Introduction to Analysis: Sequences

Sequences

Dedekind cuts are a way to define $\mathbb{R}$ via partitions on the order $<$.

We will now develop some theory about **sequences** of rationals and reals.

With this theory of sequences, we develop another definition of the real numbers.

Sequences

Definition

A **sequence** of numbers is an ordered infinite list

$$(a_1, a_2, a_3, ...),$$

a 1-1 correspondence between an ordered set of numbers and $\mathbb{N}$.

We can shorten the notation to, simply, (a_n).

This is different from the *set* of numbers

$$\{a_1, a_2, a_3, ...\}$$

which has no ordering, and cannot contain duplicates.

Limit of a Sequence; Convergent Sequences

Definition

The sequence (a_n) of real numbers has a **limit** $a \in \mathbb{R}$ if,

for any real number $\varepsilon > 0$,

there is some index N, a function of ε, in the sequence,

such that for all terms beyond N,

the value of the sequence is "close" (within ε) of the limit a.

Limit of a Sequence; Convergent Sequences

Symbolically, this fits on one line.

$$\forall \varepsilon > 0,\ \exists N = N(\varepsilon) \in \mathbb{N} :\ \forall n > N,\ |a_n - a| < \varepsilon.$$

If a sequence (a_n) has a limit a, we denote this

$$\lim_{n \to \infty} a_n = a,$$

and say (a_n) **converges to** (or **tends to**) a.

Convergent Sequences

Example

The sequence $(a_n) = (5, 5, 5, 5, \ldots)$ obviously converges to 5:

if $\varepsilon > 0$, then $N = 1$ is a possible $N(\varepsilon)$ such that, for all $n > N$,

$$|a_n - 5| = |5 - 5| = 0 < \varepsilon.$$

Convergent Sequences

Example

The sequence $(a_n) = \left(\frac{4}{n^2-0.2}\right)$ converges, with

$$\lim_{n\to\infty} a_n = 0.$$

We can prove this by showing that, if $\varepsilon > 0$, then

$$N = \left\lceil \sqrt{\frac{4}{\varepsilon} + 0.2} \right\rceil$$

is a possible $N(\varepsilon)$ such that, for all $n > N$,

$$|a_n - 0| = \left| \frac{4}{n^2 - 0.2} \right| < \varepsilon.$$

Relating Convergence and Accumulation

Example

Let A be the set of points in the sequence $(a_n) = \left(\frac{4}{n^2 - 0.2}\right)$:

$$A = \left\{ a_n = \frac{4}{n^2 - 0.2} \;\middle|\; n \in \mathbb{N} \right\}.$$

Then, since (a_n) converges, the set A has an accumulation point at the limit of the sequence.

The order of the points in the sequence describes how "taking a limit" gets "closer" to the accumulation point

$$\lim_{n \to \infty} a_n = 0.$$

Limit of a Sequence; Divergent Sequences

If (a_n) does not converge to a limit, we say (a_n) **diverges**.

A divergent sequence is **unbounded** if

$$\forall M > 0,\ \exists N = N(\varepsilon) \in \mathbb{N} :\ \forall n > N,\ |a_n| > M.$$

(We will refine the notion of unboundedness later.)

A divergent sequence may oscillate amongst multiple **convergent subsequences** with different limits.

Divergent Sequences

Example

The sequence

$$(a_n) = (n)$$

diverges. Any choice of $a \in \mathbb{R}$ as a possible limit yields,

for all $n > \lceil a \rceil + 1$, that

$$|a_n - a| > 1,$$

and so it is not true that $|a_n - a| < \varepsilon$ for any $0 < \varepsilon < 1$.

Divergent Sequences

Example

The sequence

$$(a_n) = \left(\sin\left(\frac{n\pi}{2}\right)\right)$$

diverges. It oscillates among three convergent subsequences:

$$(b_n) = (a_{2n}) = (0),\ (c_n) = (a_{4n+1}) = (1),\ (d_n) = (a_{4n+3}) = (-1).$$

For any $0 < \varepsilon < 1$, there are no $N \in \mathbb{N}$ or a such that $|a_n - a| < \varepsilon$.

Convergent Sequences: When?

We will develop some theorems to help determine whether a sequence is convergent or divergent.

A key to proving several *real analysis* theorems is *bounding*.

Since the definition of convergence allows *any* $\varepsilon > 0$,
we will use a *function* of ε that vanishes[1] to 0 as ε vanishes to 0.

Then, use the Triangle Inequality to trap absolute values by the original ε.

[1] Some common examples of functions of ε in this context are $\frac{\varepsilon}{2}$, $\frac{\varepsilon}{k}$, $|c|\varepsilon$.... You may see in the literature the phrase "$\frac{\varepsilon}{2}$ argument" as a standard, popular in analysis proofs.

Convergent Sequences: When?

Theorem

Let (s_n) *and* (a_n) *be sequences of real numbers, and* $s \in \mathbb{R}$*.*

If for some constant $k > 0$ *and* $m \in \mathbb{N}$ *we have*

$$\forall n \geq m,\ |s - s_n| \leq k|a_n|,$$

then

$$\lim_{n\to\infty} a_n = 0 \implies \lim_{n\to\infty} s_n = s.$$

Convergent Sequences: When?

Proof We go to the definition of the limit of a sequence.

Given an arbitrary $\varepsilon > 0$, we can incorporate fixed k with ε to say

$$\lim_{n \to \infty} a_n = 0$$

means $\exists N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |a_n - 0| = |a_n| < \frac{\varepsilon}{k}.$$

Let $N = \max\{m, N_1\}$. Then, for $n \geq N$,

$$|s_n - s| \leq k|a_n| < k \cdot \frac{\varepsilon}{k} = \varepsilon.$$

Therefore, $\lim_{n \to \infty} s_n = s$. $\blacksquare$

Bounded Sequence

A sequence is called **bounded** if, as a set, it has both upper and lower bounds.

Formally, (a_n) is a **bounded sequence** if $\exists L, U \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N},\; L \leq a_n \leq U.$$

Note that we can use a single value to give bounds:

$$M = \max\{|L|, |U|\} \implies |a_n| \leq M.$$

An immediate truth is that all convergent sequences are bounded.

Convergent Sequence $\implies$ Bounded Sequence

Theorem

If (a_n) is a convergent sequence, with

$$a = \lim_{n \to \infty} a_n,$$

then (a_n) is a bounded sequence.

Proof Pick $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n > N,\ |a_n - a| < \varepsilon$.

Hence, $\forall n > N$, $a - \varepsilon < a_n < a + \varepsilon$.

Setting $M = \max\{|a_1|, |a_2|, ..., |a_N|, |a|\} + \varepsilon$, we therefore have that

$$\forall n \in \mathbb{N},\ |a_n| \leq M. \quad \blacksquare$$

Convergent Sequence $\implies$ Unique Limit

Theorem

If (a_n) is a convergent sequence, then

$$\lim_{n\to\infty} a_n$$

is unique.

(This may seem obvious; the theorem allows the notation.)

Convergent Sequence $\implies$ Unique Limit

Proof Say a and b are both limits of the sequence (a_n).

Pick an arbitrary $\varepsilon > 0$. Then $\exists N_1, N_2 \in \mathbb{N}$ such that

$$\forall n > N_1,\ |a_n - a| < \frac{\varepsilon}{2} \text{ and } \forall n > N_2,\ |a_n - b| < \frac{\varepsilon}{2}.$$

Thus, by the Triangle Inequality, $\forall n > \max\{N_1, N_2\}$,

$$|a - b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore \quad a = b. \quad \blacksquare$

Limits Commute With Arithmetic

Let (a_n) and (b_n) be convergent sequences, with

$$\lim_{n\to\infty} a_n = a, \ \lim_{n\to\infty} b_n = b.$$

We can prove the following:

- linear combinations (sums, differences, constant multiples): if $c, d \in \mathbb{R}$, then the sequence $(ca_n + db_n)$ is also convergent:

$$\begin{aligned}\lim_{n\to\infty}(ca_n + db_n) &= \lim_{n\to\infty} ca_n + \lim_{n\to\infty} db_n \\ &= c \lim_{n\to\infty} a_n + d \lim_{n\to\infty} b_n = ca + db.\end{aligned}$$

Limits Commute With Arithmetic

- products: $(a_n b_n)$ is also convergent:

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = ab,$$

- quotients: if $b_n \neq 0$ and $b \neq 0$, then $\left(\frac{a_n}{b_n}\right)$ is also convergent:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{a}{b}.$$

Proof: Linear Combos of Convergent Sequences Converge

Linear Combinations:

$$c, d \in \mathbb{R} \setminus \{0\}, \quad \lim_{n\to\infty} a_n = a, \; \lim_{n\to\infty} b_n = b$$
$$\implies \lim_{n\to\infty}(ca_n + db_n) = ca + db.$$

Proof Pick $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ such that, for all $n > N$,

$$|a_n - a| < \frac{\varepsilon}{2|c|} \text{ and } |b_n - b| < \frac{\varepsilon}{2|d|}.$$

Proof: Linear Combos of Convergent Sequences Converge

Hence, by the Triangle Inequality,

$$\begin{aligned}
|(ca_n + db_n) - (ca + db)| &\leq |ca_n - ca| + |db_n - db| \\
&\leq |c||a_n - a| + |d||b_n - b| \\
&\leq \frac{\varepsilon|c|}{2|c|} + \frac{\varepsilon|d|}{2|d|} = \varepsilon. \blacksquare
\end{aligned}$$

Limits Preserve $\leq$

Theorem

If one convergent sequence's values are always greater than or equal to another's, then their limits are ordered similarly.

$$\lim_{n\to\infty} s_n = s, \ \lim_{n\to\infty} t_n = t, \ s_n \leq t_n \ \forall n \in \mathbb{N} \implies s \leq t.$$

Proof Suppose (for a contradiction) that $s > t$. Let $\varepsilon = \frac{s-t}{2} > 0$.

Then $2\varepsilon = s - t$ and $t + \varepsilon = s - \varepsilon$. Thus $\exists N_1 \in \mathbb{N}$: for all $n > N_1$,

$$s - \varepsilon < s_n < s + \varepsilon.$$

Limits Preserve $\leq$

Similarly, $\exists N_2 \in \mathbb{N}$ such that, for all $n > N_2$,

$$t - \varepsilon < t_n < t + \varepsilon.$$

Let $N = \max\{N_1, N_2\}$. Then for $n > N$ we have

$$t_n < t + \varepsilon = s - \varepsilon < s_n,$$

contradicting $s_n \leq t_n$. $\rightarrow\leftarrow$ $\quad \therefore s \leq t$. $\blacksquare$

Sequence Convergence: Ratio Test

Theorem

Suppose $s_n > 0$ *for all* $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = L.$$

If $L < 1$, *then*

$$\lim_{n\to\infty} s_n = 0.$$

Proof The previous theorem implies that $L \geq 0$. Suppose $L < 1$.

Then $\exists c \in \mathbb{R}$ such that $0 \leq L < c < 1$. Let $\varepsilon = c - L > 0$.

Sequence Convergence: Ratio Test

Then, since $\frac{s_{n+1}}{s_n} \to L$, $\exists N \in \mathbb{N}$ such that

$$\forall n > N, \ \left| \frac{s_{n+1}}{s_n} - L \right| < \varepsilon.$$

Let $k = N + 1$. Then $\forall n \geq k$ we have $n - 1 \geq N$, so that

$$\frac{s_{n+1}}{s_n} < L + \varepsilon = L + (c - L) = c.$$

Sequence Convergence: Ratio Test

Thus, it follows that

$$\forall n \geq k,\ 0 < s_n < s_{n-1}c < s_{n-2}c^2 < \cdots < s_k c^{n-k}.$$

Letting $M = \frac{s_k}{c^k}$, we obtain $0 < s_n < Mc^n$ for all $n \geq k$.

Since $0 < c < 1$, we can see that

$$\lim_{n\to\infty} c^n = 0 \implies \lim_{n\to\infty} s_n = 0. \blacksquare$$

Sequence Divergence: Infinite Limits

A sequence **diverges to** $+\infty$, i.e.

$$\lim_{n\to\infty} s_n = +\infty,$$

if and only if for any $M \in \mathbb{R}$,

$$\exists N \in \mathbb{N} \ni n \geq N \implies s_n > M.$$

Sequence Divergence: Infinite Limits

Likewise, a sequence **diverges to** $-\infty$, i.e.

$$\lim_{n\to\infty} s_n = -\infty,$$

if and only if for any $M \in \mathbb{R}$,

$$\exists N \in \mathbb{N} \ni n \geq N \implies s_n < M.$$

(Here, $+\infty$ and $-\infty$ are not real numbers and merely act as notation describing unboundedness.)

Sequence Divergence: Infinite Limits

Theorem

Suppose that $s_n \leq t_n$ for all $n \in \mathbb{N}$.

$$(a)\ \lim_{n\to\infty} s_n = +\infty \implies \lim_{n\to\infty} t_n = +\infty.$$

$$(b)\ \lim_{n\to\infty} t_n = -\infty \implies \lim_{n\to\infty} s_n = -\infty.$$

Theorem

Let $s_n > 0$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} s_n = +\infty \iff \lim_{n\to\infty} \frac{1}{s_n} = 0.$$

Increasing, Decreasing Sequences

Let (a_n) be a sequence of real numbers.

- If $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, we call (a_n) a **monotonically increasing** (or **nondecreasing**) sequence.
- If $a_{n+1} > a_n$ for all $n \in \mathbb{N}$, we call (a_n) a **strictly increasing** sequence.
- Likewise,
  $a_{n+1} \leq a_n$ is a **monotonically decreasing** (**nonincreasing**),
  and $a_{n+1} < a_n$ is a **strictly decreasing**, sequence.

Increasing, Bounded Sequence $\Longrightarrow$ Convergent Sequence

The fact that a sequence is increasing or decreasing is a property that gives a partial converse to the fact that convergent sequences are bounded.

Theorem

If (a_n) is a monotone (increasing or decreasing), bounded sequence, then (a_n) is a convergent sequence.

Proof If (a_n) is increasing, then $(-a_n)$ is decreasing, and so the "decreasing" version of the theorem follows immediately from the "increasing" version.

Increasing, Bounded Sequence $\implies$ Convergent Sequence

Suppose (a_n) is increasing. Then, as a set

$$A = \{a_n : \ n \in \mathbb{N}\},$$

A has upper and lower bounds:

- a_1 is a lower bound, and
- we will call an upper bound u.

By the Completeness Axiom, A has a sup: we will call it

$$\sup(A) = a.$$

We need to show that $\sup(A) = a$ is, in fact, the limit of (a_n).

Increasing, Bounded Sequence $\implies$ Convergent Sequence

For any $\varepsilon > 0$, $a = \sup(A)$, so $a - \varepsilon$ is not an upper bound of (a_n).

Thus,

$$\exists N \in \mathbb{N} : \; a - \varepsilon < a_N.$$

But a is an upper bound, so $a_N \leq a$.

Since (a_n) is increasing, then we have

$$\forall n > N, \quad a - \varepsilon < a_N \leq a_n \leq a.$$

This implies

$$\forall n > N, \quad |a_n - a| < \varepsilon,$$

which means that, since $\varepsilon > 0$ was chosen arbitrarily,

$$\lim_{n \to \infty} a_n = a. \quad \blacksquare$$

Monotone Convergence Theorem

Together, the two theorems

Theorem
If (a_n) is a convergent sequence, then (a_n) is a bounded sequence.

and

Theorem
If (a_n) is a monotone and bounded sequence, then (a_n) converges.

yield the

Theorem
(Monotone Convergence Theorem) *Suppose (a_n) is monotone. Then (a_n) converges $\iff$ (a_n) is bounded.*

Monotone Divergence

Theorem
Suppose (s_n) is an unbounded sequence. Then

$$\text{(a) } (s_n) \text{ increasing} \implies \lim_{n\to\infty} s_n = +\infty.$$

$$\text{(b) } (s_n) \text{ decreasing} \implies \lim_{n\to\infty} s_n = -\infty.$$

Proof We will prove (a).

If (s_n) is increasing and unbounded, then $(-s_n)$ is decreasing and unbounded, and so (b) can be deduced from (a).

Monotone Divergence

(s_n) increasing $\implies$ s_1 is a lower bound for

$$S = \{s_n : \ n \in \mathbb{N}\},$$

the set of the elements of the sequence, devoid of order.

Thus, S is unbounded above (as a set).

Hence, for any $M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $s_N > M$.

Thus, for any $n \geq N$, $s_n > S_N > M$, and so

$$\lim_{n\to\infty} s_n = +\infty. \ \blacksquare$$

Subsequences

A **subsequence** of a sequence (x_n) is a sequence (y_k) consisting of some[2] of the terms of (x_n), in the same order.

Formally, a subsequence (y_k) of (x_n) is generated by a strictly increasing function

$$k : \mathbb{N} \to \mathbb{N} : \quad y_k = x_{n(k)} \text{ for every } k = 1, 2, 3, \ldots.$$

[2] but not necessarily all

Subsequences

Example

If $(x_n) = (\frac{1}{n})$, then the subsequence consisting of only perfect square denominators is

$$(y_k): \; y_k = \frac{1}{k^2} = x_{k^2}, \; k = 1, 2, 3, ...$$

i.e. the subsequence index function is $n(k) = k^2$.

Convergent Subsequences

Theorem
Assume (a_n) converges.
Then every subsequence of (a_n) also converges, to the same limit.

It is not obvious that some divergent sequences may contain convergent subsequences.

Theorem
(x_n) *is bounded* $\implies$ (x_n) *contains a convergent subsequence.*

Convergent Subsequences

Proof Let S be the set of distinct values from the sequence.

There are two cases:

1. If S is a finite set, then at least one of the values in S is repeated infinitely often in the sequence (call it x).

   Select the subsequence $s_{n(k)} = x$ for every k.

2. If S is an infinite set, then by the Bolzano-Weierstrass Theorem, S has an accumulation point (call it y).

Convergent Subsequences

We now construct a subsequence of (y_k) converging to y.

For each $k \in \mathbb{N}$, let

$$A_k = \left(y - \frac{1}{k}, y + \frac{1}{k}\right) = N\left(y, \frac{1}{k}\right)$$

be the neighborhood about y of radius $\frac{1}{k}$.

Convergent Subsequences

Since y is an accumulation point of S, there are an infinite number of distinct points

$$s_{n(k)} \in A_k \subseteq S.$$

Pick one and call it y_k. Repeating the process, making sure

$$n(k+1) > n(k)$$

for every k, yields a sequence (y_k) that converges to y, since

$$\forall k \in \mathbb{N},\ |y_k - y| < \frac{1}{k}. \blacksquare$$

Unbounded $\implies$ Monotone Unbounded Subsequence

Theorem

Let (s_n) be an unbounded sequence.

Then (s_n) contains a monotone subsequence that converges to $+\infty$ (if unbounded above) or $-\infty$ (if unbounded below).

Proof We will prove this for (s_n) that is unbounded above, and (s_n) unbounded below follows from $(-s_n)$ being unbounded above.

Unbounded $\implies$ Monotone Unbounded Subsequence

Given any $M \in \mathbb{R}$, there are infinitely many s_n such that $s_n > M$.

Since the set of indices

$$I_M = \{n : s_n > M\} \subseteq \mathbb{N}$$

(and are nested, since $M_1 > M_2$ implies $I_{M_1} \subseteq I_{M_2}$),
then by well ordering, I_M has a smallest element.

Unbounded $\implies$ Monotone Unbounded Subsequence

Select

$$n(k) = \min(I_k \setminus \{n(1), n(2), ..., n(k-1)\})$$

for each $k \in \mathbb{N}$ as your index set.

Also, I_M is infinite, so there will always be more $n(k)$ to choose.

Then the subsequence $(s_{n(k)})$ is monotonically increasing and unbounded, and $\lim_{k\to\infty} s_{n(k)} = +\infty$. $\blacksquare$

Limit Superior

The **limit superior**, or **upper limit**, of a sequence (s_n) is the supremum of the limits of all convergent subsequences of (s_n).

Symbolically,

$$\limsup s_n = \lim_{n \to \infty} \sup_{k \geq n} s_k.$$

The sequence

$$(y_n): \;\; y_n = \sup_{k \geq n} s_k$$

is a monotone decreasing sequence.

Limit Inferior

Likewise, the **limit inferior**, or **lower limit**, of a sequence (s_n) is the infimum of the limits of all convergent subsequences of (s_n).

Symbolically,

$$\liminf s_n = \lim_{n\to\infty} \inf_{k\geq n} s_k.$$

The sequence

$$(z_n): \; z_n = \inf_{k\geq n} s_k$$

is a monotone increasing sequence.

Subsequential Limits, Oscillation

A **subsequential limit** is the limit of a subsequence of (s_n).

If S is the set of subsequential limits of (s_n), then

$$\liminf s_n = \inf S \leq \sup S = \limsup s_n.$$

If

$$-\infty < \liminf s_n = \limsup s_n < \infty,$$

then (s_n) converges and these values match $\lim_{n\to\infty} s_n$.

If

$$\liminf s_n < \limsup s_n,$$

we say that the sequence (s_n) **oscillates**.

Metric

Recall, a **metric** $d(x, y)$ generalizes the notion of distance.

We most commonly use *absolute value (the Euclidean metric)*

$$d(x, y) = |x - y| \text{ for } x, y \in \mathbb{R}$$

to describe distance, but there are other metrics.

Metric

A metric on $\mathbb{R}$ has four properties: $\forall x, y, z \in \mathbb{R}$,

- $d(x, y) \geq 0$ (distance is nonnegative)
- $d(x, x) = 0$ (distance from a point to itself is 0)
- $d(x, y) = d(y, x)$ (distance is symmetric)
- $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality)

Cauchy Sequences

Definition

A **Cauchy sequence**[3] of reals is a sequence (a_n) such that,

for any distance, there is an index in the sequence where all the following points are that close, or closer, together.

Symbolically,

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} : \forall m, n > N,\ d(a_m, a_n) < \varepsilon.$$

[3] named after Augustin-Louis Cauchy (1789-1857)

Convergent sequences are Cauchy sequences

Theorem

Every convergent sequence (a_n) is a Cauchy sequence.

Proof If

$$\lim_{n\to\infty} a_n = a,$$

then, for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\forall n > N, \quad d(a_n, a) < \frac{\varepsilon}{2}.$$

Hence, by the Triangle Inequality,

$$\forall m, n > N, \quad d(a_n, a_m) \leq d(a_n, a) + d(a, a_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare$$

Cauchy sequences are convergent sequences

Theorem

Every Cauchy sequence (a_n) is a convergent sequence.

Proof Let (a_n) be a Cauchy sequence. Fix $\varepsilon > 0$.

We use an $\frac{\varepsilon}{2}$ argument. Since (a_n) is Cauchy,

$$\exists N = N(\varepsilon) \in \mathbb{N} : \ \forall m, n > N, \quad d(a_n, a_m) < \frac{\varepsilon}{2}.$$

In particular, for all $n > N$,

$$d(a_n, a_N) < \frac{\varepsilon}{2}.$$

Cauchy sequences are convergent sequences

Thus, the sequence (a_n) is bounded, since

$$\forall n > N, \;\; a_n \in B_{\varepsilon/2}(a_N),$$

and the points $a_1, a_2, ..., a_N$ are a finite number of points.

The sequence (a_n) is bounded, so it has a convergent subsequence.

Call this subsequence $(a_{i(n)})$ for some index $i(n)$, with limit a:

$$\lim_{n\to\infty} a_{i(n)} = a.$$

Cauchy sequences are convergent sequences

We need to show that a is the limit of the entire sequence (a_n).

Since a is the limit of $(a_{i(n)})$, there are an infinite number of

$$a_{i(n)} \in B_{\varepsilon/2}(a_N).$$

Hence, since (a_n) is Cauchy, then by the triangle inequality,

$$\forall m, i(n) > N : \; d(a_m, a) \leq d(a_m, a_{i(n)}) + d(a_{i(n)}, a) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Cauchy sequences are convergent sequences

But $\varepsilon > 0$ was chosen arbitrarily, so this is precisely the definition of a convergent sequence:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} :\ \forall m > N,\ d(a_m, a) < \varepsilon.$$

$$\therefore \ \lim_{m \to \infty} a_m = a. \ \blacksquare$$

Cauchy $\iff$ convergent

Combining these two theorems, we can identify convergence of a sequence with "Cauchy-ness".

Theorem

A sequence (a_n) is Cauchy if and only if (a_n) converges.

Another definition of the real numbers: Cauchy sequences

The Cauchy criterion for sequences allows for a definition of the real numbers $\mathbb{R}$ as constructed from sequences of rationals.

Definition

The **real numbers** $\mathbb{R}$ are the set of equivalence classes of Cauchy sequences of rational numbers, whose equivalence is determined by their limits being equal.

Another definition of the real numbers: Cauchy sequences

For each Cauchy sequence of rationals (a_n), define its equivalence class by

$$[(a_n)] = \left\{(b_n) : \; b_n \in \mathbb{Q}, \, (b_n) \text{ Cauchy}, \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n\right\}.$$

Then the real numbers are defined by

$$\mathbb{R} = \{[(a_n)] : \; a_n \in \mathbb{Q}, \, (a_n) \text{ Cauchy}\}.$$

The real numbers "complete" the rationals, limit-wise.

This definition of the real numbers, like Dedekind's, is motivated by the notion of **completion** of the rationals.

There are numbers, like $\sqrt{2}$ and π, that sit in the "gaps" between rationals, representing real lengths, but are not themselves rational.

$\mathbb{R}$ is the smallest set that "fills the gaps" of $\mathbb{Q}$, making a **complete** metric space in the sense of containing its sequences' limits.[4]

[4] Note that this is not the same as being *algebraically closed*, in the sense of solving polynomials with coefficients from the same field you wish to solve with. $\mathbb{R}$ is not algebraically closed; the set of **complex numbers**, $\mathbb{C}$ is.