Introduction to Analysis: Limits and Continuity

Limit of a Real-Valued Function at a Point

Definition

Assume that a function

$$f : D \to \mathbb{R}$$

is defined in a deleted neighborhood[1] of $c \in \mathbb{R}$, i.e.

$$\exists h > 0 : \; N^*(c, h) \subseteq D.$$

The **limit** as x approaches c of the function f is the value $L \in \mathbb{R}$, denoted

$$\lim_{x \to c} f(x) = L,$$

if, the "closer x gets to c", the "closer $f(x)$ gets to L."

[1] c is an accumulation point of D, but $c \in D$ is not necessary here.

Limit of a Real-Valued Function at a Point

Formally: if

$$\forall \varepsilon > 0 \ \exists \delta = \delta(c, \varepsilon) > 0 : \ 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon,$$

then we say

$$\lim_{x \to c} f(x) = L.$$

In neighborhood notation,

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \ x \in N^*(c, \delta) \implies f(x) \in N(L, \varepsilon).$$

Limits Bound Maps of Neighborhoods

Restating the limit definition in neighborhood terms proves

Theorem

Suppose $f : D \to \mathbb{R}$ and let c be an accumulation point of D.

Then

$$\lim_{x \to c} f(x) = L$$

if and only if, for each $\varepsilon > 0$ and neighborhood $V = N(L, \varepsilon)$,

$\exists \delta > 0$ and $U = N^*(c, \delta)$ such that the image

$$f(U \cap D) \subseteq V.$$

Limits Bound Maps of Neighborhoods

Using $\varepsilon = \frac{|L|}{2}$ in the definition of the limit proves

Corollary

(sign preservation near a nonzero limit)

Suppose $f : D \to \mathbb{R}$ and let c be an accumulation point of D.
Suppose $\exists L > 0$ such that

$$\lim_{x \to c} f(x) = L.$$

Then $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies f(x) > 0$.

Sequential Criterion for Limits

Here we relate limits of functions with convergence of sequences.

Theorem

Suppose $f : D \to \mathbb{R}$ *and let* c *be an accumulation point of* D*.*

Then

$$\lim_{x \to c} f(x) = L$$

iff for any sequence (x_n) *in* $D \setminus \{c\}$,

$$\lim_{n \to \infty} x_n = c \implies \lim_{n \to \infty} f(x_n) = L.$$

Sequential Criterion for Limits

Proof ($\Longrightarrow$) Assuming

$$\lim_{x \to c} f(x) = L,$$

we need to show that, for any sequence (x_n),

$$\text{if } x_n \to c, \text{ then } f(x_n) \to L.$$

We know that for each $\varepsilon > 0$,

$$\exists \delta > 0 : \; 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

If $x_n \to c$,

$$\exists N \in \mathbb{N} : \; \forall n > N, \; |x_n - c| < \delta.$$

Sequential Criterion for Limits

Then

$$|x_n - c| < \delta \implies |f(x_n) - L| < \varepsilon,$$

which is precisely the criterion for $f(x_n) \to L$: for each $\varepsilon > 0$,

$$\exists N \in \mathbb{N} : \ \forall n > N, \ |f(x_n) - L| < \varepsilon.$$

($\Longleftarrow$) For this direction, a direct proof would be difficult. Here, we use the contrapositive. To do so, we need to show

$$\lim_{x \to c} f(x) \neq L \implies (\exists (x_n) : \lim_{n \to \infty} x_n = c \text{ and } \lim_{n \to \infty} f(x_n) \neq L).$$

Sequential Criterion for Limits

Assume

$$\lim_{x \to c} f(x) \neq L.$$

Then $\exists \varepsilon > 0$ such that

$$\forall \delta > 0, \ \exists x \in D : \ 0 < |x - c| < \delta \text{ and } |f(x) - L| \geq \varepsilon.$$

In particular, for each $n \in \mathbb{N}$,

$$\exists x_n \in D : \ |x_n - c| < \delta \text{ and } |f(x_n) - L| \geq \varepsilon.$$

This gives us a sequence (x_n) such that

$$\lim_{n \to \infty} x_n = c \text{ and } \lim_{n \to \infty} f(x_n) \neq L. \ \blacksquare$$

Limits, if they exist, are unique

Corollary

If $f : D \to \mathbb{R}$, and c is an accumulation point of D,
then if f has a limit at c, the limit is unique.

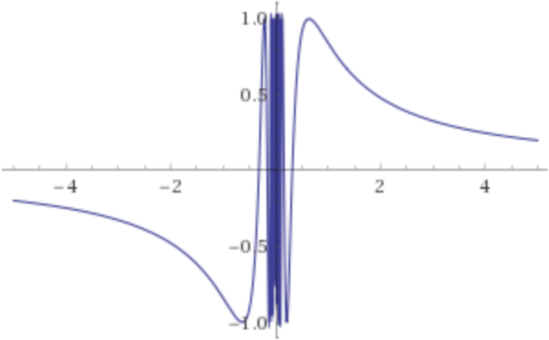
Theorem

Suppose $f : D \to \mathbb{R}$, and c is an accumulation point of D. Then f does not have a limit at c iff $\exists$ a sequence (x_n) in $D \setminus \{c\}$ such that $x_n \to c$ but $(f(x_n))$ does not converge in $\mathbb{R}$.

Limits, if they exist, are unique

Example

$f(x) = \sin(\frac{1}{x})$ defined on $D = (0, \infty)$ has no limit at $x = 0$: consider the sequence $(x_n) = (\frac{2}{n\pi})$.



Limits, if they exist, are unique

Example

The Dirichlet function

$$f(x) = 1_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

has no limit at any $x \in \mathbb{R}$: for $c \in \mathbb{R}$, with decimal expansion $c = c_0.c_1c_2c_3\ldots$, consider the sequence (x_n) defined by

$$x_n = \begin{cases} c_0.c_1\ldots c_n \in \mathbb{Q} & n \text{ even} \\ x_{n-1} + \frac{\sqrt{2}}{10^{n+1}} \in \mathbb{R} \setminus \mathbb{Q} & n \text{ odd.} \end{cases}$$

Then $x_n \to c$ but $f(x_n) \not\to 0$ or 1.

Squeeze (Sandwich) Theorem

Theorem

(Squeeze Theorem) *Let f, g, h be functions defined on a neighborhood $N(a, \varepsilon)$ of a.*

If

$$\forall x \in N^*(a, \varepsilon),\ f(x) \leq g(x) \leq h(x)$$

and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then

$$\lim_{x \to a} g(x) = L.$$

One-Sided Limits

Note that the definition of a limit goes in "both directions": positive and negative.

Definition

Assume that $f : D \to \mathbb{R}$ is a function, and $N^*(c, h) \subseteq D$ for some $h > 0$. The **one-sided limits** are defined as:

$$\textbf{left-hand limit}: \lim_{x \to c-} f(x) = L \iff$$
$$\forall \varepsilon > 0, \exists \delta_- > 0 : \; -\delta_- < x - c < 0 \implies |f(x) - L| < \varepsilon.$$

$$\textbf{right-hand limit}: \lim_{x \to c+} f(x) = L \iff$$
$$\forall \varepsilon > 0, \exists \delta_+ > 0 : \; 0 < x - c < \delta_+ \implies |f(x) - L| < \varepsilon.$$

One-Sided Limits

The limit

$$\lim_{x \to c} f(x) = L,$$

only exists if

$$\lim_{x \to c-} f(x) = L = \lim_{x \to c+} f(x).$$

There are several ways the limit may fail to exist:

- Both one-sided limits exist, but do not match (jump)
- One or both of the one-sided limits does not exist (vertical asymptote or oscillation).

In either case we often use the notation DNE ("does not exist"):

$$\lim_{x \to c} f(x) \ DNE.$$

Limits are Linear Maps

Let $f, g : D \to \mathbb{R}$ be functions defined on a neighborhood of c, such that

$$\lim_{x \to c} f(x) = L, \ \lim_{x \to c} g(x) = M.$$

The limit is a **linear map**: for any $a, b \in \mathbb{R}$,

$$\lim_{x \to c} (af(x) + bg(x)) = a \lim_{x \to c} f(x) + b \lim_{x \to c} g(x) = aL + bM.$$

Limits are Linear Maps

Proof We use an $\frac{\varepsilon}{2}$-type argument.

Since f and g have limits at c, then for any $\varepsilon_f, \varepsilon_g > 0$, $\exists \delta_f, \delta_g > 0$:

$$|x - c| < \delta = \min(\delta_f, \delta_g) \implies |f(x) - L| < \varepsilon_f,\ |g(x) - M| < \varepsilon_g.$$

Limits are Linear Maps

Let $\varepsilon > 0$ and choose ε_f, $\varepsilon_g > 0$ such that $\varepsilon = |a|\varepsilon_f + |b|\varepsilon_g$.

Then $\exists\delta = \min(\delta_f, \delta_g)$ (same as before) such that

$$\begin{aligned}
|x - c| < \delta \implies & |(af(x) + bg(x)) - (aL + bM)| \\
& \leq |af(x) - aL| + |bg(x) - bM| \\
& = |a| \cdot |f(x) - L| + |b| \cdot |g(x) - M| \\
& < |a|\varepsilon_f + |b|\varepsilon_g = \varepsilon. \quad \blacksquare
\end{aligned}$$

Limits of Products, Quotients

Products and quotients are also preserved by limits:

$$\lim_{x\to c} f(x)g(x) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right) = LM,$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M} \text{ (if } M \neq 0).$$

Examples of Limits: Powers, Polynomials

It should be clear from the linearity and product rules of limits that powers of x have limits:

$$c \neq 0,\ n > 0 \implies \lim_{x \to c} x^n = c^n$$

From here it is easy to see that polynomials (with degree $n \in \mathbb{N}$) with real coefficients $a_i \in \mathbb{R}$, i.e.

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0,$$

have limits for all $x \in \mathbb{R}$:

$$\lim_{x \to c} p(x) = a_n c^n + a_{n-1} c^{n-1} + ... + a_1 c + a_0.$$

Examples of Limits

Example

$$f(x) = \frac{x^2 + 2x - 5}{x^2 + 3x - 5} \implies \lim_{x \to 2} f(x) = \frac{3}{5}.$$

Example

$$g(x) = \begin{cases} 3x^4 - x^2 + 3, & x > 1 \\ 26, & x = 1 \\ 5x, & x < 1 \end{cases} \implies \lim_{x \to 1} g(x) = 5.$$

Continuity at a Point

Definition

A function $f : D \to \mathbb{R}$ defined at $c \in D$ is called **continuous at c** if for any $\varepsilon > 0$,

$$\exists \delta = \delta(c, \varepsilon) > 0 : \; |x - c| < \delta, \, x \in D \implies |f(x) - f(c)| < \varepsilon.$$

This should look strikingly familiar to the definition of the limit of $f(x)$ as x approaches c.

If this property is not met, c is a **point of discontinuity** of f.

If f is continuous $\forall c \in S \subseteq D$, we call f **continuous on S**, and if f is continuous $\forall c \in D$, we call f a **continuous function**.

Continuity at an Isolated Point is Automatic

Note that, if c is an isolated point of D (not an accumulation point), then f is trivially continuous at c since $\exists \delta > 0$ such that D contains no other points near c, i.e.

$$c \text{ isolated in } D \implies \exists \delta > 0 : \; 0 < |x - c| < \delta \implies x \notin D.$$

Example

Let $D = (1, 6) \cup \{8, 24.5, -3\}$. Then $f : D \to \mathbb{R}$ defined by

$$f(x) = 5x - 4$$

is a continuous function on D.

Sequential, Neighborhood Criteria of Continuity

Theorem

Let $f : D \to \mathbb{R}$ and $c \in D$. Then the following are equivalent:

(a) *f is continuous at c.*

(b) *If (x_n) is a sequence in D such that $\lim_{n\to\infty} x_n = c$, then*

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(c).$$

(c) *For any $\varepsilon > 0$,*

$$\exists\delta > 0 : \; x \in N(c,\delta) \implies f(x) \in N(f(c),\varepsilon).$$

Furthermore, if c is an accumulation point of D, then (a)-(c) are all equivalent to

$$\lim_{x\to c} f(x) = f\left(\lim_{x\to c} x\right) = f(c).$$

Examples of Continuity: Powers, Polynomials

It should be clear from the linearity and product rules of limits that powers of x are continuous functions:

$$c \neq 0, \ n > 0 \implies \lim_{x \to c} x^n = c^n$$

From here it is easy to see that polynomials (with degree $n \in \mathbb{N}$) with real coefficients $a_i \in \mathbb{R}$, i.e.

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0,$$

are continuous for all $x \in \mathbb{R}$:

$$\lim_{x \to c} p(x) = a_n c^n + a_{n-1} c^{n-1} + ... + a_1 c + a_0 = p(c).$$

Examples of Continuity / Discontinuity

Example

$$f(x) = \frac{x^2 + 2x - 5}{x^2 + 3x - 5} \implies \lim_{x \to 2} f(x) = \frac{3}{5} = f(2).$$

Hence, f is continuous at $x = 2$.

Example

$$g(x) = \begin{cases} 3x^4 - x^2 + 3, & x > 1 \\ 26, & x = 1 \\ 5x, & x < 1 \end{cases} \implies \lim_{x \to 1} g(x) = 5 \neq g(1).$$

Hence, g is not continuous at $x = 1$.

Examples of Continuity / Discontinuity

Example

Define $f(x) = x\sin(\frac{1}{x})$ if $x \neq 0$ and $f(0) = 0$. Then, for all $x \neq 0$, f is continuous, and

$$|f(x) - f(0)| = \left| x\sin\left(\frac{1}{x}\right)\right| = |x|\left|\sin\left(\frac{1}{x}\right)\right| \leq |x| = |x - 0|.$$

Hence, for any $\varepsilon > 0$, the choice of $\delta = \varepsilon$ satisfies

$$0 < |x - 0| < \delta \implies |f(x) - f(0)| < \varepsilon,$$

which proves that f is continuous at $x = 0$.

Example

The Dirichlet function $1_{\mathbb{Q}}(x)$ has no limits for any $x \in \mathbb{R}$. Hence, it is discontinuous for every $x \in \mathbb{R}$.

Different Types of Discontinuity

There are many ways a function can be discontinuous at a point:

- **removable jump discontinuity**: the limit value $L \in \mathbb{R}$, and

  $$\lim_{x \to c} f(x) = L, \text{ but } f(c) \neq L.$$

  The function f can be redefined at this point to make a continuous function: if f has a removable jump at c, then

  $$g(x) = \begin{cases} f(x) & x \neq c \\ L & x = c \end{cases}$$

  is continuous at c.

- **nonremovable jump discontinuity**: $\exists K, L \in \mathbb{R}$ such that

  $$\lim_{x \to c+} f(x) = K \neq L = \lim_{x \to c-} f(x).$$

Different Types of Discontinuity

- f is defined at c (i.e. $f(c)$ exists), but $\lim_{x\to c} f(x)$ does not exist, due to **oscillations** of f near c
(see, for example, $f(x) = \sin(\frac{1}{x})$ if $x \neq 0$, $f(0) = 0$).

- f has a **vertical asymptote** at $x = c$:

$$\lim_{x\to c+} f(x) = \infty \text{ or } -\infty, \quad \text{or } \lim_{x\to c-} f(x) = \infty \text{ or } -\infty.$$

- If f is undefined at c, then clearly f cannot be continuous at c, even if the limit exists.
However, as with a removable jump discontinuity, if the limit exists, we can redefine the function at c to be continuous.

Sequential Criterion of Discontinuity

Theorem

Let $f : D \to \mathbb{R}$ and $c \in D$.

Then f is discontinuous at c iff $\exists$ a sequence (x_n) in D that converges to c but $(f(x_n))$ does not converge to $f(c)$.

Sequential Criterion of Discontinuity

Example

We modify the Dirichlet function to get a function that is continuous only at irrational values. Let

$$f(x) = \begin{cases} \frac{1}{n} & x \in \mathbb{Q}, x = \frac{m}{n} \text{ in lowest terms} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

If $c \in \mathbb{Q}$, a sequence (x_n) of irrationals converging to c (like, say, $x_n = \frac{\sqrt{2}}{n} + c$) all satisfy $f(x_n) = 0$, but $f(c) > 0$.

Hence, f is discontinuous at c.

However, if $c \notin \mathbb{Q}$, then any sequence (x_n) converging to c has $f(x_n) \to f(c) = 0$. Hence, f is continuous at irrational c.

Linear Combos, Products, Quotients of Cont. Functions

Let $f, g : D \to \mathbb{R}$ be continuous at c. Then:

- Any linear combination of f and g is continuous at c, i.e. for any $a, b \in \mathbb{R}$, $af + bg$ is continuous at c.
- Products and quotients of continuous functions are also continuous:

$$\lim_{x \to c} f(x)g(x) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right) = f(c)g(c),$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{f(c)}{g(c)} \text{ (if } g(c) \neq 0\text{).}$$

Max, Min, Compositions of Cont. Functions

Let $f, g : D \to \mathbb{R}$ be continuous at c. Then:

- The maximum function
  $f \vee g(x) = \max(f, g)(x) = \max(f(x), g(x))$ and
- the minimum function
  $f \wedge g(x) = \min(f, g)(x) = \min(f(x), g(x))$
  are both continuous.
- If g is continuous on D and f is continuous on $g(D)$, then the composition $f \circ g(x) = f(g(x))$ is continuous on D.

Open Sets Determine Continuity

The following theorem holds in the much more general setting of *topological spaces*, not just in the special case of the real numbers.

Theorem

$f : D \to \mathbb{R}$ is continuous on D $\iff$ for every open set G in $\mathbb{R}$ there exists an open set H in $\mathbb{R}$ such that $H \cap D = f^{-1}(G)$.

Corollary

$f : \mathbb{R} \to \mathbb{R}$ is continuous $\iff$ $f^{-1}(G)$ is open if G is open.

Bounded Functions

Definition

$f : D \to \mathbb{R}$ is called a **bounded function** if $\exists M > 0$ such that

$$\forall x \in D,\ |f(x)| \leq M,$$

i.e. the range $f(D) \subseteq \mathbb{R}$ is a bounded set.

Note

D bounded does not imply $f(D)$ bounded.
Try $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$.

Theorem

$f : D \to \mathbb{R}$ is continuous, and D is compact $\implies f(D)$ is compact.

Continuous image of a compact set is compact

Theorem

$f : D \to \mathbb{R}$ is continuous, and D is compact $\implies f(D)$ is compact.

Proof Let $\mathcal{G} = \{G_\alpha\}$ be an open cover of $f(D)$.

We will show that $\mathcal{G}$ has a finite subcover.

By the previous theorem, since f is continuous on D, we have that

$$\text{for each } G_\alpha \in \mathcal{G}, \exists H_\alpha \subseteq \mathbb{R} \text{ open such that } H_\alpha \cap D = f^{-1}(G_\alpha).$$

Continuous image of a compact set is compact

Since $f(D) \subseteq \bigcup_\alpha G_\alpha$, it follows that

$$D \subseteq \bigcup_\alpha f^{-1}(G_\alpha) \subseteq \bigcup_\alpha H_\alpha.$$

Thus, the collection $\{H_\alpha\}$ is an open cover of D.

D is compact, so there is a finite subcover

$$\{H_{\alpha_1}, H_{\alpha_2}, ..., H_{\alpha_n}\}$$

such that

$$D \subseteq \bigcup_{i=1}^{n} H_{\alpha_i}.$$

Continuous image of a compact set is compact

Note that

$$D \subseteq \bigcup_{i=1}^{n} H_{\alpha_i} \implies D \subseteq \bigcup_{i=1}^{n} (H_{\alpha_i} \cap D) \implies f(D) \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}.$$

Hence, $\{G_{\alpha_i}\}_{i=1}^{n}$ is a finite subcover of $\mathcal{G}$ for $f(D)$.

$\therefore$ $f(D)$ is compact. $\blacksquare$

The Attainment of Extreme Values

As a corollary we get the well-known

Corollary

Extreme Value Theorem (EVT) Let f be continuous on $[a, b]$.

Then f attains its minimum (greatest lower bound, infimum) and maximum (least upper bound, supremum) on $[a, b]$.

Proof f is continuous on $[a, b]$, and $[a, b]$ is compact, so the image $f([a, b])$ is compact. Thus, $f([a, b])$ contains its min and max:

$\exists x_m, x_M \in [a, b]$ and $m, M \in f([a, b])$ such that

$$\forall x \in [a, b],\ f(x_m) = m \leq f(x) \leq M = f(x_M). \blacksquare$$

Sign Preservation

Theorem

(Sign Preservation) *Let* $f : [a, b] \to \mathbb{R}$ *be continuous,* $c \in (a, b)$*.*

Then, if $f(c) \neq 0$*,* $\exists$ *a neighborhood* $N = N(c, \varepsilon)$ *of* c *such that*

$$\forall x \in N(c, \varepsilon),\ f(x) \text{ and } f(c) \text{ have the same sign.}$$

Sign Preservation

Proof WLOG assume $f(c) > 0$.

Since f is continuous at c, and $f(c) = y_0 > 0$, then for any $\varepsilon > 0$,

$$\exists \delta > 0 : \; 0 < |x - c| < \delta \implies |f(x) - y_0| < \varepsilon.$$

Choosing $\varepsilon = \frac{y_0}{2} > 0$, we get some $\delta > 0$ such that

$$x \in (c - \delta, c + \delta) \implies f(x) \in \left(\frac{y_0}{2}, \frac{3y_0}{2}\right). \; \therefore \; f(x) > 0.$$

If $f(c) < 0$, consider $g(x) = -f(x)$. $\blacksquare$

The Intermediate Value Theorem (version 0)

Lemma

Intermediate Value Theorem (IVT) (through 0)

Suppose f is continuous on $[a, b]$, and that $f(a) < 0 < f(b)$.

Then $\exists c \in (a, b)$ such that $f(c) = 0$.

Proof Let[2]

$$S = \{x \in [a, b] : \ f(x) \leq 0\}.$$

Let $c = \sup S$. We claim that $f(c) = 0$.

[2] In pre-image terms, $S = [a, b] \cap f^{-1}((-\infty, 0])$.

The Intermediate Value Theorem (version 0)

For a contradiction, suppose that $f(c) < 0$.

Then, by sign preservation, $\exists$ a neighborhood U of c such that

$$\forall x \in U \cap [a, b],\ f(x) < 0.$$

Since $f(c) < 0 < f(b)$, $\exists p \in U$ such that $c < p < b$.

But $f(p) < 0$ since $p \in U$.

Thus, $p \in S$, which contradicts $c = \sup S$. $\rightarrow\leftarrow \quad \therefore f(c) \geq 0$.

The Intermediate Value Theorem (version 0)

Likewise, suppose for another contradiction that $f(c) > 0$.

Then, again by sign preservation, $\exists V = V(c, \varepsilon)$ for some $\varepsilon > 0$:

$$\forall x \in V \cap [a, b],\ f(x) > 0.$$

Then, since $f(a) < 0 < f(c)$, $\exists q \in V$ such that $a < q < c$.

Hence, $f(q) > 0$, and so q is an upper bound for S with $q < c$, contradicting $c = \sup S$. $\rightarrow\leftarrow$ $\quad \therefore f(c) \leq 0$.

Therefore, $f(c) = 0$, and $c \neq a$ and $c \neq b \implies c \in (a, b)$. $\quad \blacksquare$

The Intermediate Value Theorem

Theorem

Intermediate Value Theorem (IVT)

Suppose f is continuous on $[a, b]$, and that $f(a) \neq f(b)$.

Then, as x varies from a to b, $f(x)$ takes on every value k between $f(a)$ and $f(b)$.

Intuition If f is continuous on $[a, b]$, you don't need to lift your pencil when drawing the graph of f.

In addition, every horizontal line $y = k$ for k between $f(a)$ and $f(b)$ crosses the graph at least once.

The Intermediate Value Theorem

Proof WLOG, assume $f(a) < f(b)$. We show

$$\forall k \in (f(a), f(b)),\ \exists x_k \in (a, b) :\ f(x_k) = k.$$

Apply the previous lemma to the continuous function

$$g(x) = f(x) - k,$$

since

$$g(a) = f(a) - k < 0 \text{ and } g(b) = f(b) - k > 0.$$

Thus $\exists c \in (a, b)$ such that

$$g(c) = f(c) - k = 0 \implies f(c) = k.$$

The $f(a) > f(b)$ case repeats the argument on $h(x) = -g(x)$. ■

Cont. image of a compact interval is a compact interval

We now refine the theorem from the beginning of the section:

Theorem

Suppose f is continuous on $[a, b]$. Then f is bounded on $[a, b]$.

can be focused to the more immediately-applicable

Theorem

Suppose $f : D \to \mathbb{R}$ is continuous, and D is a compact interval. Then $f(D)$ is a compact interval.

Cont. image of a compact interval is a compact interval

Proof Since D is compact and f is continuous,
by the EVT $f(D)$ attains its min m and max M.

Thus, $f(D) \subseteq [m, M]$, and

$$\exists x_m, x_M \in D : \; f(x_m) = m, \; f(x_M) = M.$$

If $m = M$, then f is a constant function and we are done.

Else, $m < M$. Clearly, $m, M \in f(D)$.

Let $k \in (m, M)$. By the IVT, $\exists x_k \in D : \; f(x_k) = k$.

Since k is arbitrarily chosen, this implies $(m, M) \subseteq f(D)$.

Thus, $[m, M] \subseteq f(D) \subseteq [m, M] \implies f(D) = [m, M]$. ■

Uniform continuity

Recall the definition of continuity at a point:

$f : D \to \mathbb{R}$ is **continuous** at $c \in D$ if, for any $\varepsilon > 0$,

$$\exists \delta = \delta(c, \varepsilon) : \forall x \in D, \; |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon,$$

and f is continuous on D if f is continuous for each $c \in D$.

Note that each δ here could (but does not necessarily) depend on the choice of c. (Also, I've added the $\forall x \in D$ for emphasis.)

Uniform continuity

Uniform continuity on a set does not depend on a point c, only on the set D.

Definition

$f : D \to \mathbb{R}$ is **uniformly continuous** on D if, for any $\varepsilon > 0$,

$$\exists \delta = \delta(D, \varepsilon) : \forall x, c \in D,\ |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon.$$

Clearly, f uniformly continuous on $D \implies f$ continuous on D.

Uniform continuity: Examples

Example

Define $f : (1, \infty) \to \mathbb{R}$ by $f(x) = 4\sqrt{x} + 2$.

Then f is uniformly continuous on $(1, \infty)$: let $\varepsilon > 0$.

Choosing $\delta = \frac{\varepsilon}{2}$, we get

$$
\begin{aligned}
|x - c| < \delta \implies |f(x) - f(c)| &= |4\sqrt{x} + 2 - 4\sqrt{c} - 2| \\
&= 4|\sqrt{x} - \sqrt{c}| \\
\boxed{\text{for } x > 1, c > 1,\ \sqrt{x} + \sqrt{c} > 2} \quad &= 4\frac{|x - c|}{|\sqrt{x} + \sqrt{c}|} \\
&< 2|x - c| < 2\delta = \varepsilon.
\end{aligned}
$$

Not uniformly continuous?

Let $f : D \to \mathbb{R}$. Then f is *not* uniformly continuous on D if the negation of the definition of uniform continuity is true, i.e.

$$
\begin{aligned}
&f \text{ is not uniformly continuous on } D \\
&\quad\iff \neg\big[\text{for any } \varepsilon > 0, \exists \delta = \delta(D,\varepsilon) : \\
&\qquad\quad \forall x, c \in D,\ |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon\big] \\
&\quad\iff \exists \varepsilon > 0 : \neg\big[\exists \delta = \delta(D,\varepsilon) : \\
&\qquad\quad \forall x, c \in D,\ |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon\big] \\
&\quad\iff \exists \varepsilon > 0 :\ \text{for any } \delta > 0, \\
&\qquad\quad \neg\big[\forall x, c \in D,\ |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon\big] \\
&\quad\iff \exists \varepsilon > 0 :\ \text{for any } \delta > 0, \\
&\qquad\quad \exists x, c \in D :\ |x - c| < \delta \text{ and } |f(x) - f(c)| \geq \varepsilon.
\end{aligned}
$$

Not uniformly continuous?

Example

Define $g : (0, \infty) \to \mathbb{R}$ by $g(x) = 4x^2 + 2$. Then g is continuous (since g is a polynomial), but not uniformly continuous, on $(0, \infty)$.

To prove g is *not* uniformly continuous on $(0, \infty)$, we can show that $\exists \varepsilon > 0$ such that, for any $\delta > 0$,

$$\exists x, c \in (0, \infty) : \ |x - c| < \delta \text{ and } |g(x) - g(c)| \geq \varepsilon.$$

Let $\varepsilon = 1$ (although any $\varepsilon > 0$ will do); we want $|x - c| < \delta$ and

$$|g(x) - g(c)| = 4|x^2 - c^2| = 4|x + c| \cdot |x - c| \geq \varepsilon = 1.$$

Not uniformly continuous?

The $|x + c|$ term requires we select one of the inputs specifically, and set the other a certain distance away.

Pick any $\delta > 0$, and set $x = \frac{1}{\delta}$ and $c = x + \frac{\delta}{2}$ so that $|x - c| = \frac{\delta}{2}$, and $|x + c| = \frac{2}{\delta} + \frac{\delta}{2}$. Then

$$\begin{aligned} |x - c| &< \delta \text{ and} \\ |g(x) - g(c)| &= 4|x + c| \cdot |x - c| \\ &= 4 \cdot \left(\frac{2}{\delta} + \frac{\delta}{2}\right) \cdot \frac{\delta}{2} \\ &\geq 4 \cdot \left(\frac{2}{\delta}\right) \cdot \frac{\delta}{2} = 4 > 1 = \varepsilon. \end{aligned}$$

Thus, g is not uniformly continuous[3] on $(0, \infty)$.

[3] Note that $(0, \infty)$ is not compact.

Uniform continuity properties

What extra properties give us uniform continuity from continuity?

Theorem

$f : D \to \mathbb{R}$ continuous, D compact $\implies$ f unif continuous on D.

Theorem

$f : D \to \mathbb{R}$ uniformly continuous, (x_n) is a Cauchy sequence in D $\implies$ $(f(x_n))$ is a Cauchy sequence in $f(D)$.

Note

This theorem does not hold for continuity on D alone: check $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$, with the sequence $(x_n) = (\frac{1}{n})$.

Uniform continuity preserves Cauchy sequence-ness

Theorem

$f : D \to \mathbb{R}$ uniformly continuous, (x_n) is a Cauchy sequence in D $\implies$ $(f(x_n))$ is a Cauchy sequence in $f(D)$.

Proof Pick $\varepsilon > 0$. Then $\exists \delta = \delta((a,b), \varepsilon) > 0$ such that

$$\forall x, y \in (a,b),\ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Since (x_n) is Cauchy, $\exists N \in \mathbb{N}$ such that

$$\forall m, n > N,\ |x_m - x_n| < \delta.$$

Thus, $(f(x_n))$ is Cauchy:

$$\forall m, n > N,\ |f(x_m) - f(x_n)| < \varepsilon. \blacksquare$$

Uniform continuity on (a,b) iff continuity on [a,b]

Definition

A function $\tilde{f} : E \to \mathbb{R}$ is called an **extension** of the function $f : D \to \mathbb{R}$ if $D \subseteq E$ and $\forall x \in D,\ f(x) = \tilde{f}(x)$.

Theorem

$f : (a, b) \to \mathbb{R}$ uniformly continuous on (a, b)
$\iff$ f can be extended to $\tilde{f} : [a, b] \to \mathbb{R}$ continuous on $[a, b]$.

Uniform continuity on (a,b) iff continuity on [a,b]

Proof ($\Longleftarrow$) Suppose $\tilde{f} : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$.

$[a, b]$ is compact $\implies$ $\tilde{f}$ is uniformly continuous on $[a, b]$

$\implies$ $\tilde{f}$ is uniformly continuous on (a, b).

But, for $x \in (a, b)$, $\tilde{f}(x) = f(x)$.

$\therefore$ f is uniformly continuous on (a, b).

Uniform continuity on (a,b) iff continuity on [a,b]

Proof ($\Longrightarrow$) Suppose $f : (a, b) \to \mathbb{R}$ is unif continuous on (a, b).

We need to show that

$$\lim_{x \to a+} f(x) = p \text{ and } \lim_{x \to b-} f(x) = q$$

exist. After this, we define $\tilde{f} : [a, b] \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} p & x = a \\ f(x) & a < x < b \\ q & x = b \end{cases}$$

and we are done.

Uniform continuity on (a,b) iff continuity on [a,b]

WLOG we will only show that $\lim_{x \to a+} f(x) = p$, and claim that a similar argument works for $\lim_{x \to b-} f(x) = q$.

Since sequence convergence $\iff$ Cauchy-ness,
if (s_n) is a sequence of points in (a, b) that converges to a,
then (s_n) is Cauchy.

Thus, by the previous theorem, since f is uniformly continuous on (a, b), we have that the sequence $(f(s_n))$ is also Cauchy.

Uniform continuity on (a,b) iff continuity on [a,b]

Since Cauchy sequences converge,

$$\lim_{n\to\infty} f(s_n) = p$$

for some $p \in \mathbb{R}$.

By the sequential criterion for limits (since a is an accumulation point of (a, b)), this is the $p \in \mathbb{R}$ we seek, as

$$\lim_{n\to\infty} s_n = a \implies \lim_{n\to\infty} f(s_n) = \lim_{x\to a} f(x). \blacksquare$$