Combinatoric :combinaison repeatition

How many words on length 5 on the alphabet $\{0,1,\dots,9\}$ have their digits

1. **weakly increasing**

   Method 1 We proceeding by enumeration :

   - Fix all letters before the arrow :

     $0000\overset{\downarrow}{0}$ under the arrow we can have all integer values between 0 and 9, then we have $10-0=10$ words.

     $0001\overset{\downarrow}{1}$ we begin with 1 to respect the increasing and we can have all integer values between 0 and 9, then we have $10-1=9$ words.

     Thus for an $i \in [\![0,9]\!]$, $000i\overset{\downarrow}{i}$ if we set the letter before the arrow i we begin with i to respect the increasing and we can have all integer values between 0 and 9, then we have $10-i$ words.

     So if we fix the three firsts letter as 0 we have $\sum\limits_{i=0}^{9}(10-i)$ words.

   - We move our arrow in the fourth posititon

     $000\overset{\downarrow}{0}0$ if the letter before the arrow is 0 we have the previous then we have $\sum\limits_{i=0}^{9}(10-i)$ words.

     $001\overset{\downarrow}{j}j$ for each j in $[\![1,9]\!]$ we have $\sum\limits_{i=j}^{9}(10-i)$ words

     Thus for an $j \in [\![0,9]\!]$, $00j\overset{\downarrow}{i}h$ if we set the letter before the arrow j we will begin with j to respect the increasing and we can have all integer i values between j and 9, then we have $\sum\limits_{i=j}^{9}(10-i)$ words for each j. Hence the total words is $\sum\limits_{j=0}^{9}\sum\limits_{i=j}^{9}(10-i)$

   - And by the same if we move the arrow we add a new cumulative sum

     For an $k \in [\![0,9]\!]$, $0k\overset{\downarrow}{j}ih$ if we set the letter before the arrow k we begin with k to respect the increasing and we can have all integer j values between k and 9, then we have $\sum\limits_{j=k}^{9}\sum\limits_{i=j}^{9}(10-i)$ words for each j.

     Hence the total words is $\sum\limits_{k=0}^{9}\sum\limits_{j=k}^{9}\sum\limits_{i=j}^{9}(10-i)$

   - Thus to have for all weakly increasing words we move the the arrow at second position

     For an $t \in [\![0,9]\!]$, $t\overset{\downarrow}{k}jih$ if we set the letter before the arrow k we begin with k to respect the increasing and we can have all integer k values between t and 9, then we have $\sum\limits_{k=t}^{9}\sum\limits_{j=k}^{9}\sum\limits_{i=j}^{9}(10-i)$ words for each j.

     Hence the total weakly increasing words of length 5 is $\boxed{\sum\limits_{t=0}^{9}\sum\limits_{k=t}^{9}\sum\limits_{j=k}^{9}\sum\limits_{i=j}^{9}(10-i)}$

     $$S_n=\sum_{t=0}^{9}\sum_{k=t}^{9}\sum_{j=k}^{9}\sum_{i=j}^{9}(10-i)=\sum_{t=0}^{9}\sum_{k=t}^{9}\sum_{j=k}^{9}\sum_{i=j}^{9}\left[\binom{10-i+1}{2}-\binom{10-i}{2}\right]$$

$$\sum_{i=j}^{9}\left[\binom{10-i+1}{2}-\binom{10-i}{2}\right]=\binom{10-j+1}{2}-\binom{10-j}{2}+\binom{10-j}{2}-\binom{10-j-1}{2}+$$
$$\cdots+\binom{10-7}{2}-\binom{10-8}{2}+\binom{10-8}{2}-\binom{10-9}{2}=\binom{10-j+1}{2}-\binom{10-9}{2}=\binom{10-j+1}{2}$$
(Telescopic sum)

$$S_n=\sum_{t=0}^{9}\sum_{k=t}^{9}\sum_{j=k}^{9}\binom{10-j+1}{2}=\sum_{t=0}^{9}\sum_{k=t}^{9}\sum_{j=k}^{9}\left[\binom{10-j+2}{3}-\binom{10-j+1}{3}\right] \text{ (telescopic sum)}$$
$$=\sum_{t=0}^{9}\sum_{k=t}^{9}\left[\binom{10-k+2}{3}-\binom{10-9+1}{3}\right]=\sum_{t=0}^{9}\sum_{k=t}^{9}\binom{10-k+2}{3} \text{ (telescopic sum)}$$
$$=\sum_{t=0}^{9}\left[\binom{10-t+3}{4}-\binom{10-9+2}{4}\right]=\sum_{t=0}^{9}\binom{10-t+3}{4}=\binom{10-0+4}{5}-\binom{10-9+3}{5}$$
(telescopic sum)

Then $S_n=\sum_{t=0}^{9}\sum_{k=t}^{9}\sum_{j=k}^{9}\sum_{i=j}^{9}(10-i)=\binom{14}{5}=2002$

We generalize it by the using the telescopic sum for each step of sum, the total weakly increasing words of length $n\geq 2$ in the alphabet $[\![0,9]\!]$ is :

$$\boxed{\sum_{l_1=0}^{9}\sum_{l_2=l_1}^{9}\cdots\sum_{l_{n-1}=l_{n-2}}^{9}(10-l_{n-1})=\binom{10+n-1}{n}}$$

Example weakly increasing of length 2 is $\sum_{l_1=0}^{9}(10-l_1)=\binom{11}{2}=55$

We can to generalize it by the using the telescopic sum for each step of sum,the total weakly increasing words of length $n\geq 2$ in the alphabet of size A

$$\boxed{\sum_{l_1=0}^{A-1}\sum_{l_2=l_1}^{A-1}\cdots\sum_{l_{n-1}=l_{n-2}}^{A-1}(A-l_{n-1})=\binom{A+n-1}{n}}$$

Example weakly increasing of length 3 in the alphabet $\{0,1,2\}$ is

$$\sum_{l_2=0}^{2}\sum_{l_1=l_2}^{2}(3-l_1)=\binom{5}{3}=10$$

They are '222', '122', '112', '111', '022', '012', '011', '002', '001', '000'.

Example weakly increasing of length 2 in the alphabet $\{0,1,2\}$ is

$$\sum_{l_1=0}^{2}(3-l_1)=\binom{4}{2}=6$$

They are '22', '12', '11', '02', '01', '00'

2. **weakly increasing or weakly decreasing**

Since a set of n numbers has an unique weakly increasing order and an unique weakly decreasing order too. Then the number of words of length n weakly decreasing ordering digits is equal to the weakly increasing of length n .

And their intersection is the words which composed by one letter then his cardinal is the number of letters A.

Thus #**weakly increasing or weakly decreasing** $=2\binom{A+n-1}{n}-A$

Words of length 5 weakly increasing or weakly decreasing :

$2\binom{10+5-1}{5}-10=2\binom{14}{5}-10=2\times 2002-10=3994.$

Words of length 3 weakly increasing or weakly with S={0,1,2} :

$2\binom{3+3-1}{3}-3=2\binom{5}{3}-3=2\times 10-3=17.$

Which are ’222’, ’122’, ’112’, ’111’, ’022’, ’012’, ’011’, ’002’, ’001’, ’000’, ’221’,’211’ ,’220’,’210’,’110’,’200’ and ’100’.