Introduction to Analysis: Integration

A Very Brief History of Integration

- Archimedes (3rd Century BCE): method of exhaustion
- Isaac Newton (1642-1727) vs Gottfried Leibniz (1646-1716): physics, philosophy, religion
- ⋆ Bernhard Riemann (1826-1866): differential geometry, metrics, proto-quantum mechanics
- ⋆ Thomas Stieltjes (1856-1894): analytic theory, continued fractions
- Henri Lebesgue (1875-1941): measure theory

Partition of the interval $[a, b]$, Refinement of a partition

Let $[a, b]$ be an interval in $\mathbb{R}$.

A **partition** P of $[a, b]$ is a finite set of points

$$P = \{x_0, x_1, ..., x_n\} \subseteq [a, b]$$

such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

If P and Q are partitions of $[a, b]$ and $P \subseteq Q$, then Q is called a **refinement** of P.

Partition of the interval $[a, b]$, Refinement of a partition

For a partition $P = \{x_0, x_1, ..., x_n\}$, define

$$\Delta x_i(P) = x_i - x_{i-1},\ \ i = 1, 2, ..., n.$$

These are the distances between consecutive points in the partition.

We will abbreviate the notation to Δx_i if context allows.

If Q is a refinement of P, then Q not only has more Δx_i than P, but they are smaller as well.

max and min (sup and inf), upper and lower sums

Let $f : [a, b] \to \mathbb{R}$ be bounded, and P a partition on $[a, b]$. Define

$$M_i(f) = \sup\{f(x) : \; x \in [x_{i-1}, x_i]\},$$
$$m_i(f) = \inf\{f(x) : \; x \in [x_{i-1}, x_i]\}.$$

We will abbreviate the notation to M_i and m_i if context allows.

Define the **upper** and **lower Darboux sums** of f with respect to P by

$$U(P, f) = \sum_{i=1}^{n} M_i(f) \Delta x_i(P),$$
$$L(P, f) = \sum_{i=1}^{n} m_i(f) \Delta x_i(P).$$

Upper, lower integrals; Riemann integral

f bounded implies $\exists m, M \in \mathbb{R}$, $-\infty < m \leq M < \infty$ such that

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a).$$

Define the **upper** and **lower integrals** of f on $[a,b]$ by

$$U([a,b],f) = \inf\{U(P,f) : \ P \text{ is a partition of } [a,b]\},$$
$$L([a,b],f) = \sup\{L(P,f) : \ P \text{ is a partition of } [a,b]\}.$$

Upper, lower integrals; Riemann integral

If

$$-\infty < L([a,b], f) = U([a,b], f) < \infty,$$

then we say f is **Riemann integrable** on $[a,b]$, and that the common value is the **Riemann integral** of f on $[a,b]$, denoted

$$\int_a^b f, \quad \text{or} \quad \int_a^b f(x)dx.$$

Riemann-Stieltjes integral

To generalize, consider using a difference of another function g that is monotone on $[a, b]$, For the differences

$$\Delta_g x_i = g(x_i) - g(x_{i-1}),$$

define the upper and lower sums of f with respect to P and g by

$$U(P, f, g) = \sum_{i=1}^{n} M_i(f) \Delta_g x_i,$$
$$L(P, f, g) = \sum_{i=1}^{n} m_i(f) \Delta_g x_i.$$

Riemann-Stieltjes integral

Similarly, define the upper and lower integrals by

$$U([a,b],f,g) = \inf\{U(P,f,g) : \ P \text{ is a partition of } [a,b]\},$$
$$L([a,b],f,g) = \sup\{L(P,f,g) : \ P \text{ is a partition of } [a,b]\}.$$

If $U([a,b],f,g) = L([a,b],f,g)$, then we call the common value the **Riemann-Stieltjes integral** (or **Stieltjes integral**) of f with respect to g on $[a,b]$, denoted

$$\int_a^b f\,dg, \quad \text{or} \quad \int_a^b f(x)dg(x).$$

Riemann-Stieltjes integral

If g is absolutely continuous on $[a, b]$, then the differential of g is

$$dg(x) = g'(x)dx$$

and the Stieltjes integral can be rewritten as

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)dx,$$

a form which has uses in probability theory and integration by parts (which we will see shortly).

f bounded $\implies$ integral refinements are monotone

Theorem

Let f be a bounded function on $[a, b]$.

If P and Q are partitions of $[a, b]$ and Q is a refinement of P, then

$$L(P, f) \leq L(Q, f) \leq U(Q, f) \leq U(P, f).$$

Proof $L(Q, f) \leq U(Q, f)$ is immediate.

We will show

$$U(Q, f) \leq U(P, f),$$

and note that a similar argument will result in

$$L(P, f) \leq L(Q, f).$$

f bounded $\implies$ integral refinements are monotone

Suppose that $P = \{x_0, x_1, ..., x_n\}$. Pick $k \in \{1, 2, ..., n\}$. Then

$$\exists x^* \in (x_{k-1}, x_k) : \ x^* \in Q \setminus P.$$

Let

$$P^* = \{x_0, x_1, ..., x_{k-1}, x^*, x_k, ..., x_n\}.$$

Set

$$t_1 = \sup\{f(x) : \ x \in [x_{k-1}, x^*]\}, \ t_2 = \sup\{f(x) : \ x \in [x^*, x_k]\}.$$

f bounded $\implies$ integral refinements are monotone

Then, since $M_k \geq t_1$ and $M_k \geq t_2$, we have

$$\begin{aligned}
U(P,f) &= \sum_{i=1}^{n} M_i \Delta x_i \\
&\geq \sum_{i=1}^{k-1} M_i \Delta x_i + M_k(x^* - x_{k-1}) + M_k(x_k - x^*) + \sum_{i=k+1}^{n} M_i \Delta x_i \\
&\geq \sum_{i=1}^{k-1} M_i \Delta x_i + t_1(x^* - x_{k-1}) + t_2(x_k - x^*) + \sum_{i=k+1}^{n} M_i \Delta x_i \\
&= U(P^*, f).
\end{aligned}$$

Note that $U(Q,f)$ is achieved by repeating this process until all of Q's extra points above P are inserted. ■

Lower, upper integrals of a bounded function

Limiting the inequality from this theorem,

$$L(P, f) \leq L(Q, f) \leq U(Q, f) \leq U(P, f),$$

as refinements get finer and finer, results in

Theorem

Let f be a bounded function on $[a, b]$. Then

$$L([a, b], f) \leq U([a, b], f).$$

We can see various examples of this limiting by using specific sequences of refinements. For example, let

$$P_n = \left\{ a, a + \frac{1}{n}, a + \frac{2}{n}, ..., b - \frac{1}{n}, b \right\}$$

and send $n \to \infty$.

Bounded functions with shrinking $U - L$ gap are integrable

For f to be integrable, as refinements increase in cardinality,

$$U(P, f) - L(P, f) \to 0.$$

This is a theorem, that looks similar to our notions of limit existence (because that is, in fact, what it is).

Bounded functions with shrinking $U - L$ gap are integrable

Theorem

Let f be a bounded function on $[a, b]$.

Then f is integrable $\iff$ for each $\varepsilon > 0$, $\exists$ a partition P of $[a, b]$:

$$U(P, f) - L(P, f) < \varepsilon.$$

Bounded functions with shrinking $U - L$ gap are integrable

Proof

($\Longleftarrow$) First, for arbitrary $\varepsilon > 0$, suppose such a P exists that

$$U(P, f) - L(P, f) < \varepsilon.$$

Then, since ε was arbitrary, we have

$$U([a, b], f) \leq U(P, f) < L(P, f) + \varepsilon \leq L([a, b], f) + \varepsilon,$$

and so must have $U([a, b], f) \leq L([a, b], f)$.

But by the previous theorem, $L([a, b], f) \leq U([a, b], f)$. Therefore,

$$L([a, b], f) = U([a, b], f).$$

Bounded functions with shrinking $U - L$ gap are integrable

$(\implies)$ Suppose $\exists \varepsilon > 0$ such that, for any partition P, we have

$$U(P, f) - L(P, f) \geq \varepsilon.$$

Then, for any partition P,

$$U(P, f) \geq L(P, f) + \varepsilon,$$

yielding the limiting inequality

$$U([a, b], f) \geq L([a, b], f) + \varepsilon.$$

Therefore, since $U([a, b], f) \neq L([a, b], f)$, f is not integrable. $\blacksquare$

Integration of a constant function is area of a rectangle

Our first computational result tells us the value of the Riemann integral of a constant function.

Theorem

If f is the constant function $f(x) = c$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = c(b-a).$$

Integration of a constant function is area of a rectangle

Proof For any partition P of $[a, b]$,

$$M_i(f) = m_i(f) = c.$$

Thus,

$$U(P, f) = L(P, f) = \sum_{i=1}^{n} c\Delta x_i = c \sum_{i=1}^{n} \Delta x_i = c(b-a). \blacksquare$$

This value of the definite integral

$$\int_a^b f(x)\, dx = c(b-a).$$

is the area[1] of the rectangle of height c with base width $b - a$.

[1] This is called *signed* area: if $c < 0$ the graphed area is "below the x-axis".

Monotone functions are integrable

Now we will classify some types of functions as integrable.

Theorem

Let f be a monotone function on $[a, b]$. Then f is integrable.

Proof WLOG let f be an increasing function.

Clearly, f is bounded on $[a, b]$, and in fact

$$m = f(a) \leq f(b) = M.$$

Monotone functions are integrable

If $P = \{a = x_0, x_1, ..., x_n = b\}$ is a partition of $[a, b]$, then

$$m_i = f(x_{i-1}) \text{ and } M_i = f(x_i) \text{ for } i = 1, 2, ..., n.$$

Since f is bounded, then by the Archimedean property,

$$\forall \varepsilon > 0,\ \exists k = k(\varepsilon) > 0 :\ \forall i = 1, 2, ..., n,\ k(M - m) < \varepsilon.$$

Monotone functions are integrable

Thus, if we choose a partition P such that $\Delta x_i \leq k$ for all i,

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]\Delta x_i \\ &\leq k \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \\ &= k[f(b) - f(a)] = k(M - m) < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this means that by our previous theorem, $U([a,b],f) = L([a,b],f)$, and so f is integrable. ■

Continuous functions are integrable

Theorem

Let f be a continuous function on $[a, b]$. Then f is integrable.

Proof Suppose f is continuous on $[a, b]$.

Since $[a, b]$ is compact, f is uniformly continuous on $[a, b]$.

Thus, for any $\varepsilon > 0$, $\exists \delta > 0$ such that, if $x, y \in [a, b]$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

Continuous functions are integrable

Now select:

- a partition P such that $\Delta x_i < \delta$ for every i,
- $s_i, t_i \in [x_{i-1}, x_i]$ for each i such that

$$m_i = f(s_i) \text{ and } M_i = f(t_i).$$

Then

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} [M_i - m_i]\Delta x_i < \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \varepsilon,$$

and so f is integrable. ■

Non-integrable functions are not continuous or monotone

The contrapositives of the previous two theorems are logically equivalent to those theorems. Therefore,

Corollary

If f is not integrable on $[a, b]$, then f is not monotone on $[a, b]$.

Corollary

If f is not integrable on $[a, b]$, then f is not continuous on $[a, b]$.

These do *not*, however, imply that

- no discontinuous functions are integrable
  (many are, such as simple functions), nor that
- no non-monotone functions are integrable
  (there are, for example, non-monotone continuous functions).

Integration is a linear operation

Theorem

Integration on $[a, b]$ is a linear operation.

More specifically, if f and g are integrable functions on $[a, b]$ and $c_1, c_2 \in \mathbb{R}$, then the function $c_1 f + c_2 g$ is also integrable on $[a, b]$, and

$$\int_a^b (c_1 f + c_2 g) = c_1 \int_a^b f + c_2 \int_a^b g.$$

Integration is a linear operation

Proof We will prove this theorem in two pieces:

- For any $c \in \mathbb{R}$, cf is integrable, and
$$\int_a^b cf = c\int_a^b f.$$
- $f + g$ is integrable, and
$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Integration is a linear operation

- For any $c > 0$ and partition P, it is clear that

  $$U(P, cf) = cU(P, f), \; L(P, cf) = cL(P, f).$$

  The result follows since f is integrable on $[a, b]$:

  $$U(P, cf) - L(P, cf) = c(U(P, f) - L(P, f)).$$

  For $c < 0$,

  $$U(P, cf) = cL(P, f), \; L(P, cf) = cU(P, f)$$

  and the result still follows.

Integration is a linear operation

- By properties of max,

$$U(P, f+g) \leq U(P, f) + U(P, g),$$

and by properties of min,

$$L(P, f+g) \geq L(P, f) + L(P, g).$$

Thus, if f and g are integrable on $[a, b]$,

$$0 \leq U(P, f+g) - L(P, f+g) \leq U(P, f) - L(P, f) + U(P, g) - L(P, g),$$

and the result follows. $\blacksquare$

Mean Value Theorem for Integrals

Theorem

Mean Value Theorem (integrals):

If $f : [a, b] \to \mathbb{R}$ *is continuous, then* $\exists c \in [a, b]$ *such that*

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c).$$

Mean Value Theorem for Integrals

Proof f is continuous on the closed interval $[a, b]$, so by the EVT f attains its maximum M and minimum m on $[a, b]$. Thus,

$$\forall x \in [a, b],\ m \leq f(x) \leq M.$$

Hence, for any approximating sum of the Riemann integral

$$\int_a^b f(x)dx,$$

we have, for whatever choices of x_i and x_i', and lower and upper sums s_n and S_n,

$$m \sum_{i=1}^{n} \Delta x_i \leq s_n \leq \sum_{i=1}^{n} f(x_i')\Delta x_i \leq S_n \leq M \sum_{i=1}^{n} \Delta x_i.$$

Mean Value Theorem for Integrals

But, as we have seen before,

$$\sum_{i=1}^{n} \Delta x_i = b - a.$$

Therefore, we have

$$m(b-a) \le \sum_{i=1}^{n} f(x_i')\Delta x_i \le M(b-a)$$

for any approximating sum.

Mean Value Theorem for Integrals

Taking the limit, we get the same bounds on the Riemann integral:

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) \implies m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M.$$

Set

$$\mu = \frac{1}{b-a}\int_a^b f(x)dx$$

as the average (mean) value of f over $[a, b]$.

Since $m \leq \mu \leq M$, by the IVT we know

$$\exists c \in [a, b] : \; f(c) = \mu. \; \blacksquare$$

Function Dominance (integrals)

Theorem

Function Dominance (integrals): *If*

$$\forall x \in [a, b],\ \ f(x) \leq g(x),$$

and f *and* g *are integrable on* $[a, b]$*, then*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Function Dominance (integrals)

Proof If f and g are integrable, then so is $g - f$, and

$$g(x) - f(x) \geq 0 \text{ for all } x \in [a, b].$$

Thus, for any partition P of $[a, b]$,

$$0 \leq L(P, g - f) \leq U(P, g - f), \quad \text{and } L(P, 0) = U(P, 0) = 0.$$

Hence,

$$0 = \int_a^b 0\, dx \leq \int_a^b (g(x) - f(x))\, dx = \int_a^b g(x)\, dx - \int_a^b f(x)\, dx. \blacksquare$$

Integrals sum across consecutive intervals

Theorem

If f is integrable on $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Proof Let $\varepsilon > 0$. Then $\exists$ partitions P_1 of $[a, c]$, P_2 of $[c, b]$:

$$U(P_1, f) - L(P_1, f) < \frac{\varepsilon}{2},\ U(P_2, f) - L(P_2, f) < \frac{\varepsilon}{2}.$$

Then $P_1 \cup P_2$ is a partition of $[a, b]$ and

$$U(P, f) - L(P, f) = U(P_1, f) + U(P_2, f) - L(P_1, f) - L(P_2, f) < \varepsilon.$$

Thus, f is integrable on $[a, b]$. The equality follows. ■

Composition of integrable functions is integrable

Although we will not prove it here, we will need the following theorem shortly.

Theorem

If f is integrable on $[a, b]$ and g is continuous on $[c, d]$, where $f([a, b]) \subseteq [c, d]$, then $g \circ f$ is integrable on $[a, b]$.

Triangle inequality (integral version)

Theorem

Let f be integrable on $[a, b]$. Then $|f|$ is integrable on $[a, b]$ and

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Proof By the previous theorem, since $g(x) = |x|$ is continuous on $\mathbb{R}$, we have that $g \circ f(x) = |f(x)|$ is integrable on $[a, b]$.

By dominance, since $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a, b]$,

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f| \implies \left|\int_a^b f\right| \leq \int_a^b |f|. \quad \blacksquare$$

Some interval notation for integrals

Recall some notation for integrals: over the interval $[a, b]$,

$$\int_b^a f = -\int_a^b f$$

$$\int_a^a f = 0.$$

Definite Integrals: Variable of Integration

The **variable of integration**, or the **"dummy variable"**, of an integral, is the variable used *inside* the integration[2].

For example, in the notation

$$\int_a^b f(x)\,dx,$$

the variable x is the dummy variable, and can be replaced with a different, unused letter[3].

We will refer to Riemann integrals on a compact interval $[a, b]$ as **definite integrals**.

[2] This is not the same usage as "dummy variable" as an indicator function in statistics.

[3] This is why the dummy-less notation $\int_a^b f$ is acceptable.

Definite Integrals: Limits of Integration

For definite integrals, if the **limits of integration** (endpoints a, b of the interval one integrates over) are fixed constants, then

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(v)dv = \int_a^b f(u)du = ...$$

all mean the same value.

However, when a limit of integration (typically the upper limit) is a *variable*, we must avoid a notation issue.

Variable Limits of Integration

If x is intended as a variable limit of integration, the expression

$$\int_a^x f(x)dx$$

doesn't make sense, as x cannot simultaneously be used as a limit of integration and the variable of integration.

Variable Limits of Integration

Suppose we have a function defined by a variable limit of integration. If f is integrable on $[a, b]$, let, for $a \leq x \leq b$,

$$F(x) = \int_a^x f(t)dt$$

be the definite integral of f on the variable compact interval $[a, x]$.

This notation is acceptable, as long as the dummy variable (here, t) and the limit of integration / function variable (here, x) are different symbols.

With this notation, we now state and prove the two

Fundamental Theorems of Calculus (FTC),

which relate differentiation and integration.

The Derivative of an Integral: FTC I

Theorem

(Fundamental Theorem of Calculus I):

Suppose f is integrable on $[a, b]$. For $x \in [a, b]$, define

$$F(x) = \int_a^x f(t)dt.$$

Then F is uniformly continuous on $[a, b]$.

Furthermore, if f is continuous on $[a, b]$, then F is differentiable on (a, b) and

$$\forall c \in (a, b), \; F'(c) = f(c).$$

The Derivative of an Integral: FTC I

Proof First, we show that F is uniformly continuous on $[a, b]$.

f is integrable on $[a, b] \implies f$ is bounded on $[a, b]$, i.e.

$$\exists B > 0 : \ \forall x \in [a, b], \ |f(x)| \leq B.$$

The Derivative of an Integral: FTC I

Let $\varepsilon > 0$. Then $\exists \delta = \frac{\varepsilon}{B} > 0$, such that if $x, y \in [a, b]$ and $x < y$,

$$\begin{aligned} & |y - x| < \delta \\ \implies |F(y) - F(x)| &= \left| \int_a^y f(t)dt - \int_a^x f(t)dt \right| \\ &= \left| \int_x^y f(t)dt \right| \\ &\leq \int_x^y |f(t)|dt \\ &\leq \int_x^y B\, dt \\ &= B(y - x) < \varepsilon. \end{aligned}$$

The Derivative of an Integral: FTC I

Each inequality uses a separate theorem:

$$
\begin{aligned}
&f \text{ integrable on } [a,c],[c,b] && \implies \int_a^b f = \int_a^c f + \int_c^b f, \\
&f \text{ integrable on } [a,b] && \implies \left| \int_a^b f \right| \leq \int_a^b |f|, \\
&f, g \text{ integrable on } [a,b] \text{ and } f(x) \leq g(x) && \implies \int_a^b f \leq \int_a^b g, \\
&\text{ and } f \text{ constant on } [a,b],\ f(x) = c && \implies \int_a^b f = c(b-a).
\end{aligned}
$$

Thus, since $\delta = \frac{\varepsilon}{B}$ does not depend on x and y, F is uniformly continuous on $[a, b]$.

The Derivative of an Integral: FTC I

Next, suppose that f is continuous at a fixed $c \in [a, b]$.

If $\varepsilon > 0$,

$$\exists \delta > 0 : \; t \in [a, b], \; |t - c| < \delta \implies |f(t) - f(c)| < \varepsilon.$$

$f(c)$ is a constant, so we can write $f(c)$ as an integral: for $x \neq c$,

$$f(c) = \frac{1}{x - c} \int_c^x f(c)\, dt.$$

Note that it does not matter if $x < c$ or $x > c$ here, only that $x \in [a, b]$ and that $x \neq c$.

The Derivative of an Integral: FTC I

Then, if $x \in [a, b]$ such that $0 < |x - c| < \delta$,

$$\begin{aligned}
\left| D_F(x, c) - f(c) \right| &= \left| \frac{F(x) - F(c)}{x - c} - \frac{\int_c^x f(c)\,dt}{x - c} \right| \\
&= \left| \frac{\int_a^x f(t)\,dt - \int_a^c f(t)\,dt}{x - c} - \frac{\int_c^x f(c)\,dt}{x - c} \right| \\
&= \left| \frac{\int_c^x f(t)\,dt}{x - c} - \frac{\int_c^x f(c)\,dt}{x - c} \right| \\
&\leq \frac{1}{|x - c|} \int_c^x |f(t) - f(c)|\,dt \\
&< \frac{1}{|x - c|} \varepsilon |x - c| = \varepsilon.
\end{aligned}$$

$$\therefore \; F'(c) = \lim_{x \to c} D_F(x, c) = f(c). \quad \blacksquare$$

Antiderivatives (Indefinite Integrals)

A differentiable function F on an interval $[a, b]$ such that

$$\forall x \in (a, b),\ F'(x) = f(x)$$

for a function f defined on $[a, b]$ is called an **antiderivative**, or **indefinite integral**, of f.

FTC I establishes the existence of antiderivatives.

However, antiderivatives are not *unique* to a given function f.

Antiderivatives are Not Unique

If F is an antiderivative of f, and $G(x) = F(x) + C$ for some constant $C \in \mathbb{R}$, then

$$G'(x) = \frac{d}{dx}(F(x) + C) = f(x)$$

and so G is also an antiderivative of f.

We typically refer to "the antiderivative" F of a function f as a *family* of functions of the form

$$F(x) + C,$$

where we refer to C as the **constant of integration**.

The Inverse of Differentiation (Antiderivatives)

Consider a *first-order linear differential equation*, of the form

$$\frac{dy}{dx} = f(x).$$

We are to compute the antiderivative y via separation of variables and integration using differentials. First, write

$$dy = f(x)\,dx$$

considering dy and dx as *differentials*, denoting changes in y and x.

The Inverse of Differentiation (Antiderivatives)

We use the integral sign $\int$ without limits of integration to commit the inverse operation of differentiation on differentials.

Letting

$$y = \int dy,$$

we say

$$y = \int dy = \int f(x)\, dx = F(x) + C.$$

In practice, this requires some skill in pattern recognition to notice how to apply differentiation rules "in reverse".

The Inverse of Differentiation (Antiderivatives)

An antiderivative F of the function

$$y = f(x)$$

is a **general solution** to the differential equation

$$\frac{dy}{dx} = f(x).$$

With an initial or boundary condition, we get a **particular solution** of one function, rather than a "$+C$" general solution family.

Substitution on Upper Limit of Integration

Corollary

(Chain Rule for Integrals: Limits of Integration)
Let f be continuous on $[a, b]$ and g differentiable on $[c, d]$, where $g([c, d]) \subseteq [a, b]$.

For $x \in [c, d]$, define

$$F(x) = \int_a^{g(x)} f(t)\,dt.$$

Then F is differentiable on $[c, d]$ and

$$F'(x) = (f \circ g)(x) \cdot g'(x).$$

Substitution on Upper Limit of Integration

Proof Define

$$G(x) = \int_a^x f(t)\,dt,$$

so that

$$F(x) = G \circ g(x) = \int_a^{g(x)} f(t)\,dt$$

on $[c, d]$. Apply the Chain Rule for derivatives and the FTC I. $\blacksquare$

The Integral of a Derivative: FTC II

Theorem

(Fundamental Theorem of Calculus II)*:*

Suppose f is integrable on $[a, b]$, with antiderivative F. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

When computing definite integrals, we often use the notation

$$[F(x)|_a^b = F(b) - F(a)$$

to display the use of the antiderivative F as a function.

The Integral of a Derivative: FTC II

Proof Let

$$H(x) = \int_a^x f(t)dt.$$

By FTC I, $H'(x) = f(x)$. Thus, G defined by

$$G(x) = F(x) - H(x) = F(x) - \int_a^x f(t)dt$$

is a constant, since its derivative is

$$G'(x) = f(x) - H'(x) = 0.$$

The Integral of a Derivative: FTC II

Evaluating G at the endpoints a and b, we have

$$G(a) = f(a) - H(a) = f(a) \quad \text{since } H(a) = 0$$

$$G(b) = f(b) - H(b) = f(b) - \int_a^b f'(t)dt$$

$$G(b) = G(a) \quad \text{since } G \text{ is constant.}$$

$$\text{Thus, } f(a) = f(b) - \int_a^b f'(t)dt.$$

$$\therefore \quad \int_a^b f'(t)dt = f(b) - f(a). \; \blacksquare$$

Theorem

(u-substitution, i.e. Chain Rule for Indefinite Integrals): *Suppose f is continuous with antiderivative F.*
Suppose g is differentiable and g' is continuous.

Then the substitution

$$u = g(x),\ du = g'(x)\,dx$$

can be used to simplify integration of the form:

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Chain Rule for Definite Integrals: Substitution

Theorem

(u-substitution, i.e. Chain Rule for Definite Integrals): *Suppose f is continuous on $[a, b]$ with antiderivative F. Suppose g is differentiable and g' is continuous on $[\alpha, \beta]$, such that*

$$g(\alpha) = a \leq g(t) \leq b = g(\beta).$$

Then the substitution

$$u = g(x),\ du = g'(x)\, dx$$

can be used to simplify integration of the form:

$$\int_{\alpha}^{\beta} f(g(x))g'(x)dx = \int_{a}^{b} f(u)du = F(b) - F(a).$$

Mean Value Theorem for Integrals

Theorem

(Mean Value Theorem, Integrals):
If $f : [a, b] \to \mathbb{R}$ *is continuous, then* $\exists c \in (a, b)$ *such that*

$$\frac{1}{b-a}\int_a^b f(t)dt = f(c).$$

Proof An antiderivative F of f is (uniformly) continuous on $[a, b]$, and so by the MVT for derivatives, $\exists c \in (a, b)$ such that

$$\frac{F(b) - F(a)}{b-a} = F'(c).$$

Due to FTC II,

$$F'(c) = \frac{1}{b-a}\int_a^b f(t)dt = f(c). \quad \blacksquare$$

Integration By Parts

Theorem

(Integration by Parts, Indefinite Integrals)

Suppose f and g are differentiable and f' and g' are integrable. Then

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

Proof Use the product rule for derivatives to show, up to a $+C$,

$$\int [f(x)g'(x) + f'(x)g(x)]dx = \int [f(x)g(x)]'dx. \blacksquare$$

Integration By Parts

Theorem

(Integration by Parts, Definite Integrals)

Suppose f and g are differentiable on $[a, b]$, and f' and g' are integrable on $[a, b]$. Then

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx.$$

Proof Combine the product rule for derivatives and FTC II for

$$\begin{aligned}\int_a^b [f(x)g'(x) + f'(x)g(x)]dx &= \int_a^b [f(x)g(x)]'dx \\ &= f(b)g(b) - f(a)g(a). \quad \blacksquare\end{aligned}$$

Taylor's Theorem Remainder (integral form)

Recall Taylor's Theorem, which generalizes the MVT for higher derivatives:

Theorem

(Taylor's Theorem):
Let $f \in C^{n+1}([a,b])$, and let $x_0 \in [a,b]$. Then, for each $x \in [a,b]$ with $x \neq x_0$, and $n \in \mathbb{N}$, $\exists c = c(n)$ between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

Taylor's Theorem Remainder (integral form)

We can now see that the remainder term

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

can be written as an integral without reference to the MVT's c, and the theorem can be restated:

Theorem

(Taylor's Theorem): *Let $f \in C^{n+1}([a,b])$, and let $x_0 \in [a,b]$. Then, for each $x \in [a,b]$ with $x \neq x_0$, and for each $n \in \mathbb{N}$,*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x - t)^n dt.$$

Improper Integral on $(a, b]$

Let f be defined on $(a, b]$ and integrable on $[c, b]$ for every $c \in (a, b]$. If

$$\lim_{c \to a+} \int_c^b f(x)dx$$

exists, then the **improper integral** of f on $(a, b]$, is denoted by

$$\int_a^b f(x)dx.$$

Improper Integral on $(a, b]$

Certainly, if

$$\lim_{c \to a+} \int_c^b f(x)dx = L < \infty,$$

and f is defined at a, then the improper integral and proper integral on $[a, b]$ match with value L, regardless of the value $f(a)$.

In this case we say the improper integral **converges** to L.

Otherwise, the improper integral **diverges**. (Similar for $[a, b)$.)

Improper Integral on $[a, \infty)$

If f is defined on $[a, \infty)$, and integrable on $[a, c]$ for every $c > a$, then if

$$\lim_{c \to \infty} \int_a^c f(x)dx$$

exists, we call its value the **improper integral** on $[a, \infty)$, and denote it by

$$\int_a^\infty f(x)dx = \lim_{c \to \infty} \int_a^c f(x)dx.$$

(We do similarly for $(-\infty, a]$.)

Antiderivatives Without Closed Forms

Functions built out of algebraic functions (polynomials and roots), exponentials, logarithms, trigonometric functions, and inverse trigonometric functions all have derivatives; the operation

$$f \mapsto f'$$

is *closed* under these *elementary functions*. However,

$$f \mapsto \int f$$

is *not* closed under elementary functions. For example, the antiderivative (integral) of

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

has no *closed form* in terms of elementary functions.

Antiderivatives Without Closed Forms

That said, we can find *infinite series* representations of such functions. For example, recall the power series representation of e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

While we consider e^x to be an elementary function, we do not consider the power series representation as a "closed form" of e^x.

Definition

A **closed form expression** is a mathematical expression that can be evaluated in a finite number of operations.

These operations may be algebraic, trigonometric, exponential, or logarithmic, as these are accepted **elementary functions**.