Linear Algebra
$A = LDU$ Decomposition Examples

Consider the 3x3 linear system $A\vec{x} = b$ given by the equations

$$\begin{aligned} 3x + 4y + 5z &= 6 \\ x - y + 2z &= 5 \\ -6x - 8y - 8z &= -8. \end{aligned}$$

What we would like to do is the following:

(a) Solve the system of equations.

(b) Give the LDU factorization of the system's coefficient matrix.[1]

(c) Give the inverse of the system's coefficient matrix.

The decomposition of $A = LDU$ will have the following structure:

- L is a **lower triangular matrix** (all entries above the diagonal are 0), and all diagonal entries are 1: this matrix corresponds to the **down/forward elimination** part of the process;
- D is a **diagonal matrix** (all entries above and below the diagonal are 0): this matrix corresponds to the **scaling** part of the process;
- U is an **upper triangular matrix** (all entries below the diagonal are 0), and all diagonal entries are 1: this matrix corresponds to the **up/backward substitution** part of the process.

Following this elimination/scaling/substitution method, we'll do all three parts of the problem simultaneously.

Please note that, while this entire process is governed by left multiplication of A by elementary matrices, the same process could be done to the columns of A by multiplying on the right. However, our focus here is on *reducing equations*, so we use elementary matrices to correspond to row operations on A. This way, at any point in the process, you could convert your augmented matrix back into a set of equations, which is noticably "simpler" than the one you started with.

First, we convert the system of equations into the matrix equation $A\vec{x} = b$:

$$\begin{aligned} 3x + 4y + 5z &= 6 \\ x - y + 2z &= 5 \\ -6x - 8y - 8z &= -8 \end{aligned}$$

becomes $A\vec{x} = b$:

$$\begin{pmatrix} 3 & 4 & 5 \\ 1 & -1 & 2 \\ -6 & -8 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -8 \end{pmatrix}.$$

Next, we set up our **augmented matrix** for this system:

$$[A \,|\, b] =$$

Our overall goal is to convert this system, via row operations, through the process

$$[A \,|\, b] \xrightarrow{\text{elimination}} [DU \,|\, L^{-1}b] \xrightarrow{\text{scaling}} [U \,|\, D^{-1}L^{-1}b] \xrightarrow{\text{substitution}} [I \,|\, U^{-1}D^{-1}L^{-1}b = A^{-1}b],$$

yielding the solution $\vec{x} = A^{-1}b$, and, simultaneously, the factorization $A^{-1} = U^{-1}D^{-1}L^{-1}$, which gives us $A = LDU$.

[1]There is a factorization called $PA = LU$; the P is a permutation matrix, which is sometimes needed if row swaps are required. We will ignore this step here, and only see systems where no row swaps are needed. Also, the U in a more traditional $PA = LU$ factorization is our DU; we factor the scaling diagonals into their own matrix to line up with other factorizations we will see in the future.

1 Elimination

First, we eliminate downward by adding rows above to rows below, *eliminating* the lower triangle into zeroes. We'll denote row j by R_j.

$$[A\,|b] = \left[\begin{array}{ccc|c} 3 & 4 & 5 & 6 \\ 1 & -1 & 2 & 5 \\ -6 & -8 & -8 & -8 \end{array}\right]$$
$$\xrightarrow{R_2=-\frac{1}{3}R_1+R_2} \left[\begin{array}{ccc|c} 3 & 4 & 5 & 6 \\ \mathbf{0} & \mathbf{-7/3} & \mathbf{1/3} & \mathbf{3} \\ -6 & -8 & -8 & -8 \end{array}\right]$$
$$\xrightarrow{R_3=2R_1+R_3} \left[\begin{array}{ccc|c} 3 & 4 & 5 & 6 \\ 0 & -7/3 & 1/3 & 3 \\ \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{4} \end{array}\right] = [DU \mid L^{-1}b].$$

As this point we have completed elimination and acquired

$$DU = \begin{bmatrix} 3 & 4 & 5 \\ 0 & -7/3 & 1/3 \\ 0 & 0 & 2 \end{bmatrix}, \quad L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

L is lower triangular, and its diagonal entries are 1.

2 Scaling

Next, we scale the diagonal **pivots** all to 1:

$$[DU \mid L^{-1}b] = \left[\begin{array}{ccc|c} 3 & 4 & 5 & 6 \\ 0 & -7/3 & 1/3 & 3 \\ 0 & 0 & 2 & 4 \end{array}\right]$$
$$\xrightarrow{R_1=\frac{1}{3}R_1, R_2=-\frac{3}{7}R_2, R_3=\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & \frac{4}{3} & \frac{5}{3} & 2 \\ 0 & 1 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & 1 & 2 \end{array}\right] = [U \mid D^{-1}L^{-1}b].$$

We now have the scaling factor D:

$$D^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{3}{7} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \implies D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{7}{3} & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We also know one of the solution values: $z = 2$. We also have the final factor, U:

$$U = \begin{bmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix}$$

We will calculate U^{-1} during the substitution phase.

3 Substitution

Repeat the process you used in elimination, but do it upward to eliminate the upper triangle.

$$[U\,|\,D^{-1}L^{-1}b] = \left[\begin{array}{ccc|c} 1 & \frac{4}{3} & \frac{5}{3} & 2 \\ 0 & 1 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2=\frac{1}{7}R_3+R_2} \left[\begin{array}{ccc|c} 1 & 4/3 & 5/3 & 2 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1=R_1-\frac{4}{3}R_2-\frac{5}{3}R_3} \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] = [I\,|\,U^{-1}D^{-1}L^{-1}b = A^{-1}b = \vec{x}].$$

Thus, our unique solution is

$$\begin{pmatrix} 3 & 4 & 5 \\ 1 & -1 & 2 \\ -6 & -8 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -8 \end{pmatrix} \implies \vec{x} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

and our factorization of A is

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{7}{3} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix}.$$

4 An example with no solutions

An example of an inconsistent system of equations is

$$\begin{aligned} 3x + 4y + 5z &= 6 \\ 3x + 4y + 5z &= 9 \\ -6x - 8y - 8z &= -8. \end{aligned}$$

To see this, attempt to solve the system in the same fashion. Attempting elimination yields

$$[A\,|b] = \left[\begin{array}{ccc|c} 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 9 \\ -6 & -8 & -8 & -8 \end{array}\right]$$

$$\xrightarrow{R_2=-R_1+R_2} \left[\begin{array}{ccc|c} 3 & 4 & 5 & 6 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{3} \\ -6 & -8 & -8 & -8 \end{array}\right]$$

which reduces the system of equations to

$$\begin{aligned} 3x + 4y + 5z &= 6 \\ 0 &= 3 \\ -6x - 8y - 8z &= -8. \end{aligned}$$

This system is clearly nonsense, as $0 = 3$ is a contradiction. Hence, this system has no solution.

5 An example with infinitely many solutions

An example of an underdetermined system of equations is

$$\begin{aligned} 3x + 4y + 5z &= 6 \\ -6x - 8y - 10z &= -12 \\ z &= 2. \end{aligned}$$

To see this, attempt to solve the system in the same fashion. Attempting elimination yields

$$\begin{aligned} [A\,|b] &= \left[\begin{array}{ccc|c} 3 & 4 & 5 & 6 \\ -6 & -8 & -10 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right] \\ &\xrightarrow{R_2 = 2R_1 + R_2} \left[\begin{array}{ccc|c} 3 & 4 & 5 & 6 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & 2 \end{array}\right]. \end{aligned}$$

If we continue with the process, we scale, then substitute to reduce the system to

$$\begin{aligned} &\left[\begin{array}{ccc|c} 3 & 4 & 5 & 6 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & 2 \end{array}\right] \\ \xrightarrow{R_1 = \frac{1}{3}R_1} &\left[\begin{array}{ccc|c} 1 & \frac{4}{3} & \frac{5}{3} & 2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & 2 \end{array}\right] \\ \xrightarrow{R_1 = R_1 - \frac{5}{3}R_3} &\left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{\frac{4}{3}} & \mathbf{0} & -\mathbf{\frac{4}{3}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right], \end{aligned}$$

which, moving the y term to the right hand side, corresponds to the system of equations

$$\begin{aligned} x &= -\frac{4}{3}y - \frac{4}{3} \\ 0 &= 0 \\ z &= 2. \end{aligned}$$

Instead of having a "$y =$" equation for a unique solution, the $0 = 0$ equation contributes no information, and more specifically no reduction of dimension, to the solution set. Also, the system clearly has an infinite number of solutions in the variable y (which we will call a **free variable**; x and z are **pivot variables**, having pivots in the final reduction). The solution can be written as

$$A\vec{x} = b \implies \vec{x} = \begin{pmatrix} -\frac{4}{3}y - \frac{4}{3} \\ y \\ 2 \end{pmatrix} = y\begin{pmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{4}{3} \\ 0 \\ 2 \end{pmatrix}, \ y \in \mathbb{R}.$$

These two vectors have names: the scaled-by-any-y vector $\vec{x}_n = \begin{pmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{pmatrix}$ is a **special solution**, since it solves the system $A\vec{x} = 0$ (try it!), and exists in the **nullspace** $N(A)$ of the matrix A; the second vector $\vec{x}_p = \begin{pmatrix} -\frac{4}{3} \\ 0 \\ 2 \end{pmatrix}$ is the **particular solution** for the given vector b, and exists in the **row space** $C(A^t)$ of the matrix A. We'll see these terms in Chapter 3.