Curve fitting

In many branches of applied mathematics and Engineering Sciences we come across experiments and problems which involve two variables.
For example it is known that the speed v of a satelite various with horse power p of an engine according to the formula $p = a + bv^3$. Here a and b are the constants to be determined. For this purpose we take several sets of readings of speeds and the corresponding horse powers. The problem is to find the best value for a and b using the observed values of v and p. Thus the general problem is to find a suitable relation or Law that may exist between the variables x and y from the given set of observed values (x_i, y_i), $i = 1, 2, 3 \ldots$. Such a relation connecting x and y is known as empirical Law.
The processes of finding equation of the curve of best fit which may be most suitable for predicting the unknown values is known as curve fitting.
In this chapter we discuss the standard ~~and~~ and most popular method method of least squares approxim-ation

Fitting a first degree Curve: Straight Line

Suppose the data is given for the function $f(x)$ at n distinct points.

$$x_1, x_2, x_3 \dots, x_{n-1}, x_n$$

$$f_1, f_2, f_3 \dots, f_{n-1}, f_n.$$

We want to fit the given polynomial

$$P_1(x) = a_0 + a_1 x.$$

that means we need to find a_0, a_1, we have 2 normal equations

$$a_0 \sum 1 + a_1 \sum x_i = \sum f_i$$

$$a_0 \sum x_i + a_1 \sum x_i^2 = \sum f_i x_i$$

We Solve for a_0, a_1 then we get a polynomial

$$P_1(x) = a_0 + a_1 x$$

Hence We claims that $P_1(x) \approx f(x)$.

Fitting Second degree Curve: Parabola

Suppose data is given for the function $f(x)$ at n distinct points

$$x_1, \ x_2, \ x_3, \ \ldots \ x_{n+1}, x_n$$

$$f_1, \ f_2, \ f_3, \ \ldots \ f_{n+1}, f_n$$

We want to fit a polynomial

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

That means we need to find a_0, a_1, a_2

We have 3 normal Equations

$$a_0 \sum 1 + a_1 \sum x_i + a_2 \sum x_i^2 = \sum f_i$$

$$a_0 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 = \sum x f_i$$

$$a_0 \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4 = \sum x^2 f_i$$

We solve for a_0, a_1, a_2 then we get the Polynomial $P_2(x) = a_0 + a_1 x + a_2 x^2$

We claim that $P_2(x) \approx f(x)$.

Approximation

Weistrass Theorem:

Any continuous function can be approximated by a polynomial of given degree.

Even if data/information about the continuous function is given, we can approximate/fit it by a polynomial of a given degree.

$x_1, \quad x_2, \quad x_3 \; . \; . \; . \; . \; x_{n-1}, x_n$

$f_1, \quad f_2 \quad f_3 \; . \; . \; . \; . \; f_{n-1}, f_n.$

This is popularly known as curve fitting.

This curve/polynomial - may/may not satisfy the data.

But minimise the error in some sence/Norm !!!.

Error is the difference of function and polynomial approximation.

The error of approaximation defined as

$$E = \| f(x) - \phi_i(x) \| .$$ where $\| . \|$

is a we defined norm.

error is a small as possible in some sence.

By using the different norms, we obtain different types of approximations. once a criteria (or a particular norm) is fixed, the function which makes this error smallest according to this criterion is called the best approximation. Thus, minimisation of error norm solves the problem of best approximation.

Discrete Data.

Euclidean norm.

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

L^P- norm $\quad \|x\| = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad p \geq 1.$

if $P=2$

Particular case of $p=2$ and also called Square norm and written $\|x\|_2$.

Least square Principle.

Sum of the squares of the Errors is tobe minimised.

minimisation of the sum of the squares of the Errors.

minimisation of the Error in square Norm.

minimisation of the Error in L2 Norm.

Least Square Method for Discrete Data.

Suppose data is given for the function $f(x)$ at n distinct points

$$x_1, x_2, x_3 \dots x_{n-1}, x_n$$

$$f_1, f_2, f_3 \dots f_{n-1}, f_n.$$

We wanta to approximate it by the given polynomial $P_k(x)$

$$P_k(x) = a_0 + a_1(x) + a_2(x^2) + \dots a_k x^k.$$

that means we need to find $a_0, a_1, a_2 \dots, a_k$.

We find these constants using the least squares principle that is by

Minimising the sum of the squares of the errors at data points.

$$e_i = e(x_i) = f_i - P_k(x_i) \quad \text{for } i = 1, 2, 3 \dots, n.$$

$$e_1^2 + e_2^2 + e_3^2 + \dots + e_{n-1}^2 + e_n^2 = \sum_{i=1}^{n} e_i^2 = E \text{ to be Minimise.}$$

Hence We Claim that

$$P_k(x) \cancel{=} f(x)$$

$$P_k(x) \approx f(x).$$

Least Square Method.

We find these $a_0, a_1, \ldots, a_K$ $(n+1)$ unknowns by minimising

$$E = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} [f_i - P_K(x_i)]^2$$

$$E = \sum_{i=1}^{n} [f_i - (a_0 + a_1 x + \ldots a_K x_i^K)]^2$$

To minimise, We first differentiate E with respect $a_0, a_1, a_2 \ldots a_k$. and equate it to zero.

Then we will have $(k+1)$ equations to find these $k+1$ unknowns.

Minimising principle

$$\frac{\partial E}{\partial a_0} = -2 \sum_{i=1}^{n} [f_i - (a_0 + a_1 x_i + a_2 x_i^2 + \ldots a_K x_i^K] = 0$$

$$\frac{\partial E}{\partial a_1} = -2 \sum_{i=1}^{n} [f_i - (a_0 + a_1 x_i + a_2 x_i^2 - \ldots a_K x_i^K] x_i = 0$$

$$\vdots$$

$$\frac{\partial E}{\partial a_K} = -2 \sum_{i=1}^{n} [f_i - (a_0 + a_1 x_i + a_2 x_i^2 + \ldots a_K x_i^K] x_i^K$$

Normal Equations

$$\sum_{i=1}^{n} (a_0 + a_1 x_i + a_2 x_i^2 + \dots a_k x_i^k) = \sum_{i=1}^{n} f_i$$

$$\sum_{i=1}^{n} (a_0 + a_1 x_i + a_2 x_i^2 + \dots a_k x_i^k)\, x_i = \sum_{i=1}^{n} f_i x_i$$

$$\vdots$$

$$\sum_{i=1}^{n} (a_0 + a_1 x_i + a_2 x_i^2 + \dots a_k x_i^k)\, x_i^k = \sum_{i=1}^{n} f_i x_i^k.$$

Normal Equations

$$a_0 \sum 1 + a_1 \sum x_i + a_2 \sum x_i^2 + \dots a_k \sum x_i^k = \sum f_i$$

$$a_0 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 + \dots a_k \sum x_i^{k+1} = \sum f_i x_i$$

$$\vdots$$

$$a_0 \sum x_i^k + a_1 \sum x_i^{k+1} + a_2 \sum x_i^{k+2} \dots a_k \sum x_i^{2k} = \sum f_i x_i^k$$

Least Squares Method.

With these $(k+1)$ equations we find the $k+1$ unknowns $a_0, a_1, a_2 \dots a_k$

Hence we get the polynomial $P_k(x)$.

$$P_k(x) = a_0 + a_1 x + a_2 x^2 + \dots a_k x^k.$$

Which may / may not satisfy data.

But error at every point is minimised.

Hence We claim that

$$P_k(x) \approx f(x).$$

Prob. 1. Fit the straight line for the data $P_1(x) = a_0 + a_1 x$

x	-1	0	1	2
$f(x)$	1	0	1	4

Normal Equations for straight lines are.

$$a_0 \sum 1 + a_1 \sum x = \sum f_i$$

$$a_0 \sum x + a_1 \sum x^2 = \sum x f_i$$

	x	$f(x)$	x^2	xf
	-1	1	1	-1
	0	0	0	0
	1	1	1	1
	2	4	4	8
Sum $\sum$	2	6	6	8

$$4a_0 + 2a_1 = 6$$

$$2a_0 + 6a_1 = 8$$

Solving weget $a_0 = 1$, $a_1 = 1$

$$f_1(x) = a_0 + a_1 x = 1 + x$$

Prob.

Fit a Second degree polynomial (Parabola) for the given data.

x:	0	1	2
$f(x)$	1	6	17.

$p_2(x) = a_0 + a_1 x + a_2 x^2$ Normal Equations for Parabola. 3 Unknowns 3 Eqns.

$$a_0\sum 1 + a_1\sum x + a_2\sum x^2 = \sum f_i$$

$$a_1\sum x + a_2\sum x^2 + a_3\sum x^3 = \sum x f_i$$

$$a_1\sum x^2 + a_2\sum x^3 + a_3\sum x^4 = \sum x^2 f_i$$

	x	$f(x)$	x^2	x^3	x^4	xf	$x^2 f$
	0	1	0	0	0	0	0
	1	6	1	1	1	6	1
	2	17	4	8	16	34	8
Sum Σ	3	24	5	9	17	40	74

from Normal Equations

$$3a_0 + 3a_1 + 5a_2 = 24$$
$$3a_0 + 5a_1 + 9a_2 = 40$$
$$5a_0 + 9a_1 + 17a_2 = 74$$

Solving the equations $a_0 = 1,\ a_1 = 2,\ a_2 = 3$.

Hence $f(x) \approx p_2(x) = 1 + 2x + 3x^2$

Obs: For the sake of convienience and easy calculations, it is some times advisible to change the orgin and scale with the Substitution $X = \frac{x-A}{h}$, $Y = \frac{y-B}{h}$ A, B are assumed means (or middle values) of x and y series respectively and h be width of the instury

Prob: Fit a straight line for the below data.

x	1	2	3	4	5	6	7	8	9
$f(x)$	9	8	10	12	11	13	14	16	15.

$$f = a_0 + a_1 x.$$

It is better to change of orgin to get a small values for summation.

Put $X = x - 5$ $\quad y = F = f - 12.$

x	$f(x)$	X	F	XF	X^2
1	9	-4	-3	12	16
2	8	-3	-4	12	9
3	10	-2	-2	4	4
4	12	-1	0	0	1
5	11	0	-1	0	0
6	13	1	1	1	1
7	14	2	2	4	4
8	16	3	4	12	9
9	15	4	3	12	16
Sum		0	0	57	60.

Normal Equations for $F = a_0 + a_1 X$ st. line for new data.

$$a_0 \sum 1 + a_1 \sum X = \sum F_i$$

$$a_0 \sum x + a_1 \sum x^2 = \sum F_i X_i$$

$$9a_0 + 0 \cdot a_1 = 0$$

$$0 \cdot a_0 + 60 a_1 = 57$$

$a_0 = 0$ & $a_1 = 0.95$

$$F = a_0 + a_1 X = 0 + 0.95X$$

$$f - 12 = 0.95(x-5) = 7.25 + 0.95x$$

Prob: Fit a second degree parabola to the following data.

x	1989	1990	1991	1992	1993	1994	1995	1996	1997
y	352	356	357	358	360	361	361	360	359.

We shift the orgin $X = x - 1993$ $Y = y - 357$.

the equation $y = a + bx + cx^2$ becomes $Y = a + bX + cX^2$.

x	y	$X = x-1993$	$Y = y-357$	XY	X^2	X^2Y	X^3	X^4
1989	352	-4	-5	20	16	-80	-64	256
1990	356	-3	-1	3	9	-9	-27	81
1991	357	-2	0	0	4	0	-8	16
1992	358	-1	1	-1	1	1	-1	1
1993	360	0	3	0	0	0	0	0
1994	361	1	4	4	1	4	1	1
1995	361	2	4	8	4	16	8	16
1996	360	3	3	9	9	27	27	81
1997	359	4	2	8	16	32	64	256
		$\Sigma X = 0$	$\Sigma Y = 11$	$\Sigma XY = 51$	$\Sigma X^2 = 6$	$\Sigma X^2Y = -9$	$\Sigma X^3 = 0$	$\Sigma X^4 = 708$

Normal Equations

$$\Sigma Y = A\Sigma 1 + b\Sigma X + c\Sigma X^2$$
$$\Sigma XY = A\Sigma X + b\Sigma X^2 + c\Sigma X^3$$
$$\Sigma X^2Y = A\Sigma X^2 + b\Sigma X^3 + c\Sigma X^4$$

$11 = 9A + 60C$

$51 = 60B$

$B = \frac{51}{60} = \frac{17}{20}$

$-9 = 60A + 708C.$

By solving these equations $A = \frac{694}{231}$, $B = \frac{17}{20}$, $C = \frac{-247}{924}$.

$$\therefore\ Y = \frac{694}{231} + \frac{17}{20}X - \frac{247}{924}X^2.$$

$$y - 357 = \frac{694}{231} + \frac{17}{20}(x-1993) - \frac{247}{924}(x-1993)^2.$$

By simplification

$$y = -1062526.37 + 1066.37x - 0.267x^2$$

Miscellaneous examples for fitting a given curve

Fit a curve of the form given below for the data.

x_1	x_2	x_3	...	x_{n-1}	x_n
f_1	f_2	f_3	...	f_{n-1}	f_n

1. $f = a_0 + a_1 x^n$

We set $X = x^n$ then the curve becomes

$$f = a_0 + a_1 X$$

$$f = a_0 + a_1 X + a_2 X^2$$

2. $f = a_0 + a_1 \sqrt{x}$

We set $X = \sqrt{x}$ then the curve becomes.

$$f = a_0 + a_1 X$$

3 $f = a_0 + a_1 \cdot \frac{1}{x}$

We set $X = \frac{1}{x}$ then the curve becomes $f = a_0 + a_1 X$.

4. $f = a e^{bx}$. taking log and set

$a_0 = \log a \qquad a_1 = b$

$F = \log f$

then $F = a_0 + a_1 x$.

$f = a e^{b}$.

taking log both sides.

$\log f = \log a + bx \log e$

$\log f = bx + \log a$.

$F = a_1 x + a_0$

5. $f = a x^b$.

Taking log both sides. $\log_{10} y = \log_{10} a + b \log_{10} x$.

Put $X = \log x$, $F = \log y$ $\quad \log a = a_0$.

$b = a_1$.

$F = \log f$.

$$F = a_0 + a_1 X$$

6. $xy^a = b$. or $pv^r = k$.

Taking logarthums $\log x + a\log y = \log b$.

$$\log y = \frac{1}{a}\log b - \frac{1}{a}\log x.$$

Put $F = \log y$ $\quad a_0 = \frac{1}{a}\log b \quad a_1 = -\frac{1}{a}$.

then the curve becomes. $F = a_0 + a_1X$; $X = \log x$

Prob: Fit the curve $y = ax^2 + b/x$ to the following data.

x	1	2	3	4
y	-1.51	0.99	[illegible].88	7.66.

$$y = ax^2 + b/x.$$

Re writing the equation $xy = ax^3 + b$. and putting

$$x^3 = X \qquad xy = Y.$$

We get $Y = aX + b$.

Normal Equations $\Sigma Y = a\Sigma X + b\Sigma 1$

$$\Sigma XY = a\Sigma X^2 + b\Sigma X.$$

The Values of ΣX ΣY are calculated by below table.

x	y	$X = x^3$	$Y = xy$	XY	X^2
1	-1.51	1	-1.51	-1.51	1
2	0.99	8	1.98	15.84	64
3	[illegible].88	27	11.64	314.28	729
4	7.66	64	30.64	1960.96	4096.
		$\Sigma X = 100$	$\Sigma Y = 42.75$	$\Sigma XY = 2289.57$	$\Sigma X^2 = 4890$

Substituting these values in Normal equations

We get $42.75 = 100a + 4b$

$$2289.57 = 4890a + 100b$$

Solving these equations, We get $a = 0.51$ $b = -2.06$

Hence the curve of best fit is

$$Y = 0.51X - 2.06$$

ie $xy = 0.51x^3 - 2.06$

or $y = 0.51x^2 - \frac{2.06}{x}$.

Prob: Find the curve of best fit of the type $y = ae^{bx}$ to the following data by the method of least squares.

x	1	5	7	9	12
y	10	15	12	15	21.

Sol: Required curve to be fitted is $y = ae^{bx}$.

Taking log on both sides.

Setting $\log y = Y$, $x = X$ $\log a = a_0$ and $b = a_1$ then the relation (1) takes the form $Y = a_0 + a_1 x$.

The method of procedure is the same as in fitting a straight line and we ~~find~~ the following table.

$x = X$	y	$Y = \log_e y$	X^2	XY.
1	10	2.3026	1	2.3026
5	15	2.7080	25	13.54
7	12	2.4849	49	17.3943
9	15	2.7080	81	24.372
12	21	3.0445	144	36.534
Sum $\Sigma X = 34$		$\Sigma Y = 13.248$	$\Sigma X^2 = 300$	$\Sigma XY = 94.1429$

Normal Equations: $\Sigma Y = a_0 \Sigma 1 + a_1 \Sigma X$

$\Sigma XY = a_0 \Sigma X + a_1 \Sigma X^2$

Substituting these values in Normal Equations, We get

$5a_0 + 34a_1 = 13.248$

$34a_0 + 300a_1 = 94.1429$ By Solving

$a_0 = 2.2484$ $a_1 = 0.059$.

Now $a = e^{a_0} = e^{2.2484} = 9.4725$ and $b = a_1 = 0.059$.

Hence the required curve is $y = 9.4725 \cdot e^{0.059x}$

Prob: Fit the curve $y = ax^b$ to the following data.

x	100	200	300	400
y	50	30	10	25.

Given curve $y = ax^b$.

Taking ln both sides we get

$$\log_{10} y = \log_{10} a + b \log_{10} x.$$

Let $\log_{10} y = Y$ $\log_{10} a = A$ and $\log_{10} x = X$.

the the equation becomes.

$$Y = A + bX.$$

Normal Equations are $\sum Y = A\sum 1 + b\sum X$

$$\sum XY = A\sum X + b\sum X^2.$$

The values of $\sum x, \sum X^2, \sum Y$ and $\sum XY$ are given in the following table.

x	y	$X = \log_{10} x$	$Y = \log_{10} y$	X^2	XY.
100	50	2.00	1.70	4.00	3.40
200	30	2.30	1.48	5.29	3.40
300	10	2.48	1.00	6.15	2.48
400	25	2.60	1.40	6.78	3.64

Here $n = 4$ $\sum X = 9.38$ $\sum Y = 5.58$ $\sum X^2 = 22.22$ $\sum XY = 12.92$

$$5.58 = 4A + 9.38b$$

$$12.92 = 9.38A + 22.22b.$$

Solving these equations, we get $A = 3.11, b = 0.73$.

then $a = \text{Antilog } A = 1288.2496$

$$y = 1288.25\, x^{-0.73}$$

1. If p is the pull required to lift the weight by means of a pulley block, find a linear law of the form $p = a + bw$, connecting p *and* w, using the following data:

w(lb):	50	70	100	120
p(lb):	12	15	21	25

Compute p, when w = 150 lb.

2. Convert the following equations to their linear forms:

(*i*) $y = ax + bx^2$ (*ii*) $y = b/[x(x - a)]$.

3. The resistance R of a carbon filament lamp was measured at various values of the voltage V and the following observations were made:

Voltage V...	62	70	78	84	92
Resistance R...	73	70.7	69.2	67.8	66.3

Assuming a law of the form $R = \frac{a}{V} + b$, find by graphical method the best values of a and b.

4. Verify if the values of x and y, related as shown in the following table, obey the law $y = a + b\sqrt{x}$. If so, find graphically the values of a and b.

x:	500	1,000	2,000	4,000	6,000
y:	0.20	0.33	0.38	0.45	0.51

5. The following table gives the pressure p and the volume v at various instants during the expansion of steam in a cylinder. Show that the equation of the expansion is of the form $pv^n = c$ and find the values of n and c approximately.

p:	200	100	50	30	20	10
v:	1.0	1.7	2.9	4.8	5.9	10

6. The following values of T and l follow the law $T = al^n$. Test if this is so and find the best values of a and n.

$T = 1.0$	1.5	2.0	2.5
$l = 25$	56.2	100	1.56

Fit the curve $y = ae^{bx}$ to the following data:

x:	0	2	4
y:	5.1	10	31.1

The following are the results of an experiment on friction of bearings. The speed being kept constant, corresponding values of the coefficient of friction and the temperature are shown in the table:

t:	120	110	100	90	80	70	60
μ:	0.0051	0.0059	0.0071	0.0085	0.00102	0.00124	0.00148

If μ and t are given by the law $\mu = ae^{bt}$, find the values of a and b by plotting the graph for μ and t.

1. By the method of least squares, find the straight line that best fits the following data:

x:	1	2	3	4	5
y:	14	27	40	55	68

2. In some determinations of the value v of carbon dioxide dissolved in a given volume of water at different temperatures θ, the following pairs of values were obtained:

$\theta = 0$	5	10	15
$v = 1.80$	1.45	1.18	1.00

Obtain by the method of least squares, a relation of the form $v = a + b\theta$ which best fits to these observations.

3. A simply supported beam carries a concentrated load P(lb) at its midpoint. Corresponding to various values of P, the maximum deflection Y (in) is measured. The data are given below:

P:	100	120	140	160	180	200
Y:	0.45	0.55	0.60	0.70	0.80	0.85

Find a law of the form $Y = a + bP$.

4. The result of measurement of electric resistance R of a copper bar at various temperatures $t°$ C are listed below:

t:	19	25	30	36	40	45	50
R:	76	77	79	80	82	83	85

Find a relation $R = a + bt$ when a and b are constants to be determined by you.

5. A chemical company, wishing to study the effect of extraction time (t) on the efficiency of an extraction operation (e) obtained the data shown in the following table:

t:	27	45	41	19	3	39	19	49	15	31
e:	57	64	80	46	62	72	52	77	57	68

Fit a straight line to the given data by the method of least squares.

6. Find the parabola of the form $y = a + bx + cx2$ which fits most closely with the observations:

x:	– 3	– 2	– 1	0	1	2	3
y:	4.63	2.11	0.67	0.09	0.63	2.15	4.58

7. By the method of least squares, fit a parabola of the form $y = a + bx + cx^2$, to the following data:

x:	2	4	6	8	10
y:	6.07	12.85	31.47	57.38	91.29

Fit a parabola $y = a + bx + cx^2$ to the following data:

x:	1	2	3	4	5	6	7	8	9
y:	2	6	7	8	10	11	11	10	9

8. The velocity V of a liquid is known to vary with temperature T according to a quadratic law $V = a + b\,T + CT^2$. Find the best values of a, b, c for the following table:

T:	1	2	3	4	5	6	7
V:	2.31	2.01	3.80	1.66	1.55	1.46	1.41

9. The following table gives the results of the measurements of train resistance, V is the velocity in miles per hour, R is the resistance in pounds per ton:

V:	20	40	60	80	100	120
R:	5.5	9.1	14.9	22.8	33.3	46.0

If R is related to V by the relation $R = a + bV + cV^2$, find a, b and c.

1. If V (km/hr) and R(kg/ton) are related by a relation of the type $R = a + bV^2$, find by the method of least squares a and b with the help of the following table:

$V =$	10	20	30	40	50
$R =$	8	10	15	21	30

2. Using the method of least squares fit the curve $y = ax + bx^2$ to following observations:

x:	1	2	3	4	5
y:	1.8	5.1	8.9	14.1	19.8

3. Fit the curve $y = ax + b/x$ to the following data:

x:	1	2	3	4	5	6	7	8
y:	5.4	6.3	8.2	10.3	12.6	14.9	17.3	19.5

4. Estimate y at $x = 2.25$ by fitting the *indifference curve* of the form $xy = Ax + B$ to the following data:

x:	1	2	3	4
y:	3	1.5	6	7.5

5. Fit a least square *geometric curve* $y = ax^b$ to the following data:

x:	1	2	3	4	5
y:	0.5	2	4.5	8	12.5

6. Predict y at $x = 3.75$, by fitting a *power curve* $y = ax^b$ to the given data:

x:	1	2	3	4	5	6
y:	2.98	4.26	5.21	6.10	6.80	7.50

7. Obtain a relation of the form $y = kx^m$ for the following data by the method of least squares:

x:	1	2	3	4	5
y:	7.1	27.8	62.1	110	161

8. Fit the *exponential curve* $y = ae^{bx}$ to the following data:

x:	2	4	6	8
y:	25	38	56	84

9. Fit the curve of the form $y = ae^{bx}$ to the following data:

x:	77	100	185	239	285
y:	2.4	3.4	7.0	11.1	19.6