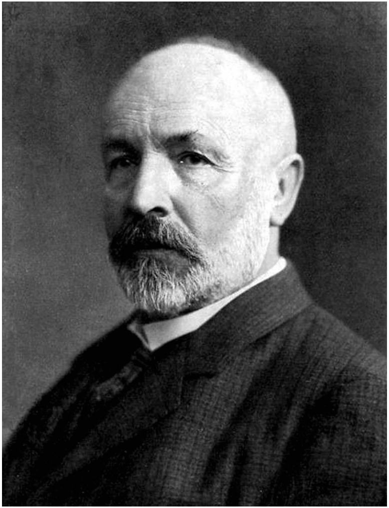
Introduction to Analysis:
Sets and Functions

Set / Element

We start with the basic definitions of set notation, as first written down by Georg Cantor (1845-1918).



Set / Element

Definition

A **set** is a collection of distinct objects, called the set's **elements**.

The elements:

- do not necessarily have an inherent order or relationship to each other (although they usually will);
- duplicates are not allowed;
- they are merely *different things in the same bag*.

Set / Element Notation

"The set A contains three elements:

'cat', 'tree', and the number 6."

This is denoted

$$A = \{\text{tree}, \text{cat}, 6\}$$

with curly brackets indicating the set.

This is an *explicit* definition of a particular set. It has 3 elements.

"6 is an element of A" is denoted $6 \in A$.

"'dog' is not an element of A" is denoted dog $\notin A$.

Set / Element Notation

"The set B consists of the even numbers *strictly* between 0 and 25" (i.e. *exclusive*) is an *implicit* definition of the set denoted

$$B = \{2, 4, 6, ..., 22, 24\}.$$

The *ellipsis* "..." means:

"we agree on, and you understand, the pattern given by context".

Set / Element Notation: "Set Builder" Notation

Another way to write

$$B = \{2, 4, 6, ..., 22, 24\}$$

is

$$B = \{x \in \mathbb{Z} \,|\, 0 < x < 25,\ x \text{ even}\},$$

where the vertical bar $|$ means "such that".

(Sometimes a colon : is used instead of the bar $|$.)

Popular Number Sets

$$\mathbb{N} = \{1, 2, 3, ...\}$$

is the set of **natural** or **counting numbers** (sometimes with 0).

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

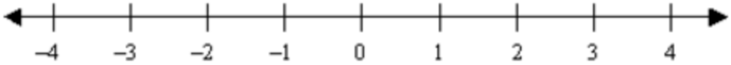
is the set of **integers** (for the German word Zahlen ("numbers")).

$$\mathbb{Q} = \left\{ \frac{m}{n} \,\middle|\, m, n \in \mathbb{Z},\ n \neq 0 \right\}$$

is the **rational numbers** (fractions, ratios; Q for "quotient").

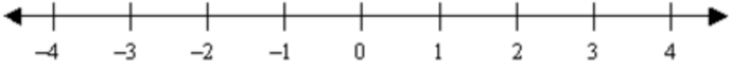
Real Number Sets

$\mathbb{R}$ is the set of **real numbers**, containing all rational and irrational numbers. This is the set of numbers along the continuous number line, containing all infinite-length decimal expansions.



Our focus in this course is sets, sequences, and functions on $\mathbb{R}$.

Intervals of Real Numbers



An **open interval** of real numbers is denoted

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

(In some texts, notably French, open interval notation is $]a, b[$.)

A **closed interval** of real numbers is denoted

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

Cardinality

The **cardinality** of a set is the number of (distinct) elements of the set. The cardinality of the set A is denoted $|A|$ or $\#A$.

Finite sets have an obvious cardinality: count the elements.

Examples: $|\{4, 6, 2, 9\}| = 4$, $|\{7, 8, ..., 100\}| = 94$.

Infinite sets are more complicated to deal with.

Empty Set

The **empty set** is the set with no elements (think: an empty bag).

It is denoted

$$\emptyset$$

and defined with curly brackets by

$$\emptyset = \{\}.$$

The cardinality of the empty set is

$$|\emptyset| = 0.$$

Venn diagrams

A **Venn diagram** (named after John Venn (1834-1923)) is a simple graphic displaying the overlap of different sets.

We use Venn diagrams to help understand problems in counting, probability, logic, and other fields.

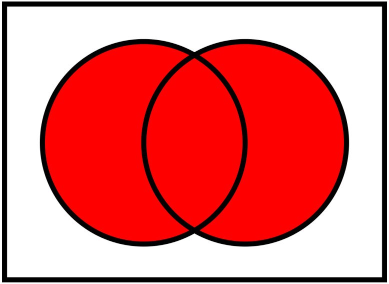
We'll sketch some diagrams for two sets A and B, sharing the same universal set U (the box surrounding them).

Union

Some basic operations we can use on sets are:

The **union** of the sets A and B is the set of all elements of A and B combined. It is denoted $A \cup B$, and defined by

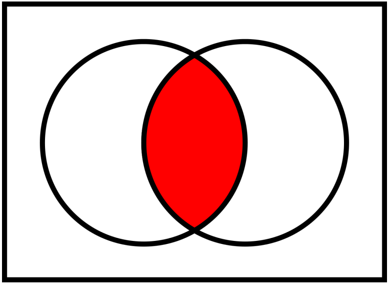
$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ (or both)}\}.$$



Intersection

The **intersection** of the sets A and B is the shared elements of A and B. It is denoted $A \cap B$ or AB, and defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

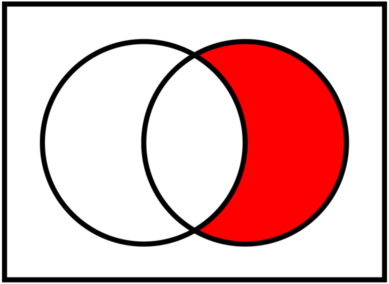


If A and B have no common elements, we say A and B are **disjoint** sets and denote this fact by $A \cap B = \emptyset$.

Set Difference

The **set difference** $B \setminus A$ (sometimes denoted $B - A$) is the elements of B with the shared elements of A removed. It is denoted

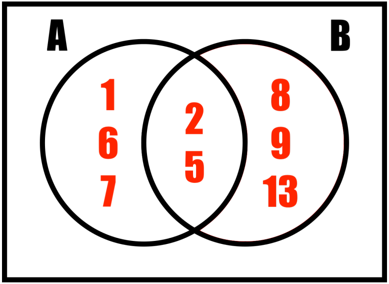
$$B \setminus A = \{x : x \in B \text{ and } x \notin A\}.$$



Set Difference

$$A = \{1, 2, 5, 6, 7\}, \quad B = \{2, 5, 8, 9, 13\}$$

$$\implies A \setminus B = \{1, 6, 7\} \text{ but } B \setminus A = \{8, 9, 13\}.$$

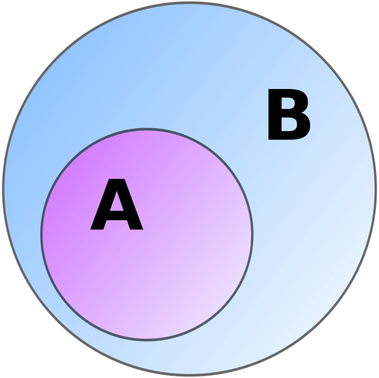


Thus, $A \setminus B \neq B \setminus A$ (an important general result).

Subset

The set A is called a **subset** of B, denoted $A \subseteq B$, if all of A's elements are in B.

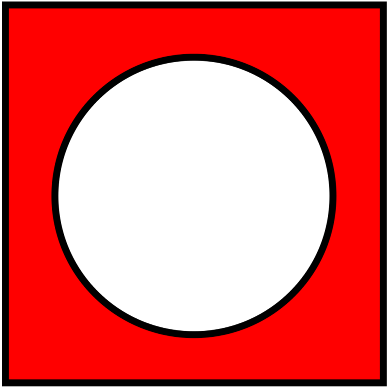
$$A \subseteq B \iff (x \in A \implies x \in B).$$



Two sets A and B are **equal** (written $A = B$) if $A \subseteq B$ and $B \subseteq A$.

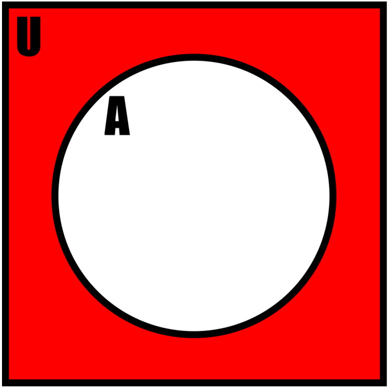
Universal Set, Complement

In certain collections of problems, we may define a **universal set** as a common "top-level set" under which all the sets described in the problem are subsets of.



Universal Set, Complement

The **complement** of a set A, relative to the universal set U, is denoted $A^C = U \setminus A$. (Some texts use A' or $\overline{A}$.)



Using this, we can define the set difference as $A \setminus B = A \cap B^C$.

Example: intervals in $\mathbb{R}$

Considering $\mathbb{R} = (-\infty, \infty)$ as the universal set[1], let

$$A = (1, \infty) = \{x \in \mathbb{R} : x > 1\}.$$

Then

$$A^C = \mathbb{R} \setminus A = (-\infty, 1] = \{x \in \mathbb{R} : x \leq 1\}.$$

[1] Note here that the "infinity" symbol ∞ in this context means

"unbounded in this direction"

and is not itself representing a number.

Unions, Intersections, Complements

Union and intersection are *commutative* operations:

$$A \cup B = B \cup A, \;\; A \cap B = B \cap A.$$

We've already seen that set difference is *not* commutative:

$$A \setminus B \neq B \setminus A.$$

Unions, Intersections, Complements

Union and intersection are also *associative*:

$$(A \cup B) \cup C = A \cup (B \cup C), \; (A \cap B) \cap C = A \cap (B \cap C).$$

Regardless of universal set, it should be clear that the complement of a complement is the original set:

$$(A^C)^C = A.$$

Distributivity

The **distributive property** describes how unions and intersections work together:

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

DeMorgan's Laws

DeMorgan's Laws describe the complements of unions and intersections. Complementation flips a union or intersection:

$$(A \cap B)^C = A^C \cup B^C$$
$$(A \cup B)^C = A^C \cap B^C$$

Note the similarity to DeMorgan's Laws in logic.

Visually, these rules make more obvious sense with Venn diagrams.

Set (Cartesian) Product, Sequences

The **Cartesian product** of two sets A and B, denoted $A \times B$, is the set whose elements are **ordered pairs** (a, b), where $a \in A$ and $b \in B$. (We will define (a, b) set-theoretically later.)

Extending, the Cartesian product of n sets $A_1, A_2, ..., A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$ or $\prod_{i=1}^{n} A_i$, is the set whose elements are **ordered n-tuples** $(a_1, a_2, ..., a_n)$, where $a_i \in A_i$ for $i = 1, 2, ..., n$.

If $A_1 = A_2 = \cdots = A_n = A$, we can write the product as A^n.

Note that *order matters* in a Cartesian product of sets: in general,

$$A \times B = B \times A \iff A = B.$$

Power Set

The **power set** of the set A, denoted 2^A or $\mathcal{P}(A)$, is the set of all subsets of A. (The notation will prove to be important.)

It is always true that $\emptyset, A \in 2^A$.

Example

$A = \{1, 4, 6\}$ has the power set

$$2^A = \{\emptyset, \{1\}, \{4\}, \{6\}, \{1, 4\}, \{1, 6\}, \{4, 6\}, A\}.$$

The set A has cardinality $|A| = 3$.

The power set of A, 2^A, has cardinality $|2^A| = 8$.

Example: Power Set

For a finite set A with cardinality $|A| = n$, what is $|2^A|$?

In other words, how many possible subsets of A are there?

Example: Power Set

For a subset $B \subseteq A$, consider the n elements of A in an order (or, simply call them $1, 2, ..., n$).

Build an ordered n-tuple of 1's and 0's via this ordering:

For each element $x \in A$, place a 1 if $x \in B$, and 0 if $x \notin B$.

How many different ordered n-tuples of 1's and 0's are there?

Example: Power Set

For example, let

$$B = \{a, d\} \subseteq A = \{a, b, c, d\}.$$

The 4-tuple corresponding to B is $(1, 0, 0, 1)$.

In general,

$$|\{0, 1\}^n| = |\{0, 1\}|^n = 2^n.$$

The number of items in the power set 2^A is $|2^A| = 2^{|A|}$.

Some Proofs About Sets

Two sets A and B are *equal* (written $A = B$) if $A \subseteq B$ and $B \subseteq A$.

That is,

$$(x \in A \iff x \in B) \iff A = B.$$

Many proofs about sets are to show set equality.

This means verifying the above iff statement.

Some Proofs About Sets

Much of modern mathematics is based in modern set theory.

To write a proof, you often need to deduce a set inclusion.

If x is an element of the domain D in question, $p(x)$ are the hypotheses, and $q(x)$ the conclusions, then you need to prove

$$x \in D \cap \{y : p(y)\} \implies x \in D \cap \{y : q(y)\},$$

i.e.

$$D \cap \{y : p(y)\} \subseteq D \cap \{y : q(y)\}$$

is the same statement as

$$\forall x \in D,\, p(x) \implies q(x).$$

Proof of a Distributive Law of Sets

Proposition

For sets A, B, C,

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

Proof We need to prove

$$x \in (A \cap B) \cup C \iff x \in (A \cup C) \cap (B \cup C).$$

This is the conjunction of the two implications

$$x \in (A \cap B) \cup C \implies x \in (A \cup C) \cap (B \cup C)$$

and

$$x \in (A \cap B) \cup C \Longleftarrow x \in (A \cup C) \cap (B \cup C).$$

Proof of a Distributive Law of Sets

Since the following argument goes both ways, we will write it with a chain of $\iff$ s (to save space).

The argument shows the translation from set notation to logic and back in a cleverly-different form.

Each step can be verified by checking the truth table for the pair of statements.

$$
\begin{aligned}
x \in (A \cap B) \cup C &\iff x \in A \cap B \text{ or } x \in C \\
&\iff (x \in A \text{ and } x \in B) \text{ or } x \in C \\
&\iff (x \in A \text{ or } x \in C) \text{ and } (x \in B \text{ or } x \in C) \\
&\iff (x \in A \cup C) \text{ and } (x \in B \cup C) \\
&\iff x \in (A \cup C) \cap (B \cup C). \ \blacksquare
\end{aligned}
$$

Set (Cartesian) Product, Sequences

The **Cartesian product** of two sets A and B, denoted $A \times B$, is the set whose elements are **ordered pairs**[2]

$$(a, b) := \{\{a\}, \{a, b\}\},$$

where $a \in A$ and $b \in B$.

Order matters in a Cartesian product of sets: in general,

$$A \times B = B \times A \iff A = B.$$

[2] This definition is by Kuratowski (1921).

Binary Relations

A **relation** R (also known as a **correspondence** or **map**) from the set A to the set B is a subset of the product set $A \times B$.

The simplest notation for a binary relation is

$$R \subseteq A \times B.$$

Note that a relation has **ordering**:

- A is the first set in the Cartesian product (called the **domain** of R), and
- B the second (called the **codomain** of R).

Binary Relations: Notation

An element being in a relation, $(a, b) \in R$, is often denoted by

$$aRb \text{ or } a \sim_R b.$$

The notation

$$R : A \to B$$

is usually reserved for a special type of relation called a *function*, that *maps* "from" the set A "to" the set B. In this case we will say that a **maps to** b under R with the notation

$$a \mapsto b.$$

Examples of Binary Relations

Relations are the primary concept tying pieces of data together in *relational databases*.

Here are some examples of relations:

- "divides": $| \subseteq \mathbb{Z} \times \mathbb{Z}$
  $(4, 8) \in |$, i.e. $4|8$, $(6, 3) \notin |$, i.e. $6 \not| 3$;
- "*a* teaches the course *b*, in Fall 2017, at Baruch College":
  from the set of all instructors to the set of all course offerings, containing ("Michael Carlisle", "MTH 4010 CMWA")
  but not ("Michael Carlisle", "MTH 4000 JMWA");
- "is greater than": $> \subseteq \mathbb{R} \times \mathbb{R}$
  $(9.24, 5) \in >$, i.e. $9.24 > 5$, but $(1, 1.2) \notin >$.

Sets involved with a binary relation

For a relation $R \subseteq A \times B$:

- The **domain** is A. Not all elements of a domain of a relation need to map to elements in the codomain.
- The **codomain** is B. Not all elements of the codomain need to be mapped to by an element of the domain.
- For any subset of the domain $C \subseteq A$, the **image** of C under R is the subset of B defined by

$$R(C) = \{b \in B : \exists a \in C, aRb\}.$$

Sets involved with a binary relation

For a relation $R \subseteq A \times B$:

- The image of A under R is called the **range** of R: $\text{range}(R) = R(A)$.
- For any subset of the range $D \subseteq B$, the **pre-image** (or **inverse image**) of D under R is the subset of A defined by

  $$R^{-1}(D) = \{a \in A : \exists b \in D, aRb\} = \{a \in A : \exists b \in D, bR^{-1}a\}.$$

Properties of Relations where domain = codomain

Let $R \subseteq A \times A$. R may have some of the following properties:

- **reflexive**: $\forall x \in A$, xRx.
- **irreflexive**: $\forall x \in A$, $\neg(xRx)$.
  Note: R cannot be both reflexive and irreflexive,
  but R could be neither.
- **symmetric**: $\forall x, y \in A$, $xRy \implies yRx$.
- **antisymmetric**: $\forall x, y \in A$, xRy and $yRx \implies x = y$.
- **asymmetric**: $\forall x, y \in A$, $\neg(xRy$ and $yRx)$.
- **transitive**: $\forall x, y, z \in A$, xRy and $yRz \implies xRz$.

Equivalence Relations

Definition

If a relation R on one set A is:

- reflexive ($\forall x \in A$, xRx),
- symmetric ($xRy \implies yRx$), and
- transitive ($xRy, yRz \implies xRz$),

then we call R an **equivalence relation**.

The concept of "equivalence relation" generalizes the notion of "equality" to accept more concepts than "is the same element".

Equivalence Relations

Some examples of equivalence relations:

- usual equality $=$ *on any set*;
- congruence *modulo n* on $\mathbb{Z}$;
- "is in this classroom today"
  on the set of students in the school today.

An Equivalence Relation Partitions a Set

An equivalence relation $R \subseteq A \times A$ *partitions* the set A into **equivalence classes**, each of which has elements which are symmetric and transitive under R (all are reflexive).

The notation of an equivalence class is $[k]$ or $cl(k)$ for $k \in A$, representing *all* elements of that class.

An Equivalence Relation Partitions a Set: Modulo

Example

Congruence modulo n partitions $\mathbb{Z}$ into n equivalence classes:

$$[0] = \{..., -2n, -n, 0, n, 2n, ...\}$$

$$[1] = \{..., -2n+1, -n+1, 1, n+1, 2n+1, ...\}$$

$$...$$

$$[n-1] = \{..., -2n-1, -n-1, -1, n-1, 2n-1, ...\}.$$

Real-Valued Functions of a Real Variable

Definition

$f : D \to \mathbb{R}$ is a **real-valued function of a real variable** if

- f is a relation from D, its **domain**
  (in this course, is a subset of $\mathbb{R}$ or $\mathbb{R}^n$), i.e. $dom(f) = D$,
- to $\mathbb{R}$, its **codomain**, i.e. $cod(f) = \mathbb{R}$,
- such that every point $x \in D$ has exactly one value $y \in \mathbb{R}$
  such that (x, y) is an element of f,
  i.e. f is **well-defined** on its domain.

Real-Valued Functions of a Real Variable

We use the **functional notation** to describe the values of f:

- x is typically used as the **independent variable**, or **argument**, from the domain, i.e. $x \in D$, and
- $y = f(x)$ is the **dependent variable**, or **value** of f at x.

The **image** of f under the subset $C \subseteq D$ is the set of target outputs in the codomain $\mathbb{R}$ whose source inputs are in C:

$$f(C) = \{y \in \mathbb{R} \mid \exists x \in C,\ y = f(x)\}.$$

The **range** of f is the image of the domain, $range(f) = f(D)$.

Functions

Certain types of functions bear their own names:

- A **constant function** $f : A \to B$ maps every element $a \in A$ to the same element $b^* \in B$: $\forall a \in A,\ f(a) = b^*$.
- An **identity function** $f : A \to A$ maps every element $a \in A$ to itself: $\forall a \in A,\ f(a) = a$.

Inverse Functions

We use the notation f^{-1} to refer to the inverse relation to the function $f : A \to B$. In general, f^{-1} is not a function.

If f^{-1} is a function, then it is defined on $range(f) = f(A) \subseteq B$, we call it the **inverse function** of f, and the following hold.

- f must be injective.
- If $f(a) = b$, then $f^{-1}(b) = a$.
- $f(f^{-1}(b)) = b$ and $f^{-1}(f(a)) = a$ for every $a \in A$, $b \in f(A)$.

Function Composition

If $f : A \to B$ and $g : B \to C$ are functions, with $f(A) = g^{-1}(C)$, then define the **function composition** $g \circ f : A \to C$ by

$$(g \circ f)(a) = g(f(a)) = g(b) = c, \text{ if } f(a) = b \in B.$$

Note that $f \circ g \neq g \circ f$ unless $A = B = C$ and $f = g$.

Function Composition Properties

- If f and g are injective, then so is $g \circ f$.
- If f and g are surjective, then so is $g \circ f$.
- Hence, if f and g are bijective, then so is $g \circ f$.
- If f^{-1} is a function, then

  $$f \circ f^{-1} : f(A) \to f(A)$$

  and

  $$f^{-1} \circ f : A \to A$$

  are identity maps.

Cardinality

The **cardinality** of a set is the number of (distinct) elements of the set. The cardinality of the set A is denoted $|A|$.

Finite sets have an obvious cardinality: count the number of elements.

Examples: $|\{4, 6, 2, 9\}| = 4$, $|\{7, 8, ..., 100\}| = 94$.

If two sets have the same cardinality, they are called *equinumerous*.

Infinite Cardinality

Infinite sets have more elements than any finite set.

The first thing to know about infinite sets is:

There is an infinite hierarchy of sizes of infinity.

Infinite Cardinality: Countable, Uncountable

The “smallest” level is called **countable** (the *cardinal number* denoting this count is $\aleph_0$); the next highest[3] is

$$\aleph_1 = 2^{\aleph_0},$$

and any $\aleph_k$ with $k \geq 1$ represent **uncountable** infinite cardinality.

We can prove:

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0; \;\; |\mathbb{R}| = \aleph_1.$$

[3] if you believe a statement called the *continuum hypothesis*

What is cardinality?, set equivalence

We want to generalize the concept of

"counting the elements of a set"

to more sets than just finite sets.

The set B has cardinality at least as large as another set A if

$$\exists g : A \hookrightarrow B$$

i.e. there exists a function g that is 1-1 (called an **injection**[4]).

We denote this fact by $|A| \leq |B|$.

[4] The notation $\hookrightarrow$ is sometimes used to denote that a function is 1-1.

What is cardinality?, set equivalence

Likewise, $|A| \geq |B|$ if

$$\exists h : A \to B$$

that is a **surjection** (onto).

Two sets A and B are called **equivalent** (or **equinumerous**, or in **1-1 correspondence**), denoted in several ways:

$$A \sim B, \; A \approx B, \; |A| = |B|,$$

if $\exists$ a **bijection** (1-1, onto function) $f : A \to B$.

Power set

Recall, the **power set** of a set A, denoted $\mathcal{P}(A)$ or 2^A, is the set of all subsets of A:

$$2^A = \{E : \ E \subseteq A\}.$$

It should be clear that, since there is at least one more subset of A than elements of A, that for finite sets A, $|2^A| > |A|$. In fact,

Theorem

$|2^A| > |A|$ *for any set A.*

We will prove this after we begin a discussion on infinite sets.

Infinite sets

A set whose cardinality is greater than n for every $n \in \mathbb{N}$ is called an **infinite set**.

The very obvious first example of this is $\mathbb{N}$ itself.

Infinite sets

The cardinality of $\mathbb{N}$ is denoted $|\mathbb{N}| = \aleph_0$ ("aleph-null", "aleph-naught", "aleph-zero"), the *first infinite cardinal*.

Note that $\aleph_0$ is different from ∞ ("infinity"), the symbol representing unboundedness in a given direction.

$\aleph_0$ refers to the infinite "size" of the set of natural numbers.

A natural question is, what other sets are of size $\aleph_0$?

Countable sets

Definition

A set S is called a **denumerable set**, or **countable set**[5], or **denumerably (countably) infinite set**, if there is a 1-1 correspondence from S to $\mathbb{N}$.

All countably infinite sets are in the same equivalence class under the cardinality equivalence relation $\sim$ as $\mathbb{N}$.

[5] Sometimes finite sets are also called "countable"; the term "countably infinite" is used to distinguish between the two types.

Countable sets

Two sets are equivalent if there exists a bijection between them.

Thus, all countably infinite sets have bijections between all of them, just like, in the case of finite sets, all sets of size 5 have bijections between them all.

The common concept is the *list*.

All sets of cardinality 5 have bijections with the set $\{1, 2, 3, 4, 5\}$, i.e. you can put the five elements on a numbered list.

Countable sets

Example

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d, e\}$, $C = \{3.4, \pi, 3e, \clubsuit, 0.4\}$.
Then $\exists f : A \to B$, $g : A \to C$ bijections, given by lists:

1. $f(1) = a, g(1) = 3.4$
2. $f(2) = b, g(2) = 3e$
3. $f(3) = e, g(3) = 0.4$
4. $f(4) = d, g(4) = \pi$
5. $f(5) = c, g(5) = \clubsuit$

is one possible pair of such bijections.
(How many bijections are there for each map?)

Infinite sets may be a bit less intuitive.

Countable sets are, at first, unintuitive.

A countable set's elements can be put on an infinitely-long list.

Proposition

$A = \{2, 3, 4, ...\} \sim \mathbb{N}$.

Proof $f : A \to \mathbb{N}$, defined by $f(n) = n - 1$, is a bijection. ■

Thus, we cannot say that A has "one less element" than $\mathbb{N}$; they have the *same* cardinality $|A| = |\mathbb{N}| = \aleph_0$, both countably infinite.

Countable sets are, at first, unintuitive. You get used to it.

Proposition

The positive evens $B = \{2, 4, 6, 8, ...\} \sim \mathbb{N}$.

Proof $f : B \to \mathbb{N}$, defined by $f(n) = n/2$, is a bijection. $\blacksquare$

Another: $g : \mathbb{N} \to B$, defined by $g(n) = 2n$, is a bijection. $\blacksquare$

Thus, we cannot say that B has "half the elements" of $\mathbb{N}$; they have the *same* cardinality $|B| = |\mathbb{N}| = \aleph_0$, both countably infinite.

Countable sets are, at first, unintuitive. You get used to it.

Proposition

$\mathbb{Z}$ is countable.

Proof $f : \mathbb{N} \cup \{0\} \to \mathbb{Z}$, defined by a "hopping back and forth" pattern:

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ -\frac{n+1}{2} & n \text{ odd} \end{cases}$$

is a bijection. Hence, $|\mathbb{N} \cup \{0\}| = |\mathbb{Z}|$.

We can easily show that $|\mathbb{N}| = |\mathbb{N} \cup \{0\}|$ via the bijection $g : \mathbb{N} \to \mathbb{N} \cup \{0\}$ defined by $g(n) = n - 1$. ■

Countable sets are, at first, unintuitive. You get used to it.

Proposition

If A_1, A_2, ..., A_n are countable sets, then their union $\bigcup_{i=1}^{n} A_i$ is also a countable set.

Proof Without loss of generality (Wlog), we assume all the elements of all n sets are distinct. If not, discard duplicates.

Since each A_i is countable, we can list the elements:

$$A_i = \{a_{i1}, a_{i2}, a_{i3}, ...\}.$$

Therefore, we can merge all n lists into one big list, showing that the union of the n sets is also countable:

$$\bigcup_{i=1}^{n} A_i = \{a_{11}, a_{21}, a_{31}, ..., a_{n1}, a_{21}, a_{22}, a_{32}, ..., a_{n2}, a_{31}, a_{32}, a_{33}, ...\}. \blacksquare$$

Countable sets are, at first, unintuitive. You get used to it.

Corollary

If $A \subseteq B \subseteq C$ and $|A| = |C|$, then $|A| = |B| = |C|$.

Proposition

The Cartesian product $\mathbb{Z} \times \mathbb{Z}$ is countable.

Proof (sketch) There is a bijection $f : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$ which can be visualized by drawing a spiral from the center of the coordinate plane outward, connecting all the points on the *integer lattice*. ■

Countable sets are, at first, unintuitive. You get used to it.

Proposition

If A and B are countably infinite or finite, then the Cartesian product $A \times B$ is countable or finite.

Proof Obviously, if both A and B are finite, with $|A| = m$ and $|B| = n$, then $|A \times B| = mn$ is also finite.

If both A and B are countable, we can list their elements:

$$A = \{a_1, a_2, a_3, ...\}, \;\; B = \{b_1, b_2, b_3, ...\}.$$

We can list the elements of the Cartesian product "diagonally":

$$\begin{aligned} A \times B = \{&(a_1, b_1), \\ &(a_1, b_2), (a_2, b_1), \\ &(a_1, b_3), (a_2, b_2), (a_3, b_1), ...\}. \blacksquare \end{aligned}$$

Countable sets are, at first, unintuitive. You get used to it.

Proposition

$\mathbb{Q}$ is countable.

Proof (sketch) We can put $\mathbb{Q}$ in 1-1 correspondence with a subset $Q \subseteq \mathbb{Z} \times \mathbb{Z}$:

$$Q = \left\{ (m, n) : m, n \in \mathbb{Z},\ \frac{m}{n} \in \mathbb{Q} \text{ and in lowest terms} \right\}.$$

Since $\mathbb{Z} \times \mathbb{Z}$ is countable by the previous theorem, and

$$\mathbb{N} \times \{1\} \subseteq Q \subseteq \mathbb{Z} \times \mathbb{Z},$$

then, since $|\mathbb{Q}| = |Q|$, we have

$$\aleph_0 = |\mathbb{N} \times \{1\}| \leq |\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}| = \aleph_0 \implies |\mathbb{Q}| = \aleph_0. \blacksquare$$

Power set is strictly bigger (finite set)

Theorem

For a finite set A, $|A| < |2^A|$.

Proof

$$|A| = n \in \mathbb{N} \implies 2^{|A|} = |2^A| = \sum_{j=0}^{n} \binom{n}{j} = 2^n > n = |A|. \blacksquare$$

Power set is strictly bigger (any set)

For infinite sets, however, we need a bit more work.

Theorem
$|A| < |2^A|$ *for any set A.*

Proof There is an injection

$$g : A \hookrightarrow 2^A$$

defined by selecting singleton sets:

$$a \mapsto g(a) = \{a\}.$$

Therefore, $|A| \leq |2^A|$.

Power set is strictly bigger

We need to show that *no* bijection exists between A and 2^A; hence,

$$|A| \neq |2^A|.$$

We prove by contradiction. Assume $f : A \to 2^A$ is a bijection.

We will show that no such f can exist.

Power set is strictly bigger

If such an f exists, then, for each $a \in A$,

- either $a \in f(a)$ (which happens with $f(a) = A$),
- or $a \notin f(a)$ (which happens with $f(a) = \emptyset$),

but not both.

Power set is strictly bigger

Let

$$B = \{b \in A : \; b \notin f(b)\}.$$

Then $B \subseteq A$ (and so $B \in 2^A$), since B is well-defined if f exists.

Since $B \in 2^A$, $\exists c \in A$ such that $B = f(c)$ since f is a bijection.

Is $c \in B$? By definition, if $c \in B$, then $c \notin f(c) = B$.

If $c \notin f(c) = B$, then $c \in B$. This is a contradiction. $\rightarrow\leftarrow$

Hence, there is no bijection between A and 2^A, and $|A| \neq |2^A|$. $\blacksquare$

Cantor diagonalization argument

The next proof is known as **Cantor's diagonalization argument**[6], named after the use of the *diagonal*

$$\{(c, f(c)) : c \in A\}.$$

An immediate consequence of this theorem is that there exists a hierarchy of "sizes of infinity": since $|A| < |2^A|$ for any set A, then $|\mathbb{N}| < |2^{\mathbb{N}}|$, and since $2^{\mathbb{N}}$ is itself a set, we have

$$|\mathbb{N}| < |2^{\mathbb{N}}| < |2^{2^{\mathbb{N}}}| < \cdots .$$

[6]Cantor published this argument in 1891. Bertrand Russell modified Cantor's diagonalization argument ten years later to show a weakness in Cantor's "naive" set theory, creating what is known as the "barber paradox".

$|\mathbb{R}| = ?$

What is the cardinality of the real numbers $\mathbb{R}$?

Certainly, $|\mathbb{R}| \geq \aleph_0$, since $\mathbb{Z} \subseteq \mathbb{R}$.

We can find the cardinality of $\mathbb{R}$ by finding a bijection between $\mathbb{R}$ and another set.

That set is $2^{\mathbb{N}}$.

Uncountable sets

We will define $\aleph_1 = |\mathbb{R}|$. We know $\aleph_0 \leq \aleph_1$.

How big is $\aleph_1$ relative to $\aleph_0$?

$$\aleph_0 = \aleph_1? \quad \aleph_0 < \aleph_1?$$

If it turns out that $\aleph_0 < \aleph_1$, are there any cardinals between them?

We call sets of cardinality $\aleph_1$ or greater **uncountable sets**.

Why we use that term will be explored shortly.

Uncountable sets are weirder.

We call a set A

- **finite** if we can list its elements on a finite-length list, i.e. $\exists$ a bijection between A and $\{1, 2, ..., n-1, n\}$.

  In this case, $|A| = n$.

- **countable** if we can list its elements on an infinite-length list, i.e. $\exists$ a bijection between A and $\mathbb{N}$.

  In this case, $|A| = \aleph_0$.

- **uncountable** if no bijection exists between A and any subset of $\mathbb{N}$.

  In this case, $|A| > \aleph_0$.

$|\mathbb{R}| > |\mathbb{N}|$ and all open intervals are the same cardinality

Theorem

$\mathbb{R}$ is uncountable, and any open interval (a, b) has the same cardinality as $\mathbb{R}$: $|(a, b)| = |\mathbb{R}|$ for any $a, b \in \mathbb{R}$, $a < b$.

Proof There are many ways to write down this argument:
I will choose one that is visually easy to understand.

First, we will start by showing that any open interval has the same cardinality as the open unit interval $(0, 1)$.

$|\mathbb{R}| > |\mathbb{N}|$ and all open intervals are the same cardinality

For $a, b \in \mathbb{R}$, $a < b$, the function $g : (a, b) \to (0, 1)$ defined by

$$g(x) = \frac{x - a}{b - a}$$

is a bijection (a line segment). Thus, $|(0, 1)| = |(a, b)|$.

The function $h : (0, 1) \to \mathbb{R}$ defined by

$$h(x) = \tan\left(\frac{(2x - 1)\pi}{2}\right)$$

is also a bijection, so $|(0, 1)| = |\mathbb{R}|$.

Cantor's diagonalization argument: $|\mathbb{R}| > |\mathbb{N}|$

Now, we will show that the open unit interval $(0,1)$ is uncountable.

For a contradiction, assume $b : \mathbb{N} \to (0,1)$ is a bijection.

Then $(0,1)$ is countable, so there is a list of the elements of $(0,1)$, which can be written as infinite-length decimal expansions:

$$\begin{aligned}
b(1) &= 0.a_{11}a_{12}a_{13}a_{14}a_{15}\dots \\
b(2) &= 0.a_{21}a_{22}a_{23}a_{24}a_{25}\dots \\
b(3) &= 0.a_{31}a_{32}a_{33}a_{34}a_{35}\dots \\
b(4) &= 0.a_{41}a_{42}a_{43}a_{44}a_{45}\dots \\
b(5) &= 0.a_{51}a_{52}a_{53}a_{54}a_{55}\dots \\
&\dots
\end{aligned}$$

where the $a_{ij} \in \{0, 1, 2, \dots, 8, 9\}$.

Cantor's diagonalization argument: $|\mathbb{R}| > |\mathbb{N}|$

From this list, select the *diagonal* digits and construct the number

$$a = 0.b_1 b_2 b_3 b_4 b_5 \ldots$$

where $b_j = (a_{jj} + 1) \mod 10$ for $j = 1, 2, 3, 4, \ldots$.

$$\begin{aligned}
b(1) &= 0.\color{red}{a_{11}}\color{black} a_{12} a_{13} a_{14} a_{15} \ldots \\
b(2) &= 0.a_{21} \color{red}{a_{22}}\color{black} a_{23} a_{24} a_{25} \ldots \\
b(3) &= 0.a_{31} a_{32} \color{red}{a_{33}}\color{black} a_{34} a_{35} \ldots \\
b(4) &= 0.a_{41} a_{42} a_{43} \color{red}{a_{44}}\color{black} a_{45} \ldots \\
b(5) &= 0.a_{51} a_{52} a_{53} a_{54} \color{red}{a_{55}}\color{black} \ldots \\
&\ldots
\end{aligned}$$

$a \neq b(j)$ for any j, i.e. a is not on the list, since a differs from $b(j)$ on digit j of its expansion. Hence, b is not a bijection. $\rightarrow\leftarrow$
Therefore, $|\mathbb{R}| = |(0, 1)| > |\mathbb{N}|$. $\blacksquare$

A quick list of diagonal contradiction arguments

In fact, we know

Theorem

$\aleph_1 = |\mathbb{R}| = |2^{\mathbb{N}}| = 2^{|\mathbb{N}|} = 2^{\aleph_0}$.

but will not prove this here.

A quick list of diagonal contradiction arguments

Here is a short list of important diagonal contradiction arguments:

- Cantor (1891): $|\mathbb{R}| > |\mathbb{N}|$
- Russell (1901): Russell's Paradox
- Gödel (1931): Incompleteness Theorems
- Turing (1936): Entscheidungsproblem (Halting Problem)
- Tarski (1936): Undefinability Theorem

The use of a diagonal-style contradiction proof is rooted in *self-reference* in the key object constructed to make the contradiction.