Interpolation

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1. Any Continuous function can be approximated by a Polynomial. (Weistras Theorem).

Even if data/information about the continuous function is given, we can approximate by a polynomial.

$x_0, x_1, x_2 \ldots x_n$

$f_0, f_1, f_2 \ldots f_n$

Suppose data is given for the interpolation $f(x)$ at $(n+1)$ distinct points

$x_0, x_1 \ldots x_{n-1}, x_n$

$f_0, f_1 \ldots f_{n-1}, f_n$

Actually speaking we have to retrive the function $f(x)$. Some times we want to find $f(x)$ at any other point. So let us find polynomial $P(x)$ which satisfies the given data.

That is values of polynomial $P(x)$ and function $f(x)$ must be same at these $(n+1)$ points. that is

$$P(x_i) = f_i \quad \text{for } i = 0, 1, 2 \ldots n$$

Hence we claim that $P_n(x) \cong f(x)$.

1.

Interpolation

Def: The polynomial which satisfies given data is called Interpolating Polynomial.

Finding the value of the function at any point inside the interval by interpolating polynomial is called Interpolation.

It is the process of finding the most appropriate estimate for missing data.

Finding the value of the function at any point out side the interval by interpolating polynomial is called Extrapolation.

Note: The maximum of this interpolating polynomial is n.

Since data/Information is given $(n+1)$ points We can think of a polynomial having $(n+1)$ variables only Hence we can think polynomial ~~of~~ degree n only.

Note: Maximum degree of this interpolating polynomial is n.

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots a_{n-1} x^{n-1} + a_n x^n.$$

These $a_0, a_1 \dots a_{n-1}, a_n$ can be obtained by the interpolation condition.

$$P_n(x_i) = f_i \quad \text{for } i = 0, 1, 2 \dots n$$

$(n+1)$ conditions for $(n+1)$ variables so we can find

Legranges Interpolation:

DATA is given for the function $f(x)$ has at $(n+1)$ distinct points

$$\begin{array}{cccccc} x_0, & x_1, & x_2, & \dots & x_{n-1}, & x_n \\ f_0, & f_1, & f_2 & \dots & f_{n-1}, & f_n \end{array}$$

We discuss Legranges approach to find n^{th} degree interpolating polynomial $P(x)$ which satisfies the given data.

That is a polynomial $P(x)$ and function $f(x)$ must be same at these $(n+1)$ points. That is

$$P_n(x_i) = f_i \text{ for } i = 0, 1, 2 \dots n.$$

We first discuss for case by case as follows:

(i) for ONE point data

(ii) for TWO point data

(iii) then the general case for $(n+1)$ points data.

Legranges Interpolation

case(i). Data is given for the function $f(x)$ at one point.
$$\begin{array}{c} x_0 \\ f_0 \end{array}$$

We discuss Legranges approach to find 0^{th} degree polynomical $P(x)$ which satisfies given data.

That is polynomial $P(x)$ and function $f(x)$ must be same at the given one point. That is

$$P_0(x_i) = f_i \quad \text{for } i = 0.$$

We take 0^{th} degree polynomial as $P_0(x) = a_0$

$$P_0(x_0) = a_0 = f_0 .$$

We get 0^{th} degree Legranges $\quad P_0(x) \approx f_0$

Interpolating polynomial as $\quad P_0(x) = f_0 \approx f(x)$.

Case (ii). Data is given for the function $f(x)$ at

TWO Points

x_0	x_1
f_0	f_1

We discusss Legranges approach to find 1^{st} degree polynomial $P(x)$ which satisfies the given data. That is polynomial $P(x)$ and function $f(x)$ must be same at these two points. That is

$$P_i(x_i) = f_i \quad \text{for } i = 0, 1.$$

We take 1^{st} degree polynomial as

$$P_1(x) \approx a_0 + a_1 x .$$

Using Co-ordinate geometry we can find the straight line passing through the given TWO points

x_0	x_1
f_0	f_1

The equation such a straight line is

$$f(x) - f_0 = \left(\frac{f_1 - f_0}{x_1 - x_0}\right)(x - x_0)$$

$$f(x) = f_0 + \left(\frac{f_1 - f_0}{x_1 - x_0}\right)(x - x_0) .$$

$$f(x) = \left[\frac{f_0(x_1 - x_0) + (f_1 - f_0)(x - x_0)}{(x_1 - x_0)}\right]$$

Straight line $= \left[\frac{f_0(x_1 - x_0) - f_0(x - x_0) + f_1(x - x_0)}{(x_1 - x_0)}\right]$.

$$P_1(x) = \left[\frac{(x - x_1)}{(x_0 - x_1)}\right] f_1 + \left[\frac{(x - x_0)}{(x_1 - x_0)}\right] f_1$$

This is the 1st degree Legranges Interpolating Polynomial $P(x)$ which satisfies the given two points $P_i(x_i) \approx f_i$ for $i = 0, 1$.

Hence we can write $P_1(x) \approx \overline{f(x)}$.

$$P_1(x) = \left[\frac{(x - x_1)}{(x_0 - x_1)}\right] f_0 + \left[\frac{(x - x_0)}{(x_1 - x_0)}\right] f_1$$

let us define Legranges Polynomial as:

$$l_0(x) = \left[\frac{(x - x_1)}{(x_0 - x_1)}\right] \qquad L_1(x) = \left[\frac{(x - x_0)}{(x_1 - x_0)}\right]$$

$P_1(x) = l_0(x) f_0 + l_1(x) f_1$ written as $P_1(x) = \sum_{i=0}^{1} l_i(x) f_i$

Hence we can write $P_1(x) = \sum_{i=0}^{1} L_i(x) f_i \approx f(x)$

Observations on these Legranges polynomials.

$$l_0(x) = \left[\frac{(x-x_1)}{(x_0-x_1)}\right] \text{ and } l_1(x) = \left[\frac{(x-x_0)}{(x_1-x_0)}\right]$$

Observations (i): These are degree 1

Obeservations (ii) $\quad l_0(x_0)=1 \quad l_0(x_1)=0$

$l_1(x_1)=1.$

$l_1(x_0)=0$

That is in general

$l_1(x_j)=0$

$l_1(x_1)=1$

Obs. (iii): $l_0(x)+l_1(x) = \left[\frac{(x-x_0)}{(x_0-x_1)}\right] + \left[\frac{(x-x_0)}{(x_1-x_0)}\right] = 1.$

Now let us take a general Case

DATA is given for the function $f(x)$ at $(n+1)$ distinct points

$x_0 \quad x_1 \quad x_2 \quad x_{n-1} \quad x_n$

$f_0 \quad f_1 \quad f_2 \quad f_{n-1} \quad f_n$ We discuss Legranges approach to find n^{th} degree polynomial $P(x)$ which satisfies the given data.

That is polynomial $P(x)$ and function $f(x)$ must be same at these $(n+1)$ points. That is

$$P_n(x_i)=f_i \quad i=0,1,2\ldots n$$

We can take n^{th} degree Legranges Interpolation polynomial as

$$P_n(x) = \sum_{i=0}^{n} l_i(x)\, f_i$$

Formula: Let $f_0, f_1, f_2 \ldots f_n$ be the values of $f(x)$ at $x_0, x_1, x_2, \ldots x_n$ (nodes) not neccessarly equal points then an interpolating polynomial $p(x)$ for $f(x)$ is given by

$$p(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f_2$$

$$+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} \cdot f_n.$$

Advantages of Legranges Interpolation

1. Easy to write the polynomial
2. Data points need not be equi-spaced.

If data points are increased by one more point we have to re-do Entire polynomial. That is previous polynomial will not be use full to get next polynomial.

For the two points data we have 1st degree polynomial Than for the Three points data we have 2nd degree polynomial.

Clearly $P_2(x)$ can not be obtained using $P(x)$ if it possible we say polynomial permanance property.

Prob: Find Legrange Interpolating polynomial.

	x_0	x_1	x_2
x	0	1	3
$f(x)$	1	3	55
	f_0	f_1	f_2

Data Points are 3, So max. Degree of Interpolating polynomial: 2

$$f(x) \approx P_2(x) = \sum_{i=0}^{2} l_i(x) f_i = L_0 f_0 + L_1 f_1 + L_2 f_2.$$

or

$$P_2(x) = \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\right] f_0 + \left[\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\right] \cdot f_1$$

$$+ \left[\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\right] f_2.$$

$$P_2(x) = \left[\frac{(x-1)(x-3)}{(0-1)(0-3)}\right] \cdot 1 + \left[\frac{(x-0)(x-3)}{(1-0)(1-3)}\right] \cdot 3 +$$

$$\left[\frac{(x-0)(x-1)}{(3-0)(3-1)}\right] 55$$

$$f(x) \approx P_2(x) = 8x^2 - 6x + 1$$

i.e, $f(0) = P_2(0) = 1$

$f(1) = P_2(1) = 3$

$f(3) = P_2(3) = 55$

Prob: Find Legranges Interpolate polynomial

	x_0	x_1	x_2	x_3
x	-1	0	1	2
$f(x)$	1	1	1	-5
	f_0	f_1	f_2	f_3.

Sol. Data is given at 4 points, max degree of interpolating polynomial: 3.

$$f(x) \approx P_3(x) = \sum_{i=0}^{3} L_i(x) f_i = L_0^{(x)} f_0 + L_1^{(x)} f_1 + L_2^{(x)} f_2 + L_3^{(x)} f_3$$

Where all these Legranges polynomials are of degree 3.

$$L_i(x) = \frac{\text{All the factors except the suffix point}}{\text{Same Numerator where } x \text{ is replaced by the forgotten point.}}$$

Further $L_i(x_i) = 1 \quad L_i(x_j) = 0$

$$f(x) \approx P_3(x) = \left[\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right] f_0 +$$

$$\left[\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right] f_1 +$$

$$\left[\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right] f_2 +$$

$$\left[\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right] f_3 .$$

$$P_3(x) = \left[\frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)}\right]\cdot 1 + \left[\frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)}\right]\cdot 1$$

$$+ \left[\frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)}\right]\cdot 1 + \left[\frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)}\right](-5)$$

$$f(x) \approx P_3(x) = -x^3 + x + 1$$

$f(x_i) = P_3(x_i)$ Clearly $i = 0, 1, 2, 3.$

Prob
Using the Interpolating polynomial find $f(6)$.

	x_0	x_1	x_2	x_3
x	1	2	7	8
$f(x)$	4	5	5	4.
	f_0	f_1	f_2	f_3

Data is given at 4 Points, max-degree of interpolating polynomial: 3

$$f(x) \approx P_3(x) = \sum_{i=0}^{3} L_i(x) f_i = L_0(x) f_1 + L_1(x) f_2 + L_2(x) f_3 + L_3(x) f_3.$$

Where all these legranges polynomial are of degree 3.

$$L_i(x) = \frac{\text{All the factors except the suffix point}}{\text{Same numerator where } x \text{ is replaced by the forgotten point}}.$$

Further $L_i(x_i) = 1 \quad L_i(x_j) = 0$.

$$P_3(x) = \left[\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}\right] \cdot f_0$$

$$+ \left[\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}\right] \cdot f_1$$

$$+ \left[\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}\right] f_2$$

$$+ \left[\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}\right] f_3.$$

$f(x) \approx P_3(x)$ so $f(6) \approx P_3(6)$.

in the above eqn.

~~Set~~ So put $x = 6$. in the above eqn.

$$
\begin{aligned}
f(6) \approx P_3(6) = {} & \left[\frac{(6-2)(6-7)(6-8)}{(1-2)(1-7)(1-8)}\right] \cdot 4 + \\
& \left[\frac{(6-1)(6-7)(6-8)}{(2-1)(2-7)(2-8)}\right] \cdot 5 + \\
& \left[\frac{(6-1)(6-2)(6-8)}{(7-1)(7-2)(7-8)}\right] \cdot 5 + \\
& \left[\frac{(6-1)(6-2)(6-7)}{(8-1)(8-2)(8-7)}\right] \cdot 4
\end{aligned}
$$

$$f(6) \approx P_3(6) = \underline{\underline{5.66}}\,.$$

Prob: Using the Legranges Interpolating Polynomial find the Partial Fractions of .

$$\phi(x) = \frac{x^2+x+3}{x^3-2x^2-x+2}$$

Observe $\phi(x) = \dfrac{x^2+x+3}{x^3-2x-x+2} = \dfrac{x^2+x+3}{(x+1)(x-1)(x-2)}$.

$$x=1 \left| \begin{array}{cccc} \multicolumn{4}{l}{x^3-2x^2-x+2} \\ 1 & -2 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ \hline 1 & -1 & -2 & \lfloor 0 \end{array} \right.$$

$x^2-x-2=0$

$x^2-2x+x-2=0$

$x(x-2)+1(x-2)=0$

$(x+1)(x-2)$.

Now take $f(x) = x^2+x+3$ and write as product of factors.

x : -1 . 1 2

$f(x)$: -3 -1 3

Data is given at 3 points, max degree of interpolating polynomial is 2:

$$P_2(x) = \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\right] f_0 + \left[\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\right] f_1 + \left[\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\right] f_2$$

$$P_2(x) = \left[\frac{(x-1)(x-2)}{(-1-1)(-1-2)}\right](-3) + \left[\frac{(x+1)(x-2)}{(1+1)(1-2)}\right](-1) + \left[\frac{(x+1)(x-1)}{(2+1)(2-1)}\right](3) + \Big[$$

$$f(x) = P_2(x) = \left[\frac{(x-1)(x-2)}{-2}\right] + \left[\frac{(x+1)(x-2)}{2}\right] + \left[\frac{(x+1)(x-1)}{1}\right]$$

$$\text{So } \phi(x) = \frac{x^2+x-3}{(x+1)(x-1)(x-2)} = \left[\frac{(x-1)(x-\cancel{2})}{(-2)(x+1)(\cancel{x-1})(x-\cancel{2})} + \right.$$

$$\left.\frac{(x+1)(x-2)}{2(x+1)(x-1)(x-2)} + \frac{(x+1)(x-1)}{1\cdot(x+1)(x-1)(x-2)}\right]$$

$$= \frac{-1}{2}\frac{1}{(x+1)} + \frac{1}{2}\cdot\frac{1}{(x-1)} + \frac{1}{1\cdot(x-2)}.$$

$$= \frac{P_2(x)}{(x-1)(x-2)(x+3)}.$$

$$\phi(x) = \frac{x^2+x-3}{(x+1)(x-1)(x-2)} = -\frac{1}{2}\frac{1}{(x+1)} + \frac{1}{2}\cdot\frac{1}{x-1} + \frac{1}{x-2}$$

Assignment on Legranges Interpolation

1. Use Lagrange's interpolation formula to find the value of y when $x = 10$, if the following values of x and y are given:

x:	5	6	9	11
y:	12	13	14	16

2. The following table gives the viscosity of oil as a function of temperature. Use Lagrange's formula to find the viscosity of oil at a temperature of 140°.

Temp°:	110	130	160	190
Viscosity:	10.8	8.1	5.5	4.8

3. Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$, find by using Lagrange's formula, the value of $\log_{10} 656$.

4. The following are the measurements T made on a curve recorded by oscilograph representing a change of current I due to a change in the conditions of an electric current.

T:	1.2	2.0	2.5	3.0
I:	1.36	0.58	0.34	0.20

Using Lagrange's formula, find I and $T = 1.6$.

5. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data:

Year:	1997	1999	2001	2002
Profit in Lakhs of Rs:	43	65	159	248

6. Use Lagrange's formula to find thee form of $f(x)$, given

x:	0	2	3	6
$f(x)$:	648	704	729	792

7. If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$, $y(6) = 132$, fin d the Lagrange's interpolation polynomial that takes the same values as y at the given point s.

8. Given $f(0) = -18$, $f(1) = 0$, $f(3) = 0$, $f(5) = -248$, $f(6) = 0$, $f(9) = 13104$, find $f(x)$.

9. Find the missing term in the following table using interpolation

x:	1	2	4	5	6
y:	14	15	5	....	9

10. Using Lagrange's formula, express the function $\dfrac{x^2 + x - 3}{x^3 - 2x^2 - x + 2}$ as a sum of partial fractions.

11. Using Lagrange's formula, express the function $\dfrac{x^2 + 6x - 1}{(x^2 - 1)(x - 4)(x - 6)}$ as a sum of partial fractions.

[*Hint.* Tabulate the values of $f(x) = x^2 + 6x - 1$ for $x = -1, 1, 4, 6$ and apply Lagrange's formula.]

1. 11.5 **2.** 6.304 **3.** 37.23. **4.** 2.3.

5. 0.2679 **6.** 1.3714.

The calculus of finite differences is an interesting topic and has wide applications in various fields. Using this concept, we deal with the changes that take place in the value of the function, the dependent variable due to finite changes in the independent variable.

Suppose a table of values (x_i, y_i) $i = 1, 2, 3 \ldots n$, of any function $y = f(x)$, the values of x being equally spaced. i.e., $x_i = x_0 + h$, $i = 0, 1, 2 \ldots n$ we are required to obtain the values of $f(x)$ for some intermediate value of x or to obtain the derivative of $f(x)$ for some x in the range $x_0 \leq x \leq x_n$. The following differences are usefull.

i. Forward Differences: If a function $y = f(x)$ is tabulated for the equally spaced arguments $x_0, x_0+h, x_0+2h \ldots x_0+nh$ giving the functional values $y_0, y_1, y_2 \ldots y_n$.
The constant difference between two consecutive arguments (x) is called the interval of differencing and is denoted by h.
The forward difference operator Δ is denoted and defined as

$$\Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1 \quad \Delta y_2 = y_3 - y_2$$

and so on $\Delta y_n = y_{n+1} - y_n$.

$\Delta y_0, \Delta y_1, \ldots \Delta y_n$ are called first forward differences.

The differences of the first forward differences are called 2^{nd} forward differenes and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots \Delta^2 y_n$, defined as.

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$$

$$= (y_2 - y_1) - (y_1 - y_0)$$

$$= y_2 - y_1 - y_1 + y_0$$

$$\Delta^2 y_0 = y_2 - 2y_1 + y_0 .$$

llly 3^{rd} forward differenes and 4^{th} forward differeus. ets.

$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_4 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

In General.

$$\Delta^n y_n = \Delta^{n-1} y_{n+1} - \Delta^{n-1} y_n .$$

The following table shows how the forward differenes of all orders canbe formed.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$.
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
				$\Delta^3 y_0$	
		Δy_1			$\Delta^4 y_0$.
x_2	y_2		$\Delta^2 y_1$		
				$\Delta^3 y_1$	
		Δy_2			
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

Backward Differences: If a function $y=f(x)$ is tabulated for the equally spaced arguments $x_0, x_0+h, x_0+2h \dots x_0+nh$ giving functional values $y_0, y_1 \dots y_n$. The backward difference operator ∇ is defined as

$$\nabla y_1 = y_1 - y_0 \quad \nabla y_2 = y_2 - y_1 \quad \nabla y_3 = y_3 - y_2, \nabla y_n = y_n - y_{n-1}$$

are called the first difference operator.

In a similar way, we can define the backward differences of higher orders

Thus $\nabla^2 y_2 = \nabla(\nabla y_2) = \nabla(y_2 - y_1)$.

$$= \nabla y_2 - \nabla y_1$$

$$= (y_2 - y_1) - (y_1 - y_0)$$

$$\nabla^2 y_2 = y_2 - 2y_1 + y_0$$

Illy $\nabla^2 y_3 = y_3 - 3y_2 + 3y_1 - y_0$ etc..

The following table shows how the backward differences of all orders can be formed.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		Δy_1			
x_1	y_1		$\Delta^2 y_1$		
		Δy_2		$\Delta^3 y_1$	
x_2	y_2		$\Delta^2 y_2$		$\Delta^4 y_1$.
		Δy_3		$\Delta^3 y_2$	
x_3	y_3		$\Delta^2 y_3$		
		Δy_4			
x_4	y_4				

Central Differences: Some times it is very usefull to employ another system of differences is known as Central differences. The Central difference operator δ (delta) is defined by the relation

$$\delta_{1/2} = y_1 - y_0 ; \quad \delta y_{3/2} = y_2 - y_1 \; \cdots \; \delta y_{n/2} = y_n - y_{n-1}.$$

Illy, the central differences of higher orders are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1 , \quad \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2 \cdots$$

Central Difference Table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		δy_1		
		$\delta y_{3/2}$		$\delta y_{3/2}$	
x_2	y_2		δy_2		$\delta^4 y_2.$
		$\delta y_{5/2}$		$\delta y_{5/2}$	
x_3	y_3		δy_3		
		$\delta y_{7/2}$			
x_4	y_4				

Shift operator: The shift operator E is defined as

$$E f(x) = f(x+h).$$
$$E^2 f(x) = f(x+2h).$$
$$\vdots$$
$$E^n f(x) = f(x+nh).$$

The inverse operator E^{-1} is defined as.

$$E^{-1} f(x) = f(x-h),$$
$$E^{-2} f(x) = f(x-2h)$$
$$\vdots$$
$$E^{-n} f(x) = f(x-nh).$$

Average Operator:

The averaging operator μ is defined as.

$$\mu f(x) = \frac{1}{2}\left[f(x+h/_2) + f(x-h/_2) \right].$$

Relation Between Operators.

Relation between Δ, ∇, E.

$$\Delta y_x = y_{x+h} - y_x.$$

$$= E y_x - y_x.$$

$$\Delta y_x = (E-1) y_x.$$

or $\Delta = E-1$

$$\boxed{E = 1+\Delta} \quad \text{—} (1)$$

$$\nabla y_x = y_x - y_{x-h}.$$

$$= y_x - E^{-1} y_x.$$

$$\nabla y_x = (1-E^{-1}) y_x.$$

$$\nabla = 1 - E^{-1}$$

$$\nabla = 1 - \frac{1}{E}. \quad = \frac{1}{E} = 1-\nabla.$$

$$\boxed{E = \frac{1}{1-\nabla}} \quad \text{—} (2)$$

from these two (1 & 2)

$$1+\Delta = \frac{1}{1-\nabla}.$$

$$\Delta = \frac{1}{1-\nabla} - 1 = \frac{\nabla}{1-\nabla}$$

$$\boxed{\Delta = \frac{\nabla}{1-\nabla}}$$

Relation between δ and E.

We know that

$$\delta y_x = y_{x+h/2} - y_{x-h/2}.$$

$$\delta y_n = E^{1/2} y_n - E^{-1/2} y_n.$$

$$\delta y_x = [E^{1/2} - E^{-1/2}] y_n.$$

$$\boxed{\delta = E^{\frac{1}{2}} - E^{-1/2}}$$

Relation on μ & E.

$$\mu f(x) = \frac{1}{2}\left[f(x+h/2) + f(x-h/2)\right]$$

$$= \frac{1}{2}\left[E^{\frac{1}{2}} f(x) + E^{-1/2} f(x)\right]$$

$$\mu f(x) = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x)$$

$$\boxed{\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]}$$

Relation between Δ, ∇, E and δ

$$E(\nabla f(x)] = E(f(x) - f(x-h))$$

$$= E(f(x) - E f(x-h)$$

$$= f(x+h) - f(x).$$

$$= \Delta f(x).$$

$$\boxed{E\nabla = \Delta}$$

$$\nabla(E f(x)) = \nabla(f(x+h)$$

$$= f(x+h) - f(x)$$

$$= \Delta f(x)$$

$$\boxed{\nabla E = \Delta}$$

Prove: 1. $E = e^{hD}$ 2. $\Delta - \nabla = \delta^2$

3. $\Delta = \delta^2/2 + \delta\sqrt{1+\delta^2/4}$ 4). $\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{\Delta}{2}$

5. $\Delta . \nabla = \Delta - \nabla$.

Evaluate. $\Delta^3 (1-x)(1-2x)(1-3x)$.

let $f(x) = (1-x)(1-2x)(1-3x)$.

$f(x) = -6x^3 + 11x^2 - 6x + 1$.

So that the polynomial $f(x)$ is of order 3.

$\therefore \quad \Delta^3 f(x) = \Delta^3(-6x^3) + \Delta^3(11x^2) - \Delta^3(-6x) + \Delta^3(1)$

$= -6.3!$

$= \underline{\underline{-36}}$.

Prob: $\Delta \tan^{-1}x$.

$\Delta \tan^{-1}x = \tan^{-1}(x+h) - \tan^{-1}x$.

$= \tan^{-1}\left\{\frac{x+h-x}{1+(x+h)x}\right\} = \tan^{-1}\left\{\frac{h}{1+hx+x^2}\right\}$

Rob: (i) $\Delta^2\left(\frac{5x+12}{x^2+5x+16}\right)$.

$\Delta^2\left(\frac{5x+12}{x^2+5x+16}\right) = \Delta^2\left(\frac{5x+12}{(x+2)(x+3)}\right)$.

$= \Delta^2\left(\frac{2}{x+2} + \frac{3}{x+3}\right)$.

$= \Delta\left\{\Delta\left(\frac{2}{x+2} + \frac{3}{x+3}\right)\right\}$

$= \Delta\left\{2\left(\frac{1}{x+3} - \frac{1}{x+2}\right) + 3\left(\frac{1}{x+4} - \frac{1}{x+3}\right)\right\}$

$$= -2\Delta\left\{\frac{1}{(x+2)(x+3)}\right\} - 3\Delta\left\{\frac{1}{(x+3)(x+4)}\right\}.$$

$$= -2\left\{\frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)}\right\} - 3\left\{\frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)}\right\}$$

$$= \frac{4}{(x+2)(x+3)(x+4)} + \frac{6}{(x+3)(x+4)(x+5)}.$$

$$= \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}.$$

Prob: find $\Delta(2^x)$

$$\Delta(2^x) = 2^{x+h} - 2^x = 2^x(2^h - 1).$$

Prob: Find the n^{th} difference of e^x.

$$\Delta e^x = e^{x+h} - e^x.$$
$$= e^x(e^h - 1)$$
$$\Delta^2 e^x = \Delta(\Delta e^x)$$
$$= \Delta(e^x(e^h-1)).$$
$$= (e^h-1)\Delta e^x.$$
$$= (e^h-1)(e^x(e^h-1)).$$
$$\Delta^2 e^x = e^x(e^h-1)^2$$

Illy $\quad \Delta^3 e^x = e^x(e^h-1)^3$

Proceeding like this we get

$$\Delta^n e^x = e^x(e^h-1)^n.$$

Preparation of forward difference table.

1. Numbers of entries get reduced in columns.
2. Table is complete if there is only one entry in column
3. Any mistake in table will give wrong answer.

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$.
x_0	f_0			
		$f_1 - f_0$		
x_1	f_1		$(f_2-f_1)-(f_1-f_0)$	
		$f_2 - f_1$		$[(f_3-f_2)-(f_2-f_1)] - [(f_2-f_1)-(f_1-f_0)]$.
x_2	f_2		$(f_3-f_2)-(f_2-f_1)$	
		$f_3 - f_2$		
x_3	f_3			

Backward Difference table:

x	$f(x)$	∇f	$\nabla^2 f$	∇^3
x_0	f_0			
		$f_1 - f_0$		
x_1	f_1		$(f_2-f_1)-f_1-f_0$	
		$f_2 - f_1$		$[(f_3-f_2)-(f_2-f_1)] - [(f_2-f_1)-(f_1-f_0)]$.
x_2	f_2		$(f_3-f_2)-(f_2-f_1)$	
		$f_3 - f_2$		
x_3	f_3			

1. Construct a forward difference table for the data given below.

x	10	20	30	40
y	1.1	2.0	4.4	7.9

x	y	Δ	Δ^2	Δ^3
10	1.1			
		0.9 Δy_0		
			1.5 $\Delta^2 y_0$	
20	2.0			-0.4 $\Delta^3 y_0$
		2.4 Δy_1	1.1 $\Delta^2 y_1$	
30	4.4			
		3.5 Δy_2		
40	7.9			

Thus $\Delta^3 y_0 = \Delta^3 y_{10} = \underline{\underline{-0.4}}$.

2. Find the missing values in the table.

x	45	50	55	60	65
$f(x)$	3	—	2	—	-2.4

Table.

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
45	3				
		$A-3$			
			$5-2A$		
50				$3A+B-9$	
		$2-A$			$-4A-4B+12$
			$A+B-4$		
55	2			$3.6-A-3B$	
		$B-2$			
			$-0.4-2B$		
60	B				
		$-2.4-B$			
65	-2.4				

Data is 3 points max degree is 2. So third forward difference should zero. $3A+B-9=0$ and $-3.6-A-3B=$

Solving we get $A = 2.925$ $B = 0.225$.

Prob: Find the missing term is the table.

x	2	3	4	5	6
y	45.0	49.2	54.1	—	67.4.

Let the missing value be a. Then the difference table is as follows.

x	y		Δy.	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	45.0	y_0	4.2	0.7	a−59.7	240.2−49
3	49.2	y_1	4.9	a−59.0	180.6−3a	
4	54.1	y_2	a−54.1	121.5−2a		
5	a	y_3	67.4−a			
6	67.4	y_4				

The given 4 entries degree of polynomial is 3.

So 4th (degree) differences as zero.

We know that $\Delta^4 y = 0$.

$$240.9 - 4a = 0.$$

$$a = \frac{240.9}{4} = \underline{\underline{60.05}}.$$

$\therefore$ missing term $= \underline{\underline{60.05}}$

Prob: The following table gives the values of y which is a polynomial of degree five. It is known that $f(3)$ is in error. Correct the error.

x	0	1	2	3	4	5	6
y	1	2	33	$\boxed{254}$	1025	3126	7777

Let the correct value of y when $x=3$ be a. Then the difference table is as follows.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1					
		1				
1	2		30			
		31		$a-94$		
2	33		$a-64$		$1216-4a$	
		$a-33$		$1122-3a$		$-2320+10a$
3	a		$1058-2a$		$-1104+6a$	
		$1025-a$		$18+3a$		$2560-10a$
4	1025		$1076+a$		$1456-4a$	
		2101		$1474-a$		
5	3126		2550			
		4651				
6	7777					

$\Delta^6 y$

$4880-20a.$

Since y is a polynomial of fifth degree the sixth difference $\Delta^6 y = 0$

$$4880-20a=0.$$

$$20a=4880 \Rightarrow a=\frac{488\cancel{0}}{2\cancel{0}}=244$$

Hence the error $= 254-244$

$= \underline{10}$

Newton's Forward Interpolation formula.

Let the function $y = f(x)$ take the values $y_0, y_1, \cdots y_n$ Corresponding to the values $x_0, x_0+h, x_0+2h \cdots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$ where p is any number. [Equispaced step size].

For any real number p, we have defined E (Shift operator) Such that

$f(x_0) = y_0$

$$E^p f(x) = f(x+ph).$$

$$y_p = f(x_0+ph) = E^p f(x_0)$$

$\because E = 1+\Delta.$

$$= E^p y_0.$$

$$y_p = (1+\Delta)^p y_0$$

$$\therefore y_p = \left\{ 1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots \right\} y_0$$

Using bionomial theorem.

ie, $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

If $y = f(x)$ is a polynomial of the n^{th} degree, then $\Delta^{n+1} y_0$ and higher differences will be zero. Hence Newton Forward Difference formula defined as.

$$\boxed{y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)\cdots(p-\overline{n-1})}{n!}\Delta^n y_0.}$$

§ This is the formula is used for interpolating the values of y near the begining of a set of tabulated values and extrapolating values of y a little backward. (i.e., to the left of y_0)

§ The first two terms of this formula give the linear interpolation while the first three terms give a parabolic interpolation and so on.

Newton Back Word Difference Interpolation

Let us the function $y=f(x)$ take the values $y_0, y_1, y_2 \cdots$ Corresponding to the values $x_0, x_0+h, \ldots x_0+ph$ of x [Equispaced Step Size]. Suppose it is required to evaluate $f(x)$ for $x = x_0+Ph$, where P is any real numbers. Then we have

$$y_p = f(x_n+Ph) = E^p f(x_n) = (1-\nabla)^p y_0 \qquad \because E^{-1} = 1-\nabla$$

$$= E_p f(x_n) = (1-\nabla)^p y_0$$

$$= \left[1 + P\nabla + \frac{P(P+1)}{2!}\nabla^2 + \frac{P(P+1)(P+2)}{3!}\nabla^3 + \cdots\right] y_n$$

$$y_p = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \cdots$$

it is called Newton's Backward interpolation formula Contains y_n and backward differences of y_n.

§ This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little a head to the right of y_n.

Find the interpolating polynomial by Newton's Forward Interpolation formula.

x : 0 1 2 3

$f(x)$: 1 2 1 10

Hence or otherwise evaluate $f(4)$.

Data is given at 4 points, maximum degree of interpolating polynomial degree is : 3.

Step size $h=1$.

The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1 $f(0)$			
		1 $\Delta f(0)$		
			-2 $\Delta^2 f(0)$	
				12. $\Delta^3 f(0)$
1	2			
		-1		
			10	
2	1			
		9		
3	10			

We take $x_0 = 0$ and $P = \frac{x - x_0}{h} = \frac{x-0}{1} = x$.

$\because h = 1$.

$\therefore$ Using Newton's Forward Interpolation formula, we get

$$f(x) = y_0 + P\Delta y_0 + P(P-1)\Delta^2 y_0 + \cdots\cdots$$

$$= y_0 + P\Delta f(0) + P(P-1)\Delta^2 f(0) + \cdots\cdots$$

$$f(x) = 1 + x \cdot 1 + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6} 12.$$

$$= 1 + x + x(x-1)(-1) + x(x-1)(x-2) \cdot 2$$

$$= 2x^3 - 7x^2 + 6x + 1. \quad f(x) \approx P_3(x) = 2x^3 - 7x^2 + 6x + 1$$

Which is the required polynomial

To Compute $f(4)$, We take $x_n = 3$ $x = 4$

$$P = \frac{x - x_n}{h} = 1. \quad \because h = 1.$$

Using the backward interpolation formula.

$$f(4) = f(3) + P \nabla f(3) + \frac{P(P+1)}{1 \cdot 2} \nabla^2 f(3) + \frac{P(P+1)(P+2)}{3!} \nabla^3 f(3)$$

$$= 10 + 9 + 10 + 12 = 41.$$

We can Verify

$f(0) = P_3(0) = 1.$

$f(1) = P_3(1) = 0$

$f(2) = P_3(2) = 1$

$f(3) = P_3(3) = 10.$

Prob: The table gives the distance is natical miles of the visible horizon for the given heights in feet above the earth's surface.

x = height	100	150	200	250	300	350	400
$y = f(x)$ = distance	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the value of y when (i) $x = 160$ ft. (ii). $x = 410$ ft.

The difference table is as under

x	$y=f(x)$	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
			-0.39		
150	13.03 y_0			0.15	
		2.01 Δy_0			-0.07
200	15.04		-0.24 $\Delta^2 y_0$		
				0.08 $\Delta^3 y_0$	
		1.77			-0.05 $\Delta^4 y_0$
			-0.16		
250	16.81			0.03	
		1.61			$\boxed{-0.01}$ $\nabla^4 y_n$
			-0.13		
300	18.42			$\boxed{0.02}$ $\nabla^3 y_n$	
		1.48			
			$\boxed{-0.11}$ $\nabla^2 y_n$		
350	19.90				
		$\boxed{1.37}$ ∇y_n			
400	$\boxed{21.27}$ y_n				

If we take $x = 160$ then $y_0 = 13.03$ $\Delta y_0 = 2.01$

$\Delta^2 y_0 = -0.24$ $\Delta^3 y_0 = 0.08$ $\Delta^4 y_0 = -0.05$.

and $h = 50$.

Since $x = 160$ $x_0 = 150$.

$$\therefore\ p = \frac{x - x_0}{h} = \frac{160 - 150}{50} = \frac{10}{50} = 0.2.$$

$\therefore$ Using Newton's Forward Interpolation formula we get

$$y_{x} = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots$$

$$y_{160} = 13.03 + 0.2(2.01) + \frac{(0.2)(0.2-1)\cdot(-0.24)}{2} + \frac{(0.2)(0.2-1)(0.2-2)}{6}(0.08) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{4!}(-0.05)$$

$$= 13.03 + 0.402 + 0.192 + 0.0384 + 0.00168$$

$$= 13.46 \text{ nautical miles.}$$

(ii) find at $x = 410$.

Since $x = 410$ is near the end of the table, we use Newtons Backword Interpolation formula.

$\therefore$ take $x_n = 400$, $\quad P = \frac{x - x_n}{h} = \frac{410-400}{50}$

$$= \frac{10}{50} = 0.2.$$

Using the line of backward interpolation formula.

$$y_{410} = y_{400} + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n + \frac{P(P+1)(P+2)(P+3)}{4!}\nabla^4 y_n + \cdots$$

$$= 21.47 + 0.2(1.37) + \frac{(0.2)(1.2)(-0.11)}{2!} + \frac{(0.2)(1.2)(2.2)}{3!}(0.02) + \frac{(0.2)(1.2)(2.2)(3.2)}{4!}(0.01)$$

$$= 21.47 + 0.274 - 0.0132 + 0.0018 - 0.0007 = 21.53$$

Prob: Find number of students who got marks ~~less than 45~~ by NFIF. between 40 and 45.

Range	30-40	40-50	50-60	60-70	70-80
$f(x)$	31	42	51	35	31

Marks $<X$	$f(x)=y_x$	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	Δ^4
40	31				
		42 Δy_x			
50	73		9 $\Delta^2 y_x$		
		51		−25 $\Delta^3 y_x$	
60	124		−16		37 $\Delta^4 y_x$
		35		12	
70	159		−4		
		31			
80	190				

We shall find y_{45}, i.e., the number of students with marks less than 45. Taking $x_0=40$. $x=45$ We have

$$P=\frac{x-x_0}{h}=\frac{45-40}{10}=0.5 \qquad \because h=10.$$

Using Newton's forward Interpolation formula, we get

$$y_{45}=y_{40}+P\Delta y_{40}+\frac{P(P-1)}{2!}\Delta^2 y_{40}+\frac{P(P-1)(P-2)}{3!}\Delta^3 y_{40}+\frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_{40}+\cdots$$

$$=31+0.5\times 42+\frac{(0.5)(-0.5)}{2}\times 9+\frac{(0.5)(-0.5)(-1.5)}{6}(-25)$$

$$+\frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}\times 37.$$

$$=31+21-1.125-1.5625-1.4453$$

$$=47.87 \text{ on simplification}$$

The number of students with marks less than ~~40~~ 45 is 47.87 i.e., 48.

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 = 48 − 31 = 17.

—•—

Use Newton Forward Interpolation formula if you are finding $f(x)$ near first point.

Use Newton Backward Interpolation formula if you are finding $f(x)$ near to last point.

Newtons forward and Backward ~~Central difference~~ formula are applicable for interpolating near the begining and end of the tabulated values. However, they are not applicable to interpolate near the central value of the table because they contain forward and backward differences which are far away from the central line.

To obtain more accurate estimation near the middle of a table we introduce Central difference interpolation.

Consider the function $y = f(x)$ whose values at a collection of equally spaced points are given. Denote the middle point as x_0 so that the set of equally spaced points are given by

$$\ldots, x_0-3h, x_0-2h, x_0-h, x_0, x_0+h, x_0+2h, x_0+3h, \ldots$$

With this notation the table giving the values of the function takes the following.

x	$\ldots$	x_0-2h	x_0-h	x_0	x_0+h	$x_0+2h \ldots$
$f(x)$	$\ldots$	y_{-2}	y_{-1}	y_0	y_1	$y_2 \ldots$

We now form a difference table in the case of several values.

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0-3h	y_{-3}						
		Δy_{-3}					
x_0-2h	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_0-h	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_0+h	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_0+2h	y_2		$\Delta^2 y_1$				
		Δy_2					
x_0+3h	y_3.						

The entries in this table can be related to the entries of the previous table by using the operator relation

$$\delta = \Delta E^{-1/2}.$$

3. With the usual notations, show that

(*i*) $\Delta = 1 - e^{-hD}$ (*ii*) $D = \frac{2}{h}\sinh^{-1}\left(\frac{\delta}{2}\right)$

(*iii*) $(1+\Delta)(1-\nabla) = 1$ (*iv*) $\Delta\nabla = \nabla\Delta = \delta^2$

4. Prove that

(*i*) $\delta = \Delta(1+\Delta)^{-1/2} = \nabla(1-\nabla)^{-1/2}$

(*ii*) $\mu^2 = 1 + \frac{\delta^2}{4}$ (*iii*) $\delta\left(E^{1/2} + E^{-1/2}\right) = \Delta E^{-1} + \Delta$

5. Show that

(*i*) $\Delta = \Delta E^{-1/2} = \Delta E^{1/2}$

(*ii*) $\mu\Delta = \frac{1}{2}(\Delta + \nabla)$ (*iii*) $1 + \delta^2/2 = \sqrt{(1 + \delta^2\mu^2)}$

6. Show that

(*i*) $\Delta = \mu\delta + \frac{\delta^2}{2}$ (*ii*) $E^{1/2} = \left(1 + \frac{\delta^2}{4}\right)^{1/2} + \frac{\delta^2}{2}$

(*iii*) $E^r = \left(\mu + \frac{1}{2}\delta\right)^{2r}$ (*iv*) $\mu = \frac{2+\Delta}{2\lambda(1+\Delta)} = \frac{2+\nabla}{2\lambda(1+\nabla)}$

7. Prove that

(*i*) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$ (*ii*) $\nabla = \Delta E^{-1} = E^{-1}\Delta = 1 - E^{-1}$

10. Estimate the missing term in the following table:

x:	0	1	2	3	4
$f(x)$	1	3	9	–	81

11. Find the missing terms in the following table:

x:	1	1.5	2	2.5	3	3.5
y:	6	?	10	20	?	1.5

12. Find the missing values in the following table:

0	1	2	3	4	5	6
5	11	22	40	...	140	...

13. Estimate the production for 2004 and 2006 from the following data:

Year:	2001	2002	2003	2004	2005	2006	2007
Production:	200	200	260	...	350	...	430

Gauss Forward Interpolation Formula.

The Newton's Forward Interpolation formula is,

$$y_p = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots \quad (1)$$

where y_p is the value of y at $x = x_p$

$x_p = x_0 + ph$

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

i.e., $\Delta^2 y_0 = \Delta^3 y_{-1} + \Delta^2 y_{-1} = \Delta^2 y_{-1} + \Delta^3 y_{-1}$

lly $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$

$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$ etc...

Also $\Delta^4 y_{-2} = \Delta^3 y_{-1} - \Delta^3 y_{-2}$

i.e., $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$

lly $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc.

Substituting for $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$ values in (1)

We get

$$y_p = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{P(P-1)(P-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$

$$+ \frac{P(P-1)(P-2)(P-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

Now substituting the value of $\Delta^4 y_{-1}$ in the above equation we have

$$y_p = y_0 + P\Delta y_0 + P(P-1)(\Delta^2 y_{-1}) + \frac{P(P+1)(P-1)}{3!}(\Delta^3 y_{-1}) +$$

$$\frac{P(P+1)(P-2)(P-1)}{4!}\Delta^4 y_{-2} + \cdots$$

This version of Newton's forward Interpolation formula is known as as the GAUSS'S Forward inter

§ ~~Ddeexaplayes~~.

Gauss's Backward Interpolation formula.

The Newton's Forward Interpolation Formula

$$y_p = y_0 + p\Delta y_0 + p(p-1)\Delta^2 y_0 + \cdots \quad \text{—— (1)}$$

We have $\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$

i.e., $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$

|||ly $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$

$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ etc

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

i.e., $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$.

|||ly $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc.

Substituting for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 \ldots$ in (1)

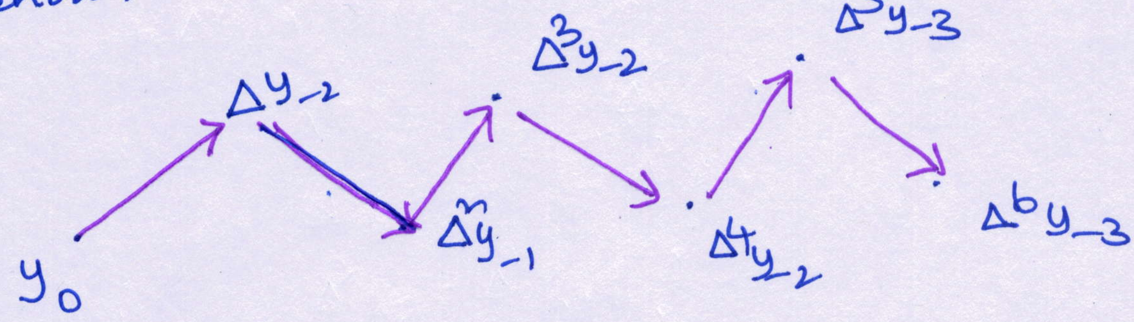
$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1} + \frac{p(p+1)(p-1)(p-2)}{4!}\Delta^4 y_{-1} + \cdots$$

Obs: It employs odd differences just below the central line and even differences on the central difference line as shown below.

$$y_0 \searrow \Delta y_0 \nearrow \Delta^2 y_{-1} \searrow \Delta^3 y_{-1} \nearrow \Delta^4 y_{-2} \searrow \Delta^7 y_{-3}.$$

$\Delta^4 y_{-2} \longrightarrow$ Central line.

Obs. 2. This formula contains odd differences above the central line and even differences on the central line as shown below:



Prob: Use Gauss forward interpolation formula find $f(3.3)$ from the following data.

x	1	2	3	4	5
$y = f(x)$	15.30	15.10	15.00	14.50	14.00

The difference table for the given data below.

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1 x_{-2}	15.30 y_{-2}				
		-0.20 Δy_{-2}			
2 x_{-1}	$\boxed{15.10\ y_{-1}}$		$\boxed{0.10}$ $\Delta^2 y_{-1}$.		
		$\boxed{-0.10}$ Δy_0		$\boxed{-0.50}$ $\Delta^3 y_{-1}$	
3 x_0	15.00 y_0		-0.40		0.90.
		-0.50		0.40	
4 x_1	14.50 y_1		0.00		
		-0.50			
5 x_2	14.00 $_{-2}$.				

Prob: Use Gauss Forward Interpolation Formula find $f(3.3)$ from the following data.

x	1	2	3	4	5
$f(x)$	15.30	15.10	15.0	14.50	14.00

The difference table for the given data below.

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_{-2} 1	y_{-2} 15.30				
		-0.20			
			0.10 $\Delta^2 y_{-2}$		
x_{-1} 2	y_{-1} 15.10			-0.50	
		-0.10	$\Delta^2 y_{-1}$		$\boxed{0.90}$
x_0 3	$\boxed{y_0\ 15.0}$	Δy_0	$\boxed{-0.40}$	$\boxed{0.40}$	
		$\boxed{-0.50}$			
			0.00		
x_1 4	y_1 14.50		$\Delta^2 y_0$		
		-0.50			
x_2 5	y_2 14.00				

From the table we note that $\Delta^2 y_{-1}=-0.40$ $\quad \Delta^3 y_{-1}$ $\quad \Delta^4 y_{-2}$.

$y_0=15.0$, $\Delta y_0=-0.50$, $x_0=3$ $\quad P=\frac{x-x_0}{h}$

at $x_p=3.3$ ie $x=3.3$, $\quad =\frac{3.3-3}{1}=0.3$.

By Gauss Forward difference formula.

$$y(x)=y_0+P\Delta y_0+\frac{P(P-1)}{2!}\Delta^2 y_{-1}+\frac{P(P-1)(P+1)}{3!}\Delta^3 y_{-1}+\cdots$$

$$f(3.3)=15+(0.3)(-0.50)+\frac{(0.3)(0.3-1)}{2!}(-0.40)+\frac{(0.3)(0.09-1)}{6}0.40+\frac{0.3(0.09-1)(0.3-2)}{24}(0.90)$$

$$=14.9$$

Prob: Use Gauss's forward interpolation formula find the value of y at $x=30$ using the following data.

x	21	25	29	33	37
y	18.4708	17.8144	17.1070	16.3422	15.5154.

let us prepare difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_{-2} 21	y_{-2} 18.4708				
		-0.6564			
			-0.0510		
x_{-1} 25	y_{-1} 17.8144			-0.054	
		-0.7074			$\boxed{-0.02}$
x_0 29	$\boxed{y_0\ 17.1070}$		$\boxed{-0.0564}$	$\boxed{-0.0076}$	
		$\boxed{-0.7638}$			
			-0.0640		
x_1 33	y_{+1} 16.3422				
		-0.8278			
x_2 37	y_2 15.5154				

From the table we find that $y_0=17.1070$, $\Delta y_0=-0.7638$, $\Delta^2 y_{-1}=-0.0564$ $\Delta^3 y_{-1}=-0.0076$, $\Delta^4 y_{-2}=-0.02$

$$P=\frac{x-x_0}{h}=\frac{30-29}{4}=0.25$$

By Gauss Forward Interpolation formule.

$$f(30)=y_p=y_0+P\Delta y_0+\frac{P(P-1)}{2!}\Delta^2 y_{-1}+\frac{(P+1)P(P-1)}{3!}\Delta^3 y_{-1}+\cdots$$

$$=17.1070+(0.25)(-0.7638)+\frac{(0.25)(0.25-1)}{2!}(-0.0564)$$

$$+\frac{(0.25)(0.0625-1)}{6}(-0.0076)+\frac{(0.25)(0.0625-1)(0.25-2)}{24}(-0.002)$$

$f(33) = 16.9213$ approx.

<u>Prob</u>

Find The value of y at $x=1986$ by using Gauss backward interpolating formula. (Population of a town is ~~lakhs~~ thousands).

x	1951	1961	1971	1981	1991	2001
y	12	15	20	27	39	52

let us take $x_0 = 1981$ and construct the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	Δ^4	Δ^5
x_{-3} 1951	12 y_{-3}					
		3				
x_{-2} 1961	15 y_{-2}		2			
		5		0		
x_{-1} 1971	20 y_{-1}		2		3	
		$\boxed{7}$		$\boxed{3}$		$\boxed{-10.}$
x_0 1981	$\boxed{27}$ y_0		$\boxed{5}$		$\boxed{-7}$	
		12		-4		
x_1 1991	39 y_1		1			
		13				
x_2 2001	52. y_2					

from the table, we have $y_0 = 27$, $\Delta y_{-1} = 7$, $\Delta^2 y_{-1} = 5$

$\Delta^3 y_{-2} = 3$, $\Delta^4 y_{-2} = -7$, $\Delta^5 y_{-3} = -10$.

let $x_0 = 1981$ $x_p = 1986$. then $p = \frac{x_p - x_0}{h}$.

$$= \frac{1986-1981}{10} = \underline{\underline{0.5}}$$

By The Gauss Backward Interpolation formula.

$$y_p = y_0 + p\,\Delta y_{-1} + \frac{p(p+1)}{2!}\,\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\,\Delta^3 y_{-1} +$$

$$\frac{(p+2)(p+1)p(p-1)}{4!}\,\Delta^4 y_{-2} + \cdots$$

$$y_{(1986)} = 27 + 0.5\,(7) + \frac{(0.5)(0.5+1)}{2}(5) + \frac{(0.5)(0.25-1)}{6}(3)$$

$$+ \frac{(0.5)(0.25-1)(0.5+2)}{24}(-7) + \frac{(0.5)(0.25-1)(0.25-4)(-10)}{120}.$$

$$= 32.345$$

$\therefore$ y at $1986 = \underline{32.345}$.

Prob: Given that $\sqrt{6500} = 80.6223$, $\sqrt{6510} = 80.6846$

$\sqrt{6520} = 80.7456$, $\sqrt{6530} = 80.8084$ find $\sqrt{6526}$

By Using Gauss's Backward formula.

Here the given function is of the form $f(x) = \sqrt{x}$.

Let us take $x_0 = 6520$ and construct the difference table

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_{-2}	6500	80.6223			
			0.0623		
				-0.0004	
x_{-1}	6510	80.6846			$\boxed{0.0004}$
			$\boxed{0.0619}$	$\boxed{0}$	
x_0	$\boxed{6520}$	$\boxed{80.7456}$ y_0			
			0.0619		
x_1	6530	80.8084			

From the table $y_0 = 80.7456$, $\Delta y_{-1} = 0.0619$ $\Delta^2 y_{-1} = 0$

$\Delta^3 y_{-2} = -0.0004$

Let $x = 6526$ then $p = \dfrac{x - x_0}{h} = \dfrac{6526 - 6520}{10} = 0.6$.

Substitute these values in Gauss's Backward Interpolation formula.

We have

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2}$$
$$+ \frac{p(p+1)(p-1)(p+2)}{4!}\Delta^4 y_{-2}$$

$$y_{\sqrt{6526}} = 80.7456 + 0.6(0.0619) + \frac{(0.6)(1.6)}{2!}(0) + \frac{(0.6)(0.36-1)}{6}(0.0004)$$

$= 80.7836$, $\sqrt{6526} = 80.7826$

Numerical Differentiation.

Assume that data points only the data of function is given in distinct points.

$$x_0 \quad x_1 \quad x_2 \quad \cdots \quad x_{n-1},\ x_n$$

$$f_0 \quad f_1 \quad f_2 \quad \cdots \quad f_{n-1},\ f_n$$

How to find derivatives numerically ?

Case 1. At any Un-tabulated points :

Find the insterpolating polynomial (by any formula) then differentiate it to find derivatives of function at any un-tabulated points.

$$P_n(x) \approx f(x).$$

Then $P_n'(x) \approx f'(x)$

$$P_n''(x) \approx f''(x). \quad \text{etc} \ldots$$

Assume that data points are EQI-SPACED with Step size as h $\quad x_{i+1} - x_i = h$

$$x_0 \quad x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_{n-1},\ x_n$$

$$f_0 \quad f_1 \quad f_2 \quad f_3 \quad \cdots \quad f_{n-1},\ f_n$$

Case 2. Derivatives at tabulated points.

We know that from NFFDF

$$E f_i = f(x_{i+1}) = f(x_i + h)$$

By Taylor series.

$$= f_i + h f_i' + \frac{h^2}{2} f_i'' + \frac{h^3}{3} f_i''' + \cdots$$

$$Ef_i = f_i + hf_i' + \frac{h^2}{\lfloor 2} f_i'' + \frac{h^3}{\lfloor 3} f_i''' + \cdots$$

$$= f_i + hDf_i + \frac{h^2}{\lfloor 2} D^2 f_i + \frac{h^3}{\lfloor 3} D^3 f_i + \cdots$$

$$Ef_i = \left[1 + hD + \frac{h^2}{2!} D^2 + \cdots\right] f_i$$

$$Ef_i = \left[e^{hD}\right] f_i$$

Now $\boxed{E = e^{hD}}$

Derivatives using forward Differences

$$E = e^{hD}$$

$$\log E = hD$$

Since $E = 1 + \Delta$.

$$\therefore \quad hD = \log E$$

$$hD = \log(1+\Delta)$$

$$= \left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} + \cdots\right]$$

$$f_i' = Df_i = \frac{1}{h}\left[\Delta f_i - \frac{\Delta^2}{2} f_i + \frac{\Delta^3}{3} f_i + \cdots\right]$$

$$f_i'' = D^2 f_i = \frac{1}{h^2}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots\right]^2 f_i$$

$$= \frac{1}{h^2}\left[\Delta^2 f_i - \Delta^3 f_i + \frac{11}{12}\Delta^4 f_i \cdots\right]$$

$$f_i^n = D^n f_i = \frac{1}{h^n}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} + \cdots\right]^n f_i$$

$$= \frac{1}{h^2}\left[\Delta f_i - \frac{\Delta^2}{2} f_i + \frac{\Delta^3}{3} f_i + \cdots\right]^n$$

$$= \frac{1}{h^2}\left[\Delta^n f_i - n\frac{\Delta^{n-1}}{2} f_i + \frac{n(3n+5)}{24}\Delta^{n-2} f_i \cdots\right]$$

Forward Difference Table.

x	f	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	f_0					
		Δf_0				
			$\Delta^2 f_0$			
				$\Delta^3 f_0$		
x_1	f_1	Δf_1			$\Delta^4 f_0$	$\Delta^5 f_0$.
			$\Delta^2 f_1$	$\Delta^3 f_1$	$\Delta^4 f_1$	
x_2	f_2	Δf_2				
			$\Delta^2 f_2$	$\Delta^3 f_2$		
x_3	f_3	Δf_3	$\Delta^2 f_3$			
x_4	f_4	Δf_4				
x_5	f_5					

Derivatives Using Backward differences.

$$E = e^{hD} \Rightarrow hD = \log E. \qquad \because E = (1-\nabla)^{-1}$$

$$hD = \log(1-\nabla)^{-1}.$$

$$= \left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \cdots\right] \cancel{f_i}$$

$$f_i' = Df_i = \frac{1}{h}\left[\nabla f_i + \frac{\nabla^2}{2} f_i + \frac{\nabla^3}{3} f_i + \cdots\right].$$

$$f_i'' = D^2 f_i = \frac{1}{h^2}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} \cdots\right]^2 f_i$$

$$= \frac{1}{h^2}\left[\nabla^2 f_i + \nabla^3 f_i + \frac{11}{12}\nabla^4 f_i + \cdots\right]$$

$$f_i^n = D^n f_i = \frac{1}{h^2}\left[\nabla + \nabla^2 + \frac{\nabla^3}{3} + \cdots\right]^n f_i$$

$$f_i^n = \frac{1}{h^n}\left[\nabla^n f_i + \frac{n\nabla^{n-1} f_i}{2} + \frac{n(3n+5)\nabla^{n+2} f_i}{24} \cdots\right]$$

Back Ward Difference Table.

x	f	∇	∇^2	∇^3	∇^4	∇^5
x_0	f_0					
		∇f_1				
			$\nabla^2 f_2$			
x_1	f_1			$\nabla^3 f_3$		
		∇f_2			$\nabla^4 f_4$	
			$\nabla^2 f_3$			$\nabla^5 f_5$.
x_2	f_2			$\nabla^3 f_4$		
		∇f_3			$\nabla^4 f_5$	
			$\nabla^2 f_4$			
x_3	f_3			$\nabla^3 f_5$		
		∇f_4				
			$\nabla^2 f_5$			
x_4	f_4					
		∇f_5				
x_5	f_5					

Derivatives Using Central Differences.

$$\delta = \left[E^{1/2} - E^{-1/2}\right] = \left[e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}\right]$$

$$= 2\sinh\left(\frac{hD}{2}\right).$$

$$\Rightarrow hD = 2\sinh^{-1}(\delta/2)$$

$$= 2\left[\delta/2 - \frac{\delta^2}{3!2^3} + \cdots\right.$$

$$f_i^{1} = D f_i = \frac{1}{h}\left[\delta h_i - \frac{\delta^3 f_i}{(3!)2^2} + \cdots\right]$$

$$f_i^{11} = D^2 f_i = \frac{1}{h^2}\left[\delta^2 f_i + \frac{\delta^4 f_i}{12} + \frac{11}{90}\delta^6 f_i + \cdots\right]$$

$$\therefore f_i^{n} = D^n f_i = \frac{1}{h^n}\left[\delta^n f_i + n\frac{\delta^{n+2}}{24} f_i + \frac{n(5n+22)\delta^{n+4} f_i}{5760} + \cdots\right.$$

Find the $f'(3)$ and $f''(3)$ from the following table

x	3	3.2	3.4	3.6	3.8	4
$f(x)$	-14	-10.032	-5.296	0.256	6.672	14

Sol: Construct the difference table.

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
3	-14				
		$\boxed{3.9687}$			
			$\boxed{0.768}$	$\boxed{0.48}$	$\boxed{0}$
3.2	-10.032				
		4.732			
			0.816		
3.4	-5.296			0.48	0.
		5.552			
			0.864		
3.6	0.256			0.48	
		6.416			
			0.912		
3.8	6.672				
		7.328			
4.0	14				

$h = 0.2$

$$f'(3) = f'_0 = Df_0 = \frac{1}{h}\left[\Delta f_0 - \frac{\Delta^2 f_0}{2} + \frac{\Delta^3 f_0}{3}\right]$$

$$= \frac{1}{0.2}\left[3.9687 - \frac{0.768}{2} + \frac{0.48}{3}\right]$$

$$f'(3). = 18.$$

$$f''(3) = f''_0 = D^2 f_0 = \frac{1}{h^2}\left[\Delta^2 f_0 - \Delta^3 f_0 + \frac{11}{12}\Delta^4 f_0\right]$$

$$= \frac{1}{(0.2)^2}\left[0.768 - 0.048 + 0\right]$$

$$f''(3) = 18.$$

From the following table of values of x and y obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.2$.

x	1.0	1.2	1.4	1.6	1.8	2.0.
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891

Construct the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.0	2.7183					
		0.6018				
1.2	$\boxed{3.3201}$		0.1333			
		$\boxed{0.7351}$		0.0294		
1.4	4.0552		$\boxed{0.1627}$		0.0067	
		0.8978		$\boxed{0.0361}$		0.0013
1.6	4.9530		0.1988		$\boxed{0.0080}$	
		1.0966		$\boxed{0.0441}$		
1.8	6.0496		$\boxed{0.2429}$			
		$\boxed{1.3395}$				
2.0	7.3891					

here $x_0 = 1.2$ $y_0 = 3.3201$ and $h = 0.2$

$$\therefore \left.\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{h}\left[\Delta f_0 - \frac{\Delta^2}{2} f_0 + \frac{\Delta^3}{3} f_0 - \frac{\Delta^4}{4} f_0 + \cdots\right]$$

$$= \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.008\right.$$

$$= 3.320.$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1.2} = \frac{1}{h^2}\left[\Delta^2 f_0 - \Delta^3 f_0 + \frac{11}{12}\Delta^4 f_0 + \cdots\right]$$

$$= \frac{1}{0.04}\left[0.1627 - 0.0361 + \frac{11}{12}\, 0.0080\right]$$

$$= 3.318.$$

Calculate the first and second derivatives of the function tabulated at $x = 2.0$.

at $x = 2.0$. $\qquad h = 0.2$.

$$\left.\frac{dy}{dx}\right|_{x=2.0} = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \cdots\right] f_n.$$

$$= \frac{1}{0.2}\left[1.3395 + \frac{0.2429}{2} + \frac{1}{3}0.0441 + \frac{1}{4}(0.0080)\right]$$

$$= 7.389.$$

$$\left.\frac{d^2y}{dx^2}\right|_{2.0} = \frac{1}{h^2}\left[\nabla^2 f_i + \nabla^3 f_i + \frac{11}{12}\nabla^4 f_i + \frac{5}{6}\Delta^5 f_i\right]$$

$$= \frac{1}{0.04}\left[0.2429 + 0.0441 + \frac{11}{12}(0.080) + \frac{5}{6}(0.0080)\right]$$

$$= 7.32 \text{ approx.}$$

Assignment

3. Express the value of θ in terms of x using the following data:

x:	40	50	60	70	80	90
θ:	184	204	226	250	276	304

Also find θ at $x = 43$.

4. Given sin 45° = 0.7071, sin 50° = 0.7660, sin 55° = 0.8192, sin 60° = 0.8660, find sin 52° using Newton's forward formula.

5. From the following table:

x:	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$:	2.68	3.04	3.38	3.68	3.96	4.21

find $f(0.7)$ approximately.

6. The area A of a circle of diameter d is given for the following values:

d:	80	85	90	95	100
A:	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105

7. From the following table:

$x°$:	10	20	30	40	50	60	70	80
cos x:	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1737

Calculate cos 25° and cos 73° using the Gregory-1 Newton formula.

8. A test performed on a *NPN* transistor gives the following result:

Base current f (mA)	0	0.01	0.02	0.03	0.04	0.05
Collector current I_C (mA)	0	1.2	2.5	3.6	4.3	5.34

Calculate (*i*) the value of the collector current for the base current of 0.005 mA.
(*ii*) the value of base current required for a collector correct of 4.0 mA.

9. Find $f(22)$ from the following data using Newton's backward formulae.

x:	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

10. Find the number of men getting wages between Rs. 10 and 15 from the following data:

Wages in Rs:	0—10	10—20	20—30	30—40
Frequency:	9	30	35	42

11. From the following data, estimate the number of persons having incomes between 2000 and 2500:

Income	Below 500	500–1000	1000–2000	2000–3000	3000–4000
No. of persons	6000	4250	3600	1500	650

12. Construct Newton's forward interpolation polynomial for the following data:

x:	4	6	8	10
y:	1	3	8	16

Hence evaluate y for $x = 5$.

13. Find the cubic polynomial which takes the following values:
$y(0) = 1$, $y(1) = 0$, $y(2) = 1$ and $y(3) = 10$.

Hence or otherwise, obtain $y(4)$.

14. Construct the difference table for the following data:

x:	0.1	0.3	0.5	0.7	0.9	1.1	1.3
$f(x)$:	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Evaluate $f(0.6)$

15. Apply Newton's backward difference formula to the data below, to obtain a polynomial of degree 4 in x:

x:	1	2	3	4	5
y:	1	-1	1	-1	1

16. The following table gives the population of a town during the last six censuses. Estimate the increase in the population during the period from 1976 to 1978:

Year:	1941	1951	1961	1971	1981	1991
Population: (*in thousands*)	12	15	20	27	39	52

17. In the following table, the values of y are consecutive terms of a series of which 12.5 is the fifth term. Find the first and tenth terms of the series.

x:	3	4	5	6	7	8	9
y:	2.7	6.4	12.5	21.6	34.3	51.2	72.9

18. Using a polynomial of the third degree, complete the record given below of the export of a certain commodity during five years:

Year:	1989	1990	1991	1992	1993
Export: (*in tons*)	443	384	—	397	467

19. Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$, find u_2 and u_4.

20. If $u_{-1} = 10$, $u_1 = 8$, $u_2 = 10$,$u_4 = 50$, find u_0 and u_3.

21. Given $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 200$, $y_4 = 100$, $y_5 = 8$, without forming the difference table, find $\Delta^5 y0$.

7.4 Central Difference Interpolation Formulae

In the preceding sections, we derived Newton's forward and backward interpolation formulae which are applicable for interpolation near the beginning and end of tabulated values. Now we shall develop central difference formulae which are best suited for interpolation near the middle of the table.

If x takes the values $x_0 - 2h$, $x_0 - h$, x_0, $x_0 + h$, $x_0 + 2h$ and the corresponding values of $y = f(x)$ are y_{-2}, y_{-1}, y_0, y_1, y_2, then we can write the difference table in the two notations as follows:

x	y	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*
$x_0 - 2h$	y_{-2}				
		$\Delta y-_2\ (= \Delta y-_{3/2})$			
$x_0 - h$	y_{-1}		$\Delta^2 y-_2 (= \Delta^2 y_{-1})$		
		$\Delta y_{-1}\ (= \Delta y_{-1/2})$		$\Delta^3 y_{-2}\ (= \Delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1}\ (= \Delta^2 y_0)$		$\Delta^3 y_{-2} (= \Delta^4 y_0)$
		$\Delta y_0\ (= \Delta y_{1/2})$		$\Delta^3 y_{-1}\ (= \Delta^3 y_{1/2})$	
$x_0 + h$	y_1		$\Delta^2 y_0 (= \Delta^2 y_1)$		
		$\Delta y_1 (= \Delta y_{3/2})$			
$x_0 + 2h$	y_2				

1. Find the$y$ (25), given that $y_{20} = 24$, $y_{24} = 32$, $y_{28} = 35$, $y_{32} = 40$, using Gauss for ward difference formula.

2. Using Gauss's forward formula, fin d a polynomial of degree four which takes the following values of the function $f(x)$:

x:	1	2	3	4	5
$f(x)$:	1	– 1	1	– 1	1

3. Using Gauss's forward formula, evaluate $f(3.75)$ from the table:

x:	2.5	3.0	3.5	4.0	4.5	5.0
Y:	24.145	22.043	20.225	18.644	17.262	16.047

4. From the following table:

x:	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x:	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

Find $e^{1.17}$, using Gauss forward formula.

5. Using Gauss's backward formula, estimate the number of persons earning wages between Rs. 60 and Rs. 70 from the following data:

Wages (Rs.):	Below 40	40—60	60—80	80—100	100—120
No. of persons: (*in thousands*)	250	120	100	70	50

6. Apply Gauss's backward formula to find sin 45° from the following table:

$\theta°$:	20	30	40	50	60	70	80
sin θ:	0.34202	0.502	0.64279	0.76604	0.86603	0.93969	0.98481

2. Find the first, second and third derivatives of $f(x)$ at $x = 1.5$ if

x:	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$:	3.375	7.000	13.625	24.000	38.875	59.000

3. Find the first and second derivatives of the function tabulated below, at the point $x = 1.1$:

x:	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$:	0	0.128	0.544	1.296	2.432	4.00

4. Given the following table of values of x and y

x:	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y:	1.000	1.025	1.049	1.072	1.095	1.118	1.140

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (*a*) $x = 1.05$. (*b*) $x = 1.25$ (*c*) $x = 1.15$.

5. For the following values of x and y, find the first derivative at $x = 4$.

x:	1	2	4	8	10
y:	0	1	5	21	27

6. Find the derivative of $f(x)$ at $x = 0.4$ from the following table:

x:	0.1	0.2	0.3	0.4
$f(x)$:	1.10517	1.22140	1.34986	1.49182

7. From the following table, find the values of dy/dx and d^2y/dx^2 at $x = 2.03$.

x:	1.96	1.98	2.00	2.02	2.04
y:	0.7825	0.7739	0.7651 0.	7563	0.7473

8. Given $\sin 0° = 0.000$, $\sin 10° = 0.1736$, $\sin 20° = 0.3420$, $\sin 30° = 0.5000$, $\sin 40° = 0.6428$,

(*a*) find the value of $\sin 23°$,

(*b*) find the numerical value of $\cos x$ at $x = 10°$

(*c*) find the numerical value of d^2y/dx^2 at $x = 20°$ for $y = \sin x$.

9. The population of a certain town is given below. Find the rate of growth of the population in 1961 from the following table

Year:	1931	1941	1951	1961	1971
Population: (*in thousands*)	40.62	60.80	71.95	103.56	132.68