Gauss Seidel Method

To solve AX=B where A is nxn matrix.
We have two methods Direct Methods like
Cramers – Gauss elimination method / Gauss Jordan methods.
In these methods by a certain procedure we try to arrive at the final solution. There are some numerical Methods where, we start with an approximate solution and in successive steps by performing some iterations, we improve upon the solution and finally try to arrive at an approximately "finer" solution. The procedure adopted are called indirect or iterative methods.

Sometimes if very small changes occur in the elements of the coefficient matrix A. by using the methods we may arrive at a solution not very much different from that we got earlier. Problems involving such systems are called well conditioned or well posed problem.

Sometimes if very small changes occur in the coefficient matrix A, by using the methods we arrive at a solution which is far different from the actual solution. Such problems are called ill-conditioned or ill posed problems.

Gauss Seidel Method is the iterative method
Repeat the iterations till we get satisfactory solution or two consecutive approximations are almost same that is their difference is negligible.

Vector Matrix Form.

Find $x_1, x_2 \ldots x_n$ from

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{3n} \\ & & & \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \\ b_n \end{bmatrix}$$

$$AX = B.$$

Diagonally Dominant Property: Every Main Diagonal element is dominant over the sum of the elements in that row (in magnatude)

If Diagonally Dominence property is satisfied we can guarantee the convergence of this Method (Gauss Seidel method). If neccessary we can interchange or re arrange the equations to satisfy this property.

$$|a_{ii}| \geqslant \sum_{j \neq 1} |a_{ij}| \text{ for } i, j = 1, 2, \ldots \text{ and } j \neq i.$$

Using this method we start with initial guess as ZERO and solve iteratively.

We terminate the iteration procedure when the magnatudes of ~~of~~ the differences between the two successive iterates of all the variables are smaller than a given accuracy or an error bound ϵ that is.

$$|x_i^{k+1} - x_i^{k}| \leq \epsilon \text{ for all } i.$$

Gauss Seidel Method.

To find $x_1, x_2, \dots x_n$ from.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n.$$

Intially start with $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = \dots = x_n^{(0)} = 0$

Then

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - (a_{12}x_2^{(0)} + a_{13}x_3^{(0)} + \dots a_{1n}x_n^{(0)}\right] =$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - (a_{21}x_1^{(1)} + a_{23}x_3^{(0)} + \dots a_{2n}x_n^{(0)}\right] = 0$$

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - (a_{31}x_1^{(1)} + a_{32}x_2^{(1)} + a_{34}x_4^{(0)} + \dots a_{3n}x_n^{(0)}\right] =$$

$\vdots$

$$x_i^{(1)} = \frac{1}{a_{ii}}\left[b_i - (a_{i1}x_1^{(1)} + \dots a_{ii-1}x_{i-1}^{(1)} + a_{ii+1}x_{i-1}^{(0)} + \dots a_{in}x_n^{(0)}\right].$$

$\vdots$

$$x_{n}^{(1)} = \frac{1}{a_{nn}}\left[b_n - (a_{n1}x_1^{(1)} + a_{n2}x_2^{(1)} + \dots + a_{nn-1}\right].$$

For Second iteration

$x_1^{(1)} =$; $x_2^{(1)} =$; $\dots x_n^{(1)} =$.

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - (a_{i1}x_1^{k+1} + \cdots a_{ii-1}x_{i-1}^{k+1} + a_{ii+1}x_{i+1}^{k} + \cdots a_{in}x_n^{(k)}\right.$$

Iteratively repeat till we get satisfactory solution (i.e., desired accurancy) &

Two consecutive approximations are almost same.

Example 1. Find x_1, x_2, x_3 from.

$$4x_1 - x_2 - x_3 = 3$$
$$-x_1 + x_2 + 7x_3 = -6$$
$$-2x_1 + 6x_1 + x_3 = 9$$

If we represent in the form of $AX = B$.

$$\begin{bmatrix} 4 & -1 & -1 \\ -1 & 2 & 7 \\ -3 & 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} +3 \\ -6 \\ 9 \end{bmatrix}$$

System is ~~di~~ Not a diagonally dominant.

[The diagonally dominent property is sufficient to get a guarantee solution using this Gauss Seidal Method].

So we have to check the D.D. property.

If it is not a diagonally dominent by inter changing or rearrange the equations make it diagonally dominent.

Inter changing the second and third equations

$$x_1^{(2)} = \frac{1}{a_{11}}\left[b_1 - (a_{12}x_2^{(1)} + a_{13}x_3^{(1)} + \dots + a_{1n}x_n^{(1)}\right]$$

$$x_2^{(2)} = \frac{1}{a_{22}}\left[b_2 - (a_{21}x_1^{(2)} + a_{23}x_3^{1} + \dots + a_{2n}x_n^{(1)}\right]$$

$$x_3^{(2)} = \frac{1}{a_{33}}\left[b_3 - (a_{31}x_1^{(2)} + a_{32}x_2^{(2)} + a_{34}x_4^{(1)} + \dots a_{3n}x_n^{(1)}\right]$$

$\vdots$

$$x_i^{(2)} = \frac{1}{a_{ii}}\left[b_i - (a_{i1}x_1^{2} + a_{i2}x_2^{(2)} + \dots a_{ii-1}x_{i-1}^{(2)} + a_{ii+1}x_{i+1}^{(1)} + \dots a_i x_n^{(1)}\right].$$

$$x_n^{(2)} = \frac{1}{a_{nn}}\left[b_n - (a_{n1}x_1^{2} + a_{n2}x_2^{2} + \dots + a_{n-1}x_{n-1}^{(2)}\right].$$

and so on

In general with $x_1^{(0)} = x_2^{(0)} = \dots = x_n^{(0)}$ then for $k = 0, 1, 2, \dots$

$$x_1^{k+1} = \frac{1}{a_{11}}\left[b_1 - [a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \dots a_{1n}x_n^{(k)}\right]$$

$$x_2^{k+1} = \frac{1}{a_{22}}\left[b_1 - (a_{11}x_1^{(k+1)} + a_{23}x_3^{(k)} + \dots a_{2n}x_n^{(k)}\right]$$

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - (a_{i1}x_1^{(k+1)} + a_{i2}x_2^{(k+1)} + \dots a_{ii-1}x_{i-1}^{(k+1)} + a_{ii+1}x_{i+1}^{(k)} + \dots a_i x_n^{(k)}\right]$$

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - (a_{i1}x_1^{k+1} + a_{i2}x_2^{k+1} + \dots a_{ii-1}x_{i-1}^{k+1} + a_{ii+1}x_{i+1}^{(k)} + \dots a_i x_n^{k}\right]$$

$\vdots$

$$x_n^{k+1} = \frac{1}{a_{nn}}\left[b_n - [a_{n1}x_1^{(k+1)} + a_{n2}x_2^{k+1} + \dots + a_{nn-1}x_{n-1}^{(k+1)}\right]$$

This means updated values are to be used.

Start with initial choice as

$$x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = \dots = x_n^{(0)} = 0.$$

then find for $k = 0, 1, 2, 3 \dots$ $i = 1, 2, 3 \dots$

$$4x_1 - x_2 - x_3 = 3$$
$$-2x_1 + 6x_1 + x_3 = 9$$
$$-x_1 + x_2 + 7x_3 = -6.$$

$$4 \geq |-1| + |-1|.$$
$$6 \geq |-2| + |1|.$$
$$7 \geq |-1| + |1|$$

$$\begin{bmatrix} 4 & -1 & -1 \\ -2 & 6 & 1 \\ -1 & 1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix}$$

Now the System is diagonally dominant.
So write the equations as ith equation from ith variable.

$$x_1 = \frac{1}{4}\left[3 + x_2 + x_3\right]$$
$$x_2 = \frac{1}{6}\left[9 + 2x_1 - x_3\right]$$
$$x_3 = \frac{1}{7}\left[-6 + x_1 - x_2\right]$$

First Iteration Start with $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$ then.

$$x_1^{(1)} = \frac{1}{4}\left[3 + x_2^{(0)} + x_3^{(0)}\right] = \frac{3}{4} = 0.75$$
$$x_2^{(1)} = \frac{1}{6}\left[9 + 2x_1^{(1)} - x_3^{(0)}\right] = \frac{1}{6}\left[9 + 2(0.75) - 0\right]$$
$$= \frac{1}{6}(9 + 1.50)$$
$$= \frac{10.50}{6} = 1.75$$
$$x_3^{(1)} = \frac{1}{7}\left[-6 + x_1^{(1)} - x_2^{(1)}\right]$$
$$= \frac{1}{7}\left[-6 + 0.75 - 1.75\right] = -\frac{7.75}{7} = -1.$$

Now with $x_1^{(1)} = 0.75$ $x_2^{(1)} = 1.75$ $x_3^{(1)} = -1$

Second iteration

$$x_1^{(2)} = \frac{1}{4}\left[3 + x_2^{(1)} + x_3^{(1)}\right] = \frac{1}{4}\left[3 + 0.75 - 1\right]$$

$$= 0.938$$

$$x_2^{(2)} = \frac{1}{6}\left[9 + 2x_1^{(1)} - x_3^{(1)}\right]$$

$$= \frac{1}{6}\left[9 + 2(0.938) + 1\right] = \frac{1}{6}\left[10 + 1.876\right]$$

$$= \frac{11.876}{6} = 1.979.$$

$$x_3^{(2)} = \frac{1}{7}\left[-6 + x_1^{(2)} - x_2^{(2)}\right]$$

$$= \frac{1}{7}\left[-6 + 0.938 - 1.979\right]$$

$$= -1.006.$$

In general start with $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$ then for $k = 0, 1, 2 \ldots$

$$x_1^{k+1} = \frac{1}{4}\left[3 + x_2^{(k)} + x_3^{(k)}\right]$$

$$x_2^{k+1} = \frac{1}{6}\left[9 + 2x_1^{k+1} - x_3^{k}\right]$$

$$x_3^{k+1} = \frac{1}{7}\left[-6 + x_1^{k+1} - x_2^{k+1}\right]$$ all are updated Values.

k	x_1	x_2	x_3
0	0	0	0
1	0.75	1.75	−1
2	0.938	1.979	−1.006
3			
Converges to	1	2	−1.

Answer $x_1 = 1$
$x_2 = 2$
$x_3 = -1.$

Prob: Find x_1, x_2, x_3, x_4 from

$$10x_1 - x_2 + 2x_3 = 6$$
$$3x_2 - x_3 + 8x_4 = 15$$
$$2x_1 - x_2 + 10x_3 - x_4 = -11$$
$$-x_1 + 11x_2 - x_3 + 3x_4 = 25$$

Write in the form of $AX = B$ and check for Diagonally Dominant.

$$\begin{bmatrix} 10 & -1 & 2 & 0 \\ 0 & 3 & -1 & 8 \\ 2 & -1 & 10 & -1 \\ -1 & 11 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ -11 \\ 25 \end{bmatrix}$$

Check for D.D.?

System is not a diagonally dominent
So inter change the second and fourth equation so then.

$$10x - x_2 + 2x_3 = 6.$$
$$-x_1 + 11x_2 - x_3 + 3x_4 = 25$$
$$2x_1 - x_2 + 10x_3 - x_4 = -11$$
$$3x_2 - x_3 + 8x_4 = 15.$$

Now $AX=B$ is

$$\begin{bmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 25 \\ -11 \\ 15 \end{bmatrix}$$

Clearly $10 \geq |-1| + |2|$ $\quad 11 \geq |-1| + |-1| + |3|.$

$$10 \geq |2| + |-1| + |-1|$$
$$8 \geq 0 + 3 + |-1|.$$

Now Diagonally Dominance property satisfied the System of equations.

Write the equations as ith variable from ith equation

$$x_1 = \frac{1}{10}\left[6 + x_2 - 2x_4\right]$$

$$x_2 = \frac{1}{11}\left[25 + x_1 + x_3 - 3x_4\right]$$

$$x_3 = \frac{1}{10}\left[-11 - 2x_1 + x_2 + x_4\right]$$

$$x_4 = \frac{1}{8}\left[15 - 3x_2 + x_3\right]$$

To start iteration

We start with $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 0$ then.

$$x_1^{(1)} = \frac{1}{10}\left[6 + x_2^{(0)} - 2x_1^{(0)}\right] = \frac{6}{10} = 0.6$$

$$x_2^{(1)} = \frac{1}{11}\left[25 + x_1^{(1)} + x_3^{(0)} - 3x_4^{(0)}\right]$$

$$= \frac{1}{11}\left[25 + 0.6 - 0 - 3\cdot 0\right] = \frac{25.6}{11}$$

$$= 2.32727.$$

$$x_3^{(1)} = \frac{1}{10}\left[-11 - 2x_1^{(1)} + x_2^{(1)} + x_4^{(0)}\right]$$

$$= \frac{1}{10}\left[-11 - 2(0.6) + (2.32727) + 0\right]$$

$$= -0.987273$$

$$x_4^{(1)} = \frac{1}{8}\left[15 - 3x_2^{(1)} + x_3^{(1)}\right]$$

$$= \frac{1}{8}\left[15 - 3(2.32727) - 0.987273\right]$$

$$= 0.878864$$

for Second Iteration

with $x_1^{(1)} = 0.6 \quad x_2^{(1)} = 2.32727 \quad x_3^{(1)} = -0.98727$

$x_4^{(1)} = 0.878864.$

$$x_1^{(2)} = \frac{1}{10}\left[6 + x_2^{(1)} - 2x_3^{(1)}\right]$$

$$= \frac{1}{10}\left[6 + 2.32727 - 2(-0.987273)\right]$$

$$= 1.03018.$$

$$x_2^{(2)} = \frac{1}{11}\left[25 + x_1^{(2)} + x_3^{(1)} - 3x_4^{(1)}\right]$$

$$= \frac{1}{11}\left[25 + 1.03018 - 0.987273 - 3(0.878864)\right]$$

$$= 2.03694$$

$$x_3^{(2)} = \frac{1}{10}\left[-11 - 2x_1^{(2)} + x_2^{(2)} + x_4^{(1)}\right].$$

$$= \frac{1}{10}\left[-11 - 2(1.03018) + 2.03694 + 0.878864\right].$$

$$= -1.01446.$$

$$x_4^{(2)} = \frac{1}{8}\left[15 - 3x_2^{(2)} + x_3^{(2)}\right]$$

$$= \frac{1}{8}\left[15 - 3(2.03694) + (-1.01446)\right]$$

$$= 0.984341.$$

Now start with $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 0$ then $K = 1, 2, 3 \ldots$

$$x_1^{k+1} = \frac{1}{10}\left[6 + x_2^{(k)} - 2x_1^{(k)}\right]$$

$$x_2^{k+1} = \frac{1}{11}\left[25 + x_1^{(k+1)} + x_3^{(k)} - 3x_4^{(k)}\right]$$

$$x_3^{k+1} = \frac{1}{10}\left[-11 - 2x_1^{(k+1)} + x_2^{k+1} + x_4^{(k)}\right]$$

$$\therefore\ x_4^{k+1} = \frac{1}{8}\left[15 - 3x_2^{(k+1)} + x_3^{(k+1)}\right]$$

K	x_1	x_2	x_3	x_4
0	0.	0	0	0
1	0.6	2.32727	−0.987273	0.878864
2	1.03018	2.03694	−1.0146	0.98341.
3				
⋮				
Converges to	1.	2	−1	1.

Answer
$x_1 = 1$
$x_2 = 2$
$x_3 = -1$
$x_4 = \underline{\underline{1}}$

The power method is used for finding largest eigen value (in magnitude) and finding the corresponding eigen vector of the eigen value problem $AX = \lambda X$.

To find the largest of eigen values among $\lambda_1 \lambda_2 \dots \lambda_n$.. we need to recall the some preliminies of cha. matrix

Chareteristic Equation

For a given square matrix $A_{n \times n}$, Characteristic Equation is given by $|A - \lambda I| = 0$.

$$a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots a_{n-1} \lambda + a_n = 0.$$

Cleary the characteristic polynomial is of degree n so nxn matrix will have exactly n roots.

Eigen Values: Roots of Characteristic Equations are called Eigen Values. Clearly the cha. polynomial is of degree n. So nxn matrix have n roots, denoted by $\lambda_1, \lambda_2, \dots \lambda_n$.

Eigen Vectors:

A non-zero vector X_i which satisfies the following equation is called Eigen Vector corresponding Eigen Value λ_i

$$AX_i = \lambda_i X_i$$

If λ is the eigen value of A then A^{-1} Eigen Value is $\frac{1}{\lambda} = \lambda^{-1}$.

So if λ^{-1} largest eigen value of A^{-1} then λ is the Smallest Eigen Value of A.

Power Method for finding Largest Eigen Value.

We want to find the Largest Eigen Value from these eigen values $\lambda_1 \lambda_2 \dots \lambda_n$.

let λ_1 is the Largest eigen value, that is

$$\lambda_1 > \lambda_2 > \lambda_3 > \dots > \lambda_n$$

Any non zero vector can be expressed as linear Combination of eigen vectors.

$$X^{(0)} = c_1 X_1 + c_2 X_2 + c_3 X_3 \dots + c_n X_n.$$

Define

Premultiplying by A both sides.

$$X^{(1)} = AX^{(0)} = c_1 AX_1 + c_2 AX_2 + \dots + c_n AX_n.$$

But we know that

$$= c_1 \lambda_1 X_1 + c_2 \lambda_2 X_2 \dots + c_n \lambda_n X_n. \quad AX_i = \lambda_i X_i$$

$$X^{(2)} = AX^{(1)} = c_1 \lambda_1 AX_1 + c_2 \lambda_2 AX_2 + \dots + c_n \lambda_n AX_n \quad AX^1 = A(AX^0) = (A^2 X^0)$$

$$= c_1 \lambda_1^2 X_1 + c_2 \lambda_2^2 X_2 \dots + c_n \lambda_n^2 X_n.$$

Illy

$$X^K = AX^{K-1} = c_1 \lambda_1^{K-1} AX_1 + c_2 \lambda_2^{K-1} AX_2 \dots + c_n A \lambda_n^{K-1} X_n$$

$$= c_1 \lambda_1^K X_1 + c_2 \lambda_2^K X_2 \dots + c_n \lambda_n^K X_n.$$

$$X^{(K)} = [c_1 \lambda_1^K X_1 + c_2 \lambda_2^K X_2 \dots c_n \lambda_n^K X_n] \quad \because AX = \lambda X.$$

$$= \lambda_1^K \left[c_1 X_1 + c_2 \frac{\lambda_2^K}{\lambda_1^K} X_2 \dots c_n \frac{\lambda_n^K}{\lambda_1^K} X_n \right]$$

$$X^{(K)} = \lambda_1^K \left[c_1 X_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^K X_2 \dots c_n \left(\frac{\lambda_n}{\lambda_1}\right)^K X_n \right]$$

For Large k; $\left(\frac{\lambda_i}{\lambda_1}\right)^k \to 0$ Since each $\left(\frac{\lambda_i}{\lambda_1}\right) < 1$.

Since λ_1 is large one.

So, clearly; $X^{(k)} = \lambda_1^k C X_1$ also $X^{k+1} = \lambda_1^{k+1} C X_1$.

Now Consider

$$\frac{[X^{(k)}]^T X^{(k+1)}}{[X^{(k)}]^T X^{(k)}} = \frac{\lambda_1^k C X_1^T \lambda_1^{k+1} C X_1}{\lambda_1^k C X_1^T \lambda_1^k C X_1} = \frac{\lambda_1 X_1^T X_1}{X_1^T X_1} = \lambda_1$$

Formula $\lambda_1 = \frac{[X^{(k)}]^T X^{k+1}}{[X^{(k)}]^T X^{(k)}}$.

Procedure to find the largest Eigen Value of a matrix A.

We start with some non zero vector

$X^0 = [\ ]$

$X^1 = A X^{(0)}$

$X^2 = A X^{(1)}$

$X^3 = A X^{(2)}$

....

$X^{(k+1)} = A X^k$.

then $\lambda_1 = \frac{[X^{(k)}]^T X^{(k+1)}}{[X^{(k)}]^T X^{(k)}}$.

For example $\frac{[X^{(3)}]}{[X^{(3)}]} \cdot \frac{X^{(4)}}{X^{(3)}}$.

Rob: Find the Largest Eigen Value of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}$

We are start with some nonzero vector.

$$X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$X^{(1)} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$X^{(2)} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 28 \\ 18 \end{bmatrix}$$

$$X^{(3)} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 14 \\ 28 \\ 18 \end{bmatrix} = \begin{bmatrix} 124 \\ 248 \\ 228 \end{bmatrix}$$

$$X^{(4)} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 124 \\ 248 \\ 228 \end{bmatrix} = \begin{bmatrix} 1304 \\ 2608 \\ 2088 \end{bmatrix}$$

$$\lambda_1 = \frac{[X^{(3)}]^T X^{(4)}}{[X^{(3)}]^T X^{(3)}}$$

$$= \frac{\begin{bmatrix} 124 & 248 & 228 \end{bmatrix} \begin{bmatrix} 1304 \\ 2608 \\ 2088 \end{bmatrix}}{\begin{bmatrix} 124 & 248 & 228 \end{bmatrix} \begin{bmatrix} 124 \\ 248 \\ 228 \end{bmatrix}}.$$

$$\lambda = \underline{\underline{9.868}}.$$

Prob: Find the smallest Eigen Value of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

The smallest Eigen Value of A is equal to the Largest Eigen Value of A^{-1}.

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

We start with some non zero vector.

$$X^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$X^{(1)} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6 \\ 8 \\ 6 \end{bmatrix}$$

$$X^{(2)} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 6 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 10 \\ 14 \\ 10 \end{bmatrix}$$

$$X^{(3)} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 10 \\ 14 \\ 10 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 17 \\ 24 \\ 17 \end{bmatrix}$$

$$X^{4} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 17 \\ 24 \\ 17 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 29 \\ 41 \\ 29 \end{bmatrix}$$

$$\lambda_1 = \frac{[X^{(3)}]^T X^4}{[X^{(3)}]^T X^3}$$

$$= \frac{\begin{bmatrix} 17 & 24 & 17 \end{bmatrix}\begin{bmatrix} 29 \\ 41 \\ 29 \end{bmatrix}}{\begin{bmatrix} 17 & 24 & 17 \end{bmatrix}\begin{bmatrix} 17 \\ 24 \\ 17 \end{bmatrix}} = 1.7069.$$

The Largest Eigen Value of A^{-1} is the Smallest Eigen Value of $A = \frac{1}{1.7069} = 0.5858$

Solve the following system of equations using the Gauss-Jacobi iteration method.

1. $20x + y - 2z = 17,$
   $3x + 20y - z = -18,$
   $2x - 3y + 20z = 25.$ (*A.U. Nov/Dec 2006*)

2. $27x + 6y - z = 85,$
   $x + y + 54z = 110,$
   $6x + 15y + 2z = 72.$ (*A.U. May/June*

3. $x + 20y + z = -18,$
   $25x + y - 5z = 19,$
   $3x + 4y + 8z = 7.$

4. $10x + 4y - 2z = 20,$
   $3x + 12y - z = 28,$
   $x + 4y + 7z = 2.$

Solve the following system of equations using the Gauss-Seidel iteration method.

5. $27x + 6y - z = 85,$
   $x + y + 54z = 110,$
   $6x + 15y + 2z = 72.$
   (*A.U. May/June 2006*)

6. $4x + 2y + z = 14,$
   $x + 5y - z = 10,$
   $x + y + 8z = 20.$
   (*A.U. Apr/May*

7. $x + 3y + 52z = 173.61,$
   $x - 27y + 2z = 71.31,$
   $41x - 2y + 3z = 65.46.$ Start with $x = 1, y = -1, z = 3.$ (*A.U. Apr/May*

8. $20x - y - 2z = 17,$
   $3x + 20y - z = -18,$
   $2x - 3y + 20z = 25.$ (*A.U. Nov/Dec*

9. $x + 20y + z = -18,$
   $25x + y - 5z = 19,$
   $3x + 4y + 8z = 7.$

10. $10x + 4y - 2z = 20,$
    $3x + 12y - z = 28,$
    $x + 4y + 7z = 2.$

Determine the largest eigen value in magnitude and the corresponding eigen vector of the following matrices by power method. Use suitable initial approximation to the eigen vector.

1. $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$. (*A.U. Nov/Dec 2003*)

2. $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ (*A.U. Apr/May 2005*)

3. $\begin{bmatrix} 35 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix}$.

4. $\begin{bmatrix} 20 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

5. $\begin{bmatrix} 6 & 1 & 0 \\ 1 & 40 & 1 \\ 0 & 1 & 6 \end{bmatrix}$.

6. $\begin{bmatrix} 15 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{bmatrix}$.

7. $\begin{bmatrix} 3 & 1 & 5 \\ 1 & 0 & 2 \\ 5 & 2 & -1 \end{bmatrix}$.

8. $\begin{bmatrix} 65 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Exercise 1.4

In all problems, we have taken $\mathbf{v}^{(0)} = [1, 1, 1]$. The results obtained after 8 iterations are giv
Solutions are the transposes of the given vectors.

1. $|\lambda| = 11.66$, $\mathbf{v} = [0.02496, 0.42180, 1.0]$.
2. $|\lambda| = 6.98$, $\mathbf{v} = [0.29737, 0.06690, 1.0]$.
3. $|\lambda| = 35.15$, $\mathbf{v} = [1.0, 0.06220, 0.02766]$.
4. $|\lambda| = 20.11$, $\mathbf{v} = [1.0, 0.05316, 0.04759]$.
5. $|\lambda| = 40.06$, $\mathbf{v} = [0.02936, 1.0, 0.02936]$.
6. $|\lambda| = 15$, $\mathbf{v} = [1.0, 0.00002, 0.0]$.
7. $|\lambda| = 6.92$, $\mathbf{v} = [1.0, 0.35080, 0.72091]$.
8. $|\lambda| = 65.02$, $\mathbf{v} = [1.0, 0.0, 0.01587]$.

Exercise 1.3

In all the Problems, values for four iterations have been given. *Solutions are the transposes of the given vectors.*

1. [0, 0, 0], [0.85, – 0.9, 1.25], [1.02, – 0.965, 1.03], [1.00125, – 1.0015, 1.00325], [1.00040, – 0.99990, 0.99965]. Exact: [1, – 1, 1].
2. [0, 0, 0], [3.14815, 4.8, 2.03704], [2.15693, 3.26913, 1.88985], [2.49167, 3.68525, 1.9365], [2.40093, 3.54513, 1.92265].
3. Exchange the first and second rows. [0, 0, 0], [0.76, – 0.9, 0.875], [0.971, – 0.981, 1.04], [1.00727, – 1.00055, 1.00175], [1.00037, – 1.00045, 0.99755]. Exact: [1, – 1, 1].
4. [0, 0, 0], [2.0, 2.33333, 0.28571], [1.12381, 1.85714, – 1.33333], [0.99048, 1.941, – 0.93605], [1.03628, 2.00771, – 0.96508]. Exact: [1, 2, – 1].
5. [0, 0, 0], [3.14815, 3.54074, 1.91317], [2.43218, 3.57204, 1.92585], [2.42569, 3.5729, 1.92595], [2.42549, 3.57301, 1.92595].
6. [0, 0, 0], [3.5, 1.3, 1.9], [2.375, 1.905, 1.965], [2.05625, 1.98175, 1.99525], [2.0103, 1.99700, 1.99909]. Exact: [2, 2, 2].
7. Interchange first and third rows. [1, – 1, 3], [1.32829, – 2.36969, 3.44982], [1.2285, – 2.34007, 3.45003], [1.22999, – 2.34000, 3.45000], [1.23000, – 2.34000, 3.45000].
8. [0, 0, 0], [0.85, – 1.0275, 1.01088], [0.89971, – 0.98441, 1.01237], [0.90202, – 0.9846, 1.01210], [0.90200, – 0.98469, 1.01210].
9. Interchange first and second rows. [0, 0, 0], [0.76, – 0.938, 1.059], [1.00932, – 1.003, 0.99821], [0.99978, – 0.99990, 1.00003], [1.0, – 1.0, 1.0].
10. [0, 0, 0], [2.0, 1.83333, – 1.04762], [1.05714, 1.98175, – 0.99773], [1.00775, 1.998 [illegible], [1.00068, 1.99982, – 0.99989]. Exact: [1, 2, – 1].