Introduction to Analysis: Differentiation

Difference Quotient of a Real-Valued Function

Definition

Let $f : (a, b) \to \mathbb{R}$ be a function, and $c \in (a, b)$.

The **difference quotient** $D_f(x, c)$ of f between x and c is defined as the ratio

$$D_f(x, c) = \frac{f(x) - f(c)}{x - c}$$

for any $x \in (a, b)$, $x \neq c$.

Difference Quotient of a Real-Valued Function

The difference quotient is the slope of the secant line between the two points $(x, f(x))$ and $(c, f(c))$ on the graph of f.

It represents an *average rate of change* of the function f between the points x and c.

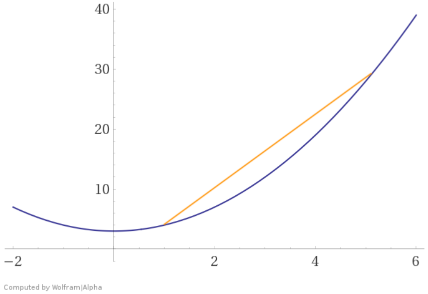

Figure: $f(x) = x^2 + 1$, with secant line on f from $x = 1$ to $x = 5$

Derivative of a Real-Valued Function

Definition

Let $f : (a, b) \to \mathbb{R}$ be a function, and $c \in (a, b)$.

The limit, if it exists, of the difference quotient $D_f(x, c)$ as $x \to c$ is called the **derivative** of f at c:

$$f'(c) = \lim_{x \to c} D_f(x, c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Derivative of a Real-Valued Function

The derivative $f'(c)$ is the slope of the tangent line to the curve f at the point $(c, f(c))$, representing the *instantaneous rate of change* of the function f at the point c.

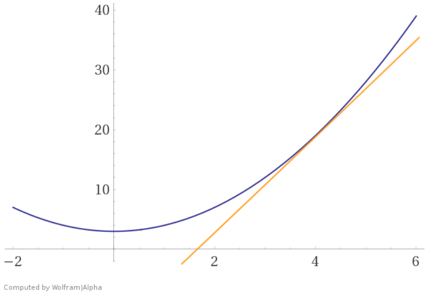

Figure: $f(x) = x^2 + 1$, with tangent line on f at $x = 4$

Derivative of a Real-Valued Function

We may also be familiar with this definition written in the form

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

where $y = f(x)$, and possibly using $h = \Delta x$.

Derivative of a Real-Valued Function

Other notations for the derivative include

$$\frac{dy}{dx} = \frac{d}{dx}f(x) = y' = f'(x).$$

If $f'(c)$ exists for $c \in (a, b)$, we say f is **differentiable** at c.

If $f'(c)$ exists $\forall c \in S \subseteq (a, b)$, we say f is **differentiable** on S.

Derivative of a Real-Valued Function

Note that the derivative is a limit, meaning there are two one-sided limits that must match: the **one-sided derivatives** of f at c are

$$f'_-(x) = \lim_{\Delta x \to 0-} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$
$$f'_+(x) = \lim_{\Delta x \to 0+} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Thus, for $f'(x)$ to exist, we require $f'_-(x) = f'_+(x)$.

In the case of endpoints of an interval, say,

$$f : [a, b] \to \mathbb{R},$$

we only consider the one available direction, i.e. $f'_+(a)$ and $f'_-(b)$.

Example: Basic Derivative

Example

A standard first derivative example is the derivative of a parabola. Let $f(x) = x^2 + 3x - 6$. Then

$$\begin{aligned} D_f(x,c) = \frac{f(x) - f(c)}{x - c} &= \frac{x^2 + 3x - 6 - c^2 - 3c + 6}{x - c} \\ &= \frac{x^2 - c^2 + 3(x - c)}{x - c} = x + c + 3 \end{aligned}$$

for $x \neq c$, which implies

$$f'(c) = \lim_{x \to c} (x + c + 3) = 2c + 3.$$

Sequential Criterion for Differentiability

Theorem

Let $f : (a, b) \to \mathbb{R}$ be a function, and $c \in (a, b)$.

Then f is differentiable at $c \iff$ for every sequence (x_n) in (a, b) such that $x_n \to c$, we have

$$\lim_{n\to\infty} D_f(x_n, c) = \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} = L.$$

Then $L = f'(c)$.

This theorem is particularly useful in the contrapositive;
to show that f does not have a derivative at c, show

$\exists$ a sequence (x_n) converging to c : $D_f(x_n, c)$ does not have a limit.

Example: Absolute Value

Example

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

does not have a derivative at $x = 0$:

$$\begin{aligned} f'_{-}(0) &= \lim_{\Delta x \to 0-} \frac{f(0+\Delta x) - f(0)}{\Delta x} = -1 \\ f'_{+}(0) &= \lim_{\Delta x \to 0+} \frac{f(0+\Delta x) - f(0)}{\Delta x} = +1 \end{aligned}$$

$f'_{-}(0) \neq f'_{+}(0)$. $\quad \therefore f'(0)$ DNE.

Example: Absolute Value

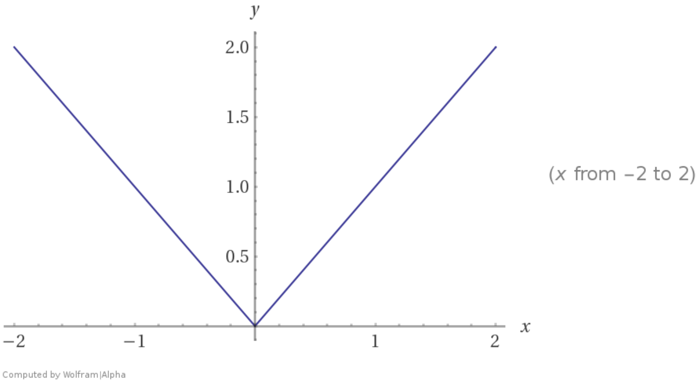

Figure: $f(x) = |x|$

f differentiable $\implies$ f continuous

Our next theorem relates differentiability and continuity:

Theorem

A function f is differentiable at $c \implies f$ is continuous at c.

f differentiable $\implies$ f continuous

Proof If $x \neq c$,

$$\begin{aligned} f(x) &= f(c) + f(x) - f(c) \\ &= f(c) + \frac{f(x) - f(c)}{x - c}(x - c) = f(c) + D_f(x, c)(x - c). \end{aligned}$$

Thus, since f is differentiable at c,

$$\begin{aligned} \lim_{x \to c} f(x) &= \lim_{x \to c} [f(c) + D_f(x, c)(x - c)] \\ &= f(c) + \cancel{f'(c)(c - c)} = f(c). \blacksquare \end{aligned}$$

Basic Derivative Rules

Consider f and g differentiable on (a, b), and let $c_1, c_2 \in \mathbb{R}$.

Recall some basic differentiation rules:

- Differentiation is a **linear operation**:

$$\frac{d}{dx}\left(c_1 f(x) + c_2 g(x)\right) = c_1 f'(x) + c_2 g'(x).$$

- **Product Rule**:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x).$$

Basic Derivative Rules

- **Chain Rule (Composition)**: If $(g \circ f)(x) = g(f(x))$,
$$\frac{d}{dx}\left(g(f(x))\right) = g'(f(x))f'(x).$$
- **Quotient Rule**: if $g(x) \neq 0$,
$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Example: Power Rule ($\mathbb{N}$)

Proposition

Let $n \in \mathbb{N}$. Then the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$:

Proof

$$D_f(x,c) = \frac{x^n - c^n}{x - c} = \frac{(x-c)\left(\sum_{k=0}^{n-1} x^k c^{n-1-k}\right)}{x - c} = \sum_{k=0}^{n-1} x^k c^{n-1-k}$$

$$\implies f'(c) = \lim_{x \to c} \sum_{k=0}^{n-1} x^k c^{n-1-k} = \sum_{k=0}^{n-1} \left(\lim_{x \to c} x^k\right) c^{n-1-k}$$

$$= \sum_{k=0}^{n-1} c^k c^{n-1-k} = \sum_{k=0}^{n-1} c^{n-1} = nc^{n-1}. \blacksquare$$

(Another proof uses induction and the product rule.)

Chain Rule

Theorem

(Chain Rule) *Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be differentiable, and let the image $f(I) \subseteq J$. Then the composite function $g \circ f : I \to \mathbb{R}$,*

$$(g \circ f)(x) = g(f(x)),$$

is differentiable, and

$$\frac{d}{dx}[g(f(x))] = g'(f(x))f'(x).$$

You may be familiar with the Leibniz notation for the chain rule:

$$y = g(u),\ u = f(x) \implies \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Chain Rule

Proof First, we'll construct the difference quotient $D_{g \circ f}$:

$$D_{g \circ f}(x, c) = \frac{g(f(x)) - g(f(c))}{x - c}.$$

Being careful to avoid a zero denominator, for $x \neq c$ such that $f(x) \neq f(c)$, we say

$$D_{g \circ f}(x, c) = \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}.$$

Chain Rule

Since we need to take a limit as $x \to c$, we cannot necessarily avoid the situation $f(x) = f(c)$, so we notice that

$$\lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} = g'(f(c)).$$

We define the function

$$h(y) = \begin{cases} \frac{g(y)-g(f(c))}{y-f(c)} = D_g(y, f(c)) & y \neq f(c) \\ g'(f(c)) & y = f(c), \end{cases}$$

and see that h is continuous at $y = f(c)$ (by definition!).

Chain Rule

f is continuous at c, so since composition of continuous functions is continuous,

$$\lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} = g'(f(c)) = h(f(c)) = \lim_{x \to c} h(f(x)).$$

Next, note that if $x \in I$, then $y = f(x) \in J$. Since

$$g(y) - g(f(c)) = h(y) \cdot (y - f(c))$$

for any $y \in J$, then for all $x \in I$, we have

$$g(f(x)) - g(f(c)) = h(f(x)) \cdot (f(x) - f(c)).$$

Chain Rule

Hence, for $x \in I$, $x \neq c$, we have

$$\frac{g(f(x)) - g(f(c))}{x - c} = h(f(x)) \cdot \frac{f(x) - f(c)}{x - c} = h(f(x)) \cdot D_f(x, c).$$

All this rearranging gets us the limit

$$\begin{aligned}(g \circ f)'(x) &= \lim_{x \to c} D_{g \circ f}(x, c) \\ &= \lim_{x \to c} h(f(x)) \cdot D_f(x, c) = g'(f(c)) \cdot f'(c). \blacksquare\end{aligned}$$

Relative/Absolute Extrema

Definition

Let $f : D \to \mathbb{R}$ be a function, and $c \in D$.

We say f has a **relative (local) maximum** at c if

$$\exists h > 0 \ \exists N(c,h) : \ \forall x \in N(c,h), \ f(x) \leq f(c).$$

f has a **relative (local) minimum** at c if

$$\exists h > 0 \ \exists N(c,h) : \ \forall x \in N(c,h), \ f(x) \geq f(c).$$

Relative/Absolute Extrema

The relative (local) maxima and minima of f are called its

relative (local) extrema.

Replacing the neighborhood N with the full set D results in the

absolute (global) extrema.

of f.

Relative Extremum & Differentiable $\implies$ Derivative Zero

Theorem

If $f : (a, b) \to \mathbb{R}$ has a (relative) extremum at $c \in (a, b)$ and f is differentiable, then $f'(c) = 0$.

Proof WLOG say $f(c)$ is a relative maximum. Then

$$\exists \delta > 0 : \ |x - c| < \delta \implies f(x) \leq f(c).$$

Relative Extremum & Differentiable $\implies$ Derivative Zero

Hence, the one-sided derivatives of f at c are such that

$$(x < c) \quad f'_-(c) = \lim_{x \to c-} D_f(x, c) = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c} \geq 0$$

$$(x > c) \quad f'_+(c) = \lim_{x \to c+} D_f(x, c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c} \leq 0.$$

If f is differentiable at c, then

$$f'(c) = f'_-(c) = f'_+(c) \implies f'(c) = 0. \blacksquare$$

(Another proof uses sequences.)

Relative Extremum $\not\Longrightarrow$ Differentiable

f having a relative extremum at c does not guarantee that f is differentiable at c.

It may be the case that $f'(c)$ does not exist.

Example

$f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = |x - 4|$$

is defined for all $x \in \mathbb{R}$, and has a relative minimum at $x = 4$, but f is not differentiable at $x = 4$:

$$f'(x) = \begin{cases} 1 & x > 4 \\ -1 & x < 4 \\ DNE & x = 4 \end{cases}$$

Derivative Zero $\not\Longrightarrow$ Relative Extremum

It is also not true that $f'(c) = 0$ for some $c \in (a, b)$ means f has a relative extremum at c.

Example

$f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = (x + 6)^3$$

is defined for all $x \in \mathbb{R}$, and has $f(-6) = 0$ and $f'(-6) = 0$:

$$f'(x) = 3(x + 6)^2.$$

However, in any neighborhood of -6, i.e. for any $\delta > 0$,

$$f(-6 - \delta) = -\delta^3 < 0, \; f(-6 + \delta) = \delta^3 > 0.$$

$f'(c) = 0$ here is a *necessary*, but not *sufficient*, condition.

Rolle's Theorem

Theorem

(Rolle's Theorem) *If* $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$ *(one-sided at endpoints) and differentiable on* (a, b)*, then*

$$f(a) = f(b) \implies \exists c \in (a, b) : \; f'(c) = 0.$$

Proof If $\forall x \in [a, b]$, $f(x) = k$ for some constant k, this is obvious.

Thus, WLOG assume $\exists x_1, x_2 \in (a, b)$: $f(x_1) < f(x_2)$.

Rolle's Theorem

f is continuous on the closed interval $[a, b]$, and so by the EVT, attains its absolute maximum M and minimum m.

Since $f(a) = f(b)$, and we assume $f(x) > f(a)$ for some $x \in (a, b)$.

Since f is differentiable on (a, b), and

$$\exists c \in (a, b): \; f(c) = M, \text{ a maximum,}$$

then by the previous theorem, $f'(c) = 0$. ■

Mean Value Theorem For Derivatives (Law of the Mean)

Theorem

(Mean Value Theorem) *If* $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$ *and is differentiable on* (a, b), *then*

$$\exists c \in (a, b) : \ \frac{f(b) - f(a)}{b - a} = f'(c).$$

Mean Value Theorem For Derivatives (Law of the Mean)

We interpret this fact geometrically as saying:

Any function differentiable on a closed interval $[a, b]$

contains a point whose

derivative (tangent line slope, instantaneous rate of change)

is equal to the

average rate of change over the interval,

i.e. the secant line slope between the endpoints of the interval.

Mean Value Theorem For Derivatives (Law of the Mean)

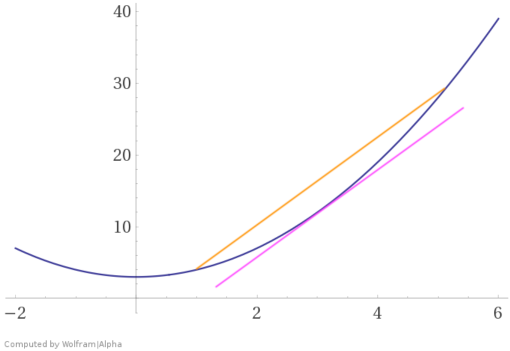

Figure: $f(x) = x^2 + 1$, with secant line between $x = 1$ and $x = 5$, and a parallel tangent line

Proof of the Mean Value Theorem For Derivatives

Proof of MVT: Define

$$h(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a).$$

Since h is continuous on $[a, b]$ and differentiable on (a, b), and

$$h(a) = h(b) = 0,$$

then by Rolle's Theorem

$$\exists c \in (a, b) : \; h'(c) = 0.$$

But

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Therefore,

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \blacksquare$$

Corollaries of the Mean Value Theorem For Derivatives

Two corollaries of the MVT tell us, without needing integrals, that

1. if $f'(x) = 0$, then f is a constant;
2. if $f'(x) = g'(x)$, then $f = g$ up to a constant.

Corollary

If $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b), and

$$\forall x \in (a, b),\ f'(x) = 0,$$

then $\exists c \in \mathbb{R}$ such that $f(x) = c$ for all $x \in [a, b]$ (i.e. f is constant on $[a, b]$).

Proof For any $a \leq x_1 < x_2 \leq b$, the MVT hypotheses are satisfied, and so $\exists w \in (x_1, x_2)$:

$$0 = f'(w) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \implies f(x_1) = f(x_2).$$

But x_1, x_2 were chosen arbitrarily. $\therefore$ f is constant $\forall a \leq x \leq b$. $\blacksquare$

Corollary

If $f, g : [a, b] \to \mathbb{R}$ is differentiable on (a, b), and

$$\forall x \in (a, b),\ f'(x) = g'(x),$$

then

$$\exists C \in \mathbb{R} :\ \forall x \in [a, b],\ f(x) = g(x) + C.$$

Strictly, Monotone Increasing/Decreasing

A function f is **strictly (monotone) increasing** on an interval I if

$$x_1 < x_2 \implies f(x_1) < f(x_2), \ (\leq)$$

and **strictly (monotone) decreasing** on I if

$$x_1 < x_2 \implies f(x_1) > f(x_2), \ (\geq)$$

Strictly, Monotone Increasing/Decreasing

Theorem

Let f be differentiable on an interval I. Then:
$f'(x) > 0$ *for all* $x \in I \implies f$ *is strictly increasing on* I*.*
$f'(x) < 0$ *for all* $x \in I \implies f$ *is strictly decreasing on* I*.*

Proof By the MVT, $\exists c \in (x_1, x_2) \subseteq I$ such that $x_1 < x_2$ and

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Thus, $f(x_2) - f(x_1)$ and $f'(c)$ have the same sign. ■

Intermediate Value Theorem (IVT) For Derivatives

Theorem

Intermediate Value Theorem (IVT) *Suppose f is differentiable on $[a, b]$, and that k is between $f'(a) \neq f'(b)$. Then*

$$\exists c \in (a, b) : \ f'(c) = k.$$

Proof WLOG $f'(a) < k < f'(b)$.

Let $g(x) = f(x) - kx$. Then g is differentiable on $[a, b]$ and

$$g'(a) < 0 < g'(b).$$

Intermediate Value Theorem (IVT) For Derivatives

g is continuous on $[a, b]$, which is compact, so g assumes its minimum at some point $c \in [a, b]$. We need to show $c \in (a, b)$.

Note that

$$g'(b) = \lim_{x \to b} \frac{g(x) - g(b)}{x - b} = \lim_{x \to b} \frac{f(x) - f(b) - k(x - b)}{x - b}$$

$$= f'(b) - k > 0,$$

so there is a deleted neighborhood U near b such that

$$\forall x \in U \cap [a, b], \ \frac{g(x) - g(b)}{x - b} > 0.$$

Intermediate Value Theorem (IVT) For Derivatives

Thus, for $x \in U \cap [a, b]$, if $x < b$, we must have $g(x) < g(b)$.

Hence, $g(b)$ is not the minimum of g on $[a, b]$, so $c \neq b$.

Likewise, since $g'(a) < 0$, a similar argument shows that $c \neq a$.

Therefore $c \in (a, b)$, and so by Rolle's Theorem, $g'(c) = 0$.

Hence,

$$f'(c) = g'(c) + k = k. \blacksquare$$

Inverse Function Theorem

Theorem
(Inverse Function Theorem, Derivatives)

Suppose that f is differentiable on an interval I and

$$\forall x \in I,\ f'(x) \neq 0.$$

Then f is injective, f^{-1} is differentiable on $f(I)$, and

$$y = f(x) \implies (f^{-1})'(y) = \frac{1}{f'(x)}.$$

Proof of Inverse Function Theorem

Proof $\forall x \in I$, $f'(x) \neq 0$, so $f'(x)$ must be the same sign $\forall x \in I$.

WLOG say $f'(x) > 0$.

Thus, f is strictly increasing. Hence, f is injective (and so the inverse function f^{-1} exists).

Proof of Inverse Function Theorem

Next, we show f^{-1} is differentiable on $f(I)$.

Let $y \in f(I)$ and let (y_n) be any sequence in $f(I) \setminus \{y\}$ such that

$$y_n \to y.$$

Then

$$x = f^{-1}(y)$$

and, since f^{-1} is continuous and injective,

$$(x_n = f^{-1}(y_n))$$

is a sequence in $I \setminus \{x\}$ converging to x.

Proof of Inverse Function Theorem

$f'(x) \neq 0$ for all $x \in (a, b)$, so

$$\lim_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - x} = f'(x) \neq 0.$$

Taking reciprocals and translating x values into y values,

$$\frac{1}{f'(x)} = \lim_{n\to\infty} \frac{x_n - x}{f(x_n) - f(x)}$$

$$= \lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y)}{y_n - y} = (f^{-1})'(y). \blacksquare$$

Indeterminate Forms

Definition

Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are functions, and $c \in D$. If

$$\lim_{x \to c} f(x) = L \text{ and } \lim_{x \to c} g(x) = M,$$

with $L = M = 0$, then the limit

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

is called an **indeterminate form**.

A shorthand notation for this form is "$\frac{0}{0}$".

Indeterminate Forms

There are other indeterminate forms, such as

- $0 \cdot \infty$,
- $\frac{\infty}{\infty}$,
- ∞^0,
- 0^0,
- 1^∞,
- $\infty - \infty$,

that can be rewritten into $\frac{0}{0}$ form.

L'Hôpital's Rule is a calculus-based way to evaluate such forms.

Cauchy MVT (Generalized Law of the Mean)

Theorem

(Cauchy Mean Value Theorem)

Suppose f and g are continuous on $[a, b]$ and differentiable on (a, b). Then $\exists c \in (a, b)$ such that

$$[f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c).$$

Note

If $g(x) = x$, then this reduces to the usual MVT:

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Cauchy MVT (Generalized Law of the Mean)

Proof Let

$$h(x) = [f(b) - f(a)]\, g(x) - [g(b) - g(a)]\, f(x).$$

Then h is continuous on $[a, b]$ and differentiable on (a, b), and

$$h(a) = f(b)g(a) - g(b)f(a) = h(b).$$

By the MVT, $\exists c \in (a, b)$ such that $h'(c) = 0$. Hence,

$$[f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c). \blacksquare$$

Theorem
(L'Hôpital's Rule)

Suppose f, g are continuous on $[a, b]$ and differentiable on (a, b).

Suppose $c \in [a, b]$ such that $f(c) = g(c) = 0$, and that

$$\exists \delta > 0 : \ \forall x \in U = (a, b) \cap N^*(c, \delta), \ g'(x) \neq 0.$$

Then

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \implies \lim_{x \to c} \frac{f(x)}{g(x)} = L.$$

L'Hôpital's Rule

Proof Let (x_n) be a sequence in U such that $x_n \to c$ and $x_n \neq c$.

For each $n \in \mathbb{N}$, let

$$a_n = \min(x_n, c) \text{ and } b_n = \max(x_n, c).$$

Then, by the Cauchy MVT on f and g on the intervals $[a_n, b_n]$, there exists sequence (c_n) with $a_n < c_n < b_n$ such that

$$[f(x_n) - f(c)]\, g'(c_n) = [g(x_n) - g(c)]\, f'(c_n).$$

$g'(x) \neq 0$ for all $x \in U$, and $g(c) = 0$, so by the contrapositive of Rolle's Theorem, we have $g(x_n) \neq 0$ for all n.

Since $f(c) = g(c) = 0$, we have $\forall n \in \mathbb{N}$,

$$f(x_n)g'(c_n) = g(x_n)f'(c_n) \implies \frac{f'(c_n)}{g'(c_n)} = \frac{f(x_n)}{g(x_n)}.$$

$x_n \to c$ and c_n is always between x_n and c, so $c_n \to c$.[1]

Thus, by the sequential criterion for limits,

$$\lim_{n\to\infty} \frac{f'(c_n)}{g'(c_n)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = L \implies \lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \lim_{x\to c} \frac{f(x)}{g(x)} = L. \blacksquare$$

[1]This is true by the *squeeze*, or *sandwich theorem*: if $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, then $\lim_{n\to\infty} c_n = L$.

Limits of Functions of a Real Variable at Infinity

Assume $\exists a \in \mathbb{R}$ such that $(a, \infty) \subseteq D$, and suppose $f : D \to \mathbb{R}$.

The **long-term** or **asymptotic** behavior of f, or, simply, the **limit** of f as $x \to \infty$, is denoted by

$$\lim_{x \to \infty} f(x)$$

(with obvious similar definition for $x \to -\infty$).

Limits of Functions of a Real Variable at Infinity

If the limit exists, i.e. $\exists L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0,\ \exists M > 0 :\ x > M \implies |f(x) - L| < \varepsilon,$$

then we say

$$\lim_{x \to \infty} f(x) = L$$

and that f has a **horizontal asymptote** at $y = L$.

Otherwise, if f increases without bound, we say f **tends to** ∞, and write

$$\lim_{x \to \infty} f(x) = \infty$$

if

$$\forall \alpha \in \mathbb{R}, \exists N = N(\alpha) : \; x > N \implies f(x) > \alpha.$$

(We say f tends to $-\infty$ if f decreases without bound).

L'Hôpital's Rule (limit at ∞)

Theorem

(L'Hôpital's Rule, limit at ∞)

Suppose f and g are differentiable on (a, ∞). Suppose also that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty,$$

and that $g'(x) \neq 0$ for $x \in (a, \infty)$. Then

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \implies \lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

Higher-Order Derivatives

If f' is differentiable at c, then we call the **second derivative** of f at c the first derivative of f' at c, denoted

$$f''(c) = (f')'(c).$$

If $f''(c)$ exists, we say f is **twice differentiable** at c.

In general, if a function f can be differentiated n times on a domain D (for $n \in \mathbb{N}$), we say f has nth derivative

$$f^{(n)}(x) = (f^{(n-1)})'(x),$$

and if $f^{(n)}$ is continuous on D, we say that $f \in C^n(D)$.[2]

[2]We'll use the "zeroth derivative" convention $f^{(0)}(x) = f(x)$.

Taylor's Theorem (Lagrange)

Taylor's Theorem generalizes the MVT for nth derivatives.

Theorem

(Taylor's Theorem)

Let $f \in C^{n+1}([a,b])$, and let $x_0 \in [a,b]$.

Then, for each $x \in [a,b]$ with $x \neq x_0$, $\exists c$ between x and x_0:

$$\begin{aligned} f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}, \end{aligned}$$

where $n! = n(n-1)\cdots 2 \cdot 1$ for $n \in \mathbb{N}$ ("n **factorial**").[3]

[3] By convention, we define $0! = 1$.

Taylor Polynomial, Remainder

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

has nth degree polynomial approximation

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

called the **Taylor polynomial** of order n of f at x_0. Its error from $f(x)$ is the **remainder** term, for some $c \in (a, b)$, of

$$R_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

If $f \in C^\infty([a,b])$, i.e. f is continuously differentiable infinitely many times, or **smooth**, on $[a,b]$, then the infinite sum produced by this method is called the **Taylor series** of f at $x_0 \in [a,b]$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Taylor's Theorem (Lagrange)

Proof of Taylor's Theorem:

Fix $x \in [a, b]$ with $x \neq x_0$ and let M be the unique solution of

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &\quad + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + M(x - x_0)^{n+1}. \end{aligned} \tag{1}$$

(This equation has a unique solution for M because, with $x \neq x_0$, and x and x_0 fixed, this is a linear equation in M.)

We will prove that

$$M = \frac{f^{(n+1)}(c)}{(n+1)!}$$

for some c between x and x_0.

Taylor's Theorem (Lagrange)

Define

$$F(t) = f(t) + f'(t)(x-t) + ... + \frac{f^{(n)}(t)}{n!}(x-t)^n + M(x-t)^{n+1}.$$

F is a polynomial in t, and so is continuous on $[a, b]$ and differentiable on (a, b).

Since $a \leq x \leq b$ and $a \leq x_0 \leq b$, these results are also true for the interval between x and x_0.

Taylor's Theorem (Lagrange)

Next, note that

$$F(x) = f(x)$$

since all but the first term cancel out, and

$$F(x_0) = f(x)$$

since M is the solution to (1).

Thus, by the MVT, $\exists c$ between x and x_0 such that

$$F'(c) = \frac{F(x) - F(x_0)}{x - x_0} = 0.$$

Taylor's Theorem (Lagrange)

Taking the derivative of $F(t)$, we see a telescoping sum (thanks to the product rule of differentiation):

$$
\begin{aligned}
F'(t) &= \left(f(t) + f'(t)(x-t) + \ldots + \frac{f^{(n)}(t)}{n!}(x-t)^n + M(x-t)^{n+1} \right)' \\
&= f'(t) + f''(t)(x-t) - f'(t) + \ldots \\
&\quad + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n-1)}(t)}{(n-1)!}(-(n-1)(x-t)^{n-2}) \\
&\quad + \frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{f^{(n)}(t)}{n!}(-n(x-t)^{n-1}) \\
&\quad - M(n+1)(x-t)^n \\
&= \left[\frac{f^{(n+1)}(t)}{n!} - M(n+1) \right] (x-t)^n.
\end{aligned}
$$

Taylor's Theorem (Lagrange)

Therefore,

$$F'(t) = \left[\frac{f^{(n+1)}(t)}{n!} - M(n+1) \right] (x-t)^n$$

and $F'(c) = 0$ implies

$$M = \frac{f^{(n+1)}(c)}{(n+1)!}. \quad \blacksquare$$

A Taylor series is a type of a **power series**, which is an infinite-series representation of a function, in the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

A power series of a function is a Taylor series around $x_0 = 0$, with coefficients

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Taylor Series vs Power Series vs Generating Function

All polynomials with degree N, for example, are power series such that $a_m = 0$ for all $m > N$ and $a_N \neq 0$.

A power series is a **generating function** that is also an actual function (not just a formal series).

Some functions have infinitely many nonzero derivatives.

Taylor Series vs Power Series Example

For example, consider the function

$$f(x) = 3x^2 - 5x + 7.$$

This function is a polynomial function, written in power series form:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n; \quad a_0 = 7,\ a_1 = -5,\ a_2 = 3,\ a_n = 0\ \forall n > 2.$$

What is the Taylor series of f at $x_0 = 2$?

Taylor Series vs Power Series Example

The Taylor series of f centered at $x_0 = 2$ would have form

$$f(x) = \frac{f(2)}{0!}(x-2)^0 + \frac{f'(2)}{1!}(x-2)^1 + \frac{f''(2)}{2!}(x-2)^2 + \frac{f''(3)}{3!}(x-2)^3 + \ldots$$

First, we can easily compute the derivatives of f:

$$f'(x) = 6x - 5,\ f''(x) = 6,\ f^{(n)}(x) = 0\ \forall n > 2.$$

Taylor Series vs Power Series Example

Thus, we can limit ourselves to just the first three terms

$$f(x) = \frac{f(2)}{0!}(x-2)^0 + \frac{f'(2)}{1!}(x-2)^1 + \frac{f''(2)}{2!}(x-2)^2.$$

The values of derivatives of f at $x_0 = 2$ are

$$f(2) = 9,\ f'(2) = 7,\ f''(2) = 6.$$

Thus, the Taylor series can be written

$$f(x) = 9 + 7(x-2) + 3(x-2)^2.$$

Simplifying this expression returns us to the Taylor series form

$$f(x) = 7 - 5x + 3x^2.$$

Some Well-Known Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{(geometric series; } |x| < 1\text{)}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (x \in \mathbb{R})$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad (x \in \mathbb{R})$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad (x \in \mathbb{R})$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \qquad (x \in (-1, 1])$$

Taylor Approximations

The **Taylor polynomials** of f at x_0,

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

can be used to approximate $f(x)$ close to x_0.

We will revisit series in a later section.