Interpolation Interpolation

1. Any Continuous function canbe approximated by a Polynomial. (Weistras Theorem).

Even if data/information about the Continuous function is given, we can approximate by a polynomial.

$$x_0, x_1, x_2 \ldots x_n$$
$$f_0, f_1, f_2 \ldots f_n$$

Suppose data is given for the interpolation $f(x)$ at $(n+1)$ distinct points

$$x_0, x_1 \ldots x_{n-1}, x_n$$
$$f_0, f_1 \ldots f_{n-1}, f_n$$

Actually speaking we have to retrive the function $f(x)$.
Some times we want to find $f(x)$ at any other point.
So let us find polynomial $P(x)$ which satisfies the given data.

That is values of polynomial $P(x)$ and function $f(x)$ must be same at these $(n+1)$ points. that is

$$P(x_i) = f_i \quad \text{for } i = 0, 1, 2 \ldots n$$

Hence We claim that $P_n(x) \cong f(x)$.

Interpolation

Def: The polynomial which satisfies given data is called Interpolating Polynomial.

Finding the value of the function at any point inside the interval by interpolating polynomial is called Interpolation.

It is the process of finding the most appropriate estimate for missing data.

Finding the value of the function at any point outside the interval by interpolating polynomial is called Extrapolation.

Note: The maximum of this interpolating polynomial is n.

Since data/Information is given (n+1) points we can think of a polynomial having (n+1) variables of degree n only. Hence we can think polynomial ~~of~~ degree n only

Note: Maximum degree of this interpolating polynomial is n.

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots a_{n-1}x^{n-1} + a_n x^n.$$

These $a_0, a_1 \dots a_{n-1}, a_n$ can be obtained by the interpolation condition.

$$P_n(x_i) = f_i \quad \text{for } i = 0, 1, 2 \dots n$$

(n+1) conditions for (n+1) variables so we can find.

Legranges Interpolation:

DATA is given for the function $f(x)$ has at $(n+1)$ distinct points

$$\begin{array}{cccccc} x_0, & x_1, & x_2, & \dots & x_{n-1}, & x_n \\ f_0, & f_1, & f_2 & \dots & f_{n-1}, & f_n \end{array}$$

We discuss Legranges approach to find n^{th} degree interpolating polynomial $P(x)$ which satisfies the given data.

That is a polynomial $P(x)$ and function $f(x)$ must be same at these $(n+1)$ points. That is

$$P_n(x_i) = f_i \quad \text{for } i = 0, 1, 2 \dots n.$$

We first discuss for case by case as follows:

(i) for ONE point data

(ii) for TWO point data

(iii) then the general case for $(n+1)$ points data.

Legranges Interpolation

case(i). Data is given for the function $f(x)$ at one point. $\begin{array}{c} x_0 \\ f_0 \end{array}$

We discuss Legranges approach to find 0^{th} degree polynomical $P(x)$ which satisfies given data.

That is polynomial $P(x)$ and function $f(x)$ must be same at the given one point. That is

$$P_0(x_i) = f_i \quad \text{for } i = 0.$$

We take 0^{th} degree polynomial as $P_0(x) = a_0$

$$P_0(x_0) = a_0 = f_0 .$$

We get 0^{th} degree Legranges $\quad P_0(x) \approx f_0$

Interpolating polynomial as $\quad P_0(x) = f_0 \approx f(x)$.

Case (ii). Data is given for the function $f(x)$ at

TWO Points

x_0	x_1
f_0	f_1

We discusss Legranges approach to find 1^{st} degree polynomial $P(x)$ which satisfies the given data. That is polynomial $P(x)$ and function $f(x)$ must be same at these two points. That is

$$P_i(x_i) = f_i \quad \text{for } i = 0, 1.$$

We take 1^{st} degree polynomial as

$$P_1(x) \approx a_0 + a_1 x .$$

Using co-ordinate geometry we can find the straight line passing through the given TWO points

x_0	x_1
f_0	f_1

The equation such a straight line is

$$f(x) - f_0 = \left(\frac{f_1 - f_0}{x_1 - x_0}\right)(x - x_0)$$

$$f(x) = f_0 + \left(\frac{f_1 - f_0}{x_1 - x_0}\right)(x - x_0) .$$

$$f(x) = \left[\frac{f_0 (x_1 - x_0) + (f_1 - f_0)(x - x_0)}{(x_1 - x_0)} \right]$$

Straight line

$$= \left[\frac{f_0 (x_1 - x_0) - f_0 (x - x_0) + f_1 (x - x_0)}{(x_1 - x_0)} \right].$$

$$P_1(x) = \left[\frac{(x - x_1)}{(x_0 - x_1)} \right] f_1 + \left[\frac{(x - x_0)}{(x_1 - x_0)} \right] f_1$$

This is the 1st degree Legranges Interpolating Polynomial $P(x)$ which satisfies the given two points $P_i(x_i) \approx f_i$ for $i = 0, 1$.

Hence we can write $P_1(x) \approx f(x)$.

$$P_1(x) = \left[\frac{(x - x_1)}{(x_0 - x_1)} \right] f_0 + \left[\frac{(x - x_0)}{(x_1 - x_0)} \right] f_1$$

Let us define Legranges Polynomial as :

$$l_0(x) = \left[\frac{(x - x_1)}{(x_0 - x_1)} \right] \qquad L_1(x) = \left[\frac{(x - x_0)}{(x_1 - x_0)} \right]$$

$$P_1(x) = l_0(x) f_0 + l_1(x) f_1$$ written as $$P_1(x) = \sum_{i=0}^{1} l_i(x) f_i$$

Hence we can write $P_1(x) = \sum_{i=0}^{1} L_i(x) f_i \approx f(x)$

Observations on these Legranges polynomials.

$$l_0(x) = \left[\frac{(x-x_1)}{(x_0-x_1)}\right] \text{ and } l_1(x) = \left[\frac{(x-x_0)}{(x_1-x_0)}\right]$$

Observations(i): These are degree 1

Obesewalions(ii) $\quad l_0(x_0) = 1 \quad l_0(x_1) = 0$

$\quad l_1(x_1) = 1.$

$\quad l_1(x_0) = 0$

That is in general

$\quad l_1(x_j) = 0$

$\quad l_1(x_1) = 1$

Obs. (iii): $l_0(x) + l_1(x) = \left[\frac{(x-x_0)}{(x_0-x_1)}\right] + \left[\frac{(x-x_0)}{(x_1-x_0)}\right] = 1.$

Now letus take a general Case

DATA is given for the function $f(x)$ at $(n+1)$ distinct points

$x_0 \quad x_1 \quad x_2 \quad x_{n-1} \quad x_n$

$f_0 \quad f_1 \quad f_2 \quad f_{n-1} \quad f_n$ We discuss Legranges approach to find n^{th} degree polynomial $P(x)$ which satisfies the given data.

That is polynomial $P(x)$ and function $f(x)$ must be same at these $(n+1)$ points. That is

$$P_n(x_i) = f_i \quad i = 0, 1, 2 \ldots n$$

We can take n^{th} degree Legranges Interpolation polynomial as

$$P_n(x) = \sum_{i=0}^{n} l_i(x) f_i$$

Formula: Let $f_0, f_1, f_2 \ldots f_n$ be the values of $f(x)$ at $x_0, x_1, x_2, \ldots x_n$ (nodes) not neceessarly equal points then an interpolating polynomial $P(x)$ for $f(x)$ is given by

$$P(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f_2$$

$$+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} \cdot f_n.$$

Advantages of Legranges Interpolation

1. Easy to write the polynomial
2. Data points need not be equi-spaced.

If data points are increased by one more point we have to re-do Entire polynomial. That is previous polynomial will not be usefull to get next polynomial.

For the two points data we have 1^{st} degree polynomial Than for the Three points data we have 2^{nd} degree polynomial.

Clearly $P_2(x)$ can not be obtained using $P(x)$ if it possible we say polynomial permanance property.

Prob: Find Legrange Interpolating polynomial.

	x_0	x_1	x_2
x	0	1	3
$f(x)$	1	3	55
	f_0	f_1	f_2

Data Points are 3, So max. Degree of Interpolating polynomial: 2

$$f(x) \approx P_2(x) = \sum_{i=0}^{2} l_i(x) f_i = L_0 f_0 + L_1 f_1 + L_2 f_2.$$

or

$$P_2(x) = \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\right] f_0 + \left[\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\right] \cdot f_1$$

$$+ \left[\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\right] f_2.$$

$$P_2(x) = \left[\frac{(x-1)(x-3)}{(0-1)(0-3)}\right] \cdot 1 + \left[\frac{(x-0)(x-3)}{(1-0)(1-3)}\right] \cdot 3 +$$

$$\left[\frac{(x-0)(x-1)}{(3-0)(3-1)}\right] 55$$

$$f(x) \approx P_2(x) = 8x^2 - 6x + 1$$

i.e, $f(0) = P_2(0) = 1$

$f(1) = P_2(1) = 3$

$f(3) = P_2(3) = 55$

Prob: Find Legranges Interpolate polynomial

	x_0	x_1	x_2	x_3
x	-1	0	1	2
$f(x)$	1	1	1	-5
	f_0	f_1	f_2	f_3.

Sol. Data is given at 4 points, max degree of interpolating polynomial: 3.

$$f(x) \approx P_3(x) = \sum_{i=0}^{3} L_i(x) f_i = L_0^{(x)} f_0 + L_1^{(x)} f_1 + L_2^{(x)} f_2 + L_3^{(x)} f_3$$

Where all these Legranges polynomials are of degree 3.

$$L_i(x) = \frac{\text{All the factors except the suffix point}}{\text{Same Numerator where } x \text{ is replaced by the forgotten point}}.$$

Further $L_i(x_i) = 1 \quad L_i(x_j) = 0$

$$f(x) \approx P_3(x) = \left[\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right] f_0 +$$

$$\left[\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right] f_1 +$$

$$\left[\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right] f_2 +$$

$$\left[\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right] f_3 .$$

$$P_3(x) = \left[\frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)}\right]\cdot 1 + \left[\frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)}\right]\cdot 1$$

$$+ \left[\frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)}\right]\cdot 1 + \left[\frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)}\right](-5)$$

$$f(x) \approx P_3(x) = -x^3 + x + 1$$

$f(x_i) = P_3(x_i)$ Clearly $i = 0, 1, 2, 3$.

Prob

Using the Interpolating polynomial find $f(6)$.

	x_0	x_1	x_2	x_3
x	1	2	7	8
$f(x)$	4	5	5	4.
	f_0	f_1	f_2	f_3

Data is given at 4 Points, max-degree of interpolating polynomial: 3

$$f(x) \approx P_3(x) = \sum_{i=0}^{3} L_i(x) f_i = L_0(x) f_1 + L_1(x) f_2 + L_2(x) f_3 + L_3(x) f_3.$$

Where all these legranges polynomial are of degree 3.

$$L_i(x) = \frac{\text{All the factors except the suffix point}}{\text{Same numerator where } x \text{ is replaced by the forgotten point}}.$$

Further $L_i(x_i) = 1 \quad L_i(x_j) = 0$.

$$P_3(x) = \left[\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}\right] \cdot f_0$$

$$+ \left[\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}\right] \cdot f_1$$

$$+ \left[\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}\right] f_2$$

$$+ \left[\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}\right] f_3.$$

$f(x) \approx P_3(x)$ So $f(6) \approx P_3(6)$.

So put $x = 6$. in the above eqn.

$$f(6) \approx P_3(6) = \left[\frac{(6-2)(6-7)(6-8)}{(1-2)(1-7)(1-8)}\right] \cdot 4 +$$

$$\left[\frac{(6-1)(6-7)(6-8)}{(2-1)(2-7)(2-8)}\right] \cdot 5 +$$

$$\left[\frac{(6-1)(6-2)(6-8)}{(7-1)(7-2)(7-8)}\right] \cdot 5 +$$

$$\left[\frac{(6-1)(6-2)(6-7)}{(8-1)(8-2)(8-7)}\right] \cdot 4$$

$$f(6) \approx P_3(6) = \underline{\underline{5.66}}\,.$$

Prob: Using the Legranges Interpolating Polynomial find the Partial Fractions of .

$$\phi(x) = \frac{x^2+x+3}{x^3-2x^2-x+2}$$

Observe $\phi(x) = \frac{x^2+x+3}{x^3-2x-x+2} = \frac{x^2+x+3}{(x+1)(x-1)(x-2)}$.

$$
\begin{array}{r|rrrr}
 & x^3 & -2x^2 & -x & +2 \\
x=1 & 1 & -2 & -1 & 2 \\
 & 0 & 1 & -1 & 2 \\
\hline
 & 1 & -1 & -2 & \lfloor 0
\end{array}
$$

$x^2-x-2=0$

$x^2-2x+x-2=0$

$x(x-2)+1(x-2)=0$

$(x+1)(x-2)$.

Now take $f(x)=x^2+x+3$ and write as product of factors.

x :	-1	1	2
$f(x)$:	-3	-1	3

Data is given at 3 points, max degree of interpolating polynomial is 2:

$$P_2(x) = \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\right] f_0 + \left[\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\right] f_1 + \left[\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\right] f_2$$

$$P_2(x) = \left[\frac{(x-1)(x-2)}{(-1-1)(-1-2)}\right](-3) + \left[\frac{(x+1)(x-2)}{(1+1)(1-2)}\right](-1) + \left[\frac{(x+1)(x-1)}{(2+1)(2-1)}\right](3)$$

$$f(x) = P_2(x) = \left[\frac{(x-1)(x-2)}{-2}\right] + \left[\frac{(x+1)(x-2)}{2}\right] + \left[\frac{(x+1)(x-1)}{1}\right]$$

$$\text{So } \phi(x) = \frac{x^2+x-3}{(x+1)(x-1)(x-2)} = \frac{(x-1)(x-2)}{(-2)(x+1)(x-1)(x-2)} +$$

$$\left.\frac{(x+1)(x-2)}{2(x+1)(x-1)(x-2)} + \frac{(x+1)(x-1)}{1\cdot(x+1)(x-1)(x-2)}\right]$$

$$= \frac{-1}{2}\frac{1}{(x+1)} + \frac{1}{2}\cdot\frac{1}{(x-1)} + \frac{1}{1\cdot(x-2)}.$$

$$= \frac{P_2(x)}{(x-1)(x-2)(x-3)}.$$

$$\phi(x) = \frac{x^2+x-3}{(x+1)(x-1)(x-2)} = -\frac{1}{2}\frac{1}{(x+1)} + \frac{1}{2}\cdot\frac{1}{x-1} + \frac{1}{x-2}$$

Assignment on Legranges Interpolation

1. Use Lagrange's interpolation formula to find the value of y when $x = 10$, if the following values of x and y are given:

x:	5	6	9	11
y:	12	13	14	16

2. The following table gives the viscosity of oil as a function of temperature. Use Lagrange's formula to find the viscosity of oil at a temperature of 140°.

Temp°:	110	130	160	190
Viscosity:	10.8	8.1	5.5	4.8

3. Given $\log_{10}654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10}661 = 2.8202$, find by using Lagrange's formula, the value of $\log_{10} 656$.

4. The following are the measurements T made on a curve recorded by oscilograph representing a change of current I due to a change in the conditions of an electric current.

T:	1.2	2.0	2.5	3.0
I:	1.36	0.58	0.34	0.20

Using Lagrange's formula, find I and $T = 1.6$.

5. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data:

Year:	1997	1999	2001	2002
Profit in Lakhs of Rs:	43	65	159	248

6. Use Lagrange's formula to find thee form of $f(x)$, given

x:	0	2	3	6
$f(x)$:	648	704	729	792

7. If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$, $y(6) = 132$, fin d the Lagrange's interpolation polynomial that takes the same values as y at the given point s.

8. Given $f(0) = -18$, $f(1) = 0$, $f(3) = 0$, $f(5) = -248$, $f(6) = 0$, $f(9) = 13104$, find $f(x)$.

9. Find the missing term in the following table using interpolation

x:	1	2	4	5	6
y:	14	15	5	....	9

10. Using Lagrange's formula, express the function $\dfrac{x^2 + x - 3}{x^3 - 2x^2 - x + 2}$ as a sum of partial fractions.

11. Using Lagrange's formula, express the function $\dfrac{x^2 + 6x - 1}{(x^2 - 1)(x - 4)(x - 6)}$ as a sum of partial fractions.

[*Hint*. Tabulate the values of $f(x) = x^2 + 6x - 1$ for $x = -1, 1, 4, 6$ and apply Lagrange's formula.]

1. 11.5 **2.** 6.304 **3.** 37.23. **4.** 2.3.

5. 0.2679 **6.** 1.3714.