# Linear Algebra and Matrix Methods
# Determinants

# Determinant answers "Is this square matrix invertible?"

The **determinant** of a square* matrix $A$ is a number that simplifies the question of whether or not the matrix is invertible.

Denoted $det(A)$ or $|A|$, the matrix $A$ is invertible iff $det(A) \neq 0$:

$$det(A^{-1}) = (det(A))^{-1} = \frac{1}{det(A)}.$$

If $det(A) = 0$, then not being allowed to divide by zero signals a lack of invertibility of $A$.

*We only compute determinants for square matrices.

# Determinants of 2x2 matrices

The determinant of the 2x2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$det(A) = ad - bc.$$

If $ad \neq bc$, then the inverse of $A$ is

$$A^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

# Properties of determinants of matrices

- $det(I) = 1$ for any size identity matrix.
- For a permutation matrix $P$, $det(P) = (-1)^s$, where $s$ is the number of row swaps from $I$ that makes $P$.
- If $A$ is triangular (lower or upper, or diagonal - both!), then $det(A)$ is the product of its diagonal entries:

$$det(A) = \prod_{i=1}^{n} a_{ii}.$$

- Scaling an $n \times n$ matrix $A$ by a constant $c$ scales $det(A)$ by $c^n$:

$$det(A) = D \implies det(cA) = c^n D.$$

# Properties of determinants of matrices

- Scaling only one row of $A$ scales the determinant by just that amount. For example,

$$det \begin{pmatrix} 2 & 4 \\ 5 & -9 \end{pmatrix} = -18 - 20 = -38$$
$$\implies det \begin{pmatrix} 2 & 4 \\ -10 & 18 \end{pmatrix} = -2(-38) = 36 + 40 = 76.$$

- $det(A)$ is a linear function of each row separately:

$$det \begin{pmatrix} a + a' & b + b' \\ c & d \end{pmatrix} = det \begin{pmatrix} a & b \\ c & d \end{pmatrix} + det \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}.$$

- Thus, in general,

$$det(A + B) \neq det(A) + det(B).$$

# Properties of determinants of matrices

- For example,

$$det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 \neq det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 + 1 = 2.$$

- If two rows are equal, then $det(A) = 0$.
  (Think: equal rows in $A$ causes "0=0" or "0=1" in $A\vec{x} = b$.)
- Generalizing, if one row is a linear combination of any of the other rows of $A$, then $det(A) = 0$. (Same reason.)
- An all-zero row makes $det(A) = 0$. (Same reason.)
- Row addition (*not* scaling) operations do *not* change $det(A)$.
- $det(A) = det(A^t)$: any "row" property is also for "columns".
- $det(AB) = det(A) \cdot det(B)$.

# In general, determinants are hard to compute.

Computing a $2 \times 2$ determinant is easy, but in general, an $n \times n$ determinant takes a large number of computations to find.

There are ways to make it simpler:

- **Factorization** yields **pivots** to multiply easily.
- Computing **recursively** via **cofactors** allows simpler computations.
- Otherwise, directly calculating via the "big formula" with **permutations** uses $n!$ terms for an $n \times n$ matrix.

# Computing a determinant via pivots

We have previously seen that, if a square matrix $A$ is invertible, then it has a factorization

$$PA = LU$$

where $P$ is a permutation matrix, $L$ is lower triangular with diagonal 1s, and $U$ is upper triangular. If row swaps are not needed, this can be written

$$A = LU,$$

or

$$A = LDU$$

if we factor the diagonal $D$ off of $U$. All of these forms make it very simple to compute $det(A)$.

# Computing a determinant via pivots

In the form $PA = LU$, we have

$$det(A) = \frac{det(L)det(U)}{det(P)}$$

where

- $det(P) = (-1)^s$ if $s$ row swaps are needed for factorization,
- $det(L) = 1^n = 1$,
- $det(U) = \prod_{i=1}^{n} u_{ii}$ if $u_{ii}$ are the diagonal elements of $U$.

# Computing a determinant via pivots

In the form $A = LDU$, with triangular $L$ and $U$ having 1 diagonals, and $D$ a diagonal matrix,

- $det(L) = det(U) = 1^n = 1$
- $det(D) = \prod_{i=1}^{n} d_{ii}$

$$\implies det(A) = det(L)det(D)det(U) = det(D) = \prod_{i=1}^{n} d_{ii}.$$

# The "big formula" for determinants

To construct the "big formula" for the determinant of an $n \times n$ matrix $A$, first note that there are $n!$ **permutations** $\sigma$ of the elements $\{1, 2, ..., n\}$:

$$\sigma : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$$

We denote the set of all permutations† on $\{1, 2, ..., n\}$ by $S_n$. There are $n!$ different permutation matrices of size $n \times n$, corresponding to the $n!$ elements of $S_n$. Thus, we will denote the permutation matrix corresponding to $\sigma$ by $P_\sigma$.

The formula is

$$det(A) = \sum_{\sigma \in S_n} det(P_\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

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†also known as the **symmetric group** on $n$ elements

# The "big formula" for determinants: $3 \times 3$

Here is the complete $3 \times 3$ case. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

There are $3! = 6$ permutations of $\{1, 2, 3\}$:

- $\frac{3!}{2} = 3$ "even" permutations, with 0 or 2 swaps:

$$\begin{pmatrix} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \end{pmatrix}, \begin{pmatrix} 1 \to 2 \\ 2 \to 3 \\ 3 \to 1 \end{pmatrix}, \begin{pmatrix} 1 \to 3 \\ 2 \to 1 \\ 3 \to 2 \end{pmatrix},$$

- and $\frac{3!}{2} = 3$ "odd" permutations, with 1 swap:

$$\begin{pmatrix} 1 \to 2 \\ 2 \to 1 \\ 3 \to 3 \end{pmatrix}, \begin{pmatrix} 1 \to 3 \\ 2 \to 2 \\ 3 \to 1 \end{pmatrix}, \begin{pmatrix} 1 \to 1 \\ 2 \to 3 \\ 3 \to 2 \end{pmatrix}.$$

# The "big formula" for determinants: $3 \times 3$

Thus,

$$\begin{aligned} det(A) = & \ a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}. \end{aligned}$$

The $4 \times 4$ case has $4! = 24$ terms; the $5 \times 5$ has $5! = 120$ terms....

# Cofactors: Determinants via recursion

The vast number of ways to factor terms in the big formula hints at the fact that we can isolate smaller blocks in a matrix, called **cofactor matrices**, and define the determinant recursively.

Define $M_{ij}$ as the $n-1 \times n-1$ submatrix of $A$ constructed by deleting row $i$ and column $j$, and let $C_{ij} = (-1)^{i+j} det(M_{ij})$.

$det(M_{ij})$ is called the $(i,j)$-**minor** of $A$; $C_{ij}$ is the $(i,j)$-**cofactor**.

Then, choose one row, row $i$, to delete, and index through all columns. We have the **cofactor formula**

$$det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}.$$

# The cofactor formula for determinants: $3 \times 3$

If we choose row 1 to delete, then the big formula can be rewritten

$$M_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}, \; M_{12} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}, \; M_{13} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$\begin{aligned} det(A) &= (-1)^{1+1} a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\ &\quad + (-1)^{1+2} a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + (-1)^{2+2} a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}. \end{aligned}$$

Note that, since $det(A) = det(A^t)$, we could also fix a column to delete, and index over rows.

# Cramer's Rule: Cofactors

Throughout this section, we assume $det(A) \neq 0$, so our square system has a unique solution.

**Cramer's Rule** is a method by which the solution of $A\vec{x} = b$ looks more like one-dimensional algebra than the $LU$ decomposition we saw earlier.

If $ax = b$ is a standard basic algebra product, then the **cofactor** of the factor $a$ is $x = \frac{b}{a}$. We generalize this method of solution with the cofactors seen earlier.

# Cramer's Rule: Cofactors

Let $X_j$ be the square matrix that replaces column $j$ in $I$ with $\vec{x}$.

Example

$$X_2 = \begin{pmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{pmatrix}$$

Let $B_j$ be the square matrix that replaces column $j$ in $A$ with $b$.

Example

$$A = \begin{pmatrix} 1 & -4 & 5 \\ -3 & 2 & 0 \\ 6 & -11 & 9 \end{pmatrix}, \; b = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix} \implies B_3 = \begin{pmatrix} 1 & -4 & 10 \\ -3 & 2 & 20 \\ 6 & -11 & 30 \end{pmatrix}.$$

Cramer's Rule is a way to solve $A\vec{x} = b$ via determinants.

# Cramer's Rule for solving $A\vec{x} = b$

Each term $x_j$ in the solution vector $\vec{x}$ can be found via taking determinants of the matrix equation

$$AX_j = B_j.$$

Since $det(X_j) = x_j$, this gives us

**Cramer's Rule I**: If $A\vec{x} = b$ has a unique solution, the solution is

$$x_j = \frac{det(B_j)}{det(A)},\ j = 1, 2, ..., n.$$

Attempting to solve this system directly requires computing $n + 1$ determinants, each at the cost of summing $n!$ products.

# Cramer's Rule for finding $A^{-1}$

We can also use a cofactor-based method for computing $A^{-1}$.

**Cramer's Rule II**: If $det(A) \neq 0$, and $C_{ij}$ are $A$'s cofactors, then the entries of $A^{-1}$ are

$$(A^{-1})_{ij} = \frac{C_{ji}}{det(A)}.$$

Setting $C$ as the **cofactor matrix** with $C_{ij}$ as entry $(i, j)$, we have

$$A^{-1} = \frac{1}{det(A)} C^t.$$

# Application: cross product

Let $u, v \in \mathbb{R}^3$. The **cross product** of $u$ and $v$, denoted $u \times v$, is defined by

$$u \times v = (||u|| \cdot ||v|| \cdot \sin(\theta)) \cdot n,$$

where $n \in \mathbb{R}^3$ has the following properties:

- $\theta$ is the angle between $u$ and $v$
- $n \perp u$
- $n \perp v$
- $||n|| = 1$
- $n$ is in the direction based on the "right hand rule" of physics.

Compare this definition to the **dot product** of $u$ and $v$:

$$u \cdot v = ||u|| \cdot ||v|| \cdot \cos(\theta).$$

# Application: cross product

Note that the three standard basis vectors in $\mathbb{R}^3$, when used in some physics and engineering contexts, are written as

$$i = e_1, \; j = e_2, \; k = e_3,$$

with the following relationship via the cross product[‡]:

$$i \times j = k, \; j \times i = -k.$$

[‡] This relationship involves a deeper mathematical meaning, via the *quaternions*, which extend the complex numbers $\mathbb{C}$ to three "imaginary" units. The cross product defines unit quaternion multiplication, which is not commutative.

# Application: cross product

The cross product can be computed by using the cofactor method of determinant computation across row 1 of the following “matrix”:

$$u \times v = \text{“det”} \begin{pmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

$$= det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} i - det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} j + det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} k.$$

# Application: cross product, area

The length of the cross product of $u \times v$ is the area of the parallelogram formed by the vectors $u$ and $v$, with diagonal $u + v$.

$$A_{||} = ||u \times v|| = ||u|| \cdot ||v|| \cdot |\sin(\theta)|$$

Thus, the area of the triangle formed by $u$ and $v$ is

$$A_{\triangle} = \frac{1}{2}||u \times v|| = \frac{1}{2}||u|| \cdot ||v|| \cdot |\sin(\theta)|.$$

# Applications: cross product, volume

**Theorem**

*In general, if $A$ is an $m \times n$ matrix, then the volume of the parallelpiped $P$ formed in $\mathbb{R}^n$ by the $m$ rows of $A$ is computed via*

$$vol(P) = \sqrt{det(AA^t)}.$$

# Applications: cross product, area

If $P$ is the parallelogram in $\mathbb{R}^n$ formed by $u$ and $v$, and $\theta$ is the angle between them, then

$$
\begin{aligned}
A = \begin{pmatrix} u \\ v \end{pmatrix} \implies AA^t &= \begin{pmatrix} ||u||^2 & u \cdot v \\ u \cdot v & ||v||^2 \end{pmatrix} \\
\implies vol(P) = \sqrt{det(AA^t)} &= \sqrt{||u||^2||v||^2 - (u \cdot v)^2} \\
&= ||u|| \cdot ||v|| \sqrt{1 - \left( \frac{u \cdot v}{||u|| \cdot ||v||} \right)^2} \\
&= ||u|| \cdot ||v|| \sqrt{1 - \cos^2(\theta)} \\
&= ||u|| \cdot ||v|| \cdot |\sin(\theta)|.
\end{aligned}
$$

In $\mathbb{R}^3$, this is $area(P) = ||u \times v||$.