Liner Algebra

$A = QR$ Decomposition Example

Consider the 3x3 matrix A of independent column vectors in $\mathbb{R}^3$:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}.$$

What we would like to do is the following:

(a) Apply the **Gram-Schmidt orthogonalization** process to the columns $\{a_1, a_2, a_3\}$ of A (which are a basis of $\mathbb{R}^3$) to get an *orthonormal* basis of $\mathbb{R}^3$, $\{q_1, q_2, q_3\}$, which preserve the directions of the a_i as best as possible (in the order given).

(b) Give the QR factorization of A, built from the G-S orthogonalization.

(c) Show how $A = QR$ can be used to solve a system $A\vec{x} = b$.

1 Gram-Schmidt Orthogonalization

The G-S process is as follows: using the vector labels b_i as intermediate vectors between the original vectors a_i and the target vectors q_i:

- First, set $b_1 = a_1$ and $q_1 = \frac{1}{||b_1||} b_1$. Thus, q_1 is unit length and in the direction of a_1.
- Next, set
$$b_2 = a_2 - \frac{b_1 \cdot a_2}{b_1 \cdot b_1} b_1.$$
You may recognize the vector $\frac{b_1 \cdot a_2}{b_1 \cdot b_1} b_1 = proj_{b_1}(a_2)$ as the projection vector of a_2 onto the line spanned by b_1. Thus, the vector b_2 is the error vector between this projection and a_2. As such, $b_2 \perp b_1$.
Scale b_2 to unit length by setting $q_2 = \frac{1}{||b_2||} b_2$. Thus, q_1 and q_2 are unit length and $q_1 \perp q_2$.
- Finally, set b_3 as the remainder from a_3 after subtracting off the projections of a_3 onto the lines of b_1 and b_2:
$$b_3 = a_3 - \frac{b_1 \cdot a_3}{b_1 \cdot b_1} b_1 - \frac{b_2 \cdot a_3}{b_2 \cdot b_2} b_2 = a_3 - proj_{b_1}(a_3) - proj_{b_2}(a_3).$$
Finally, scale b_3 to unit length: $q_3 = \frac{1}{||b_3||} b_3$.

The resultant set of vectors $\{q_1, q_2, q_3\}$ are unit length and pairwise orthgonal; hence, this is an *orthonormal set* of vectors.

To do the computation for our example, we first write out the column vectors

$$a_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \ a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ a_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix},$$

and compute all the necessary dot products as we compute new vectors:

$$\begin{array}{llll} b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}: & \begin{aligned} b_1 \cdot b_1 &= 6 \implies ||b_1|| = \sqrt{b_1 \cdot b_1} = \sqrt{6} \\ b_1 \cdot a_2 &= 1 \\ b_1 \cdot a_3 &= 6 \end{aligned} & \implies q_1 = \dfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ b_2 = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}: & \begin{aligned} b_2 \cdot b_2 &= \tfrac{6}{9} \implies ||b_2|| = \sqrt{b_2 \cdot b_2} = \tfrac{\sqrt{6}}{3} \\ b_2 \cdot a_3 &= \tfrac{7}{3} \end{aligned} & \implies q_2 = \dfrac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \\ b_3 = \begin{pmatrix} -\frac{3}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix}: & b_3 \cdot b_3 = \tfrac{18}{4} \implies ||b_3|| = \sqrt{b_3 \cdot b_3} = \tfrac{3\sqrt{2}}{2} & \implies q_3 = \dfrac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \end{array}$$

Note that $||q_i|| = 1$ for $i = 1, 2, 3$, and $q_i \cdot q_j = 0$ if $i \neq j$.

2 QR Decomposition

The decomposition of $A = QR$ will have the following structure:

- Q is an **orthogonal matrix**, meaning $Q^t Q = I$ and the columns of Q are unit length; in addition, since Q is square, Q is invertible with $Q^{-1} = Q^t$;
- R is an **upper triangular matrix**.

From the G-S orthogonalization, we have

$$Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & -1 & -\sqrt{3} \\ \sqrt{2} & 2 & 0 \\ \sqrt{2} & -1 & \sqrt{3} \end{pmatrix}.$$

Since Q is a square orthogonal matrix, we can easily compute R via

$$A = QR \implies R = Q^{-1}A = Q^t A = \frac{1}{\sqrt{6}} \begin{pmatrix} 3\sqrt{2} & \sqrt{2} & 6\sqrt{2} \\ 0 & 2 & 0 \\ 0 & 0 & -\sqrt{3} \end{pmatrix}.$$

Since R is upper triangular with nonzero diagonal entries, R is invertible.

Having this decomposition makes solving a system using A relatively easy: to solve $A\vec{x} = b$,

$$A^{-1} = (QR)^{-1} = R^{-1}Q^{-1} = R^{-1}Q^t,$$

so

$$A\vec{x} = b \implies \vec{x} = A^{-1}b = R^{-1}Q^t b.$$