Homework 2

CSE 202 Fall 2024

Due at: 11:59 PM Monday, October 21

For each problem, provide a high-level description of your algorithm. Please make sure to include the necessary details that are crucial for its correctness and efficiency. Prove its correctness and analyze its time complexity.

Background: graph reachability, BFS, DFS, bridges, cut vertices, DAG, topological ordering, 2-SAT, Dijkstra's algorithm, heap

Graded Problems

1. (20 points) You are given a directed graph (in adjacency list format) with n vertices and m edges. Each edge is colored either R, G or B. Additionally, you are provided with a source node s and a target node t. Design an algorithm to determine whether there exists a path from s to t such that no two consecutive edges along the path share the same color.

High Level Description: To determine whether there exists a path from s to t such that no two consecutive edge along the path share the same color, we can design an algorithm by modifying breadth-first search. We keep track of each node respect to the color of the incoming edge starting from s, and explore its adjacent nodes while ensuring that no consecutive edge has the same incoming color.

Algorithm: Each node in the graph has state represented by tuple (u, c), where u is the current node being explore, and c is the color of the edge that were used to reach the current node. We then use a queue for states to perform BFS. Begin with the starting node s with state (s, None), we append it to the queue. While the queue is not empty, we pop the state (u, c) of node u from the front of the queue. For every adjacent edge (u, v) such that (v, c'), the state of node v is unvisited, if the color c' of its state (v, c') is different from the color c in (u, c), we append its state (v, c') to the queue and mark (v, c') as visited and continue doing so. When the queue is empty, if the target node t is marked visited, that indicate there exists a valid path from node s to t. If node t is not visited, there is no such a path.

Algorithm 1 Find Valid Path with Alternating Edge Colors

```
1: procedure FINDPATHWITHCOLORS(G, s, t)
        Q \leftarrow \text{Empty Queue}
 2:
        Visited \leftarrow \text{Empty Set}
 3:
        Enqueue (s, None) into Q
 4:
 5:
        Add (s, None) to Visited
        while Q is not empty do
 6:
           (u,c) \leftarrow \text{Dequeue from } Q
 7:
           if u = t then
 8:
               return There exist valid path
 9:
           end if
10:
           for each neighbor (v, c') of u do
11:
               if c \neq c' then
12:
                   if (v,c') \notin Visited then
13:
                       Enqueue (v, c') into Q
14:
                       Add (v, c') to Visited
15:
16:
                   end if
               end if
17:
           end for
18:
        end while
19:
        return No valid path
20:
21: end procedure
```

Runtime Analysis: Initialization takes O(1), since BFS explore each nodes and its neighbors, as we considering all colors for each node, each node can be processed at most 3 times, that still takes O(n). We also traverse through all edges once, which is O(m). Hence, the total time complexity is O(n+m)

Proof of Correctness:

Base Case:

At the start, the BFS initializes the queue with (s, None), which represents the source node s without any prior edge color constraint. Thus, the first edge taken from s can be in any color. Therefore the algorithm can correctly start exploring all edges from s.

Inductive Steps:

During BFS, the algorithm explores each node u and checks for its adjacent nodes v. For each v, the algorithm ensure the edge from u to v is different in color from the edge to get node u.

Since our algorithms are using BFS, it explore each node by the distance from s, it will find the shortest valid path with alternative color if it exist. If such path exist, the algorithm will eventually explore node t and marked its state as visited. Using BFS also makes sure the algorithm explore all possible path from s to t while satisfy the condition of no two consecutive edges color along the path.

The algorithm will terminate when the queue is empty. Once queue is empty, if there is a valid path, the state of target node t should be marked visited. If the queue is empty without visited the state of t, that means we can't get to t from s Using BFS under the constraint, there is no valid path.

Therefore, the algorithm is correct and will find a valid path from s to t if such path exists.

- 2. (20 points) There are m constraints on n boolean variables x_1, \ldots, x_n . Each constraint is given by three indices i, j, k and three boolean values b_1, b_2, b_3 . (The boolean values can be different for different constraints.) The constraint is satisfied if and only if at least two of the following conditions are met:
 - (a) $x_i = b_1$
 - (b) $x_i = b_2$
 - (c) $x_k = b_3$

To make the problem simpler, we can assume that i, j, k are always distinct. Design an algorithm that decide whether there exists an assignment such that all the constraints can be satisfied.

We are given three-variable constraints of the form:

$$(x_i = b_1), (x_i = b_2), (x_k = b_3)$$

The constraint is satisfied if at least two of these conditions hold. Our task is to represent this as a 2-SAT problem by constructing implications.

For each three-variable constraint, we derive the implications for the possible cases:

Case 1:
$$x_i = b_1$$
 and $x_j = b_2$

If either $x_i \neq b_1$ or $x_j \neq b_2$, then $x_k = b_3$ must hold. The implications are:

$$\neg (x_i = b_1) \to (x_j = b_2) \land (x_k = b_3)$$

$$\neg(x_j = b_2) \to (x_i = b_1) \land (x_k = b_3)$$

Case 2: $x_i = b_1$ and $x_k = b_3$

If either $x_i \neq b_1$ or $x_k \neq b_3$, then $x_j = b_2$ must hold. The implications are:

$$\neg (x_i = b_1) \to (x_i = b_2) \land (x_k = b_3)$$

$$\neg (x_k = b_3) \to (x_i = b_1) \land (x_j = b_2)$$

Case 3: $x_i = b_2$ and $x_k = b_3$

If either $x_i \neq b_2$ or $x_k \neq b_3$, then $x_i = b_1$ must hold. The implications are:

$$\neg(x_j = b_2) \to (x_i = b_1) \land (x_k = b_3)$$

$$\neg (x_k = b_3) \to (x_i = b_1) \land (x_j = b_2)$$

Then for each constraint, we have:

$$f = ((x_j = b_2) \lor (x_k = b_3)) \land ((x_i = b_1) \lor (x_k = b_3)) \land ((x_i = b_1) \lor (x_j = b_2))$$

For all constraints, we have:

$$F = \prod_{c=1}^{m} \left((x_{c_i} = b_{c_2}) \lor (x_{c_k} = b_{c_3}) \right) \land \left((x_{c_i} = b_{c_1}) \lor (x_{c_k} = b_{c_3}) \right) \land \left((x_{c_i} = b_{c_1}) \lor (x_{c_i} = b_{c_2}) \right)$$

Now, for each constraint, whenever there are at least two conditions are met, the constraints can be satisfied.

With the conjunctive expression for all constraint clauses, we can convert the solution of the original question to a **2SAT** solution.

3. (15 points) You are given an array A of n integers. Your goal is to construct a min-heap of A: i.e., an array B that consists of every element of A, and satisfies

$$B[i] \le B[2i]$$
 and $B[i] \le B[2i+1]$,

whenever the indices exist. (We are using 1-indexing in this problem.) Show a linear-time algorithm that completes this task.

Hint: you may use the fact that the infinite series $\sum_{n=0}^{\infty} n2^{-n}$ converges to a constant.

For array A, We traverse the array A from index n/2 to 1 to call the heapify method, since it's 1-indexing. We assume the latter half of array A is the leaf node that does not need to be iterated. In other words, we start the iteration from the bottom of the heap.

Algorithm 2 Min-Heap

```
1: procedure \text{HEAPIFY}(B, i)
        left \leftarrow 2 \times i
2:
        right \leftarrow 2 \times i + 1
3:
        smallest \leftarrow i
4:
        if B[left] < B[smallest] then
5:
            smallest \leftarrow left
6:
        end if
7:
        if B[right] < B[smallest] then
8:
            smallest \leftarrow right
9:
        end if
10:
        if smallest \neq i then
11:
            Swap B[I] and B[smallest]
12:
            Call Heapify (B, smallest)
13:
        end if
14:
15: end procedure
```

We can easily know that there are approximately $\frac{n}{2^{h+1}}$ nodes at height h. For each node at height h, the time complexity of the heapify operation is O(h), since the heapify process may adjust nodes up to that height.

The total time complexity is the sum of the heapify costs for all nodes, which can be expressed as:

Total Complexity =
$$\sum_{h=0}^{\log n} \left(\frac{n}{2^{h+1}} \cdot h \right)$$

The hint says such an infinite series is convergent. So we can conclude that such an algorithm has a linear time complexity.

- 4. (25 points) There are n segments sitting on the real axis. Each segment L_i is of the form $[a_i, b_i]$ where $a_i \leq b_i$, i.e., a closed interval. We say two segments L_i, L_j intersect if $[a_i, b_i] \cap [a_j, b_j]$ is non-empty. You do not know what a_i, b_i are for every segment, but you learn m conditions, each of which tells you exactly one of the following for some pair of indices $1 \leq i, j \leq n$:
 - either L_i and L_j intersect, or
 - L_i and L_j do not intersect and $b_i < a_j$, i.e. L_i lies before L_j

Design an algorithm that takes as input these m conditions, and decides whether such n segments exist (i.e., there exist $L_i = [a_i, b_i]$ for all $1 \le i \le n$ satisfying all the m conditions).

We shall solve this problem by reducing it to a topological sort problem on a directed graph. First, we shall create a directed graph based on m given conditions.

Directed graph creation

Node creation: For each segment L_i , we create two nodes in the graph:

- a_i : Represents the start of the segment L_i .
- b_i : Represents the end of the segment L_i .

Edge creation: For each segment L_i , we add a directed edge from a_i to b_i :

$$a_i \to b_i$$

This edge enforces the constraint that the start of the segment must occur before the end of the segment.

For each intersection condition between segments L_i and L_j , we add two directed edges:

$$a_i \to b_j$$
 (Segment L_i starts before segment L_j ends)
 $a_j \to b_i$ (Segment L_j starts before segment L_i ends)

These edges ensure that both segments overlap, as required by the intersection condition.

For each non-intersection (comes before) condition $b_i < a_j$, we add a directed edge:

$$b_i \to a_j$$

This edge enforces that segment L_i must end before segment L_j starts, ensuring that the two segments do not intersect.

Reduction to topological sort:

Yes to Yes: The Graph is a DAG for a Valid Assignment of Intervals

We now prove that if there is a valid assignment of n intervals, the resulting graph must be a Directed Acyclic Graph (DAG), and we will be able to sort it topologically.

Proof: Suppose there exists a valid assignment of intervals such that all n intervals are arranged on the real line without violating any of the given conditions (either intersection or "comes before"). We claim that the resulting directed graph must be a DAG.

Revisiting the structure of the graph:

- For each interval $L_i = (a_i, b_i)$, we add a directed edge $a_i \to b_i$, enforcing that the start of an interval must precede its end.
- For every pair of intersecting intervals L_i and L_j , we add two directed edges: $a_i \to b_j$ and $a_j \to b_i$, representing the mutual constraint that the intervals must overlap.
- For every non-intersecting ("comes before") condition, we add a directed edge $b_i \to a_j$, enforcing that L_i must end before L_j starts.

Assume for the sake of contradiction that the graph contains a cycle. A cycle implies that we have a sequence of directed edges that eventually loops back to a starting node. This would correspond to a contradiction in the assignment of intervals because it would mean that some interval both starts before and ends after another interval, and vice versa, violating either the intersection or the "comes before" condition. Such a contradiction would imply that no valid assignment exists.

Since we assumed that a valid assignment of intervals exists, no such cycle can form in the graph. Therefore, the graph must be acyclic.

Once we have established that the graph is a DAG, we know that it can be topologically sorted.

Thus, if there is a valid assignment of intervals, the resulting graph must be a DAG, and we will be able to sort it topologically. QED

No to No: A DAG Implies a Valid Assignment of Intervals

Proof by contradiction: Assume that the graph is a DAG, and we have performed a successful topological sort of the nodes. Suppose, for the sake of contradiction, that it is not possible to assign a valid set of n intervals that satisfies all the given conditions.

This means there must be some condition in the set of m conditions (either intersection or "comes before") that cannot be satisfied in the current assignment of intervals.

Let's consider the topological sort process. For a node u representing an interval, the sort processes u only when its incoming edge count is zero, meaning all constraints related to u have been satisfied by previously processed nodes. If there were a condition m that couldn't be satisfied, it would imply that the topological sort encountered a node whose constraints were not respected, which leads to a contradiction:

- Intersection Condition: If there's an intersection condition between nodes u and v, the directed edges $u \to v$ and $v \to u$ ensure that both intervals overlap. When node u is processed, its incoming edge count being zero guarantees that node v, and any other nodes it depends on, have already been processed. This ensures that the intersection condition is satisfied. If this weren't true, it would contradict the fact that the topological sort only processes a node when all incoming edges (dependencies) are resolved.
- "Comes Before" Condition: If there's a "comes before" condition (i.e., u must come before v), the edge $u \to v$ enforces this order. When node u is processed, all preceding nodes (those with edges pointing into u) have already been placed, ensuring that u respects the order imposed by the "comes before" condition. Any violation of this would again contradict the correctness of the topological sort, where nodes are processed only after their dependencies are fully resolved.

Since the topological sort ensures that each node is processed only when all its constraints are satisfied (represented by incoming edge count = 0), the existence of a condition that cannot be satisfied forms a contradiction.

Thus, a successful topological sort guarantees that there is at least one valid way to assign all n intervals, satisfying all the conditions. QED

Algorithm 3 Check Valid Interval Assignment

```
1: procedure ValidAssignment(n, m, conditions)
        G \leftarrow \text{Empty directed graph}
 2:
        in\_degree \leftarrow array of size 2n initialized to 0
 3:
        for i \in \{1, ..., n\} do
 4:
 5:
            Add node a_i (start of interval L_i) to G
            Add node b_i (end of interval L_i) to G
 6:
            Add directed edge (a_i \to b_i) to G
 7:
            in\_degree[b_i] \leftarrow in\_degree[b_i] + 1
 8:
        end for
 9:
        for all (L_i, L_j, \text{type}) \in conditions do
10:
            if type = "intersection" then
11:
12:
                Add directed edge (a_i \to b_i) to G
                in\_degree[b_i] \leftarrow in\_degree[b_i] + 1
13:
                Add directed edge (a_i \to b_i) to G
14:
                in\_degree[b_i] \leftarrow in\_degree[b_i] + 1
15:
            else if type = "comes before" then
16:
                Add directed edge (b_i \to a_i) to G
17:
               in\_degree[a_j] \leftarrow in\_degree[a_j] + 1
18:
            end if
19:
        end for
20:
        Perform Topological Sort on the Graph
21:
22:
        Check if topological sort succeeds
        if Topological sort fails then
23:
            return False
24:
25:
        else
            return True
26:
        end if
27:
28: end procedure
```

Runtime Analysis

The overall time complexity of the algorithm is O(n+m), where n is the number of intervals and m is the number of conditions. The breakdown is as follows:

Graph Construction: For each interval L_i , we create two nodes a_i and b_i , resulting in 2n nodes. For each "comes before" condition, we add one edge, and for each "intersection" condition, we add two edges, giving m edges in total.

Time Complexity: O(n+m)

Topological Sort: The topological sort processes all 2n nodes and m edges.

Time Complexity: O(n+m)

Validity Check: After the sort, we check if all nodes were processed to detect cycles.

Time Complexity: O(n)

Total Time Complexity: The total runtime is O(n+m) + O(n+m) + O(n) = O(n+m)

O(n+m).

5. (20 points) You are given an undirected graph with n vertices and m edges (in adjacency list format), as well as a source node s and a target t. Each edge has length either 1 or 2. Give a linear time¹ algorithm that computes the length of the shortest path from s to t.

Let's first consider the simplest case: finding the shortest path in an undirected graph with edge weights of 1 (or equivalently scaled, so that all edge weights are equal). We can use BFS for this purpose. Starting from the source node (the source node is the first layer, corresponding to distance 0), for all nodes at each layer, we can determine that the shortest distance from the source node to these nodes is equal to the path length of the BFS traversal.

Proof:

By contradiction, for a pair of nodes (u, v), assume node v is reached from node u during the i-th phase of BFS. The shortest path from u to v should be of length i. Now suppose there exists a path from u to v with a length less than i. In the BFS algorithm, all intermediate nodes along this path must have been visited between the 0-th and the (i-1)-th phases. Therefore, node v should have been reached before the i-th phase, which contradicts our assumption. Thus, the proof is complete.

When dealing with a more complex case, namely finding the shortest path in an undirected graph with edge weights of 1 or 2, we can use a similar approach as in the equal-weight undirected graph, specifically BFS, to track the shortest path from the source node to each node. An intuitive transformation is to add a virtual node in the middle of an edge with weight 2. Then, we can do global BFS and record their distance at the same time.

However, the implementation of above virtual nodes is complicated so we use two queues to simulate the process. For nodes that can be reached from the current node

¹Here, linear time means O(n+m).

via an edge of weight 1, we place them in the queue for the next expansion. For nodes that can be reached from the current node via an edge of weight 2, we place them in the queue for the expansion after the next one (note that the next expansion queue includes nodes that can be reached from the current node via edges of weight 1, while the subsequent expansion queue includes nodes that can be reached from the next nodes via edges of weight 1 as well as nodes that can be reached from the current node via edges of weight 2).

The correctness of this algorithm is that even if we revisit the node visited from the current node along the edge with weight 2 in the next expansion, it will not cause it to be reentered into the queue because the conditional judgment is not satisfied (the condition is dist[node] + w < dist[nei]). The condition judgment here actually acts like a visited array, so that for all nodes, we will only queue them up once.

The above case analysis ensures that no node needs to be re-enqueued during BFS. For every two expansion queues, we can apply a similar case analysis to ensure that each node is visited only once. Therefore, in an undirected graph with edge weights of 1 and 2, BFS can be used to find the shortest path, as shown by the algorithm in the next page.

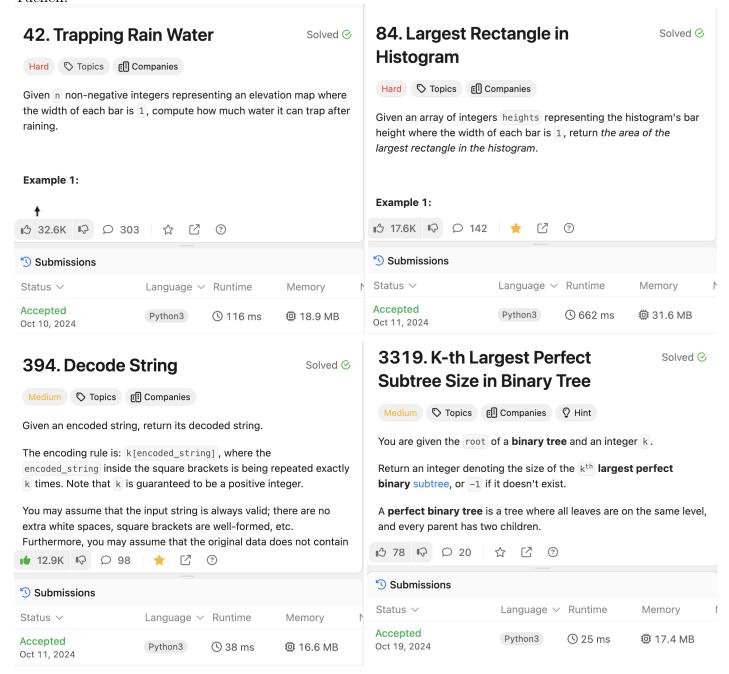
Regarding time complexity, since every node is enqueued and dequeued only once in all the queues (in practice, we only use two queues and, after each expansion, directly place the old subsequent execution queue in the current phase's next execution queue), the algorithm runs in linear time. Essentially, in this special type of graph with edge weights of 1 and 2, we utilize the specific conditions to replace the insert-search operations (which take $O(\log m)$) in traditional shortest path algorithms with simple enqueue-dequeue operations. This reduces the time complexity from $O((n+m)\log n)$ to O(n+m).

Algorithm 4 Shortest Path in Graph with Weights 1 and 2

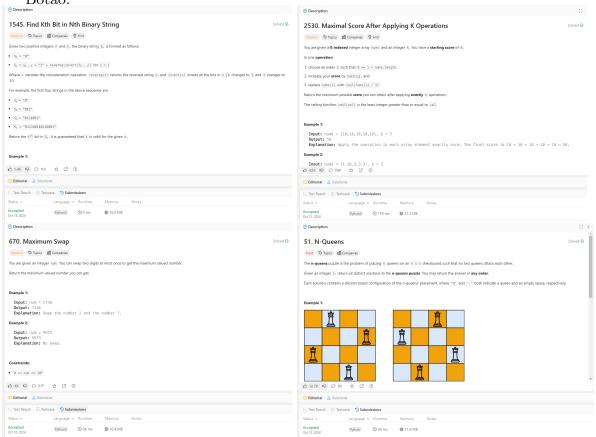
```
1: procedure ShortestPath(n, edges, s, t)
       Initialize adjacency list adj for n nodes
       for each edge (u, v, w) in edges do
 3:
 4:
           Append (v, w) to adj[u]
           Append (u, w) to adj[v]
                                                                  ▷ Since the graph is undirected
 5:
       end for
 6:
 7:
       Initialize distance array dist with size n, set all values to \infty
 8:
       Set dist[s] \leftarrow 0
       Initialize queues q_1 and q_2
 9:
10:
       Enqueue s to q_1
       while q_1 is not empty or q_2 is not empty do
11:
           Initialize new queues new_q1 and new_q2
12:
           size \leftarrow size of q_1
13:
           for i \leftarrow 1 to size do
14:
               node \leftarrow Dequeue from q_1
15:
               for each neighbor (nei, w) in adj[node] do
16:
                   if dist[node] + w < dist[nei] then
17:
                       dist[nei] \leftarrow dist[node] + w
18:
                       if w = 1 then
19:
                           Enqueue nei to new\_q1
20:
                       else
21:
22:
                           Enqueue nei to new_q2
                       end if
23:
                   end if
24:
               end for
25:
           end for
26:
27:
           q_1 \leftarrow new\_q1
28:
           while q_2 is not empty do
29:
               Enqueue from q_2 to q_1
           end while
30:
31:
           q_2 \leftarrow new\_q2
       end while
32:
       return dist[t]
33:
34: end procedure
```

Leetcode Question:

Yuchen:



Botao:



Yuheng:

