Introduction to Analysis:
The Real Numbers

Well ordering principle

Every nonempty set of natural numbers S,

$$\emptyset \neq S \subseteq \mathbb{N},$$

has a smallest element.

Note

The empty set $\emptyset$, having no elements, has no smallest element.

Otherwise, this seems like an obvious statement, because you've been trained on the *well-ordering* property of the natural numbers $\mathbb{N}$ since you were very young.

Well ordering principle: countable sets

Proposition

Any subset $C \subseteq \mathbb{N}$ is countable (finite or infinite).

Proof By the Well Ordering Principle, select the smallest element of C, call it entry 1 (c_1) in the list.

The next smallest is entry 2 (c_2). Continue. ■

Well ordering principle: Fractions have lowest terms

Theorem
Every positive rational number $q \in \mathbb{Q}^+ = \{q \in \mathbb{Q} : q > 0\}$ *has a representation in* **lowest terms**, *i.e. a form*

$$q = \frac{m}{n}, \; m, n \in \mathbb{N}$$

where m and n have no common factors.

Proof We will prove by contradiction.

Let C be the set of positive integers that are numerators m of fractions $\frac{m}{n}$ that do *not* have a form in lowest terms.

Well ordering principle: Fractions have lowest terms

$$C = \left\{ m \in \mathbb{N} : \ \frac{m}{n} \in \mathbb{Q}, \ \frac{m}{n} \text{ has } \textit{no} \text{ lowest terms} \right\}$$

Assume C is nonempty.

Then, by Well Ordering, there is a smallest element $m_0 \in C$.

(This is considered a counterexample to the proposition.
We will show that there are none.)

Well ordering principle: fractions have lowest terms

Let $n_0 \in \mathbb{N}$ be a denominator such that $\frac{m_0}{n_0}$ has no lowest terms.

Thus, $\frac{m_0}{n_0}$ is not in lowest terms, so a common factor can be divided out. Call a possible common factor $p \geq 2$. Thus,

$$\frac{\frac{m_0}{p}}{\frac{n_0}{p}} = \frac{m_0}{n_0}.$$

Thus, $\frac{\frac{m_0}{p}}{\frac{n_0}{p}}$ also has no lowest terms form, and so $\frac{m_0}{p} \in C$.

But $\frac{m_0}{p} < m_0$, and we assumed m_0 was the smallest nonnegative integer in C. This is a contradiction $\rightarrow\leftarrow$. Therefore, $C = \emptyset$. $\blacksquare$

Well ordering principle-based proofs, Induction

Proofs using the well ordering principle often go for a contradiction, showing that there are no elements that satisfy a certain property.

On the other hand, the **Principle of Mathematical Induction** (or, simply, **induction**) is a family of techniques for direct, often constructive, discrete mathematics[1].

[1] which is why the technique lends itself so well to computer science

Well ordering principle-based proofs, Induction

The idea: $p(n)$ is a predicate for $n \in I$, some discrete index set:

$I = \{k, k+1, k+2, ...\}$ for some $k \in \mathbb{Z}$. (Usually, $k = 0$ or 1.)

You wish to prove

$$\forall n \geq k,\ p(n).$$

Induction

How we do this:

1. Start with a *base case*, $p(k)$, which should be easy to prove. (It might even seem *too* easy.)

2. Take the *inductive step*: Assume $p(n)$ for a fixed $n \geq k$. (This is called the *inductive hypothesis*.)

   Use $p(n)$ to prove $p(n+1)$.

Induction

The result is the Method of Induction starting at $k \in \mathbb{Z}$ for $p(n)$:

$$p(k) \wedge (\forall n \geq k,\ p(n) \implies p(n+1)) \implies \forall n \geq k,\ p(n).$$

We can prove that induction works using Well Ordering.

We'll assume for simplicity that $k = 1$.

Induction is Provable via Well Ordering

Theorem

(Induction) *Let $p(n)$ be a predicate for $n \in \mathbb{N}$. Then*

$$p(1) \wedge (\forall n \geq 1,\ p(n) \implies p(n+1)) \implies \forall n \geq 1,\ p(n).$$

Proof Let

$$S = \{n \in \mathbb{N} :\ p(n) \text{ is false}\}.$$

We assume that $S \neq \emptyset$, and the following implication is true:

$$p(1) \wedge (\forall n \geq 1,\ p(n) \implies p(n+1)).$$

We will prove, for a contradiction, that $S = \emptyset$.

Induction is Provable via Well Ordering

$S \subseteq \mathbb{N}$, so by Well Ordering, S has a smallest element. Call it m.

$m \geq 2$ since we assume $p(1)$ is true, so $1 \notin S$.

If $p(m)$ is false, and the implication $p(m-1) \implies p(m)$ is true, then by the truth table for $p(m-1) \implies p(m)$, we know that $p(m-1)$ must also be false[2].

Hence, $m - 1 \in S$, contradicting m as the smallest element of S. $\rightarrow\leftarrow$ $\quad \therefore$ S is empty. $\blacksquare$

[2] Check this directly by writing out the truth table.

Induction: Gauss' trick

The canonical first example in induction is **Gauss' trick**.

Theorem

$$\forall n \in \mathbb{N}, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Proof We prove by induction. First, we show the base case $n = 1$:

$$\sum_{i=1}^{1} i = 1 = \frac{1(2)}{2}. \checkmark$$

Now, assume the inductive hypothesis $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Then,

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}. \quad \blacksquare$$

Strong induction

Strong induction is another form of induction (logically equivalent to "regular" induction) which allows you to assume *all* predicates before the given "new" one in your inductive step. That is,

1. Start with a *base case*, $p(k)$, which should be easy to prove. (It might even seem *too* easy.)

2. Take the *strong inductive step*: Assume

$$p(k), p(k+1), ..., p(n-1), p(n)$$

   for a fixed $n \in I$.
   (This is called the *strong inductive hypothesis*.)

   Use as many of these as you like to prove $p(n+1)$.

Strong induction

This is represented logically by

$$\left(p(k) \wedge \left[(p(k) \wedge p(k+1) \wedge \cdots \wedge p(n)) \implies p(n+1)\right]\right)$$
$$\implies \forall n \geq k,\ p(n).$$

Strong induction: $n \in \mathbb{N}$, $n > 1$ has a prime factorization

Proposition

$\forall n \in \mathbb{N},\ n > 1 \implies n$ is a product of primes.

Proof We will use strong induction.
First, the base case: $n = 2$ is clearly a (product of) prime(s).
Next, the inductive step: assume that $2, 3, \ldots, n-1, n$ are all products of primes.

There are two possible cases for $n + 1$:

- $n + 1$ is prime itself: nothing to show.
- $n + 1$ is composite: then $\exists m, q \in \mathbb{N} : 2 \leq m \leq q < n + 1$ such that $n + 1 = mq$. Since $m, q \leq n$, then each is known to be a product of primes. Therefore, $n + 1$ is a product of primes since it is a product of m and q. $\blacksquare$

Well ordering principle vs induction

Using the well ordering principle is also equivalent to using induction, but they have a difference in style.

Well ordering principle-based proofs deal with finding a *counterexample* $n \in I$ to a predicate $p(n)$ being true.

In particular, attempt to find the minimum possible n such that $p(n)$ is false.

Finding that one does not exist results in the conclusion that $p(n)$ is always true.

Well ordering principle vs induction

Induction proofs first show that a base case is true, then show that each successive predicate is also true because of it.

Guideline: If you want to show a contradiction, use well ordering (i.e. “there are no values that do *not* satisfy this property”).

If you wish to prove directly, use induction
(i.e. “each of these values satisfies this property”).

Well ordering principle vs induction: Gauss' trick

We will re-prove **Gauss' trick** using the well ordering principle.

Proposition

$\forall n \in \mathbb{N}, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof We prove by well ordering.
Let S be the set of $n \in \mathbb{N}$ such that Gauss' trick does not work.

Assume (for a contradiction) that S is nonempty and,
by well ordering, let m be the smallest element of S.

Thus,

$$\sum_{i=1}^{m} i \neq \frac{m(m+1)}{2}.$$

Well ordering principle vs induction: Gauss' trick

Clearly, $m > 1$ since $1 = \frac{1(2)}{2}$. But in that case,

$$\sum_{i=1}^{m-1} i = \frac{(m-1)(m)}{2}$$

$$\sum_{i=1}^{m} i = \sum_{i=1}^{m-1} i + m$$

$$= \frac{(m-1)(m)}{2} + m = \frac{m(m+1)}{2}.$$

Thus, $m \notin S$. $\rightarrow\leftarrow$ $\blacksquare$

$\mathbb{R}$ = "the set of lengths with a direction (sign)"

The set of real numbers $\mathbb{R}$ can be described as

"all the lengths possible (positive and negative) on the number line, relative to a fixed point called zero (0)."

This induces a **total ordering**[3] on the set, in which every pair of two distinct elements can be compared: this ordering is called

"less than" $(<)$.

[3] This is in comparison with a *partial ordering*, under which all pairs of elements can not necessarily be compared.

$\mathbb{R}$ = "the only complete ordered field"

The set of **real numbers** is **the only complete ordered field**.

- "only" in the sense that any other complete ordered field can be set in 1-1 correspondence with $\mathbb{R}$;
- "complete" in the sense of containing limits of sequences (we will address this in the next section);
- "ordered" in the sense of the total order $<$;
- "field" will be defined now, in terms of the **field axioms**, which must hold for any set to be called a **field** (in abstract algebra terms).

Field Axioms of Real Numbers

The real numbers $\mathbb{R}$ have two binary operations, $+$ ("addition") and $\cdot$ ("multiplication") which have the following properties:

(A,M1) $+$ and $\cdot$ are **closed** in $\mathbb{R}$, and consistent:

$$x, y \in \mathbb{R} \implies x + y, xy \in \mathbb{R}$$

$$x = w,\ y = z \implies x + y = w + z,\ xy = wz$$

(A,M2) $+$, $\cdot$ are **commutative** in $\mathbb{R}$:

$$x + y = y + x,\ \ xy = yx$$

Field Axioms of Real Numbers

(A,M3) $+$, $\cdot$ are **associative** in $\mathbb{R}$:

$$(x+y)+z = x+(y+z),\ (xy)z = x(yz)$$

(DL) **distributive law**: $x(y+z) = xy + xz$

Field Axioms of Real Numbers

(A4) $\exists!$ number[4] $0 \in \mathbb{R}$, the **additive identity**, such that

$$\forall x \in \mathbb{R},\ x + 0 = 0 + x = x$$

(M4) $\exists!$ number $1 \in \mathbb{R}$, the **multiplicative identity**, such that

$$1 \neq 0 \text{ and } x \cdot 1 = 1 \cdot x = x$$

[4] $\exists!$ denotes *unique existence*; only one of this type of object exists in $\mathbb{R}$!

Field Axioms of Real Numbers

(A5) For each $x \in \mathbb{R}$, $\exists a \in \mathbb{R}$, the **additive inverse** of x, denoted $a = -x$, such that

$$x + a = a + x = 0$$

(M5) For each $x \in \mathbb{R} \setminus \{0\}$, $\exists b \in \mathbb{R}$, the **multiplicative inverse** of x, denoted $b = \frac{1}{x} = x^{-1}$, such that

$$xb = bx = 1$$

Inverse Operations of Real Numbers

We then define the inverse operations of addition and multiplication (called **subtraction** and **division**, of course) by *non-commutative* combination with inverse second elements.

The operation of "subtraction" of $x \in \mathbb{R}$ by $y \in \mathbb{R}$ is defined by

$$x - y = x + (-y).$$

The operation of "division" of $x \in \mathbb{R}$ by $y \in \mathbb{R} \setminus \{0\}$ is defined by

$$x \div y = x \cdot y^{-1}.$$

Order Axioms of the Real Numbers

In addition to the field axioms, the real numbers have **order axioms** that apply to the relation we have called $<$.

(O1) **Trichotomy**
$\forall x, y \in \mathbb{R}$, exactly one of these three statements is true:

$$x < y, \; x = y, \; y < x.$$

We will also define the notation $x > y$ for the case $y < x$, i.e. $>$ is the inverse relation of $<$.

Order Axioms of the Real Numbers

(O2) **Transitivity**

$$\forall x, y, z \in \mathbb{R},\ x < y \text{ and } y < z \implies x < z.$$

(O3) **Shift Invariance**

$$\forall x, y, z \in \mathbb{R},\ x < y \implies x + z < y + z.$$

(O4) **Positive Scale Invariance**

$$\forall x, y, z \in \mathbb{R},\ x < y \text{ and } 0 < z \implies xz < yz.$$

Order Axioms of the Real Numbers

A real number $x \in \mathbb{R}$ is called

- **positive** if $x > 0$, **nonnegative** if $x \geq 0$
- **negative** if $x < 0$, **nonpositive** if $x \leq 0$

where the relations $\leq$ and $\geq$ are defined in the obvious way:

- $x \leq y \iff (x < y)$ or $(x = y)$
- $x \geq y \iff (y < x)$ or $(x = y)$

$\mathbb{R}$ is not the only field.

A number system F with the following properties is called a **field**:

(A,M1) F has two binary arithmetic operations (which we'll call $+$ and $\cdot$) under which F is *closed*, meaning that, if $a, b \in F$, then $a + b, a \cdot b \in F$ as well.

(A,M2) $+$ and $\cdot$ are commutative.

(A,M3) $+$ and $\cdot$ are associative.

(DL) $+$ and $\cdot$ are distributive.

(A,M4) F has a unique **additive identity** (0) and **multiplicative identity** (1).

(A,M5) F contains **additive inverses** and **multiplicative inverses** for all its nonzero elements.

Here are some properties of real number inequalities:

Proposition

$a < b \implies 0 < b - a.$

Proof By (O3),

$$a < b \implies a + (-a) < b + (-a) \implies 0 < b - a. \blacksquare$$

Proposition

If $a < b$ and $c < d$, then $a + c < b + d$.

Proof By (O3),

$$a < b \implies a + c < b + c \text{ and } c < d \implies c + b < d + b.$$

But, by (A2),

$$b + c = c + b \text{ and } d + b = b + d.$$

Therefore,

$$a + c < b + c < b + d. \blacksquare$$

Inequalities: the ordering of $\mathbb{R}$

Proposition

$x > y \iff -x < -y$.

Proof Suppose $x \neq y$. Further, suppose $x > y$ and $-x > -y$. Then, by the previous proposition,

$$x + (-x) > y + (-y) \implies 0 > 0,$$

which is a contradiction. Hence, one of the inequalities is false. ■

Corollary

$x > 0 \implies -x < 0$.

More inequality properties

Proposition

$x \neq 0 \implies x^2 > 0.$

Proof Two cases:

$$\begin{aligned} I.\ x > 0 &\implies x^2 > x(0) = 0 \text{ by positive scale invariance (O4).} \\ II.\ x < 0 &\implies -x > 0 \text{ by the previous proposition} \\ &\implies (-x)(-x) = x^2 > 0(-x) = 0.\ \blacksquare \end{aligned}$$

More inequality properties

Proposition

$0 < 1$.

Proof We are given $0 \neq 1$ in (M4).

If $0 > 1$, then $-0 = 0 < -1$, which implies by the previous proposition that $0 < (-1)(-1) = 1$.

$0 > 1 \implies 0 < 1$ is a contradiction of trichotomy.

$\therefore 0 \not> 1$, and so $0 < 1$ is the only one of the three that holds. $\blacksquare$

Even more inequality properties

Proposition

$u > v > 0 \implies u^2 > v^2$.

Proof By scale invariance,

$$u > v \implies u^2 > uv \text{ and } uv > v^2.$$

Combining these two via transitivity, we get $u^2 > v^2$. ■

Even more inequality properties

Proposition

$x > 0 \implies \frac{1}{x} > 0.$

Proof $\frac{1}{x} \neq 0$, so for a contradiction, assume $x > 0$ and $\frac{1}{x} < 0$.

Then $x \cdot \frac{1}{x} = 1 < 0.$ $\rightarrow\leftarrow$ ■

Even more inequality properties

Proposition

If $xy = 0$ and $y \neq 0$, then $x = 0$.

Proof $y \neq 0$ and $y \in \mathbb{R} \implies y^{-1} \in \mathbb{R}$, $y^{-1} \neq 0$. Then,

$$xy = 0 \implies xyy^{-1} = 0y^{-1} \implies x \cdot 1 = x = 0. \blacksquare$$

A helpful analysis bounding theorem

Theorem

Let $x, y \in \mathbb{R}$ such that $\forall \varepsilon > 0$, $x \le y + \varepsilon$. Then $x \le y$.

Proof We will prove the contrapositive. Suppose $x > y$.

(We need to show $\exists \varepsilon > 0$ such that $x > y + \varepsilon$.)

Let $\varepsilon = \frac{x-y}{2}$. We know that

$$x > y \text{ and } \frac{1}{2} > 0 \implies x - y > 0 \implies \varepsilon > 0.$$

Thus,

$$x = \frac{x+x}{2} > \frac{x+y}{2} = \frac{x-y}{2} + y = \varepsilon + y = y + \varepsilon. \blacksquare$$

Modulus (Absolute Value), Triangle Inequality

Definition

The **modulus** (absolute value) of $x \in \mathbb{R}$ is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Theorem

(Triangle Inequality) $\quad |x+y| \leq |x| + |y| \quad \forall x, y, \in \mathbb{R}.$

Note

$|x-y| \geq |x| - |y|$ *and* $|x-y| \geq \big||x| - |y|\big|$ *follow quickly.*

Upper, Lower Bounds

Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$. We define **bounds** of S by the following:

- $l \in \mathbb{R}$ is called a **lower bound** of S if $\forall s \in S$, $l \leq s$.
- $u \in \mathbb{R}$ is called an **upper bound** of S if $\forall s \in S$, $u \geq s$.

Upper, Lower Bounds

It should be obvious that these bounds are not unique:

- If, say, l is a lower bound for S,
  then, for any $\varepsilon > 0$, $l - \varepsilon$ is also a lower bound for S.
- Likewise, if u is an upper bound for S,
  then $u + \varepsilon$ is also an upper bound for S for any $\varepsilon > 0$.

Also, note that most bounds of S are *not* elements of S.

Upper, Lower Bounds

Example

The half-open, half-closed interval $(a, b]$ has lower bounds:

$$a,\ a-2,\ a-0.5,\ a-0.0000001,\ a-10,004,345,\ \text{etc.}$$

$(a, b]$ also has upper bounds: $b,\ b+3,\ b+54,430$, etc.

Example

The union of intervals $(a, b] \cup [c, d]$, with $a < b < c < d$, has all the lower bounds of $(a, b]$, and upper bounds

$$d,\ d+1,\ d+25,409,\ \text{etc.}$$

Example

The half-line (a, ∞) has all the lower bounds of $(a, b]$. However, (a, ∞) has no upper bound. (We call such an interval *unbounded*.)

Example

It should be obvious that $\mathbb{R} = (-\infty, \infty)$ is unbounded.

Upper, Lower Bounds

The interval notation makes it easy to determine bounds, but implicitly defined functions require some work.

Example

$$P = \{x \in \mathbb{R} : \; x \text{ is a (positive) prime number}\}$$

is a subset of $\mathbb{N}$ which we know has no upper bound.

However, any subset of $\mathbb{N}$, by the Well Ordering Principle, has a lower bound of 0 (and anything less than zero).

Example

$$S = \{x \in \mathbb{R} : \; x^2 - 3x + 2 > 0\}$$

is unbounded, but we must discover this fact via algebra:

$$x^2 - 3x + 2 = (x-1)(x-2) > 0 \implies x < 1 \text{ or } x > 2,$$

which implies $S = (-\infty, 1) \cup (2, \infty)$.

Least Upper Bound, Greatest Lower Bound

It is natural to ask, if a set is bounded, is there a bound we can use as the *extreme*[5] bound for that side of the set?

- If c is an upper bound for S, then we call c the **supremum (**$\sup(S)$**)**, or **least upper bound (**$LUB(S)$**)**, of S if $c \leq u$ for any other upper bound u of S.
- Likewise, if d is a lower bound for S, then we call d the **infimum (**$\inf(S)$**)**, or **greatest lower bound (**$GLB(S)$**)**, of S if $d \geq l$ for any other lower bound l of S.

[5]Note the use of the definite article "the" in these definitions; the infimum and supremum, if they exist for a set S, are *unique*.

Extrema

If $c = \sup(S)$ and $c \in S$, we call c S's **maximum** $\mathbf{(}c = \max(S)\mathbf{)}$.

If $d = \inf(S)$ and $d \in S$, we call d S's **minimum** $\mathbf{(}d = \min(S)\mathbf{)}$.

Example

The interval $I = [a, b)$ obviously has $\inf(I) = a$ and $\sup(I) = b$.

$a = \min(I)$, but $\max(I)$ does not exist.

Example

$$S = \left\{ \frac{1}{n} : \; n \in \mathbb{N} \right\}$$

has both an inf and a sup: since we can explicitly write S as

$$S = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\},$$

it is clear that $1 = \sup(S)$, as it is an upper bound of S, and $1 \in S$, meaning $1 = \max(S)$.

$0 = \inf(S)$, but $0 \notin S$, so $\min(S)$ does not exist. That $0 = \inf(S)$ requires a more subtle proof (which we cannot yet deliver).

Axiom of Continuity

Axiom of Continuity:

Suppose that all real numbers are separated into two sets, denoted L and R, such that

- $x \in \mathbb{R} \implies x \in L$ or $x \in R$, but not both.
- Each of L and R is nonempty. (These two properties make the pair L, R a **partition** of $\mathbb{R}$ of size two.)
- If $a \in L$ and $b \in R$, then $a < b$.

Axiom of Continuity

Then, $\exists c \in \mathbb{R}$ such that

$$L = \{x \in \mathbb{R} : \ x < c\}$$

and

$$R = \{x \in \mathbb{R} : \ x \geq c\}.$$

In other words,

$$\exists c \in \mathbb{R} : \ L = (-\infty, c), \ R = [c, \infty).$$

In this circumstance, this partition is called a **Dedekind cut** (named after Richard Dedekind, 1831-1916), and c is called the **cut number**.

The real numbers $\mathbb{R}$ are all of the Dedekind cut numbers.

We can define the **field of real numbers** $\mathbb{R}$ as the (only) ordered field that satisfies the Axiom of Continuity for *all possible Dedekind cuts* of rational numbers (leaving no "gaps", in Dedekind's terminology).

This is deeply related to the notion of the **completeness** of the real numbers.

Completeness Axiom

There is a vast generalization of the Well Ordering Principle for the real numbers called the **Completeness Axiom**.

Completeness Axiom: Let S be a nonempty subset of $\mathbb{R}$.

- If S has an upper bound, then S has a sup.
- If S has a lower bound, then S has an inf.

Note that the Completeness Axiom does not claim that the sup or inf is an *element* of S, only that the extreme exists.

Axiom of Continuity $\iff$ Completeness Axiom

One proof can show that

$$\text{Axiom of Continuity} \implies \text{Completeness Axiom},$$

and another proof can show that

$$\text{Completeness Axiom} \implies \text{Axiom of Continuity}.$$

Thus, the two are equivalent statements, and so either is usable as a foundation to define the real numbers.

We shall revisit the Completeness Axiom to define the real numbers via **Cauchy sequences** in the next section.

Archimedean Property of $\mathbb{R}$

There are several equivalent ways to state the **Archimedean Property**, which says that there is always a larger number in $\mathbb{R}$.

Archimedean Property of $\mathbb{R}$: The following are equivalent:

(a) $\mathbb{N}$ is unbounded above.

(b) For any $z \in \mathbb{R}$, $\exists n \in \mathbb{N}$: $n > z$.

(c) If $a, b \in \mathbb{R}$ such that $a > 0$ and $b > 0$, then $\exists n \in \mathbb{N}$: $na > b$.

(d) If $a \in \mathbb{R}$ such that $a > 0$, then $\exists n \in \mathbb{N}$: $\frac{1}{n} < a$.

Note that (d) proves that $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ has $\inf(S) = 0 \notin S$.

Rationals are Dense in the Reals

Recall,

$$\mathbb{Q} = \left\{ \frac{m}{n} \,\middle|\, m, n \in \mathbb{Z},\ n \neq 0 \right\} \subseteq \mathbb{R}$$

is the set of **rational numbers**, a subset of the real numbers.

Proposition

The rationals $\mathbb{Q}$ are **dense** in themselves, and thus in $\mathbb{R}$; that is,

$$\forall r, s \in \mathbb{Q} \text{ such that } r > s,\ \exists q \in \mathbb{Q} :\ r > q > s.$$

Rationals are Dense in the Reals

Proof We prove directly, splitting the proof into two parts.

First, note that $q = \frac{r+s}{2}$ is between r and s:

$$r > s \implies \frac{1}{2}r > \frac{1}{2}s \implies \frac{1}{2}r + \frac{1}{2}s > \frac{1}{2}s + \frac{1}{2}s = s,$$

$$\text{and } r = \frac{1}{2}r + \frac{1}{2}r > \frac{1}{2}r + \frac{1}{2}s.$$

Next, if $r = \frac{m}{n}$ and $s = \frac{p}{q}$, with $m, n, p, q \in \mathbb{Z},\ n, q \neq 0$, then

$$\frac{r+s}{2} = \frac{r}{2} + \frac{s}{2} = \frac{mq+np}{2nq} \in \mathbb{Q}. \ \blacksquare$$

Irrational Numbers Exist

Next, we show that irrational numbers, in fact, *exist*.

Definition

An **irrational number** is a real number that cannot be expressed as a ratio of integers.

$$x \in \mathbb{I} \iff x \in \mathbb{R} \setminus \mathbb{Q},\ i.e.\ \neg\left(\exists m, n \in \mathbb{Z} : \frac{m}{n} = x\right).$$

Our first view of irrational numbers? Square roots.

Square Roots Exist

Proposition

$$\exists \alpha \in \mathbb{R} : \ \alpha^2 = 2.$$

Proof (via geometric construction) The **unit square** with side length 1 has a diagonal of length α satisfying the Pythagorean Theorem:

$$1^2 + 1^2 = \alpha^2.$$

We call $\alpha = \sqrt{2}$. ■

Square Roots Exist

Proposition

For any prime $p \in \mathbb{N}$, $\exists \sqrt{p} \in \mathbb{R}$:

$$\exists \alpha \in \mathbb{R} : \alpha > 0, \alpha^2 = p.$$

Proof (via set theory) Let

$$S = \{r > 0 \text{ and } r^2 < p\}.$$

Since $1^2 = 1 < p$, $1 \in S$, so S is nonempty.

For any prime $p \in \mathbb{N}$, $\exists \sqrt{p} \in \mathbb{R}$

Since $p > 1$, $p^2 > p$, so p is an upper bound for S.

Therefore, by the Completeness Axiom, S has a supremum.

Call it $\alpha = \sup S$.

We need to show that $\alpha^2 = p$.

We do this by proving $\alpha^2 < p$ and $\alpha^2 > p$ lead to contradictions.

For any prime $p \in \mathbb{N}$, $\exists \sqrt{p} \in \mathbb{R}$

Start with $\alpha^2 < p$. Then $p - \alpha^2 > 0$. For any $n \in \mathbb{N}$,

$$\begin{aligned}\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &= \alpha^2 + \frac{1}{n}\left(2\alpha + \frac{1}{n}\right) \leq \alpha^2 + \frac{1}{n}\left(2\alpha + 1\right).\end{aligned}$$

For any prime $p \in \mathbb{N}$, $\exists \sqrt{p} \in \mathbb{R}$

By the Archimedean Property, we can choose $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \frac{p - \alpha^2}{2\alpha + 1} \implies \frac{2\alpha + 1}{n} < p - \alpha^2.$$

This "controls" the $\frac{1}{n}$ term. Then

$$\left(\alpha + \frac{1}{n}\right)^2 \leq \alpha^2 + \frac{1}{n}(2\alpha + 1) < \alpha^2 + p - \alpha^2 = p.$$

Therefore, $\alpha + \frac{1}{n} \in S$, and so $\alpha \neq \sup S$.

For any prime $p \in \mathbb{N}$, $\exists \sqrt{p} \in \mathbb{R}$

We can similarly argue that, if $\alpha^2 > p$, then by the Archimedean Property, $\exists m \in \mathbb{N}$ such that $\alpha - \frac{1}{m}$ is an upper bound for S.

This means $\alpha \neq \sup S$ since α is not the *least* upper bound of S.

Therefore, $\alpha^2 = p$ is our only option left, by trichotomy. $\blacksquare$

$\sqrt{2} \notin \mathbb{Q}$

Proposition

$\sqrt{2} \notin \mathbb{Q}$.

Proof We prove by contradiction.[6]

Assume $\sqrt{2} \in \mathbb{Q}$. Then $\exists m, n \in \mathbb{N}$ such that $\sqrt{2} = \frac{m}{n}$.

WLOG[7] let $\frac{m}{n}$ be in lowest terms. Then squaring both sides yields

$$2 = \frac{m^2}{n^2} \implies 2n^2 = m^2.$$

[6] This argument can be generalized to prove $\sqrt{p} \notin \mathbb{Q}$ if p is prime.

[7] *without loss of generality*: "we can assume this simplification without simplifying or breaking the argument"

$\sqrt{2} \notin \mathbb{Q}$

But

$$2 = \frac{m^2}{n^2} \implies 2n^2 = m^2$$

implies that m^2 is even, and hence m is even.

Thus, $\exists k \in \mathbb{N}$ such that $m = 2k$.

Plugging in $m = 2k$ yields

$$2n^2 = (2k)^2 = 4k^2 \implies n^2 = 2k^2,$$

meaning that n^2, and hence n, is also even.

But if m and n are *both* even, then $\frac{m}{n}$ was not in lowest terms to begin with. $\rightarrow\leftarrow$ $\blacksquare$

Relationships between $a \in \mathbb{Q}$ and $b \in \mathbb{I}$

A couple more relationships between rationals and irrationals:

Proposition

Let $a \in \mathbb{Q}$ and $b \in \mathbb{I}$. Then

(i) $a + b \in \mathbb{I}$

(ii) $a \neq 0 \implies ab \in \mathbb{I}$.

Proof (i) Assume $a + b \in \mathbb{Q}$.
Then $\exists m, n, p, q \in \mathbb{Z}$ such that $a + b = \frac{m}{n}$, $a = \frac{p}{q}$. Thus,

$$b = (a + b) - a = \frac{m}{n} - \frac{p}{q} \in \mathbb{Q}. \rightarrow\leftarrow$$

(ii) Left as an exercise. ■

Relationships between $a \in \mathbb{Q}$ and $b \in \mathbb{I}$

Finally, we prove that $\mathbb{I}$ is dense in $\mathbb{R}$.

Proposition

$\forall a, b \in \mathbb{R}$, $a < b$, $\exists x \in \mathbb{I}$: $a < x < b$.

Proof There are two cases: (i) $a \in \mathbb{Q}$ and (ii) $a \in \mathbb{I}$.

(i) $a \in \mathbb{Q}$ implies, by the previous proposition, that if $y \in \mathbb{I}$, then $a + y \in \mathbb{I}$. Pick $n \in \mathbb{N}$ such that, by Archimedes, $n > \frac{\sqrt{2}}{b-a}$. Then

$$0 < \frac{\sqrt{2}}{n} < b - a \implies a < a + \frac{\sqrt{2}}{n} < b \text{ and } \frac{\sqrt{2}}{n} \in \mathbb{I}.$$

(ii) $a \in \mathbb{I}$ implies $a < a + \frac{1}{n} < b$ by the same argument; $\frac{1}{n} \in \mathbb{Q}$. ∎

Point Sets on a Line (the real axis $\mathbb{R}$), Neighborhoods

Let $x \in \mathbb{R}$. We consider x as a point on the number line, and $\mathbb{R}$ is the universal set under consideration.

Definition

An ε-**neighborhood** of $x \in \mathbb{R}$ is the set of all points

$$N(x; \varepsilon) = \{y \in \mathbb{R} : \ |x - y| < \varepsilon\}$$

for the given **radius** $\varepsilon > 0$ around the **center** x.

Point Sets on a Line (the real axis $\mathbb{R}$), Neighborhoods

We know this neighborhood $N(x;\varepsilon)$ as the **open interval**

$$N(x;\varepsilon) = (x-\varepsilon, x+\varepsilon).$$

A **deleted neighborhood** around x leaves out the center x:

$$N^*(x;\varepsilon) = \{y \in \mathbb{R} :\ 0 < |x-y| < \varepsilon\} = N(x;\varepsilon) \setminus \{x\}.$$

Interior, Boundary Points

Definition

x is called an **interior point** of $S \subseteq \mathbb{R}$ if $\exists$ a neighborhood N of x such that $N \subseteq S$. We denote the **interior** of S by int S.

Definition

x is called a **boundary point** of $S \subseteq \mathbb{R}$ if for every neighborhood N of x,

$$N \cap S \neq \emptyset \text{ and } N \cap (\mathbb{R} \setminus S) \neq \emptyset.$$

We denote the set of boundary points of S by bd S or ∂S.

Open, Closed Sets in $\mathbb{R}$

Boundary points of S are decidedly *not* interior points:

$$int\, S \cap bd\, S = \emptyset.$$

Definition

A set $S \subseteq \mathbb{R}$ is called **closed** if bd $S \subseteq S$.

Definition

A set $S \subseteq \mathbb{R}$ is **open** if bd $S \subseteq \mathbb{R} \setminus S$.

(Recall, the **complement** of a set $S \subseteq \mathbb{R}$ is denoted $S^C = \mathbb{R} \setminus S$.)

Open, Closed Sets in $\mathbb{R}$

Note that *open* and *closed* sets are *not opposites*:

if S is not open, that does not imply that S is closed, or vice versa.

For example, $\mathbb{R}$ and $\emptyset$ are both open and closed; $[a, b)$ is neither.

Theorems on Open, Closed Sets

Theorem

(i) A set S is open if and only if $S = int\, S$.
(ii) A set S is closed if and only if S^C is open.

Theorem

(i) The union of any collection of open sets is an open set.
(ii) The intersection of a finite collection of open sets is open.

This theorem results in a corresponding theorem, via DeMorgan's laws, for complements:

Theorem

(i) The intersection of any collection of closed sets is a closed set.
(ii) The union of a finite collection of closed sets is a closed set.

Accumulation Points

Definition

A point $x \in \mathbb{R}$ is called an **accumulation point (limit point)** of $S \subseteq \mathbb{R}$ if for each $\varepsilon > 0$, and deleted neighborhood $N^*(x; \varepsilon)$, $\exists s \in S$ such that $s \in N^*(x; \varepsilon)$.

We denote the set of accumulation points of S by S'.

Note

An accumulation point of S is not necessarily a point in S. A simple example is an endpoint of an open interval: a is an accumulation point of (a, b) since, for each $\varepsilon > 0$, the point $a + \frac{\varepsilon}{2} \in (a, b) \cap N^*(a, \varepsilon)$. However, $a \notin (a, b)$.

Note

Finite sets have no accumulation points.

Isolated Points

Definition

A point $x \in \mathbb{R}$ is called an **isolated point** of $S \subseteq \mathbb{R}$ if $x \in S \setminus S'$.

Note

Finite sets have only isolated points.

Definition

The **closure** of S is $\mathrm{cl}\, S = S \cup S'$, the union of S with its accumulation points.

Theorem

$S \subseteq \mathbb{R}$ is closed $\iff$ S contains all its accumulation points.

Proof This proof requires two directions: $\Longleftarrow$ and $\Longrightarrow$.

$(\Longleftarrow)$ Suppose S contains all its accumulation points.

If S is not closed, its complement S^C is not open, so, for a contradiction, suppose S^C is not open.

Then $\exists y \in S^C$ such that, for every $h > 0$, $(y - h, y + h) \not\subseteq S^C$.

In other words, y is not an interior point of S^C.

Closed $\Longleftarrow$ Contains all accumulation points

Then, for each $h > 0$,

$$\exists x_h \in (y - h, y + h),\ x_h \neq y,$$

such that $x_h \in S$ (since $x_h \notin S^C$). But this is precisely the definition of an accumulation point of S; therefore, y is an accumulation point of S.

But this contradicts the supposition that S contains all its accumulation points (since $y \in S^C$). $\rightarrow\leftarrow$

Hence, S^C is open, and so S is closed.

Closed $\implies$ Contains all accumulation points

$(\implies)$ Assume S is closed, and let x be an accumulation point of S. (We will show that $x \in S$.)

For each $h > 0$, $\exists x_h \in S$, $x_h \neq x$, such that $x_h \in N^*(x, h)$.

For a contradiction, assume $x \notin S$. Then $x \in S^C$. But S^C is open, so every point $y \in S^C$ has a neighborhood contained in S^C.

Thus, $\exists h^* > 0$ such that $(x - h^*, x + h^*) \subseteq S^C$. But this contradicts the fact that x is an accumulation point for S, since there is no $x_{\frac{h^*}{2}} \in (x - \frac{h^*}{2}, x + \frac{h^*}{2})$ such that $x_{\frac{h^*}{2}} \in S$. $\rightarrow\leftarrow$

Hence, the supposition $x \in S^C$ is false, and so $x \in S$. $\blacksquare$

Theorems about closed sets

Theorem

- *cl S is a closed set.*
- *S is closed $\iff$ $S = cl\ S$.*
- *$cl\ S = S \cup bd\ S$.*

Open Cover

Let $S \subseteq \mathbb{R}^n$. Suppose we have a collection of open sets A_i (indexed by some set I, which may be finite, countable, or uncountable) such that

$$S \subseteq \bigcup_{i \in I} A_i.$$

Then we call the collection $\{A_i : i \in I\}$ an **open cover** of S.

Note that these sets A_i may overlap themselves.

Open Subcover

A **subcover** of an open cover $\{A_i : i \in I\}$ of S is a subset of the cover, $\{A_i : i \in J\}$, with $J \subseteq I$, such that the subset still covers S.

That is,

$$S \subseteq \bigcup_{i \in J} A_i \subseteq \bigcup_{i \in I} A_i.$$

It is a reasonable question to ask, when is it possible to have only a *finite* number of A_i cover a set S?

Compact Set

Definition

A set $S \subseteq \mathbb{R}^n$ is called **compact** if, for any open cover of S, there exists a finite subcover.

It seems reasonable that if a finite cover of “small” sets exists to cover S, then somehow S is “small” in the sense that it can be covered by a finite number of small sets.

We make this rigorous in the **Heine-Borel**[8] **Theorem**.

[8] from Eduard Heine and Émile Borel

Heine-Borel Theorem

Theorem

Let $S \subseteq \mathbb{R}$. Then S is compact $\iff$ S is closed and bounded.

We will prove one direction and leave the other for reading.

Proof ($\implies$) Suppose S is compact.
Then any open cover $\{A_i : i \in I\}$ of S has a finite subcover.

One possible cover of S covers all of $\mathbb{R}$: let $I_n = (-n, n)$.
Then $\{I_n : n \in \mathbb{N}\}$ covers S. (This, in fact, is a countable cover.)

Heine-Borel Theorem

Since S is compact, there is a finite subcover of $\{I_n\}$ that cover S. Thus, there is some maximum index N in the finite subcover.

Therefore, $S \subseteq (-N, N)$ for some $N \in \mathbb{N}$, and so S is bounded.

We still need to show that S is closed, meaning it contains all its accumulation points.

Heine-Borel Theorem

Suppose S is not closed. Then there is an accumulation point $p \in \text{cl } S \setminus S$. We must show that this yields a contradiction. Let

$$U_n = \mathbb{R} \setminus \left[p - \frac{1}{n}, p + \frac{1}{n}\right].$$

Then U_n is open, and $\{U_n : n \in \mathbb{N}\}$ is an open cover of S.

Heine-Borel Theorem

Note that the U_n are *nested*: if $m \leq n$, then $U_m \subseteq U_n$.

But S is compact, so there is a finite subcover of S, meaning, again, a maximum index N such that

$$S \subseteq \bigcup_{i=1}^{N} U_i,$$

so by nesting, $S \subseteq U_N$.

Heine-Borel Theorem

Thus there is a gap of length at least $\frac{1}{2N}$ around p with no points of S, i.e.

$$S \cap N^*\left(p; \frac{1}{2N}\right) = \emptyset,$$

meaning p is *not* an accumulation point of S. $\rightarrow\leftarrow$

Thus, cl $S \setminus S$ is empty and so S contains all its accumulation points. $\therefore S$ is closed. $\blacksquare$

Finite Intersection Property of Compact Sets

Theorem

If $\mathcal{F} = \{K_\alpha : \alpha \in A\}$ is a collection of compact sets with index set A, such that for any finite subset $B \subseteq A$,

$$\bigcap_{\alpha \in B} K_\alpha \neq \emptyset.$$

Then

$$\bigcap_{\alpha \in A} K_\alpha \neq \emptyset.$$

Proof (hint) Use the finite intersection property of closed sets.

Finite Intersection Property of Compact Sets

Corollary

(Nested Intervals Theorem) Let $\mathcal{F} = \{K_n : n \in \mathbb{N}\}$ be a countable collection of nested compact intervals, i.e. $\forall n \in \mathbb{N},\ K_{n+1} \subseteq K_n$. Then

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

In particular, if $K_n = [a_n, b_n]$ are nested and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c,$$

then

$$\bigcap_{n=1}^{\infty} K_n = \{c\}.$$

Bolzano-Weierstrass Theorem

Recall, a set $S \subseteq \mathbb{R}$ is called **bounded** if it has both upper and lower bounds: i.e. if $\exists L, U \in \mathbb{R}$ such that

$$\forall s \in S, \; L \le s \le U.$$

In other words, S is contained in the interval $[L, U]$: $S \subseteq [L, U]$.

Theorem

Bolzano-Weierstrass Theorem *Suppose S is a bounded, infinite set. Then S has at least one accumulation point.*

Proof S is bounded, so pick $a < b$ such that $S \subseteq I_1 = [a, b]$. We employ a *divide and conquer*[9] routine to find an accumulation point.

Cut I_1 in half by setting $I_{21} = [a, \frac{a+b}{2}]$ and $I_{22} = [\frac{a+b}{2}, b]$.

Then one of these subintervals contains an infinite number of points of S; call that subinterval I_2.

If both have an infinite number of points of S, then select $I_{21} = I_2$.

[9] For those computer science-oriented, think bisection recursion.

Continue in this fashion: split I_n into two subintervals of length $\frac{b-a}{2^{n-1}}$ down its middle, called $I_{(n+1)1}$ and $I_{(n+1)2}$.

Then one of these two intervals contains an infinite number of points of S. Call it I_{n+1} and continue.

By the Nested Intervals Theorem, there is a unique point

$$x \in \bigcap_{n=1}^{\infty} I_n.$$

This point x is, by definition, an accumulation point of S, since any neighborhood of x contains a point from S. ■

Point Sets of Reals in Higher Dimensions

When discussing real numbers as points on a line, we are considering real numbers in *one dimension*.

Discussing sets of real numbers as points on a plane, or space, or a higher-dimensional abstract structure requires generalizations:

- **neighborhood**
- **open set**
- **closed set**
- **boundary**

Neighborhoods in Higher Dimensions

For a point $x \in \mathbb{R}^n$, the n-**dimensional real numbers**:

$$x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R},$$

a **neighborhood** of x may be defined in more than one way.

In $\mathbb{R}^n$, the point (0,0,...,0) is always called 0.

Distance Formula

Recall the **Distance Formula**[10] in two dimensions:

the Euclidean distance between two points

$$P = (x_1, y_1),\ \ Q = (x_2, y_2) \in \mathbb{R}^2$$

is defined as

$$d(P, Q) = |Q - P| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

[10] also known as the **Pythagorean Theorem**

Neighborhoods in Higher Dimensions: Discs in $\mathbb{R}^2$

The **circular neighborhood**, or **open disc**, centered at $P = (x_1, y_1)$ of radius r is the set of points

$$B_r(P) = \{(x, y) \in \mathbb{R}^2 : \ (x - x_1)^2 + (y - y_1)^2 < r^2\}$$

and the **closed disc** centered at $P = (x_1, y_1)$ of radius r is the set

$$\overline{B_r(P)} = \{(x, y) \in \mathbb{R}^2 : \ (x - x_1)^2 + (y - y_1)^2 \leq r^2\}.$$

Neighborhoods in Higher Dimensions: Spheres in $\mathbb{R}^n$

For sets $S \subseteq \mathbb{R}^n$, where points have n coordinates, i.e.

$$x \in \mathbb{R}^n \iff x = (x_1, x_2, ..., x_n),\ x_1, x_2, ..., x_n \in \mathbb{R},$$

the **open ball** of radius r, centered at c, is the neighborhood

$$B_r(c) = \left\{ (x_1, x_2, ..., x_n) \in \mathbb{R}^n :\ \sum_{j=1}^{n} (x_j - c_j)^2 < r^2 \right\}$$

and the **closed ball** $\overline{B_r(c)}$ replaces $<$ with $\leq$.

Interior Points, Open and Closed Sets in Higher Dimensions

$x \in S$ is called an **interior point** of $S \subseteq \mathbb{R}^n$ if there exists a circular neighborhood (open sphere) of x completely contained in S, i.e.

$$\exists r > 0 : \ B_r(x) \subseteq S.$$

$S \subseteq \mathbb{R}^n$ is called an **open set** if every point in S is interior.

$S \subseteq \mathbb{R}^n$ is called a **closed set** if its **complement**

$$S^C = \mathbb{R}^n \setminus S$$

is an open set.

Boundary Points in Higher Dimensions

$x \in S$ is called a **boundary** point of $S \in \mathbb{R}^n$ if every (circular) neighborhood of S contains both points in S and points in S^C.

The boundary of S is typically denoted bd S or ∂S.

Example

The boundary of the open sphere $B_r(x)$ of radius r centered at x is the **sphere** (in $\mathbb{R}^2$, **circle**)[11] of radius r centered at x:

$$S_r(x) = \partial B_r(x) = \{y \in \mathbb{R}^n : \ |x - y| = r\}.$$

[11] Note that "sphere" and "circle" refer to the boundary points only; "ball" and "disc" are the interior points.

Boundary Points in Higher Dimensions

Example

The set $S \subseteq \mathbb{R}^2$ consisting of the closed square of side length 4 centered at 0 with the open unit (radius 1) circle removed is

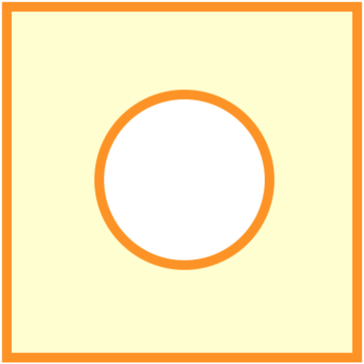
$$S = \{(x, y) \in \mathbb{R}^2 : \ -2 \leq x \leq 2, -2 \leq y \leq 2\} \setminus B_1(0),$$

which has the boundary

$$\begin{aligned} \partial S = &(\{-2\} \times [-2, 2]) \cup (\{2\} \times [-2, 2]) \\ &\cup ([-2, 2] \times \{-2\}) \cup ([-2, 2] \times \{2\}) \cup S_1(0). \end{aligned}$$

Note that S is a closed set; it contains its boundary points.

Boundary Points in Higher Dimensions



Neighborhoods in Higher Dimensions: Square

A **square neighborhood** around $x \in \mathbb{R}^n$ is, for any $\delta > 0$,

$$(x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 + \delta) \times \cdots \times (x_n - \delta, x_n + \delta).$$

Either definition of neighborhood works to describe interior points, and therefore for describing open sets:

$S \subseteq \mathbb{R}^n$ is open

$$\begin{aligned} &\iff \forall x \in S,\ \exists \text{ a circular neighborhood of } x \text{ contained in } S \\ &\iff \forall x \in S,\ \exists \text{ a square neighborhood of } x \text{ contained in } S. \end{aligned}$$

Boxes in Several Dimensions

In general, an **open rectangular region (box)** in $\mathbb{R}^n$ is an open set

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n).$$

Likewise, a **closed rectangular region (box)** in $\mathbb{R}^n$ is a closed set

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

Volume in Several Dimensions

For a rectangular region (box)

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

define the **volume** of R as

$$vol(R) = \prod_{i=1}^{n} (b_i - a_i).$$

- In $\mathbb{R}$, this is interval length.
- In $\mathbb{R}^2$ this is rectangular area.
- In $\mathbb{R}^3$ this is usual volume.
- In $\mathbb{R}^n$ for any $n \in \mathbb{N}$, this is known as **Lebesgue measure**.

Accumulation Points, Theorems in Higher Dimensions

$x \in \mathbb{R}^n$ is called an **accumulation point** of $S \subseteq \mathbb{R}^n$ if any neighborhood around x contains a point $s \in S$ where $s \neq x$.

This is the same definition as for $\mathbb{R}$.

Accumulation Points, Theorems in Higher Dimensions

In $\mathbb{R}^n$, we can talk about accumulation points using circular or square neighborhoods, and many of the results are similar.

- $S \subseteq \mathbb{R}^n$ is closed $\iff$ S contains its accumulation points.
- If R_1, R_2, ... is a sequence of closed rectangular regions such that the sequence (R_n) is a **nest**, i.e.

  $$\lim_{n\to\infty} vol(R_n) = 0,$$

  then

  $$\bigcap_{n=1}^{\infty} R_n$$

  is a singleton set.
- **Bolzano-Weierstrass Theorem**: A bounded, infinite set $S \subseteq \mathbb{R}^n$ contains at least one of its accumulation points.

Neighborhoods in Higher Dimensions: Norm

Definition

A **norm** on a vector space[12] A is a function

$$||\cdot|| : A \to \mathbb{R},$$

generalizing the notion of "length", that satisfies the following properties: for any $x, y \in A$, and $c \in \mathbb{R}$,

- $||0|| = 0$ (a point with length 0 is the zero vector)
- $||x|| \geq 0$ (lengths are positive)
- $||cx|| = |c| \cdot ||x||$ (scaling a vector is linear)
- $||x + y|| \leq ||x|| + ||y||$ (Triangle Inequality)

[12] leaving out some technical details; see linear algebra notes

Neighborhoods in Higher Dimensions: Metric

Definition

A **metric** on a vector space A is a function $d : A \times A \to \mathbb{R}$ (generalizing "distance") that satisfies, for any $x, y, z \in A$,

- $d(x, x) = 0$ (a point is distance 0 from itself)
- $d(x, y) \geq 0$ (distances are positive)
- $d(x, y) = d(y, x)$ (symmetry)
- $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality)

Metric from Norm

A norm $||\cdot||$ induces a metric by

$$d(x, y) = ||x - y||.$$

Neighborhoods are defined by a distance called a **radius** and a point called the **center**.

The most common norm used, the L^2 ("Euclidean") norm, induces the Euclidean metric[13]

$$d(x, y) = ||x - y||_2 = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}$$

and generates circular neighborhoods.

[13] and the Pythagorean Theorem

Different Metrics from Different Norms

- The L^∞ ("supremum") norm induces the metric

$$d(x,y) = ||x-y||_\infty = \sup_{i=1,\ldots,n} \{|x_i - y_i|\}$$

  and generates square neighborhoods.

- The L^1 ("taxicab", "Manhattan") norm induces the metric

$$d(x,y) = ||x-y||_1 = \left(\sum_{i=1}^{n} |x_i - y_i|\right)$$

  and generates a different kind of square neighborhood.

Neighborhoods in Higher Dimensions: Different Metrics

A metric is induced by a norm is **shift-invariant**:

$$\forall x, y, z \in A,\ d(x, y) = d(x - z, y - z).$$

In general, for $p > 0$, the L^p metric is defined by

$$d(x, y) = ||x - y||_p = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}.$$

The space denoted $L^p(\mathbb{R}^n)$ is generated by open "balls" of the form

$$B_\varepsilon(x) = \{y \in \mathbb{R}^n :\ d(x, y) < \varepsilon\} = \left\{y \in \mathbb{R}^n :\ ||x - y||_p^p < \varepsilon^p\right\}.$$

Topology

We can generalize the concept of "open set" away from distance.

Definition

A **topology** on a set A is a set $\mathcal{T}$ of subsets of A satisfying:

- $\emptyset, A \in \mathcal{T}$,
- any union of elements of $\mathcal{T}$ is in $\mathcal{T}$: for any[14] index set I,

$$\{B_i\}_{i \in I} \subseteq \mathcal{T} \implies \bigcup_{i \in I} B_i \in \mathcal{T}.$$

- finite intersections of elements of $\mathcal{T}$ are in $\mathcal{T}$:

$$B_1, B_2, \ldots, B_n \in \mathcal{T} \implies \bigcap_{i=1}^{n} B_i \in \mathcal{T}.$$

[14] finite, countable, or uncountable

Topology

If $\mathcal{T}$ is a topology on A, we call the elements of $\mathcal{T}$ **open sets**.

Example

The **standard topology**[15] on $\mathbb{R}^n$ is the one generated by any of the sets of neighborhoods discussed earlier.

[15] due to Felix Hausdorff (1868-1942)