipendent on the ray direction, whilst that of P waves depends on the ray direction (which switches sign at the reflection point)

Zero-offset modelling



$$d(t) = w(t) * \frac{Z_2 - Z_1}{Z_2 + Z_1} \approx w(t) * \frac{d}{dt}\ln(Z(t))$$

Also known as **convolutional modelling**