Numerical Solutions of Algebraic and Transcendental Equations:

An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + \dots a_n$
where a's are constants $(a_0 \neq 0)$ and n is a +ve integer,
is called a polynomial of x of degree n. the polynomial
$f(x) = 0$ is called algebraic equation of degree n.

If $f(x)$ contains some other functions such as
trigonometric, ~~loga~~ arithmatic, exponential etc. then
$f(x) = 0$ is called transcendental equation.

The value α of x which satisfies $f(x) = 0$ is called
the root of $f(x) = 0$. Geometrically a root of $f(x) = 0$
is that of x where the graph of $y = f(x)$ crosses
the x-axis.

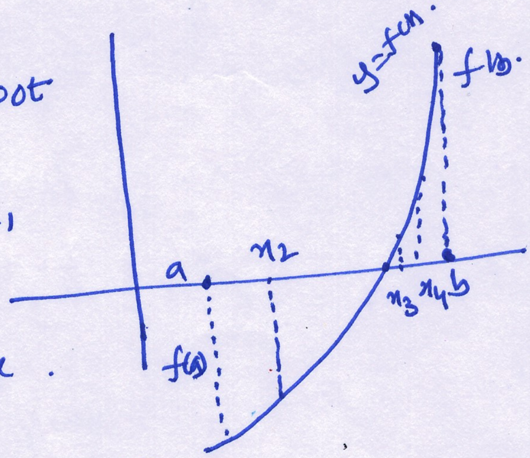
The process of finding the roots of an equation is
known as the solution of that equation.
This is a problem of basic importance in applied
Mathematics.

If $f(x)$ is quadratic, cubic, or a biquadratic expression
algebraic solutions of equations are available.
But the need often arises to solve higher degree
or transcendental equations for which no direct
methods exists.

If $f(x)=0$ find a and b such that $f(a)$ & $f(b)$ are opposite signs. that is $f(a)\cdot f(b)<0$.
then by the intermediate value theorem then there exists a $\xi \in (a,b)$ such that $f(\xi)=0$. So,
Choose initial guess as any $x_0 \in (a,b)$.

Bisection Method:

This method is based on the repeated application of the intermediate value property.
Let the function $f(x)$ be continuous between a and b.
For definiteness, let $f(a)$ be nagative and $f(b)$ be positive then the first approximation to the root is $x_1 = \frac{1}{2}(a+b)$.



If $f(x_1)=0$, then x_1 is ~~the~~ root of $f(x)=0$, otherwise, the root lies between a and x_1 or x_1 and b according as $f(x_1)$ is positive or nagative.

So start with initial approximation as

$$x_0 = \frac{a+b}{2}$$

Now check whether $f(x_0)=0$?
If not sign will decide the which sub interval contains root.

Recursively bisecting (Interval halving) the interval into two small sub intervals leads to one point ξ

Repeat till we get two consecutive approximations are almost same.

Prob: Find a real root of the equation $f(x) = x^3 - x - 1$

Initial guess $f(0) = -1$, $f(1) = -1$, $f(2) = 5$.

$f(1) f(2) \leq 0$ So $\xi \in [1, 2]$.

Since $f(1)$ is nagative and $f(2)$ is positive, a root lies between 1 & 2 and therefore,

We take $x_0 = \frac{1+2}{2} = \frac{a+b}{2} = \frac{3}{2} = 1.5$.

$$f(x_0) = f(1.5) = f(3/2) = \frac{27}{8} - \frac{3}{2} - 1 = \frac{7}{8}$$

$$= \frac{7}{8} > 0$$

So root lies between $[1, 1.5]$.

and we obtain

$$x_1 = \frac{1+1.5}{2} = 1.25$$

$$f(x_1) = f(1.25) = (1.25)^3 - (1.25) - 1$$

$$= -19/64. \text{ which is } -ve.$$

therefore, we conclude that root lies between 1.25 and 1.5 it follows that

$$x_2 = \frac{1.25 + 1.5}{2} = 1.375$$

The procedure is repeated and the successive approximations are $x_3 = 1.3125$, $x_4 = 1.34375$, $x_5 = 1.328125$, etc.

Repeat till we get two consecutive approximations are ~~same~~ almost same (3 or 4 decimal places).

Prob: Find a root of the equation $x^3-4x-9=0$ Using bisection method [Preferably 3 decimal places].

let $f(x) = x^3-4x-9$.

Intial guess $f(2) = 8-8-9 = -9 < 0$ -ve

$f(3) = 27-12-9 = 6 > 0$ +ve

$\therefore$ the root lies between $f(2) \cdot f(3) < 0$ by Int. Value ten.

So $\xi \in [2,3]$

$\therefore$ We take $x_0 = \frac{2+3}{2} = \frac{5}{2} = 2.5$

$f(x_0) = (2.5)^3 - 4(2.5) - 9 = -3.375$, -ve

$\therefore$ the root lies between $[2.5, 3]$

Thus the Second approximation to the root $x_1 = \frac{x_0+3}{2}$

$x_1 = \frac{2.5+3}{2} = 2.75$

$f(x_1) = f(2.75) = (2.75)^3 - 4(2.75) - 9 = 0.7969$ +ve

i.e., +ve

$\therefore$ the root lies between $[2.5, 2.75]$.

The third approximation to the root is

$$x_2 = \frac{2.5+2.75}{2} = 2.625.$$

$f(x_2) = f(2.625) = (2.625)^3 - 4(2.625) - 9$

$= -1.4121$ i.e., -ve

$\therefore$ the root lies between $[2.5, 2.625]$.

$$x_3 = \frac{2.5+2.625}{2} = 2.6875.$$

Repeating this process, the successive approximations are $x_5 = 2.71875$ $x_6 = 2.70313$ $x_7 = 2.71094$

$x_8 = 2.70703$ $x_9 = 2.70508$ $x_{10} = 2.70605$

$x_{11} = 2.70654$ $x_{12} = 2.70642$.

Hence root 2.706

Find a root, correct to three decimal places and lying between 0 and 0.5 of the equation

$$4e^{-x}\sin x - 1 = 0$$

$f(x) = 4e^{-x}\sin x - 1$

$f(0) = -1$ -ve and $f(0.5) = 0.163145$ +ve

There fore $x_1 = \frac{0+0.5}{2} = 0.25$.

By Int. value them $\xi \in (0, 0.5)$.

$f(x_1) = f(0.25) = 4 \cdot e^{-0.25} \cdot \sin(0.25) - 1.$

$= -0.22929.$ -ve.

it follows that the root lies between 0.25 and 0.5 $[0.25, 0.5]$.

The Second approx. $x_2 = \frac{0.25+0.5}{2} = \frac{0.75}{2} = 0.375.$

The successive approximations are given by

$x_3 = 0.3125$, $x_4 = 0.3438$, $x_5 = 0.3594$

$x_6 = 0.3672$ $x_7 = 0.3711$ $x_8 = 0.3692$

$x_9 = 0.3702$ $x_{10} = 0.3706$ $x_{11} = 0.3704$

$x_{12} = 0.3705 \ldots$

Hence the required root is 0.371, correct to the three places.

Prob: Find an approximation value of $\sqrt{3}$ correct to two decimal places by using Bisection Method.

It is clear that, the value of $\sqrt{3}$ is the +ve root of the equation $x^2 = 3$. i.e, $x = \sqrt{3}$

So to find as approximate value of $\sqrt{3}$ we have to solve the equation $x^2 - 3 = 0$ approximately

So let $f(x) = x^2 - 3$.

Since $f(1) = -2 < 0$ and $f(2) = 1 > 0$,

$f(x) = 0$ has a root which lies between $a = 1$ and 2. $[1, 2]$.

The first approximation is $x_1 = \frac{a+b}{2} = \frac{1+2}{2}$
$= 1.5$

Now, $f(1.5) = f(x_1) = -0.75 < 0$ -ve

Since $f(2)$ is +ve $f(1.5)$ is -ve

$\therefore$ the root lies between ~~2, 1.5~~ $[1.5, 2]$.

The second approximation is

$$x_2 = \frac{b + x_1}{2} = \frac{2 + 1.5}{2} = 1.75$$

$f(x_2) = f(1.75) = 0.0625 > 0$ +ve

Since $f(x_1) = f(1.5) < 0$ -ve.

So root lies between 1.5 and 1.75

The third approximation of the root is

$$x_3 = \frac{1.5 + 1.75}{2} = 1.625$$

$f(x_3) = f(1.625) = -0.359375 < 0$ -ve

Since, $f(x_2) = f(1.75) > 0$ +ve
$f(x_3) = f(1.625) < 0$ -ve.

$\therefore$ the root lies between 1.75 and 1.625
$[1.625, 1.75]$.

The fourth approximation of the root is

$$x_4 = \frac{1.75 + 1.625}{2} = 1.6875.$$

$f(x_4) = f(1.6875) = -0.15234 < 0$ -ve

$f(x_2) = f(1.75) > 0$ +ve.

$$x_5 = \frac{1.75 + 1.6875}{2} = 1.71875.$$

$f(x_5) = -$ve., x_5, x_6, ...

doing this process.

$\sqrt{3} = 1.734375$ . . approx.

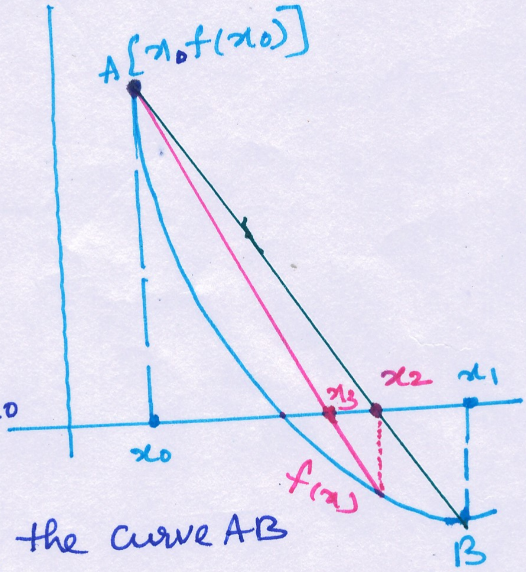
Regula falsi method

This is the oldest method of finding real root of an equation $f(x)=0$ and closely resembles the bisection Method.

Here we choose the two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are opposite signs. i.e., the graph of $y=f(x)$ crosses X-axis between these points.

This indicates that a root lies between x_0 and x_1 and consequently

$$f(x_0)\, f(x_1) < 0$$



Equation of the chord joining the points A $[x_0, f(x_0)]$ and B $[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \cdot x - x_0$$

The method consists in replacing the curve AB by means of the chord AB and taking the point $[x_1\ f(x_1)]$ of intersection of the chord with the X-axis as an approximation to the root.

So the abscissa of the point where the chord cuts the X-axis $(y=0)$ is given by

$$x_2 = x_0 + \frac{x_1 - x_0}{f(x_1) - f(x_0)}\, f(x_0) \quad \text{——} \quad (1)$$

Which is an approximation to the root.

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2 so replacing x_1 by x_2 in (1) We obtain the next approximation x_3. [The root could as well lie between x_1 and x_2 and we would obtain x_3 accordingly].

The procedure is repeated untill root found to the desired accurancy.

The iteration process based on (1) is known as the Method of false position or Regula falsi Method.

Rate of Convergence: This method has linear rate of convergence which is faster than that of the bisection Method.

Prob: Find the real root of the equation $x^3-2x-5=0$ by the method of false position correct to three decimal places.

Sol: let $f(x) = x^3-2x-5$

So that $f(2) = 8-4-5 = -1 < 0$ -ve

$f(3) = 27-6-5 = 16 > 0$ +ve.

i.e., root lies between 2 and 3.

$\therefore$ Taking $x_0 = 2$ $x_1 = 3$ $f(x_0) = f(2) = -1$

$f(x_1) = f(3) = 16$.

in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_0).$$

$x_2 = 2 + \frac{1}{17} = 2.0588.$

Now $f(x_2) = f(2.0588) = (2.0588)^3 - 2(2.0588) - 5$

$= -0.3908$

i.e., The root lies between 2.0588 and 3.

$\therefore$ Taking $x_0 = 2.0588$, $x_1 = 3$ $f(x_0) = -0.3908$

$f(x_1) = 16$ in ①, We get

$$x_3 = 2.0588 - \frac{0.9412}{19.3908}(-0.3908)$$

$= 2.0813.$ +ve.

Repeating this process, the Successive approximations are $x_4 = 2.0862$ $x_5 = 2.0915$ $x_6 = 20934$

$x_7 = 2.0941.$ $x_8 = 2.0943$

Hence the root is 2.094 Correct to three decimal places.

Prob: Find the root of the equation $\cos x = xe^x$ Using the regula-falsi method Correct to four decimal places.

Let $f(x) = \cos x - xe^x = 0$

So that $f(0) = 1$ $f(1) = \cos 1 - e = -2.17798$

i.e., the root lies between 0 and 1.

$\therefore$ Taking $x_0 = 0$, $x_1 = 1$ $f(x_0) = 1$ $f(x_1) = -2.17798$

in the regula falsi method we have

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0). \quad \text{—} (1).$$

$$= 0 + \frac{1}{3.17798} \times 1 = 0.31467.$$

Now $f(0.31467) = 0.51987$ i.e, the root lies between 0.31467 and 1.

$\therefore$ Taking $x_0 = 0.31467$ $x_1 = 1$, $f(x_0) = 0.51987$.

$f(x_1) = -2.17798$

in (1)

$$x_3 = 0.31467 + \frac{0.68533}{2.69785} \times 0.51987$$

$$= 0.44673.$$

Now $f(0.44673) = 0.20356$ i.e., the root lies 0.44673 and 1

$\therefore$ taking $x_0 = 0.44673$, $x_1 = 1$, $f(x_0) = 0.44673$

$f(x_1) = -2.17798$ in (1)

We get $$x_4 = 0.44673 + \frac{0.55327}{2.38154} \times 0.20356$$

$$= 0.49402.$$

Repeating this process the successive approximations are $x_5 = 0.50995$ $x_6 = 0.51520$ $x_7 = 0.51692$

$x_8 = 0.51748$ $x_9 = 0.51767$ $x_{10} = 0.51775$

.. etc

Here the root is 0.5177 correct to four decimal places..

Prob: Use the method of false position to find the fourth root of 32 correct to three decimal places.

Sol:

let $x = (32)^{1/4}$ so that $x^4 - 32 = 0$

Take $f(x) = x^4 - 32$ then $f(2) = -16$, and $f(3) = 49$

i.e., the root lies between 2 and 3.

$\therefore$ Taking $x_0 = 2$, $x_1 = 3$, $f(x_0) = -16$, $f(x_1) = 49$

in the false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_0) \quad \text{—— (1)}$$

$$= 2 + \frac{16}{65} = 2.2462.$$

Now $f(x_2) = f(2.2462) = -6.5438$ i.e., the root lies between 2.2462 and 3

$\therefore$ Taking $x_0 = 2.2462 = -6.5438$ i.e., the root lies between 2.2462 and 3.

$\therefore$ Taking $x_0 = 2.2462$, $x_1 = 3$ $f(x_0) = -6.5438$

$f(x_1) = 49$ in (1) we get.

$$x_3 = 2.2462 - \frac{3 - 2.2462}{49 + 6.5438}(-6.5438) = 2.335$$

Now $f(x_3) = f(2.335) = -2.2732$ i.e., the root lies between 2.335 and 3

$\therefore$ Taking $x_0 = 2.335$ and $x_1 = 3$ $f(x_0) = -2.2732$

and $f(x_1) = 49$ in (1) we obtain

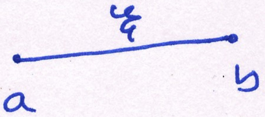
$$x_4 = 2.335 - \frac{3 - 2.335}{49 + 2.2732}(-2.2732) = 2.3645.$$

Repeating this process, the successive approximations are $x_5 = 2.3770$, $x_6 = 2.3779$ etc. Since $x_5 = x_6$ upto three [illegible] We take $(32)^{1/4} = 2.378$.

Newton-Raphsons Method.

Given a non linear equation $f(x)=0$ we want to find ξ such that $f(\xi)=0$

Find a and b such that $f(a)f(b)<0$



Then clearly $\xi \in (a,b)$ start with any $x_0 \in (a,b)$

To improve/find $x_1 = x_0 + h$ such that $f(x_1) \approx 0$

By Taylors Series

$$f(x_1) \approx f(x_0) + h f'(x_0) + \cdots\cdots = 0$$

Neglecting higher degree terms of h.

We get $f(x_0) + h f'(x_0) = 0$

$$h = -\frac{f(x_0)}{f'(x_0)}$$

Hence $x_1 = x_0 + h = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

recursively y, for $n = 0, 1, 2 \ldots$

In general $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$ which is known as Newton-Raphson Method or N-R formula.

Provided denominator is not zero. If it is zero we call it as repeated root. So NRM is only for simple root.

Advantage: Checking sign of function value is NOT required.

Repeat the process till we get two consecutive approximations are almost same.

Prob: Find by Newton–Raphson method the real root of the equation $3x = \cos x + 1$ correct to four decimal places.

let $f(x) = 3x - \cos x - 1$

$f(0) = -2$ $-ve$, $f(1) = 3 - 0.5403 - 1 = 1.4597$ $= +ve$.

So root of $f(x) = 0$ lies between 0 and 1.

It is nearer to 1. $f'(x) = 3 + \sin x$.

So let us take $x_0 = 0.6$

Also $f'(x) = 3 + \sin x$

$\therefore$ Newton–Raphson formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} =$$

$$= x_n - \frac{(3x_n - \cos x_n - 1)}{3 + \sin x_n}.$$

$$x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{(0.6)(\sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)}.$$

$$= \frac{0.6071 \times 0.564642 + 0.8213 + 1}{3 + 0.5646}$$

$$= 0.6071$$

Putting $n = 1$ in (i) the second approximation is

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6071 . \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)}$$

$$= 0.6071.$$

Prob: Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton Raphson Method.

let $f(x) = x^4 - x - 10$

So that $f(0) = -10$, $f(1) = -10$, $f(2) = 4$ so $f(1)f(2) < 0$

So $\xi \in (1,2)$ then start with $x_0 = 2$ $\qquad f'(x) = 4x^3 - 1$.

Newton–Raphson formula is.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{--- (1)}$$

at $n = 0$. the first approximation is x_1

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{4}{4\cdot 2^3 - 1} = 2 - \frac{4}{31} = 1.871$$

Putting $n=1$ in the second approximation is given by.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1}.$$

$$= 1.856.$$

Putting $n=2$ in (1) the third opproximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{f(1.856)}{f'(1.856)}.$$

$$= 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1}.$$

$$= 1.856 - \frac{0.010}{24.574} = 1.856$$

Here $x_2 = x_3$. (three decimal places)

Hence the desired root is 1.856 correct to 3 decimal places.

Using Newton Raphson Method find square root of N.

(N = your rollNumber).

let $\sqrt{N} = x$.

Since we donot know $\sqrt{N}$

So we make it as $N = x^2$

Now take function as $f(x) = x^2 - N = 0$.

Find a and b such that $f(a) f(b) < 0$ then clearly $\xi \in (a,b)$ start with any $x_0 \in (a,b)$.

$f(x) = x^2 - N$

then $f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n} \quad \text{for } n = 0, 1, 2, \ldots$$

Evaluate the value of $\sqrt{5}$ (corrects to four decimal Places) by Newton-Raphson Method.

Taking $N = 5$ in the above formula.

becomes $x_{n+1} = \frac{1}{2}\left[\frac{x_n^2 + 5}{x_n}\right] = \frac{1}{2}\left[x_n + 5/x_n\right]$.

Since an approximate value of $\sqrt{5}$

We take $x_0 = 2$

Then

$$x_1 = \frac{1}{2}\left[x_0 + 5/x_0\right] = \frac{1}{2}\left[2 + 5/2\right] = 2.25$$

$$x_2 = \frac{1}{2}\left[x_1 + 5/x_1\right] = 1/2\left[2 + 5/2.25\right] = 2.2361$$

$$x_3 = \frac{1}{2}\left[x_2 + 5/x_2\right] = \frac{1}{2}\left[2 + 5/2.2361\right] = 2.2361$$

Since $x_2 = x_3$ upto four decimal places, We have $\sqrt{5} = 2.2361$

$\sqrt{5} = 2.2361.$

Using The Newton Raphson Method find cube root of N

(N = your Roll Number).

To start NRM We need function, for that

let $\sqrt[3]{N} = x$ Since We donot know $\sqrt[3]{N}$

We make it as $N = x^3$

Take a function as $x^3 - N = 0$

find a and b such that $f(a) \cdot f(b) < 0$ then

Clearly $\xi \in (a,b)$ start with $x_0 \in (a,b)$.

If $f(x) = x^3 - N = 0$

then $f'(x) = 3x^2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$$x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{2x_n^3 + N}{3x_n^2}$$

$$\boxed{x_{n+1} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right].}$$

for $n = 0, 1, 2, 3.$

Perform four iterations of Newton – Raphson Method to obtain the approximate value $\sqrt[3]{17}$.

taking $N=17$ in the below formula.

$$x_{n+1} = \frac{1}{3}\left[\frac{2x_n^3 + N}{x_n^2}\right]$$

Taking $x_0 = 2$ and $N = 17$

the above formula becomes.

$$x_{0+1} = \frac{1}{3}\left[\frac{2x_0^3 + 17}{x_0^2}\right]$$

$$x_1 = \frac{1}{3}\left[\frac{2\cdot 2^3 + 17}{4}\right] = 2.75$$

$$x_2 = \frac{1}{3}\left[\frac{2x_1^3 + 17}{x_1^2}\right] = \frac{1}{3}\left[\frac{2\cdot(2.75)^3 + 17}{(2.75)^2}\right]$$

$$x_2 = 2.582645$$

$$x_3 = \frac{2x_2^3 + 17}{3x_2^2} = \left[\frac{2\cdot(2.582645)^3 + 17}{(2.582645)^2}\right]$$

$$x_3 = 2.571332$$

$$x_4 = \frac{2x_3^3 + 17}{x_3^2} = \frac{2(2.571332)^3 + 17}{(2.571332)^2}$$

$$= 2.571282$$

The exact value of six decimal places is

2.571282.

Using the NRM find M-th root of N (N = your Roll Number & M is your friends Roll No.].

to start NRM we need function

let $\sqrt[M]{N} = x$.

Since we donot know $\sqrt[M]{N}$

We make it as $N = x^M$.

take the function as: $f(x) = x^M - N = 0$

then $f'(x) = Mx^{M-1}$

find a and b. such that $f(a) \cdot f(b) < 0$ then

Clearly $\xi \in (a, b)$ start with $x_0 \in (a, b)$.

take the function as:

$$f(x) = x^M - N = 0$$

$$f'(x) = Mx^{M-1}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$$x_{n+1} = x_n - \frac{x^M - N}{N x^{M-1}}$$

$$x_{n+1} = \frac{(M-1)x_n^M + N}{M x_n^{M-1}}$$

for $n = 0, 1, 2, 3 \ldots$

Using the NRM find Inverse of N. (N = your Roll Number).

To start NRM we need a function

find a and b such that $f(a) \cdot f(b) < 0$ then clearly

$\xi \in (a,b)$ start with any $x_0 \in (a,b)$.

Choice: 1.

let $\frac{1}{N} = x$ take a function as $f(x) = Nx - 1$ then

$f'(x) = N$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{Nx_n - 1}{N} = \frac{1}{N}.$$

Scheme need to find that only! So wrong choice

of the function.

Choice 2.

Let $\frac{1}{N} = x$ take a function as $f(x) = x - \frac{1}{N}$

then $f'(x) = 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - \frac{1}{N}}{1} = \frac{1}{N}$$

Scheme need to find that only! So wrong choice of the function.

Choice 3. for function

let $\frac{1}{N} = x$ or $N^{-1} = x$.

Since we donot know N^{-1} we make it as $N = x^{-1}$

or $N = \frac{1}{x}$

Take a function as: $f(x) = N - \frac{1}{x} = 0$ then $f'(x) = -\frac{1}{x^2}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{N - \frac{1}{x_n}}{-1/x_n^2} = x_n - \left[\frac{Nx_n - 1}{x_n}\right] x_n^2$$

$$x_{n+1} = 2x_n - Nx_n^2$$

This is the scheme for finding inverse for $n = 0, 1, 2 \ldots$

So the correct choice of function is $f(x) = \frac{1}{x} - N = 0$

$$f(x) = \frac{1}{x} - N = 0$$

Prob: Evaluate $\frac{1}{31}$ correct to four decimal places by NRM. $N = \frac{1}{31}$.

let $f(x) = \frac{1}{x} - N = 0$

Taking the above formula $x_{n+1} = 2x_n - Nx_n^2$.

$N = 31$

$$x_{n+1} = 2x_n - 31x_n^2.$$

Since an approximation of value of $\frac{1}{31} = 0.03$

We take $x_0 = 0.03$.

Then $x_1 = x_0(2 - 31x_0) = 0.03(2 - 31 \cdot (0.03))$
$= 0.0321$.

$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31(0.0321))$
$= 0.032257$.

$$x_3 = x_2(2-31x_2) = 0.032257(2-31\cdot 0.032257)$$
$$= 0.03226 .$$

Since upto four decimal places x_3 and x_2 approximations

$\therefore$ We have $\frac{1}{31} = 0.0323$.

1. Find a root of the following equations, using the bisection method correct to three decimal places:

   (i) $x^3 - x - 1 = 0$ (ii) $x^3 - x^2 - 1 = 0$

   (iii) $2x^3 + x^2 - 20x + 12 = 0$ (iv) $x^4 - x - 10 = 0$.

2. Evaluate a real root of the following equations by bisection method:

   (i) $x - \cos x = 0$ (ii) $e^{-x} - x = 0$

   (iii) $e^x = 4 \sin x$.

3. Find a real root of the following equations correct to three decimal places, by the method of false position:

   (i) $x^3 - 5x + 1 =$ (ii) $x^3 - 4x - 9 = 0$

   (iii) $x^6 - x^4 - x^3 - 1 = 0$.

4. Using the regula falsi method, compute the real root of the following equations correct to three decimal places:

   (i) $xe^x = 2$ (ii) $\cos x = 3x - 1$

   (iii) $xe^x = \sin x$ (iv) $x \tan x = -1$

   (v) $2x - \log x = 7$

   (vi) $3x + \sin x = e^x$.

5. Find the fourth root of 12 correct to three decimal places by the interpolation method.

6. Locate the root of $f(x) = x^{10} - 1 = 0$, between 0 and 1.3 using the bisection method and method of false position. Comment on which method is preferable.

1. Find by Newton-Raphson method, a root of the following equations correct to three decimal places:

(*i*) $x^3 - 3x + 1 = 0$ (*ii*) $x^3 - 2x - 5 = 0$

(*iii*) $x^3 - 5x + 3 = 0$ (*iv*) $3x^3 - 9x^2 + 8 = 0$.

2. Using Newton's iterative method, find a root of the following equations correct to four decimal places:

(*i*) $x^4 + x^3 - 7x^2 - x + 5 = 0$ which lies between 2 and 3.

(*ii*) $x^5 - 5x^2 + 3 = 0$.

3. Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to 2 decimal places by Newton's method.

4. Using Newton-Raphson method, find a root of the following equations correct to three decimal places:

(*i*) $x^2 + 4 \sin x = 0$

(*ii*) $x \sin x + \cos x = 0$ or $x \tan x + 1 = 0$

(*iii*) $e^x = x^3 + \cos 25x$ which is near 4.5

(*iv*) $x \log_{10} x = 12.34$, start with $x_0 = 10$.

(*v*) $\cos x = xe^x$ (*vi*) $10^x + x - 4 = 0$.

5. The equation $2e^{-x} = \dfrac{1}{x+2} + \dfrac{1}{x+1}$ has two roots greater than -1. Calculate these roots correct to five decimal places.

6. The bacteria concentration in a reservoir varies as $C = 4e^{-2t} + e^{-0.1t}$. Using the Newton Raphson (N.R.) method, calculate the time required for the bacteria concentration to be 0.5.

7. Use Newton's method to find the smallest root of the equation $e^x \sin x = 1$ to four decimal places.

8. The current i in an electric circuit is given by $i = 10e^{-t} \sin 2\pi t$ where t is in seconds. Using Newton's method, find the value of t correct to three decimal places for $i = 2$ amp.

9. Find the iterative formulae for finding $\sqrt{N}, \sqrt[3]{N}$ where N is a real number, using the Newton-Raphson formula.

Hence evaluate:

(*a*) $\sqrt{10}$

(*b*) $\sqrt{21}$

(*c*) the cube-root of 17 to three decimal places.

10. Develop an algorithm using the N.R. method, to find the fourth root of a positive number N and hence find $\sqrt[4]{32}$

11. Evaluate the following (correct to three decimal places) by using the Newton-Raphson method.

(*i*) 1/18 (*ii*) $1/\sqrt{15}$ (*iii*) $(28)^{-1/4}$.

Answers for bisection and regula falsi method

1. (*i*) 1.321 (*ii*) 1.46 (*iii*) 2.875 (*iv*) 1.855.
2. (*i*) 0.0625. (*ii*) 0.567. (*iii*) 0.367.
3. (*i*) 2.128. (*ii*) 2.7065. (*iii*) 1.4036.
4. (*i*) 0.853. (*ii*) 0.6071. (*iii*) – 0.134 (*iv*) 2.798.
(*v*) 3.789. (*vi*) 0.3604.
5. (*i*) 1.861. **6.** (*i*) 0.99976. (*ii*) 0.99931.
7. (*i*) – 2.0625 (*ii*) 0.567 (*iii*) 3.496.
8. (*i*) 0.6071. (*ii*) 2.9428. (*iii*) 1.4973. (*iv*) 4.4346.
(*v*) 0.2591. (*vi*) 2.8625.
9. (*i*) and (*ii*) 5.4772. **10.** (*i*) & (*ii*) 1.524. **11.** 0.477.

Answers for NRM

1. (*i*) 1.532. (*ii*) 2.095. (*iii*) 1.834. (*iv*) 1.226.
2. (*i*) 1.856. (*ii*) 2.198. **3.** – 16.56.
4. (*i*) – 1.9338. (*ii*) 2.798. (*iii*) 4.545. (*iv*) 0.052.
(*v*) 0.518. (*vi*) 0.695.

5. Root in interval (– 0.8, 0.5) = 0.77009, Root in interval (0, 1) = 0.76839.
6. 6.889. **7.** 0.5886. **8.** 0.033 sec.
9. $x_{n+1} = \frac{1}{2}(x_n + N/x_n), x_{n+1} = \frac{1}{3}(2x_n + N/x_n^2)$; (*a*) 3.162. (*b*) 2.5713
10. $x_{n+1} = \frac{1}{4}\left(3x_n + \frac{N}{x_n^3}\right)$, 2.3784
11. (*i*) 0.0555. (*ii*) 0.2582. (*iii*) 0.4347. **12.** 0.51776.