Linear Algebra and Matrix Methods
Eigenvalues and Eigenvectors

Eigenvectors, Eigenvalues: "proper" directions, scalings

So far, we've seen ways to solve a system of linear equations,

$$A\vec{x} = b.$$

If $A\vec{x} = b$ has no solution, then the least squares (best fit) solution to

$$A^t A\hat{x} = A^t b$$

will suffice for "as close as we can get".

If $A\vec{x} = b$ has a solution, then $b \in C(A)$.

We are now interested in the special case where b is a scaling of $\vec{x}$.

Eigenvectors, Eigenvalues: "proper" directions, scalings

Goal: Solve the $n \times n$ (square) system

$$A\vec{x} = \lambda\vec{x}$$

for $\vec{x}$ and λ.

These are considered the "proper"* directions of the matrix A, where transformation by the matrix A is equivalent to merely scaling that direction.

*The "eigen" in these terms is German for "proper" or "characteristic".

If there are solutions to this **eigenproblem**[†]

$$A\vec{x} = \lambda\vec{x},$$

we will call the values in each pair $(\vec{x}, \lambda)$ by the names

eigenvector for each $\vec{x} \in \mathbb{C}^n$

and

eigenvalue for each $\lambda \in \mathbb{C}$.

[†] Some of the examples in this chapter require you to understand arithmetic with complex numbers, i.e. from $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}$.

Eigenvectors and eigenvalues via nullspaces

How can we solve this system?

Typically, if you want to solve an equation for a variable, you'll get all instances of that variable on the same side of the equals sign.

We'll do that here: seeing that scaling a vector by λ is the same as multiplying the vector by the scaled identity matrix λI,

$$A\vec{x} = \lambda\vec{x} \implies A\vec{x} - \lambda\vec{x} = 0 \implies (A - \lambda I)\vec{x} = 0.$$

Thus, if $\vec{x}$ is an eigenvector of A, then $\vec{x} \in N(A - \lambda I)$.

Eigenvectors and eigenvalues via determinants

$\vec{x} = 0$ is always a solution. However, the eigenvector 0 has *any* $\lambda \in \mathbb{C}$ as an eigenvalue; this tells us nothing about the matrix A.

If a nontrivial $\vec{x}$ solves the eigenproblem, we see that the columns of $A - \lambda I$ are linearly dependent; that is,

$$det(A - \lambda I) = 0.$$

Eigenvectors and eigenvalues via determinants

$$det(A - \lambda I) = 0.$$

This is the **characteristic equation** we need to solve, for $\lambda \in \mathbb{C}$.

This also explains why we say $\lambda \in \mathbb{C}$ and $\vec{x} \in \mathbb{C}^n$:

$$det(A - \lambda I) = 0$$

is an *n*th degree polynomial equation in λ, with n roots (with multiplicity) in $\mathbb{C}$.

Roots of the eigenequation: spectrum of the matrix

Unfortunately for the desire for a solution to the characteristic equation, we know[‡] that there is no general method of find roots of polynomials of degree $n \geq 5$.

That said, we can use various techniques to solve this problem.

[‡]thanks to Neils Abel (1802-1829)

Roots of the eigenequation: spectrum of the matrix

Define the **trace** of a square matrix A as the sum of its diagonal:

$$tr(A) = \sum_{i=1}^{n} a_{ii}.$$

Then, if $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{C}$ are the n roots of the characteristic equation of A (i.e. the **spectrum** of A), we have

$$det(A) = \prod_{i=1}^{n} \lambda_i \quad \text{and} \quad tr(A) = \sum_{i=1}^{n} \lambda_i.$$

Roots of the eigenequation: exponentiation

- Repeated multiplication by A on an eigenvector repeats scaling by λ: for any $k \in \mathbb{N}$,

$$A(A\vec{x}) = A(\lambda\vec{x}) \implies A^2\vec{x} = \lambda^2\vec{x} \implies A^k\vec{x} = \lambda^k\vec{x}.$$

- If A is invertible, then all of the eigenvalues of A are nonzero.

  If $\vec{x}$ is an eigenvector of A with eigenvalue λ, then $\vec{x}$ is also an eigenvector of A^{-1}, with eigenvalue λ^{-1}:

$$\begin{aligned} A\vec{x} = \lambda\vec{x} &\implies A^{-1}A\vec{x} = \lambda A^{-1}\vec{x} = \vec{x} \\ &\implies A^{-1}\vec{x} = \lambda^{-1}\vec{x}. \end{aligned}$$

- If A is singular, then $det(A) = 0$, and $\lambda = 0$ is one of its eigenvalues.

Roots of the eigenequation: triangular, projection

- If A is triangular, then A's eigenvalues are the diagonal entries.
- If A is symmetric ($A^t = A$), then A's eigenvalues are in $\mathbb{R}$.
- If $A = P$ is a projection matrix, then the idempotency of P determines λ: since $P^2 = P$, we get

$$\lambda^2 \vec{x} = P^2 \vec{x} = P\vec{x} = \lambda \vec{x} \implies \lambda \in \{0, 1\}.$$

  The eigenvector(s) $\vec{x}$ paired with
  - $\lambda = 1$ have $P\vec{x}_1 = \vec{x}_1$, and so project to themselves under P ($\vec{x}_1 \in C(P) = C(P^t)$ since P is symmetric);
  - $\lambda = 0$ have $P\vec{x}_0 = 0$, and so project to 0 under P ($\vec{x} \in N(P)$).

Roots of the eigenequation: orthogonal

- If $A = Q$ is orthogonal ($Q^{-1} = Q^t$), then

$$\begin{aligned} Q\vec{x} = \lambda\vec{x} &\implies (Q\vec{x})^t(Q\vec{x}) = (\lambda\vec{x})^t(\lambda\vec{x}) \\ &\implies \vec{x}^t Q^t Q\vec{x} = \vec{x}^t\vec{x} = ||\vec{x}||^2 = \lambda^2||\vec{x}||^2 \\ &\implies \lambda \in \{-1, 1\}. \end{aligned}$$

  In particular, if $A = R$ is a reflection matrix, R is orthogonal and symmetric, implying R is an involution ($R^2 = I$). Thus,

$$R\vec{x} = \lambda\vec{x} \implies R^2\vec{x} = \lambda^2\vec{x} = \vec{x} \implies \lambda \in \{-1, 1\}.$$

- If A is **skew-symmetric** ($A^t = -A$), then A's eigenvalues are "pure imaginary": $\lambda = bi \in \mathbb{C}$ for $b \in \mathbb{R}$.

Example: 2x2 matrix

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

$$det(A) = 4 - 4 = 0$$

and A is symmetric, so both eigenvalues are real.

Thus, A is singular, and so one of A's eigenvalues is 0.

Find the eigenvalues:

$$det(A - \lambda I) = 0 \implies det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} = 0$$

$$\implies (1-\lambda)(4-\lambda) - 4 = 0$$

$$\implies \lambda^2 - 5\lambda = 0 \implies \lambda = 0, 5.$$

Example: 2x2 matrix

To find the eigenvectors, we need to solve

$$(A - \lambda I)\vec{x} = 0$$

for each λ.

The eigenvectors associated with $\lambda = 0$ are:

$$A\vec{x} = 0 \implies \text{(compute rref(A))} \implies x_1 = -2x_2 \implies \vec{x}_n = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

The eigenvectors associated with $\lambda = 5$ are:

$$(A - 5I)\vec{x} = 0 \implies \begin{pmatrix} 1-5 & 2 \\ 2 & 4-5 \end{pmatrix} \vec{x} = 0$$

$$\implies \text{(compute rref(A-5I))} \implies x_1 = \frac{1}{2}x_2 \implies \vec{x}_n = x_2 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}.$$

Example: 3x3 matrix

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 5 & 4 \\ -6 & 3 & 7 \\ 0 & 0 & 2 \end{pmatrix}$.

Note: A is invertible:

$$det(A) = 2[1(3) - 5(-6)] = 66 \neq 0.$$

Thus, A's eigenvalues are all nonzero.

First, find the eigenvalues:

$$\begin{aligned} det(A - \lambda I) = 0 &\implies det \begin{pmatrix} 1-\lambda & 5 & 4 \\ -6 & 3-\lambda & 7 \\ 0 & 0 & 2-\lambda \end{pmatrix} = 0 \\ &\implies (2-\lambda)[(1-\lambda)(3-\lambda) + 30] = 0 \\ &\implies \lambda = 2 \text{ is an eigenvalue.} \end{aligned}$$

Example: 3x3 matrix, conjugate pair

There are two more: factoring out the $2 - \lambda$ term, we have

$$\begin{aligned}(1-\lambda)(3-\lambda)+30=0 &\implies 3-4\lambda+\lambda^2+30=0\\ &\implies \lambda^2-4\lambda+33=0\\ &\implies \lambda=\frac{4\pm\sqrt{16-4(1)(33)}}{2}=2\pm i\sqrt{29}.\end{aligned}$$

These are the other two eigenvalues.

Note that they are a conjugate pair of complex numbers.

Example: 3x3 matrix, conjugate pair

Now for the eigenvectors: solving $(A - 2I)\vec{x} = 0$ for $\lambda = 2$ yields

$$\lambda = 2: \qquad \vec{x}_n = x_3 \begin{pmatrix} \frac{31}{29} \\ -\frac{17}{29} \\ 1 \end{pmatrix}, \text{ or } x_3 \begin{pmatrix} 31 \\ -17 \\ 29 \end{pmatrix}.$$

The conjugate pair of complex eigenvalues have complex conjugate pair eigenvectors:

$$\lambda = 2 + i\sqrt{29}: \qquad \vec{x}_n = x_2 \begin{pmatrix} \frac{1}{6}(1 - i\sqrt{29}) \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = 2 - i\sqrt{29}: \qquad \vec{x}_n = x_2 \begin{pmatrix} \frac{1}{6}(1 + i\sqrt{29}) \\ 1 \\ 0 \end{pmatrix}.$$

Example: eigenvalue multiplicity

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 5 & 2 \\ 0 & 5 \end{pmatrix}$.

This one is easy: A is triangular, so clearly the eigenvalues are $\lambda = 5$ with multiplicity 2.

What are the eigenvectors?

$$(A - 5I)\vec{x} = 0 \implies 2x_2 = 0 \implies \vec{x}_n = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Diagonalization of an invertible matrix

We know an $n \times n$ matrix A has, with multiplicity, n pairs of eigenvectors and eigenvalues. Label them

$$(\lambda_1, \vec{x}_1), (\lambda_2, \vec{x}_2), ..., (\lambda_n, \vec{x}_n).$$

If the n eigenvectors of A are *linearly independent*, then we can construct an invertible **eigenvector matrix**

$$X = (\vec{x}_1 \ \vec{x}_2 \ \cdots \ \vec{x}_n)$$

of the n eigenvectors (which, naturally, are a basis of $\mathbb{R}^n$), and a corresponding diagonal **eigenvalue matrix**

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Diagonalization of an invertible matrix

Since $A\vec{x}_i = \lambda_i \vec{x}_i$ for all $i = 1, 2, ..., n$, and X is invertible, we have

$$AX = X\Lambda \implies X^{-1}AX = \Lambda,$$

or

$$A = X\Lambda X^{-1},$$

called the **diagonalization** of A.

Note that the exponentiation property of eigenvalues is easy to see:

$$A^n = (X\Lambda X^{-1})^n = (X\Lambda X^{-1})(X\Lambda X^{-1}) \cdots (X\Lambda X^{-1}) = X\Lambda^n X^{-1},$$

comfirming the notion that the eigenvalues of A^n are λ^n, with the same eigenvectors as A.

Finding eigenvalues is not a linear operation

Certain "shortcuts" to computing eigenvalues do not work:

If A and B are $n \times n$ matrices, with the sets of eigenvalues $\{\lambda_i\}_{i=1}^n$ for A and $\{\beta_i\}_{i=1}^n$ for B, then:

- $A + B$ does *not* have the eigenvalues $\lambda_i + \beta_i$ *unless* both A and B are diagonal.
- AB does *not* have the eigenvalues $\lambda_i \beta_i$.

Finding eigenvalues is not a linear operation

However,

Proposition

Assume A and B are diagonalizable, with diagonalizations

$$A = X\Lambda X^{-1},\ B = TKT^{-1}.$$

Then

$$X = T \iff AB = BA.$$

Invertibility vs Diagonalizability?

If any eigenvalue has multiplicity, then A is only diagonalizable if the **geometric multiplicity** of each eigenvalue's nullspace,

$$m_G(\lambda) = dim(N(A - \lambda I)),$$

equals the **arithmetic multiplicity** of the eigenvalue,

$$m_A(\lambda) = \text{multiplicity of } \lambda \text{ in } det(A - \lambda I) = 0.$$

Invertibility vs Diagonalizability?

Note that $m_A(\lambda) \geq m_G(\lambda)$ always holds.

If all n eigenvalues differ, then for each,

$$m_G(\lambda) = m_A(\lambda) = 1,$$

and A is diagonalizable (even if one of the eigenvalues is 0).

If the n eigenvectors of the matrix A are independent, then all eigenvalues differ, and A can be diagonalized.

Similar Matrices

If A is diagonalizable, in form

$$A = X\Lambda X^{-1},$$

there are multiple forms, depending on the scalings and orderings of the eigenvectors in X.

We will extend the conjugation paradigm to nondiagonalizable A.

Let M be an invertible matrix. Then we call

$$B = M^{-1}AM$$

similar to A.

Similar Matrices

If A is diagonalizable, then Λ is similar to A. In general,

Proposition

A and $M^{-1}AM$ have the same eigenvalues, and if $\vec{x}$ is an eigenvector of A, then $M^{-1}\vec{x}$ is an eigenvector of $M^{-1}AM$.

Proof

$$\begin{aligned} A\vec{x} = \lambda\vec{x} \implies (M^{-1}AM)(M^{-1}\vec{x}) &= M^{-1}A\vec{x} \\ &= M^{-1}\lambda\vec{x} = \lambda(M^{-1}\vec{x}). \blacksquare \end{aligned}$$

Some similarity-invariant properties

What properties are *invariant* (do not change) under the transformation $A \mapsto M^{-1}AM$?

- eigenvalues
- trace
- determinant
- rank
- # of independent eigenvectors
- Jordan form

Some similarity-variant properties

What properties *do* change under this transformation?

- eigenvectors (from $\vec{x}$ to $M^{-1}\vec{x}$)
- all four subspaces ($C(A)$ to $C(M^{-1}AM)$, etc.)
- singular values (in SVD)

Initial, long-term distributions

Once again examining powers of A, let $c \in \mathbb{R}^n$.
Then the eigenvectors of A make a basis of $\mathbb{R}^n$,
and so c can be represented via X in the form

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \ddots \\ c_n \end{pmatrix}, \; A = X\Lambda X^{-1} \implies Xc = \sum_{i=1}^{n} c_i \vec{x}_i.$$

Set $u_0 = Xc$ and define $u_k = A^k u_0$. We can quickly calculate u_k:

$$\begin{aligned} u_k = A^k u_0 &= A^k(c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n) \\ &= c_1\lambda_1^k\vec{x}_1 + c_2\lambda_2^k\vec{x}_2 + \cdots + c_n\lambda_n^k\vec{x}_n. \end{aligned}$$

Initial, long-term distributions (Markov matrices)

Consider a particle Y_t at each time t moving along the states $\{1, 2, ..., n\}$ with one-step probabilities

$$p_{ij} = P(Y_{t+1} = i \mid Y_t = j).$$

If A is a **Markov matrix**[§], whose entries are $a_{ij} = p_{ij}$, then A^k is the matrix of k-step probabilities

$$P(Y_{t+k} = i \mid Y_t = j).$$

[§] I am leaving out many important probability-based details here.

Initial, long-term distributions (Markov matrices)

If we define u_0 as the **initial distribution**[¶] of Y_0, and define u_k iteratively, as

$$u_{k+1} = Au_k,$$

then the **long-term distribution** of Y is found by sending $k \to \infty$.

[¶] a probability mass function of the initial position Y_0 of the n possible states in the system

Initial, long-term distributions (Markov matrices)

A property of Markov matrices is that all of their eigenvalues have the property $|\lambda_i| \leq 1$, and, in particular, one of them *is* 1: we label this one $\lambda_1 = 1$.

Thus, the **long-term distribution** of Y is the eigenvector $\vec{x}_1$, as we increase k in the k-step distribution u_k: recall, if $u_0 = Xc$, then

$$u_k = c_1\lambda_1^k\vec{x}_1 + c_2\lambda_2^k\vec{x}_2 + \cdots + c_n\lambda_n^k\vec{x}_n.$$

Initial, long-term distributions (Markov matrices)

As $k \to \infty$, $\lambda_i^k \to 0$ for $i > 1$ since $|\lambda_i| < 1$. But $\lambda_1 = 1$, so we get the limit

$$u_\infty = \lim_{k\to\infty} u_k = \lim_{k\to\infty} c_1\lambda_1^k\vec{x}_1 + c_2\lambda_2^k\vec{x}_2 + \cdots + c_n\lambda_n^k\vec{x}_n = c_1\vec{x}_1.$$

The scaling of u_∞ that yields a probability vector is called π, the **long-term**, or **steady state**, distribution of Y.

Markov matrix example

Consider a finite-state machine consisting of three states: 1, 2, 3.

The Markov matrix associated with this machine is

$$A = \begin{pmatrix} 0.2 & 0.1 & 0.5 \\ 0.4 & 0.5 & 0.5 \\ 0.4 & 0.4 & 0 \end{pmatrix}.$$

For example, the probability that, from state $X_k = 3$ at time k, the machine moves at time $k+1$ to state $X_{k+1} = 2$ is

$$p_{23} = P(X_{k+1} = 2 \,|\, X_k = 3) = 0.5.$$

and the probability the machine moves from state $X_k = 2$ to $X_{k+1} = 3$ is

$$p_{32} = P(X_{k+1} = 3 \,|\, X_k = 2) = 0.4.$$

Markov matrix example

If this machine keeps running, what is the long-term distribution?

In other words, far out in time, what are the probabilities the machine is in each state?

We can calculate the long-term state by examining the eigenproblem for A.

Markov matrix example

The Markov matrix associated with this machine is

$$A\vec{x} = \lambda\vec{x}$$

$$\begin{pmatrix} 0.2 & 0.1 & 0.5 \\ 0.4 & 0.5 & 0.5 \\ 0.4 & 0.4 & 0 \end{pmatrix} \vec{x} = \lambda\vec{x}$$

$$\det \begin{pmatrix} 0.2-\lambda & 0.1 & 0.5 \\ 0.4 & 0.5-\lambda & 0.5 \\ 0.4 & 0.4 & -\lambda \end{pmatrix} = 0$$

$$-\lambda^3 + 0.7\lambda^2 + 0.34\lambda - 0.04 = 0.$$

Markov matrix example

At this point, it would be tedious to compute the roots of the characteristic equation

$$-\lambda^3 + 0.7\lambda^2 + 0.34\lambda - 0.04 = 0,$$

or, multiplying both sides by -100,

$$100\lambda^3 - 70\lambda^2 - 34\lambda + 4 = 0,$$

but since A is a Markov matrix, we can easily see $\lambda_1 = 1$.

Markov matrix example

This simplifies our work significantly; we can use the trace and determinant formulas

$$\det(A) = \lambda_1\lambda_2\lambda_3, \quad \mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$$

to get the other two eigenvalues on top of $\lambda_1 = 1$:

$$\begin{aligned}
\det(A) &= 0.2(0 - 0.2) - 0.1(0 - 0.2) + 0.5(0.16 - 0.2) \\
&= -0.04 + 0.02 - 0.02 = -0.04 \\
\mathrm{tr}(A) &= 0.2 + 0.5 + 0 = 0.7
\end{aligned}$$

$$\implies \lambda_2\lambda_3 = -0.04, \; \lambda_2 + \lambda_3 = -0.3 \implies \lambda_2 = 0.1, \; \lambda_3 = -0.4.$$

Markov matrix example

Thus, our three eigenvalues are

$$\lambda_1 = 1,\ \lambda_2 = 0.1,\ \lambda_3 = -0.4,$$

and we now need the eigenvector $\vec{x_1}$ for $\lambda_1 = 1$.

$$\begin{aligned}
(A - I)\vec{x_1} &= 0 \\
\implies &\begin{pmatrix} -0.8 & 0.1 & 0.5 \\ 0.4 & -0.5 & 0.5 \\ 0.4 & 0.4 & -1 \end{pmatrix} \vec{x_1} = 0 \\
\implies &\begin{pmatrix} 1 & 0 & -\frac{5}{6} \\ 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 \end{pmatrix} \vec{x_1} = 0 \implies \vec{x} = \begin{pmatrix} 5/6 \\ 5/3 \\ 1 \end{pmatrix} x_3.
\end{aligned}$$

Markov matrix example

As our steady state long-term distribution needs to be a **probability vector** (a vector of nonnegative values that sum to 1), we choose the appropriate value for x_3 to make it so: this is

$$\frac{5}{6} + \frac{5}{3} + 1 = \frac{21}{6} \implies \vec{x_1} = \begin{pmatrix} 5/21 \\ 10/21 \\ 6/21 \end{pmatrix}.$$

Note that the other two eigenvectors have terms that sum to 0:

$$\vec{x_2} = \begin{pmatrix} 1 \\ 1/4 \\ -5/4 \end{pmatrix} x_3, \; \vec{x_3} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} x_3.$$

Markov matrix example

Let's say we start the machine at time 0 with initial distribution

$$u_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

in state 1 with probability 1.

At time 0,

$$u_0 = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

for some constants c_1, c_2, c_3.

Markov matrix example

We can compute $c_1 = 1$, $c_2 = \frac{24}{105}$, $c_3 = \frac{56}{105}$ to get

$$u_0 = \begin{pmatrix} 5/21 \\ 10/21 \\ 6/21 \end{pmatrix} + \frac{24}{105} \begin{pmatrix} 1 \\ 1/4 \\ -5/4 \end{pmatrix} + \frac{56}{105} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

At time k, the probabilities of the machine being in each state are

$$u_k = \begin{pmatrix} 5/21 \\ 10/21 \\ 6/21 \end{pmatrix} + \frac{24}{105}(0.4)^k \begin{pmatrix} 1 \\ 1/4 \\ -5/4 \end{pmatrix} + \frac{56}{105}(-0.1)^k \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

which, as $k \to \infty$, converges to $\vec{x_1} = \begin{pmatrix} 5/21 \\ 10/21 \\ 6/21 \end{pmatrix}$.

The exponential of a square matrix

We can compute power series-like objects using exponentiation.

Recall, for any $r, t \in \mathbb{R}$, the natural expoential e^{rt} can be defined as a power series:

$$e^{rt} = \sum_{n=0}^{\infty} \frac{1}{n!}(rt)^n = 1 + rt + \frac{1}{2}(rt)^2 + \frac{1}{6}(rt)^3 + \cdots .$$

Likewise, if A is a square matrix and $t \in \mathbb{R}$, we will ignore the scalar convention of writing scalars on the left, and define the matrix exponential by

$$e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots .$$

The exponential of a diagonalizable matrix

If A is diagonalizable, then $A = X\Lambda X^{-1}$ for some eigenvalue matrix Λ and eigenvector matrix X.

Then the exponential is easy to compute:

$$\begin{aligned}
e^{At} &= I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots \\
&= I + X\Lambda t X^{-1} + \frac{1}{2}(X\Lambda t X^{-1})^2 + \frac{1}{6}(X\Lambda t X^{-1})^3 + \cdots \\
&= I + X(\Lambda t)X^{-1} + \frac{1}{2}X(\Lambda t)^2 X^{-1} + \frac{1}{6}X(\Lambda t)^3 X^{-1} + \cdots \\
&= Xe^{\Lambda t}X^{-1}.
\end{aligned}$$

$\therefore$ the eigenvalues of e^{At} are $e^{\Lambda t}$, with the same eigenvectors as A.

The complex exponential as a diagonalizable matrix

Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Its powers have period 4, and e^{At} is orthogonal:

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Thus,

$$e^{At} = \begin{pmatrix} 1 - \frac{1}{2}t^2 + \frac{1}{4}t^4 - \cdots & t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \cdots \\ -\left(t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \cdots\right) & 1 - \frac{1}{2}t^2 + \frac{1}{4}t^4 - \cdots \end{pmatrix}.$$

The complex exponential as a diagonalizable matrix

We can rewrite this form of e^{At} in power series terms:

$$e^{At} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} & \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \\ -\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} & \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix},$$

the two-dimensional rotation matrix.

The complex exponential as a diagonalizable matrix

The eigenvalue/eigenvector pairs of A are

$$(\lambda_1, \vec{x}_1) = \left(i, \begin{pmatrix} 1 \\ i \end{pmatrix}\right), \ (\lambda_2, \vec{x}_2) = \left(-i, \begin{pmatrix} 1 \\ -i \end{pmatrix}\right).$$

The eigenvalue/eigenvector pairs of e^{At} are

$$(e^{\lambda_1 t}, \vec{x}_1) = \left(e^{it}, \begin{pmatrix} 1 \\ i \end{pmatrix}\right), \ (e^{\lambda_2 t}, \vec{x}_2) = \left(e^{-it}, \begin{pmatrix} 1 \\ -i \end{pmatrix}\right).$$

Note the connection to **Euler's formula**:

$$e^{it} = \cos(t) + i\sin(t).$$

Symmetric Matrices

Recall, a **symmetric matrix** A is a square matrix such that

$$A = A^t.$$

If A is symmetric and has independent columns, then A is diagonalizable, and so

$$A = X\Lambda X^{-1} \implies A^t = (X\Lambda X^{-1})^t = (X^{-1})^t \Lambda^t X^t = A.$$

Fact: $X^{-1} = X^t$, i.e. X is orthogonal.

Spectral Theorem

Theorem

Let A be a real-valued, symmetric matrix.
Then its eigenvector matrix X is orthogonal,
i.e. $X^t X = I$, and can be chosen as unit length vectors.

We will relabel this X as Q and write $A = Q\Lambda Q^t$.

Spectral Theorem: Supporting Propositions

Proposition

A real-valued, symmetric $\implies$ A's eigenvalues are real-valued.

Proof Suppose $A\vec{x} = \lambda\vec{x}$. Denote by $\overline{\lambda}$ the complex conjugate of λ: that is, if $\lambda = a + bi$, then $\overline{\lambda} = a - bi$.

We will prove that $\lambda = \overline{\lambda}$, which implies $b = 0$, and so $\lambda = a \in \mathbb{R}$.

Proof (continued)

Thus, $A = \overline{A}$ since A is real-valued, and so

$$A\vec{x} = \lambda\vec{x} \implies \overline{A\vec{x}} = \overline{\lambda\vec{x}} \implies A\overline{\vec{x}} = \overline{\lambda}\overline{\vec{x}}.$$

Transposing, we have

$$\begin{aligned} \overline{\vec{x}}^t A^t = \overline{\vec{x}}^t \overline{\lambda} &\implies \overline{\vec{x}}^t A = \overline{\vec{x}}^t \overline{\lambda} \\ &\implies \overline{\vec{x}}^t A\vec{x} = \overline{\vec{x}}^t \overline{\lambda}\vec{x} \\ &\implies \overline{\vec{x}}^t \lambda\vec{x} = \overline{\vec{x}}^t \overline{\lambda}\vec{x} \implies \lambda(\overline{\vec{x}}^t \vec{x}) = \overline{\lambda}(\overline{\vec{x}}^t \vec{x}). \end{aligned}$$

But $\overline{\vec{x}}^t \vec{x}$ is a scalar. Thus, $\lambda = \overline{\lambda}$. ■

Spectral Theorem: Supporting Propositions

Proposition

Let A be a real-valued, symmetric matrix. Then, if $\vec{x}_i$ and $\vec{x}_j$ are eigenvectors corresponding to the eigenvalues λ_i and λ_j, $i \neq j$, and $\lambda_i \neq \lambda_j$, then $\vec{x}_i \perp \vec{x}_j$.

Proof We are given $A\vec{x}_i = \lambda_i\vec{x}_i$, $A\vec{x}_j = \lambda_j\vec{x}_j$, $\lambda_i \neq \lambda_j$, and $A = A^t$. Thus,

$$\begin{aligned}
\lambda_i(\vec{x}_i^t\vec{x}_j) &= (\lambda_i\vec{x}_i^t)\vec{x}_j = (\lambda_i\vec{x}_i)^t\vec{x}_j \\
&= (A\vec{x}_i)^t\vec{x}_j = \vec{x}_i^t A^t\vec{x}_j = \vec{x}_i^t(A\vec{x}_j) \\
&= \vec{x}_i^t\lambda_j\vec{x}_j = \lambda_j(\vec{x}_i^t\vec{x}_j).
\end{aligned}$$

Therefore, $\lambda_i = \lambda_j$ or $\vec{x}_i^t\vec{x}_j = 0$. As we are given $\lambda_i \neq \lambda_j$, it must then be that $\vec{x}_i^t\vec{x}_j = 0$, i.e. $\vec{x}_i \perp \vec{x}_j$. ■

Pivots vs Eigenvalues

We know that, if A is triangular, then the eigenvalues of A are the diagonal entries, i.e. the pivots, of A.

However, this is not true for a non-triangular matrix. For a matrix $A = LDU$, the pivots of A correspond to the diagonal entries of D, which are *not* the eigenvalues of A.

Pivots vs Eigenvalues

We have the following "compromise" result:

If $d_1, d_2, ..., d_n$ are the diagonal entries of D, i.e. the pivots of A, and $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A, then

$$det(A) = \prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n} d_i = det(D).$$

Pivots vs Eigenvalues: Symmetric

If A is symmetric, then the signs of pivots and eigenvalues agree:

$$\# \text{ positive pivots of } A = A^t = \# \text{ positive eigenvalues of } A = A^t.$$

If all eigenvalues of A are positive (and hence all pivots are positive), then we call A is **positive definite** matrix.

(We will have another definition of this term shortly.)

Schur's Theorem

Theorem

If A is a square, complex-valued matrix, then A has a decomposition called **Schur form**,

$$A = QTQ^{-1},$$

where

- Q *is a* **unitary** *matrix, i.e.* $Q^{-1} = \overline{Q}^t$, *its* **conjugate transpose**, *and*
- T *is upper triangular.*

If A is a square, real-valued matrix, then the Schur form of A has Q an orthogonal matrix. If A is also symmetric, then T is diagonal.

Positive Definite Matrices

An $n \times n$ matrix A is called **symmetric positive definite (SPD)** if A is symmetric and satisfies all of the equivalent[‖] properties:

- $\vec{x}^t A \vec{x} > 0$ for any $\vec{x} \neq 0 \in \mathbb{R}^n$ (the "energy" definition)
- all n pivots $d_i > 0$
- all n eigenvalues $\lambda_i > 0$
- all n upper-left determinants (deleting successive bottom row/column pairs) > 0
- $A = R^t R$ for some R with independent columns.

If we relax the first (really, any) property to ≥ 0, any A satisfying is called **symmetric positive semi-definite (SPSD)**.

[‖] satisfying one satisfies them all!

Positive Definite Matrices: Cholesky decomposition

If all the inequalities are flipped, we call A **symmetric negative (semi)definite**.

If A has positive and negative eigenvalues, we call A **indefinite**.

The last property,

- $A = R^t R$ for some R with independent columns,

is the **Cholesky decomposition** of A; more to the point,

$$A = LDU \implies A = LDL^t = (L\sqrt{D})(L\sqrt{D})^t,$$

i.e. $R = (L\sqrt{D})^t$.

Positive Definite Matrices: two decompositions

If A is SPD, then there is a decomposition for pivots and another for eigenvalues:

$$A = LDL^t$$

gives the pivots in D (with triangular bookends), and

$$A = Q\Lambda Q^t$$

gives the eigenvalues in Λ (with orthogonal bookends).

Positive Definite Matrices: closed under addition, 2x2

For once, arithmetic intuition holds: if A and B are SPD, then so is $A + B$:

$$\vec{x}^t(A + B)\vec{x} = \vec{x}^t A\vec{x} + \vec{x}^t B\vec{x} > 0.$$

The converse, of course, is not necessarily true.

A 2x2 matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is SPD $\iff$ $a > 0$ and $ac - b^2 > 0$.

Positive Semidefinite Matrices: Quadratic Forms, Ellipses

Note the energy definition of an SPSD matrix:

- $\vec{x}^t A\vec{x} \geq 0$ for any $\vec{x} \neq 0 \in \mathbb{R}^n$.

This kind of function, $f(\vec{x}) = \vec{x}^t A\vec{x}$, is called a **quadratic form**.

If A is SPSD, then the equation

$$\vec{x}^t A\vec{x} = 1$$

defines an n-dimensional ellipsoid in $\mathbb{R}^n$, centered at the origin.

In particular, the equation in $\mathbb{R}^2$,

$$\vec{x}^t A\vec{x} = 1,$$

$$i.e.\ \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2 = 1,$$

defines an ellipse.

Positive Semidefinite Matrices: principal axis theorem

This ellipse

$$\vec{x}^t A\vec{x} = ax^2 + 2bxy + cy^2 = 1$$

is tilted and stretched from the unit circle

$$\vec{x}^t I\vec{x} = x^2 + y^2 = 1$$

in that its eigenvectors point in the directions of the minor and major axes, and the eigenvalues determine the lengths of the axes: this is stated in the **Principal Axis Theorem**.

Positive Semidefinite Matrices: principal axis theorem

Factoring the matrix $A = Q\Lambda Q^t$ yields a way to "standardize" the ellipse equation in the following way:

$$\vec{x}^t A\vec{x} = (\vec{x}^t Q)\Lambda(Q^t\vec{x}) = 1.$$

Writing $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = Q^t\vec{x}$, the change into "ellipse coordinates",

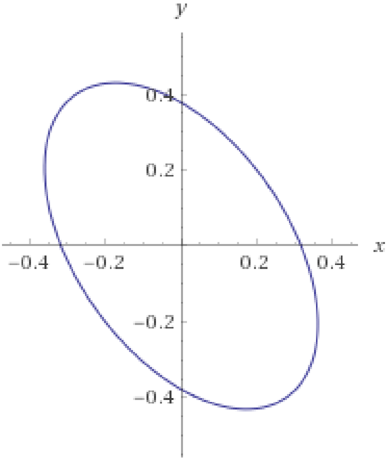
$$\vec{x}^t A\vec{x} = C^t\Lambda C = \lambda_1 c_1^2 + \lambda_2 c_2^2 = 1.$$

This implies that, if $\vec{x}_1$ and $\vec{x}_2$ the eigenvectors of A, and $\lambda_1 \leq \lambda_2$, then the semi-minor axis is in the directions of $\vec{x}_1$ with length $\frac{1}{\sqrt{\lambda_1}}$, and the semi-major axis is in the direction of $\vec{x}_2$, with length $\frac{1}{\sqrt{\lambda_2}}$.

Example: principal axis theorem

Consider the ellipse in the plane defined by

$$10x^2 + 8xy + 7y^2 = 1.$$



Example: principal axis theorem

$$10x^2 + 8xy + 7y^2 = 1$$

can be represented by the matrix equation $\vec{x}^t A \vec{x} = 1$:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 10 & 4 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1.$$

Factoring $A = Q \Lambda Q^t$ yields

$$Q = \begin{pmatrix} 0.822 & -0.570 \\ 0.570 & 0.822 \end{pmatrix}, \ \Lambda = \begin{pmatrix} 12.772 & 0 \\ 0 & 4.228 \end{pmatrix}.$$

The semi-minor axis has length 0.280 in the direction $\begin{pmatrix} 0.822 \\ 0.570 \end{pmatrix}$;

the semi-major axis is in direction $\begin{pmatrix} -0.570 \\ 0.822 \end{pmatrix}$ with length 0.486.

BONUS: Applications to Ordinary Differential Equations

Recall that

$$\frac{d}{dt}(e^{\lambda t}) = \lambda e^{\lambda t}.$$

$\frac{d}{dt}$, the **differential operator** with respect to the variable t, can be considered a *matrix* applied to a vector of functions u.

We will thus examine ODEs via linear algebra.

Ordinary Differential Equation $\frac{du}{dt} = \lambda u$: 1×1 case

The single-variable initial condition ODE

$$\frac{du}{dt} = \lambda u, \;\; u(0) = C \text{ with solution } u(t)$$

is solved by the function $u(t) = Ce^{\lambda t}$.

This is a 1×1 matrix version of a linear constant coefficient system of ODEs with initial conditions. We will generalize.

Ordinary Differential Equation $\frac{du}{dt} = Au$: $n \times n$ case

The $n \times n$ linear constant coefficient ODE system looks like

$$\frac{du}{dt} = Au, \;\; u(0) = \begin{pmatrix} u_1(0) \\ u_2(0) \\ \vdots \\ u_n(0) \end{pmatrix} \text{ with solution } u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$$

and is solvable via eigenproblem methods.

Ordinary Differential Equation $\frac{du}{dt} = Au$: $n \times n$ case

If we can solve the eigenproblem

$$A\vec{x} = \lambda\vec{x}$$

with n pairs of distinct eigenvalues and independent eigenvectors

$$(\lambda_1, \vec{x_1}), (\lambda_2, \vec{x_2}), ..., (\lambda_n, \vec{x_n}),$$

then we can rewrite the initial conditions of the ODE as

$$u(0) = \sum_{i=1}^{n} c_i \vec{x_i}$$

and solve the system with the solution

$$u(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \vec{x_i}.$$

Example: ODE $\frac{du}{dt} = Au$: $n \times n$ case

Solve the ODE system

$$\frac{du}{dt} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} u, \;\; u(0) = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} \text{ for } u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}.$$

The associated eigenproblem is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \vec{x} = \lambda \vec{x}.$$

Example: ODE $\frac{du}{dt} = Au$: $n \times n$ case

This triangular system has eigenvalues

$$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$$

The associated eigenvectors are

$$\vec{x_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \vec{x_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ \vec{x_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Example: ODE $\frac{du}{dt} = Au$: $n \times n$ case

Solving the initial condition for its coefficients c_1, c_2, c_3 yields

$$u(0) = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\implies \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\implies \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$\implies u(0) = \vec{x_1} + \vec{x_2} + 4\vec{x_3}.$$

Example: ODE $\frac{du}{dt} = Au$: $n \times n$ case

Thus, the solution to the ODE system is

$$u(t) = c_1 e^{\lambda_1 t} \vec{x_1} + c_2 e^{\lambda_2 t} \vec{x_2} + c_3 e^{\lambda_3 t} \vec{x_3}$$

$$= e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 4e^{3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t + e^{2t} + 4e^{3t} \\ e^{2t} + 4e^{3t} \\ 4e^{3t} \end{pmatrix} = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}.$$

Check: ODE $\frac{du}{dt} = Au$: $n \times n$ case

We check the solution:

$$\frac{d}{dt}u(t) = \frac{d}{dt}\begin{pmatrix} e^t + e^{2t} + 4e^{3t} \\ e^{2t} + 4e^{3t} \\ 4e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} e^t + 2e^{2t} + 12e^{3t} \\ 2e^{2t} + 12e^{3t} \\ 12e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} e^t + e^{2t} + 4e^{3t} \\ e^{2t} + 4e^{3t} \\ 4e^{3t} \end{pmatrix} = Au. \checkmark$$

Second Order ODEs

A 1×1 second order linear ODE has form

$$my'' + by' + ky = 0, \; y = y(t),$$

with constants $m, b, k \in \mathbb{R}$.

We typically solve this kind of ODE by first assuming the form $y(t) = Ce^{\lambda t}$ and checking the coefficient equation

$$m\lambda^2 + b\lambda + k = 0,$$

which comes from the derivatives

$$y' = \lambda y, \; y'' = \lambda^2 y.$$

Second Order ODEs

It should be clear that the quadratic equation yields the solutions

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m},\ \lambda_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m}.$$

Then, the solution to the second order ODE is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

where c_1 and c_2 are given by initial conditions.

Second Order ODEs

How can we use linear algebra to solve a second order linear ODE?

WLOG we will set $m = 1$ to make the calcuations easier.

Setting $u = \begin{pmatrix} y \\ y' \end{pmatrix}$, the ODE

$$y'' + by' + ky = 0$$

is transformed into the linear system

$$\frac{du}{dt} = Au, \; A = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix}.$$

Second Order ODEs

We solve the system as before.

Note that the characteristic equation of A is precisely the quadratic we used earlier:

$$\det(A - \lambda I) = \lambda^2 + b\lambda + k = 0.$$

Thus λ_1 and λ_2 are as we calculated, and the eigenvectors are

$$\vec{x_1} = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}, \ \vec{x_2} = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}.$$

Second Order ODEs

We get the system solution

$$\begin{aligned} u(t) &= c_1 e^{\lambda_1 t} \vec{x_1} + c_2 e^{\lambda_2 t} \vec{x_2} \\ &= c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \end{aligned}$$

$$\implies \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} \end{pmatrix}.$$

Note that, for this to work, we require $\lambda_1 \neq \lambda_2$.

If any eigenvalues match, we need to add solutions of form $Ct^k e^{\lambda t}$.

Example: Second Order ODE: General Solution

Solve for $y = y(t)$:

$$y'' - 6y' + 5y = 0, \; y(0) = 10, \; y'(0) = 4.$$

$$\begin{array}{rcc} \text{characteristic equation} & & \lambda^2 - 6\lambda + 5 = 0 \\ & \Longrightarrow & \lambda_1 = 1, \; \lambda_2 = 5 \\ & \Longrightarrow & \vec{x_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \vec{x_2} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ & \Longrightarrow & u(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} c_1 e^t + c_2 e^{5t} \\ c_1 e^t + 5c_2 e^{5t} \end{pmatrix}. \end{array}$$

Example: Second Order ODE: Initial Conditions

$$\text{initial conditions} \qquad u(0) = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$

$$\Longrightarrow \qquad \begin{pmatrix} c_1 + c_2 \\ c_1 + 5c_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$

$$\Longrightarrow \qquad c_1 = \frac{23}{2}, \; c_2 = -\frac{3}{2}.$$

$$\therefore \qquad y(t) = \frac{23}{2} e^{t} - \frac{3}{2} e^{5t}.$$

Check: $y'(t) = \dfrac{23}{2} e^{t} - \dfrac{15}{2} e^{5t}, \; y''(t) = \dfrac{23}{2} e^{t} - \dfrac{75}{2} e^{5t}$

$\Longrightarrow \; y'' - 6y' + 5y = 0. \; \checkmark$

Example: Second Order ODE: Matching Eigenvalues

Solve for $y = y(t)$:

$$y'' - 8y' + 16y = 0,\ \ y(0) = 10,\ \ y'(0) = 4.$$

$$\begin{aligned}
\text{char eqn} \qquad & \lambda^2 - 8\lambda + 16 = 0 \\
\Longrightarrow \qquad & \lambda_1 = \lambda_2 = 4 \\
\Longrightarrow \qquad & \vec{x_1} = \vec{x_2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\
\Longrightarrow \qquad & u(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} (c_1 + c_2 t)e^{4t} \\ (4c_1 + c_2 + 4c_2 t)e^{4t} \end{pmatrix}.
\end{aligned}$$

Example: Second Order ODE: Matching Eigenvalues

$$
\begin{aligned}
\text{init cond} \qquad & u(0) = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix} \\
\implies \qquad & \begin{pmatrix} c_1 \\ 4c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix} \\
\implies \qquad & c_1 = 10, \; c_2 = -36. \\
\therefore \qquad & y(t) = (10 - 36t)\, e^{4t}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Check:} \qquad & y'(t) = (4 - 144t)e^{4t}, \\
& y''(t) = (-128 - 576t)e^{4t} \\
\implies \qquad & y'' - 8y' + 16y = 0. \checkmark
\end{aligned}
$$

Long-Term Stability of ODE Solutions via Eigenvalues

As $t \to \infty$, what happens to $u(t)$? Does the ODE solution:

- stabilize ($u(t) \to c \in \mathbb{R}$)? ("vanish" if $c = 0$)
- explode ($|u(t)| \to \infty$)?
- cycle ($0 < |u(t)| < \infty$ but $u(t)$ continues to vary)?

Since $e^{\lambda t}$ is the structure of the terms of a typical ODE solution, we examine the eigenvalues as complex: $\lambda = r + is \in \mathbb{C}$.

- stabilize: *all* eigenvalues have $r < 0$ (exponential decay)
- explode: *any* eigenvalue has $r > 0$ (exponential growth)
- cycle: $r = 0$, $s \neq 0$