Linear Algebra and Matrix Methods
Introduction to set notation, dimension

Introduction

The point of this course is to understand,

and understand how to solve,

systems of linear equations.

Set / Element

To understand the language of linear algebra, we start with the basic definitions of set notation.

Definition

A **set** is a collection of distinct objects, called the set's **elements**.

The elements:

- do not necessarily have an inherent order or relationship to each other (although they usually will);
- duplicates are not allowed;
- they are merely *different things in the same bag*.

Set / Element Notation

"The set A contains three elements:

'cat', 'tree', and the number 6."

This is denoted

$$A = \{\text{tree}, \text{cat}, 6\}$$

with curly brackets indicating the set.

This is an *explicit* definition of a particular set. It has 3 elements.

"6 is an element of A" is denoted $6 \in A$.

"'dog' is not an element of A" is denoted dog $\notin A$.

Set / Element Notation

"The set B consists of the even numbers *strictly* between 0 and 25" (i.e. *exclusive*) is an *implicit* definition of the set denoted

$$B = \{2, 4, 6, ..., 22, 24\}.$$

The *ellipsis* "..." means:

"you understand the pattern given by the context".

Set / Element Notation

Another way to write

$$B = \{2, 4, 6, ..., 22, 24\}$$

is

$$B = \{x \in \mathbb{Z} \,|\, 0 < x < 25,\ x \text{ even}\},$$

where the vertical bar $|$ means "such that".

(Sometimes a colon $:$ is used instead of the bar $|$.)

Popular Number Sets

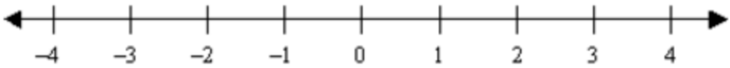
$\mathbb{N} = \{1, 2, 3, ...\}$ is the set of **natural numbers** or **counting numbers** (and sometimes contains 0).

$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ is the set of **integers** (written with a Z for the German word Zahlen ("numbers")).

$\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z},\ n \neq 0\}$ is the set of **rational numbers** (i.e. fractions, ratios; the Q stands for "quotient").

Real Number Sets

$\mathbb{R}$ is the set of **real numbers**, containing all rational and irrational numbers. This is the set of numbers along the continuous number line, containing all infinite-length decimal expansions.



An **open interval** of real numbers is denoted

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

A **closed interval** of real numbers is denoted

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

Empty Set

The **empty set** is the set with no elements (think: an empty bag).

It is denoted

$$\emptyset$$

and defined with curly brackets by

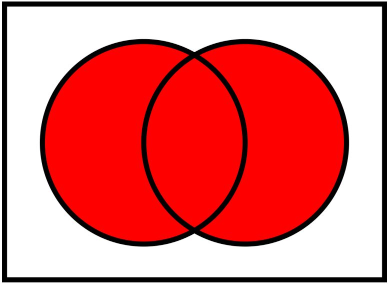
$$\emptyset = \{\}.$$

Union

Some basic operations we can use on sets are:

The **union** of the sets A and B is the set of all elements of A and B combined. It is denoted $A \cup B$, and defined by

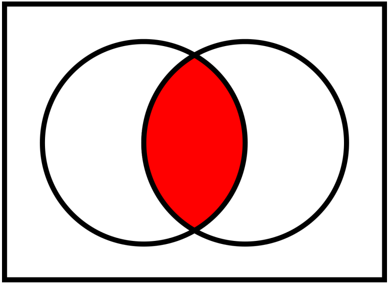
$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ (or both)}\}.$$



Intersection

The **intersection** of the sets A and B is the shared elements of A and B. It is denoted $A \cap B$ or AB, and defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

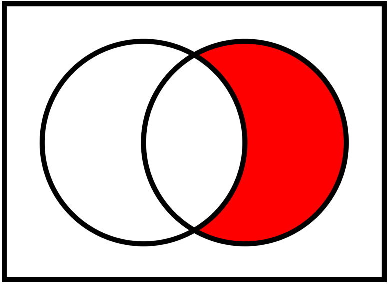


If A and B have no common elements, we say A and B are **disjoint** sets and denote this fact by $A \cap B = \emptyset$.

Set Difference

The **set difference** $B \setminus A$ (sometimes denoted $B - A$) is the elements of B with the shared elements of A removed. It is denoted

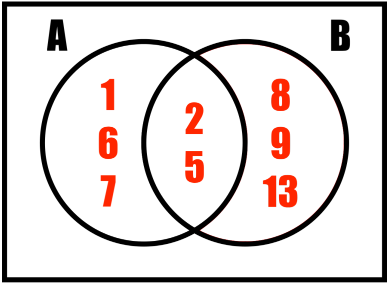
$$B \setminus A = \{x : x \in B \text{ and } x \notin A\}.$$



Set Difference

$$A = \{1, 2, 5, 6, 7\}, \quad B = \{2, 5, 8, 9, 13\}$$

$$\Longrightarrow \; A \setminus B = \{1, 6, 7\} \text{ but } B \setminus A = \{8, 9, 13\}.$$

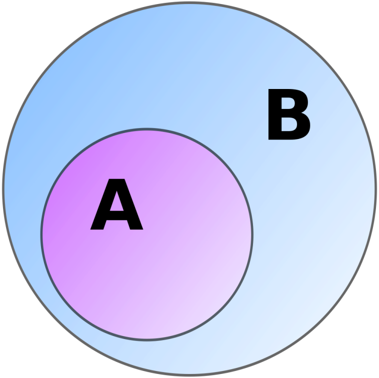


Thus, $A \setminus B \neq B \setminus A$ (an important general result).

Subset

The set A is called a **subset** of B, denoted $A \subseteq B$, if all of A's elements are in B.

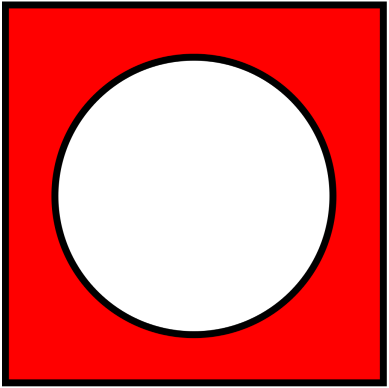
$$A \subseteq B \iff (x \in A \implies x \in B).$$



Two sets A and B are **equal** (written $A = B$) if $A \subseteq B$ and $B \subseteq A$.

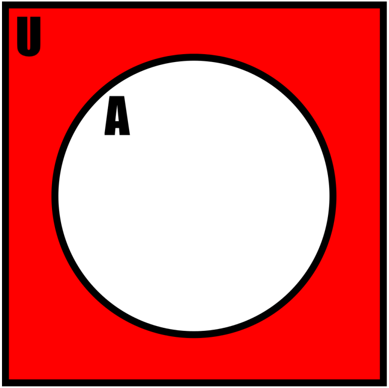
Universal Set, Complement

In certain collections of problems, we may define a **universal set** as a common "top-level set" under which all the sets described in the problem are subsets of.



Universal Set, Complement

The **complement** of a set A, relative to the universal set U, is denoted $A^C = U \setminus A$. (Some texts use A' or $\overline{A}$.)



Using this, we can define the set difference as $A \setminus B = A \cap B^C$.

Example

Considering $\mathbb{R}$ as the universal set, and let

$$A = (1, \infty) = \{x \in \mathbb{R} : x > 1\}.$$

Then

$$A^C = \mathbb{R} \setminus A = (-\infty, 1] = \{x \in \mathbb{R} : x \leq 1\}.$$

Unions, Intersections, Complements

Union and intersection are *commutative* operations:

$$A \cup B = B \cup A, \; A \cap B = B \cap A.$$

We've already seen that set difference is *not* commutative:

$$A \setminus B \neq B \setminus A.$$

Unions, Intersections, Complements

Union and intersection are also *associative*:

$$(A \cup B) \cup C = A \cup (B \cup C), \ (A \cap B) \cap C = A \cap (B \cap C).$$

Regardless of universal set, it should be clear that the complement of a complement is the original set:

$$(A^C)^C = A.$$

Tuples, Dimension

An **ordered pair** of real numbers, often written (x, y), is an element of "the two-dimensional plane", called $\mathbb{R}^2$:

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}.$$

An **ordered triple** of real numbers, often written (x, y, z), is an element of "three-dimensional space", called $\mathbb{R}^3$:

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}.$$

In general, an **ordered *n*-tuple** of real numbers, often written $(x_1, x_2, ..., x_n)$, is an element of "*n*-dimensional space", called $\mathbb{R}^n$:

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) : x_1, x_2, ..., x_n \in \mathbb{R}\}.$$

Tuples, Dimension

The number n in this context is called the **dimension** of the space we are considering. This n describes the number of **coordinates**, or separate ordered values, in the tuple.

Later, we will see a more general definition of the terms **space** and **dimension**, and how they are related.

Mathematical Space

A mathematical set with some extra structure on it, like a mathematical operation such as addition or multiplication, is called a **space**.

For example, the real numbers $\mathbb{R} = (-\infty, \infty)$ is a set, but the real numbers *with addition* $(\mathbb{R}, +)$ is a space.

Mathematical Space, Subspace

A **subspace** of a space is a subset of a space's set that still lets you use the operation(s) available to the original space.

For example, the integers $\mathbb{Z}$ is a subset of $\mathbb{R}$, but the integers *with addition* $(\mathbb{Z}, +)$ is a subspace of $(\mathbb{R}, +)$ because we can still "do addition" with just integers. ($+$ is called "closed" under $\mathbb{Z}$.)

The main objects we will examine in this course are called **vectors**. They exist in **vector spaces**.