Introduction to Analysis: Infinite Series of Real Numbers and Real-Valued Functions

Series Notation (reminder)

Let (a_n) be a sequence of real numbers. If $m \leq n$, we use the sigma (summing) notation

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_{n-1} + a_n$$

to denote the sum shown.

If we wish to ignore / suppress the indexing of the sum, we can more simply write

$$\sum a_k$$

to denote a sum indexed over the variable k.

Partial Sums, Convergence of Partial Sums

Using (a_n), we can define a new sequence (s_n) of **partial sums**, defined by

$$s_n = \sum_{k=1}^{n} a_k.$$

If the sequence of partial sums has limit

$$\lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} a_n = s,$$

we call the series **convergent** to the **sum** s.

Otherwise, the sum **diverges**, to $+\infty$ or $-\infty$ if the sum grows without bound, or **oscillates**, or otherwise has no value.

Example: Riemann-Zeta

The **harmonic series** diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

In general, the **Riemann-zeta**, or **Riemann-ζ**, function,

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} = \sum_{n=1}^{\infty} n^{-x},$$

diverges when $x \leq 1$ and converges when $x > 1$.

Summing is a Linear Operation; Series Convergence

Theorem

(linearity of sums)

Suppose $\sum a_n = s$ *and* $\sum b_n = t$*. Then, if* $c_1, c_2 \in \mathbb{R}$*,*

$$\sum (c_1 a_n + c_2 b_n) = c_1 \sum a_n + c_2 \sum b_n = c_1 s + c_2 t.$$

Theorem

(Series convergence implies terms shrink)

If $\sum a_n$ *converges, then* $\lim a_n = 0$*.*

Cauchy criterion for series

Theorem

(Cauchy criterion for series)

$\sum a_n$ converges $\iff$ for each $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq m \geq N \implies \left| \sum_{k=m}^{n} a_k \right| < \varepsilon.$$

Cauchy criterion for series

Proof ($\implies$): $\sum a_n$ converges

$\implies$ the sequence of partial sums (s_n) converges

$\implies$ (s_n) is Cauchy

$\implies$ for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq m + 1 \geq N \implies |s_n - s_{m-1}| = \left| \sum_{k=m}^{n} a_k \right| < \varepsilon.$$

Cauchy criterion for series

Proof ($\Longleftarrow$): Let $\varepsilon > 0$, and suppose $\exists N \in \mathbb{N}$ such that

$$n \geq m \geq N \implies \left| \sum_{k=m}^{n} a_k \right| < \varepsilon.$$

Then

$$|s_n - s_{m+1}| < \varepsilon,$$

which is precisely the Cauchy criterion for (s_n).

Hence, $(s_n = \sum a_n)$ converges. $\blacksquare$

Geometric Series, Alternating Geometric Series

The **geometric series** with base r is defined by

$$\sum_{n=0}^{\infty} r^n,$$

and the **alternating geometric series** is defined by

$$\sum_{n=0}^{\infty} (-1)^n r^n.$$

Geometric Series, Alternating Geometric Series

If $r \neq 1$, then the partial sums are

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}, \quad \sum_{k=0}^{n} (-1)^k r^k = \frac{1 - (-1)^{n+1} r^{n+1}}{1 + r}.$$

If $|r| \geq 1$, the sums diverge.

If $|r| < 1$, the sums converge:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}, \quad \sum_{n=0}^{\infty} (-1)^n r^n = \frac{1}{1 + r}.$$

Comparison Test (Dominated Convergence)

Now we see several results about convergence.

Theorem

(Comparison Test) *Let* $\sum a_n$ *and* $\sum b_n$ *be infinite series of nonnegative terms. Then*

(a) $\sum a_n$ *converges and* $0 \leq b_n \leq a_n \; \forall n \implies \sum b_n$ *converges.*

(b) $\sum a_n = +\infty$ *and* $0 \leq a_n \leq b_n \; \forall n \implies \sum b_n = +\infty.$

Comparison Test (Dominated Convergence)

Proof

(a) $\sum a_n = a < \infty$ and $0 \leq b_n \leq a_n \; \forall n \implies$

$$\forall n, \quad 0 \leq \sum_{k=1}^{n} b_k \leq \sum_{k=1}^{n} a_k \leq a \implies \sum b_n = b \leq a$$

by the Monotone Convergence Theorem.

(b) $(\sum a_n)$ is unbounded $\implies$ $(\sum b_n)$ is unbounded. ■

Absolute, Conditional Convergence

If $\sum |a_n|$ converges, we call the series $\sum a_n$ **absolutely convergent**.

If $\sum a_n$ converges but $\sum |a_n|$ diverges, we call the series $\sum a_n$ **conditionally convergent**.

There are relationships between these two types of convergence.

Theorem

$\sum a_n$ converges absolutely $\implies$ $\sum a_n$ converges.

Proof Triangle inequality + Cauchy criterion.

Ratio Test

Theorem

(Ratio Test) *Let* $\sum a_n$ *be a series of nonzero terms.*

(a) *If* $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, *then* $(\sum a_n)$ *converges absolutely.*

(b) *If* $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, *then* $(\sum a_n)$ *diverges.*

(c) *Otherwise,* $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$ *and the test gives no information.*

Ratio Test

Proof Let

$$\limsup \left|\frac{a_{n+1}}{a_n}\right| = L.$$

If $L < 1$, then pick r such that $L < r < 1$.

Then $\exists N \in \mathbb{N}$ such that, if $n \geq N$,

$$\left|\frac{a_{n+1}}{a_n}\right| \leq r \implies |a_{n+1}| \leq r|a_n| \implies \forall k \in \mathbb{N},\ |a_{N+k}| \leq r^k |a_N|.$$

Ratio Test

Then

$$
\begin{aligned}
\sum_{k=1}^{\infty} |a_k| &= \sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{\infty} |a_k| \\
&\leq \sum_{k=1}^{N-1} |a_k| + |a_N| \sum_{k=0}^{\infty} r^k \\
&= \sum_{k=1}^{N-1} |a_k| + \frac{|a_N|}{1-r} < \infty,
\end{aligned}
$$

and so $\sum a_n$ converges.

Ratio Test

If $\liminf |\frac{a_{n+1}}{a_n}| > 1$, then $|a_{n+1}| \geq |a_n|$ for all n sufficiently large. Then $a_n \not\to 0$ and so $\sum a_n$ must diverge.

Finally, we can give an example of two series such that

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = 1,$$

with one converging and one diverging: recalling the Riemann-zeta function, $\zeta(2)$ converges and $\zeta(0)$ diverges. $\blacksquare$

Root Test

Theorem

(Root Test) *Let* $\sum a_n$ *be a series, and let* $\alpha = \limsup |a_n|^{1/n}$.

(a) *If* $\alpha < 1$, *then* $\sum a_n$ *converges absolutely.*

(b) *If* $\alpha > 1$, *then* $\sum a_n$ *diverges.*

(c) *Otherwise,* $\alpha = 1$ *and the test gives no information.*

Root Test

Proof If $\alpha < 1$, pick r such that $\alpha < r < 1$.

Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$
\begin{aligned}
|a_n|^{1/n} \leq r \implies |a_n| \leq r^n & \\
\implies \sum_{k=1}^{\infty} |a_k| &= \sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{\infty} |a_k| \\
&\leq \sum_{k=1}^{N-1} |a_k| + \sum_{k=0}^{\infty} r^k \\
&= \sum_{k=1}^{N-1} |a_k| + \frac{1}{1-r} < \infty.
\end{aligned}
$$

Root Test

If $\alpha > 1$, then $|a_n|^{1/n} \geq 1$ for infinitely many n, so $|a_n| \geq 1$ for infinitely many n. Thus, (a_n) diverges.

We can once again give an example of a convergent series

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and a divergent sequence

$$\zeta(0) = \sum_{n=1}^{\infty} \frac{1}{n^0} = \sum_{n=1}^{\infty} 1,$$

both having $\alpha = 1$, to show its ineffectiveness in this test. $\blacksquare$

Integral Test

Theorem

Let f be continuous on $[0, \infty)$, and suppose that f is positive and decreasing. Then

$$\sum f(n) \text{ converges } \iff \lim_{n\to\infty} \left(\int_1^n f(x)dx \right) \in \mathbb{R}.$$

Integral Test

Proof Let $a_n = f(n)$ and

$$b_n = \int_n^{n+1} f(x)dx.$$

f is decreasing, so for any $n \in \mathbb{N}$,

$$f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n).$$

Thus, by the comparison test, since $0 < a_{n+1} \leq b_n \leq a_n$,

$\sum a_n$ converges $\iff$ $\sum b_n$ converges. $\blacksquare$

Alternating Series Test

Theorem

(Alternating Series Test) *If* (a_n) *is a decreasing sequence of positive numbers and*

$$\lim a_n = 0,$$

then the series

$$\sum (-1)^n a_n$$

converges.

Alternating Series Test

Proof Since (a_n) is decreasing, the differences

$$a_{2n-1} - a_{2n-2} \geq 0,$$

and every partial sum $s_n \leq a_1$ (we can show this via induction).

Hence, by the Monotone Convergence Theorem, the increasing, bounded subsequence

$$s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) \to s$$

for some $s \in \mathbb{R}$.

For the odd subsequence s_{2n+1}, it is clear that since $a_{2n+1} \to 0$, $s_{2n+1} = s_{2n} + a_{2n+1} \to s$ as well.

Therefore, the interleaved sequence $s_n \to s$. $\blacksquare$

Rearrangements

Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection, i.e. a **rearrangement** of $\mathbb{N}$.

A **rearrangement** of the series of real numbers $\sum a_n$ is the series

$$\sum b_n = \sum a_{f(n)}.$$

Theorem

(Dirichlet's Theorem): *If $\sum a_n$ converges absolutely, then any rearrangement $\sum b_n$ converges absolutely, and $\sum a_n = \sum b_n$.*

Theorem

Let $s \in \mathbb{R}$. If $\sum a_n$ converges conditionally, then there exists a rearrangement of $\sum a_n$ that converges to s.

There also exists a rearrangement that diverges.

Power Series (reminder)

Given a sequence (a_n), $n = 0, 1, 2, \ldots$, the infinite series

$$\sum_{n=0}^{\infty} a_n x^n$$

is called a **power series**. The number a_n is called the nth **coefficient** of the series.

Depending on the convergence properties of the sequence (a_n), the power series may exist as a function of the variable x, or may simply be a formal set of symbols.

(We call the power series a **generating function** in either case.)

Radius, Interval of Convergence of a Power Series

Theorem

Let $\sum a_n x^n$ *be a power series, and* $\alpha = \limsup |a_n|^{1/n}$*. Define* R *by*

$$R = \begin{cases} \frac{1}{\alpha} & \text{if } 0 < \alpha < \infty, \\ +\infty & \text{if } \alpha = 0, \\ 0 & \text{if } \alpha < +\infty. \end{cases}$$

Then the series converges absolutely whenever $|x| < R$ *and diverges whenever* $|x| > R$*.*

Proof Apply the root test to the sequence $(b_n) = (a_n x^n)$. Then

$$\beta = \limsup |b_n|^{1/n} = \limsup |a_n x^n|^{1/n} = |x|\alpha. \quad \blacksquare$$

Ratio criterion of Radius of Convergence

We call R the **radius of convergence** of the series, and the interval C around 0 of the x where $\sum a_n x^n$ converges is called the **interval of convergence**.

C may equal $(-R, R)$, $[-R, R)$, $(-R, R]$, or $[-R, R]$, depending on convergence at the endpoints.

Theorem

The radius of convergence R of the power series $\sum a_n x^n$ equals

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

if the limit exists.

Taylor series is a power series (reminder)

A power series of the form $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence centered at 0.

We can talk about more general power series centered at other values: a power series centered at x_0 has form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

This form should be familiar: if, for some function f,

$$a_n = \frac{f^{(n)}(x_0)}{n!},$$

then the power series above is the Taylor series of f centered at x_0.