Proposition For whole numbers n, k such that $n \geq k \geq 1$,

$$\binom{n}{k} = \sum_{j=k-1}^{n-1} \binom{j}{k-1}. \tag{1}$$

(In words: the number of ways to choose k of n things is the sum of all the ways to choose the first $k-1$ of them out of j things, then make the last one the nth, out of all the j that make sense (which is $j = k-1, k, k+1, ..., n-2, n-1$).

Proof We prove this via induction on n. The base case is $n = 1$, $k = 1$:

$$\binom{1}{1} = \sum_{j=0}^{0} \binom{j}{0} = 1.$$

Now, assume

$$\binom{n}{k} = \sum_{j=k-1}^{n-1} \binom{j}{k-1}.$$

is true for a given n, and all $k \leq n$. We prove the result for $n+1$, i.e. we will prove

$$\binom{n+1}{k} = \sum_{j=k-1}^{(n+1)-1} \binom{j}{k-1} = \sum_{j=k-1}^{n} \binom{j}{k-1} \tag{2}$$

using (1). This requires use of an identity we already know:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}. \tag{3}$$

We'll also assume $k > 1$ (the $k = 1$ case is obvious – what is it?). Using (3), we get

$$\begin{aligned}
\binom{n+1}{k} &= \binom{n}{k} + \binom{n}{k-1} \\
&= \sum_{j=k-1}^{n-1} \binom{j}{k-1} + \sum_{j=k-2}^{n-1} \binom{j}{k-2} \text{ using (1) twice} \\
&= \sum_{j=k-1}^{n-1} \left[\binom{j}{k-1} + \binom{j}{k-2}\right] + \binom{k-2}{k-2} \\
&= \sum_{j=k-1}^{n-1} \binom{j+1}{k-1} + \binom{k-2}{k-2} \text{ using (3)} \\
&= \sum_{l=k}^{n} \binom{l}{k-1} + \binom{k-2}{k-2} \text{ by re-indexing } l = j+1 \\
&= \sum_{l=k}^{n} \binom{l}{k-1} + \binom{k-1}{k-1} \text{ since } \binom{i}{i} = 1 \text{ for ANY } i \\
&= \sum_{l=k-1}^{n} \binom{l}{k-1} \text{ by adding that term onto the sum.}
\end{aligned}$$

Thus, the inductive step (2) is proven using the inductive hypothesis (1) and we are done.