Term Project Report

Study of Euler-Bernoulli's and Timoshenko Beam Theories for Bending and Buckling

SUBMITTED BY: GROUP 4

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1.Classical Beam Theory

1.1 Introduction to Beam Theories:

The root of classical beam theory and bending related problems can be traced way back in 16^{th} century and before. Elastic problems, and beams related problems are integral part of mechanical engineering. It's all started with Leonardo da Vinci. From the book "*The Codex Madrid*" which was published in 1492, it can be observed that the vinci was able to very accurately predict the stress and strain distribution over a beam cross-section without equipped with concepts like Hooke's Law and Calculus with was developed far down the street of history.

Then after another century *Galileo Galilei* in his book named" Dialogues *Concerning Two New Sciences*" illustrated another problem related to cantilever beam; but later rejected due to its wrong assumptions.

In the beginning of 18^{th} century Johann Bernoulli gave out a form of stress distribution which was carried out to perfection by his son Daniel Bernoulli and Johann's pupil Leonhard Euler, which was later known as "*Euler-Bernoulli Beam Theory"*, torchbearer of classical beam theory. To signify this theory let's give an example, in earlier 19th century people thought it is not possible to make a structure above 300 meters. But Eiffel tower (1889) was made considering Euler-Bernoulli theory though many objected. Its stands erect even today.

1.2 Euler-Bernoulli's Beam Theory:

For understanding a theory, first we need to know its limitation or assumptions, we are going to make. Assumption of this theories are-

i) Plane sections perpendicular to neutral axis remains perpendicular even after bending. i.e. there is to shear strain involved in plane sections.

ii) Plane sections remains plane even after bending, so our bending stress equation still holds true.

$$\sigma = \frac{My}{I}$$

In this theory, it is proposed that displacement inn z-direction is only function of x neglecting other higher order terms and displacement in y-direction is insignificant.

So, proposed displacements are:

$$u = u_s - \frac{dw}{dx}$$

$$v=0$$

$$w=w(x)$$

**non-linear components of strain are ignored in this calculation

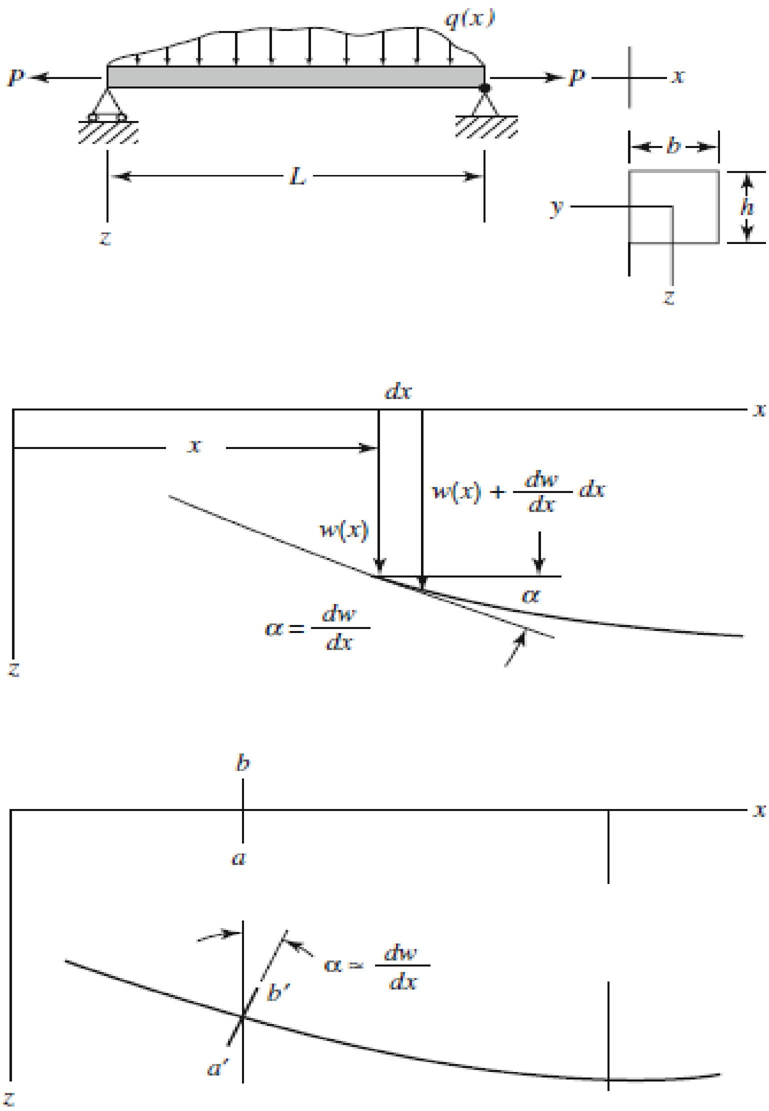

Fig.1: Bending of Euler-Bernoulli's Beam

Now, we get expressions of strains in different planes by putting the values of u,v,w in the expressions:

$$\epsilon_{xx} = \frac{du}{dx} \qquad 2\epsilon_{xy} = \frac{du}{dy} + \frac{dv}{dx}$$

$$\epsilon_{yy} = \frac{dv}{dy} \qquad 2\epsilon_{yz} = \frac{dv}{dz} + \frac{dw}{dy}$$

$$\epsilon_{zz} = \frac{dw}{dz} \qquad 2\epsilon_{xz} = \frac{dw}{dx} + \frac{du}{dz}$$

After doing following calculation we will get only expressions for ϵ_{xx}. Other terms will be zero.

$$\epsilon_{xx} = \frac{du}{dx} = \frac{du_s}{dx} - z\frac{d^2w}{dx^2}$$

Accordingly, the stress and strain values are implemented in the Principle of Virtual Work:

$$\iiint \sigma_{xx}\delta\epsilon_{xx}\mathrm{dV} = \iint q(x)\delta w dS$$

Substituting σ_{xx}= Eϵ_{xx}, and putting back strain value in the equation, we can get governing equation as:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q(x)$$

1.3 Drawbacks of Euler-Bernoulli's Beam Theory and Modification:

As the assumption in Euler Bernoulli's beam theory, we are straightway neglecting the effect of shear strain. This assumption gives nearly accurate result for thin section beam but with increasing thickness, this theory become less and less accurate as shear strain effect become more and more prominent.

In early 20th century S.P. Timoshenko came up with a solution by considering plane section displacement due to both bending as well as shear strain; this theory is later known as "*Timoshenko Beam Theory*".

2. Timoshenko Beam Theory:

2.1 Introduction:

As discussed previously in classical beam theory, shear stress along the cross section is neglected completely. But thicker the beam gets more shearing effects come into picture and CBT completely fails to predict such problems. In early 19^{th} century S.P. Timoshenko comes up this his theory involving 1^{st} degree of shear in his theory. This theory is widely known as "*Timoshenko Beam Theory*" and such kind of thicker beam is known as Timoshenko Beams.

2.2 Derivation of Governing Equation and Boundary Conditions:

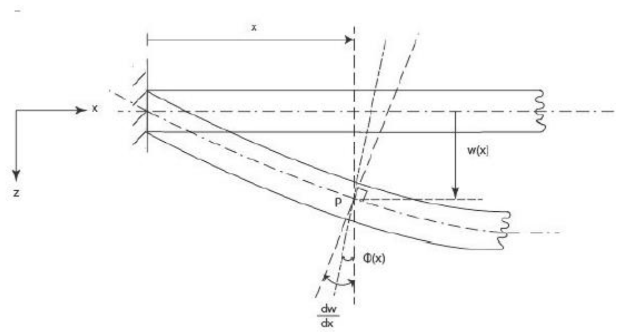

Fig. 2: Bending of Timoshenko Beam

In the above diagram,

$\Phi(x) \coloneqq$ Actual deformation angle of plane section involving shear.

$\frac{dw}{dx} \coloneqq$ Deformation angle if plane section remains perpendicular to Neutral Axis

In this theory displacement fields are given as,

$$u = u_s - z\,\Phi(x)$$

$$v = 0$$

$$w = w(x)$$

So, the strain can be calculated as,

$\epsilon_{xx} = \frac{du}{dx} = \frac{dus}{dx} - z\frac{d\Phi}{dx}$

$\epsilon_{yy} = \frac{dv}{dy} = 0,$

$\epsilon_{zz} = \frac{dw}{dz} = 0$

$2\epsilon_{xy} = \frac{du}{dy} + \frac{dv}{dx} = 0$

$2\epsilon_{yz} = \frac{dv}{dz} + \frac{dw}{dy} = 0$

$$\epsilon_{xz}= \frac{1}{2}\left(\frac{dw}{dx} + \frac{du}{dz}\right) = \frac{1}{2}\left(- \Phi + \frac{dw}{dx}\right)$$

As per we can see, except ϵ_{xx} & ϵ_{xz} every strain in beam is zero.

From lamme's stress equation we can write,

$$\sigma_{ij}=\lambda\epsilon_{kk}\delta_{ij} + 2G\epsilon_{ij} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

From (1), it can be derived that,

$$\sigma_{xx} = (\lambda + 2G)\epsilon_{xx} + \lambda(\epsilon_{yy} + \epsilon_{zz}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (2), it can be easily derived that,

$$\lambda = \frac{E\mu}{(1+\mu)(1-2\mu)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\lambda + 2G = \frac{E}{2(1+\mu)(1-2\mu)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

When μ is very close to zero then λ=0, it can be shown E=4G

$$\sigma_{xx} = E\epsilon_{xx} = E\left(\frac{dus}{dx} - z\frac{d\Phi}{dx}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$\sigma_{xz} = 2G\epsilon_{xz} = G\left(- \Phi + \frac{dw}{dx}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Now an important point to notice, in Timoshenko beam theory we have considered that shear strain is same throughout a plane section which is certainly isn't true for practical case, to compensate for that a factor k is introduced which is also known as "*Timoshenko shear coefficient*".

$$\sigma_{xz} = kG\left(- \Phi + \frac{dw}{dx}\right)$$

In order to derive Governing equations for Timoshenko, we will use Principle of Virtual Work

$$\iiint \sigma_{xx}\delta\epsilon_{xx}dV=\iint q(x)\delta w dS \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Putting the value of σ_{xz} and ϵ_{xx} in the LHS of equation (7), we get

$$\text{LHS} = \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} [E\left(\frac{dus}{dx} - z\frac{d\Phi}{dx}\right)\left(\frac{d\delta us}{dx} - z\frac{d\delta\Phi}{dx}\right) + Gk\left(-\Phi + \frac{dw}{dx}\right)\left(-\delta\Phi + \frac{d\delta w}{dx}\right) b dx dz$$

Remember that $\int_{-h/2}^{h/2} b dz = A$; $\int_{-h/2}^{h/2} bz^2 dz = I$ and with help of little bit of algebraic jugglery, we can get the following equation,

$$\Rightarrow \int_0^L EA\frac{dus}{dx}\frac{d\delta us}{dx}\mathrm{dx} + \int_0^L EI\frac{d\Phi}{dx}\frac{d\delta\Phi}{dx}dx - \int_0^L GAk\left(-\Phi+\frac{dw}{dx}\right)\delta\Phi dx +$$

$$\int_0^L GAk\left(-\Phi\ +\frac{dw}{dx}\right)\frac{d\delta\Phi}{dx}dx$$

By integration by parts we can further decompose those terms as,

$$\Rightarrow [\mathrm{EA}\frac{dus}{dx}\delta u_s] - \int_0^L \frac{d}{dx}\left(\mathrm{EA}\frac{dus}{dx}\right)\delta u_s\mathrm{dx} + [\mathrm{EI}\frac{d\Phi}{dx}\delta\Phi] - \int_0^L \frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{EI}\frac{d\Phi}{dx})\delta\Phi dx + \int_0^L GAk\left(-\Phi+\frac{dw}{dx}\right)\delta\Phi dx + [\mathrm{GAk}(-\Phi+\frac{dw}{dx})\delta w] - \int_0^L \frac{d}{dx}\left(GAk\left(-\Phi+\frac{dw}{dx}\right)\right)\delta w dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Now RHS of equation (7) can be written as a general case:

$$\mathrm{RHS}= \int_0^L q(x)\delta w dx + [\mathrm{P}\delta u_s] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Equating equation (8) & (9), we get,

$$\Rightarrow [\mathrm{EA}\frac{dus}{dx}\delta u_s] - \int_0^L \frac{d}{dx}\left(\mathrm{EA}\frac{dus}{dx}\right)\delta u_s\mathrm{dx} + [\mathrm{EI}\frac{d\Phi}{dx}\delta\Phi] - \int_0^L \frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{EI}\frac{d\Phi}{dx})\delta\Phi dx - \int_0^L GAk\left(-\Phi+\frac{dw}{dx}\right)\delta\Phi dx + [\mathrm{GAk}(-\Phi+\frac{dw}{dx})\delta w] - \int_0^L \frac{d}{dx}\left(GAk\left(-\Phi+\frac{dw}{dx}\right)\right)\delta w dx = \int_0^L q(x)\delta w dx + [\mathrm{P}\delta \mathrm{us}]$$

By taking similar terms together, we get ,

$$\Rightarrow [(\mathrm{EA}\frac{dus}{dx} - \mathrm{P})\delta u_s] - \int_0^L \frac{d}{dx}\left(\mathrm{EA}\frac{dus}{dx}\right)\delta u_s\mathrm{dx} + [\mathrm{EI}\frac{d\Phi}{dx}\delta\Phi] - \int_0^L [\frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{EI}\frac{d\Phi}{dx}) + GAk(-\Phi+\frac{dw}{dx})]\delta\Phi dx + [\mathrm{GAk}(-\Phi+\frac{dw}{dx})\delta w] - \int_0^L \left[\frac{d}{dx}\left(GAk\left(-\Phi+\frac{dw}{dx}\right)\right)+q\right]\delta w dx = 0 \quad \ldots\ldots\ldots (10)$$

From the equation (10), we get Governing Equation and Boundary Conditions:

GDE 1: $\frac{d}{dx}\left(\mathrm{EA}\frac{dus}{dx}\right) = 0$

GDE 2: $\frac{\mathrm{d}}{\mathrm{dx}}\left(\mathrm{EI}\frac{d\Phi}{dx}\right) + GAk(-\Phi+\frac{dw}{dx})] = 0$

GDE 3: $\frac{d}{dx}\left(GAk\left(-\Phi+\frac{dw}{dx}\right)\right)+q = 0$

And the determined Boundary conditions at $x = 0\ and\ x = L$ are

BC 1: Either u_s is specified or $\left(\mathrm{EA}\frac{dus}{dx} - \mathrm{P}\right) = 0$

BC 2: Either Φ is specified or $\mathrm{EI}\left(\frac{d\Phi}{dx}\right) = 0$

BC 3: Either w is specified or $kGA\left[-\Phi+\frac{dw}{dx}\right] = 0$

Now some further simplification can be made to make those complicated looking equation a bit simpler.

From GDE 2 we can get-

$$GAk\left(-\Phi + \frac{dw}{dx}\right) = -\frac{\mathrm{d}}{\mathrm{dx}}\left(\mathrm{EI}\frac{d\Phi}{dx}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Now by substituting value of Eq(10) into GDE 3, we will get

$$EI\frac{d^3\emptyset}{dx^3} = q$$

**considering bending rigidity and flexural rigidity to be constant.

From GDE 2 we can get,

$$\frac{dw}{dx} = -\frac{EI}{GAk}\frac{d^2\emptyset}{dx^2} + \emptyset$$

So, the final form of Governing Equation can be written as

$$\boldsymbol{EA\frac{d^2us}{dx^2} = 0}$$

$$\mathbf{EI\frac{d^3\emptyset}{dx^3} = q}$$

$$\boldsymbol{\frac{dw}{dx} = -\frac{EI}{GAk}\frac{d^2\emptyset}{dx^2} + \emptyset}$$

3. Comparison made in bending of simply supported beam under uniformly distributed load

3.1 Timoshenko beam theory

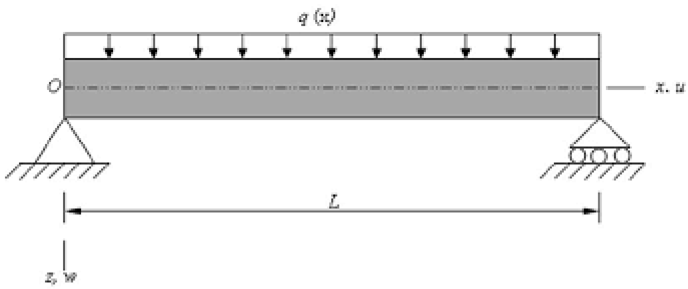

Fig3. Simply Supported beam under UDL

The governing differential equation by Timoshenko beam theory is as given below.

$$\frac{d}{dx}\left[EI\frac{d\phi}{dx}\right]+kGA\left(\frac{dw}{dx}-\phi\right)=0 \;\;\Longrightarrow\;\; EI\frac{d^2\phi}{dx^2}+kGA\left(\frac{dw}{dx}-\phi\right)=0$$

And

$$\frac{d}{dx}\left[kGA\left(\frac{dw}{dx}-\phi\right)\right]+q=0$$

$$\Longrightarrow\;\; kGA\left(\frac{d\phi}{dx}-\frac{d^2w}{dx^2}\right)=q$$

The boundary conditions on x = 0 and x = L are as follows.

(a) Either ψ specified, or $EI\frac{d\phi}{dx}=0$

(b) Either w is specified, or $kGA\left(\frac{dw}{dx}-\phi\right)=0$

After de-coupling the differential equations obtained from Euler-Lagrange method, we get

$EI\frac{d^3\phi}{dx^3}=q_0$ and $EI\frac{d^4w}{dx^4}=q_0$

From the above equations, we extract the general solution for ϕ.

$$\frac{d\phi}{dx} = \frac{1}{EI}\left[q_0\frac{x^2}{2} + C_1x + C_2\right]$$

$$\phi = \frac{1}{EI}\left[q_0\frac{x^3}{6} + C_1\frac{x^2}{2} + C_2x + C_3\right]$$

To make most of the coefficients of the equation for ψ to be zero, let's take the origin at the point where ψ becomes zero. Hence, the origin is set at the middle of the beam.

Now, equate the displacement at both ends to zero.

$$0 = \frac{1}{EI}\left[q_0\frac{0^3}{6} + C_1\frac{0^2}{2} + C_2 0 + C_3\right]$$

which gives, $C_3 = 0$.

Considering the symmetry of the loading in the beam, the slopes of deflection at the both the ends will be opposite in direction.

$$\phi|_{x=L/2} = -\phi|_{x=-L/2}$$

$$\frac{1}{EI}\left[q_0\frac{L^3}{48} + C_1\frac{L^2}{8} + C_2\frac{L}{2}\right] = \frac{1}{EI}\left[q_0\frac{L^3}{48} - C_1\frac{L^2}{8} + C_2\frac{L}{2}\right]$$

Which implies $C_1 = 0$.

According to the boundary conditions we derived from the differential equations (1) and (2), $\frac{d\phi}{dx} = 0$ at x = +L/2 and -L/2, we get

$$0 = \frac{1}{EI}\left[q_0\frac{L^2}{8} + C_2\right]$$

Which gives, $C_2 = -q_0\frac{L^2}{8}$

Hence, we get the equation for ψ as,

$$\phi = \frac{1}{EI}\left[q_0\frac{x^3}{6} - \frac{q_0L^2}{8}x\right]$$

To get the expression for the deflection w, apply the above expression for ψ into the equation (2),

$$\frac{d}{dx}\left[kGA\left(\frac{dw}{dx}-\phi\right)\right]+q=0$$

Integrate the above equation with x

$$\left[kGA\left(\frac{dw}{dx}-\phi\right)\right]=-qx$$

$$\frac{dw}{dx}=\frac{-q_0x}{kGA}+\frac{1}{EI}\left[\frac{q_0x^3}{6}-\frac{q_0L^2}{8}x\right]$$

$$w=\frac{-q_0x^2}{2kGA}+\frac{1}{EI}\left[\frac{q_0x^4}{24}-\frac{q_0L^2x^2}{16}\right]+C$$

Applying the boundary condition that the deflection at the both ends will be zero.

$$w|_{x=L/2}=0$$

$$0=\frac{-q_0L^2}{8kGA}+\frac{1}{EI}\left[\frac{q_0L^4}{384}-\frac{q_0L^4}{64}\right]+C$$

Gives

$$C=\frac{q_0L^2}{8kGA}+\frac{1}{EI}\left[\frac{q_0L^4}{64}-\frac{q_0L^4}{384}\right]=\frac{q_0L^2}{8kGA}+\frac{1}{EI}\frac{5q_0L^4}{384}$$

Putting the value of C, we get the expression for the deflection w as,

$$w=\frac{q_0}{2kGA}\left[\left(\frac{L}{2}\right)^2-x^2\right]+\frac{q_0}{EI}\left[\frac{x^4}{24}-\frac{L^2x^2}{16}+\frac{5L^4}{384}\right]$$

3.2 Euler Beam Theory

The Governing differential equation of Euler beam theory is,

$$EI\frac{d^4w}{dx^4}=q_0$$

With the boundary conditions as follows.

At x = 0 and x = L,

1. Either $EI\frac{d^2w}{dx^2} = 0$or $\frac{dw}{dx}$is specified.

2. Either $EI\frac{d^3w}{dx^3} = 0$or w is specified.

Integrating the governing differential equations with x,

$EI\frac{d^3w}{dx^3} = q_0x + C_1$

$$EI\frac{d^2w}{dx^2} = \frac{q_0x^2}{2} + C_1x + C_2$$

$$EI\frac{dw}{dx} = \frac{q_0x^3}{6} + C_1\frac{x^2}{2} + C_2x + C_3$$

$$EIw = \frac{q_0x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4$$

Setting the origin at the left end of the beam, and applying boundary conditions.

1. At x = 0, $\frac{dw}{dx}$ is not specified, which means $\frac{d^2w}{dx^2} = 0$, which implies $C_2 = 0$.

2. At x = L,$\frac{dw}{dx}$ is not specified, which means $\frac{d^2w}{dx^2} = 0$, which implies

$0 = \frac{q_0L^2}{2} + C_1L$ or, $C_1 = \frac{-q_0L}{2}$

3. At x = 0, w =0 which gives $C_4 = 0$.

4. At x = L. w = 0, which gives,

$0 = \frac{q_0L^4}{24} - \frac{q_0L}{2}\frac{L^3}{6} + C_3L$, that is $C_3 = \frac{q_0L^3}{24}$

Now, we get the equation for the deflection was,

$$w = \frac{q_0}{24EI}(x^4 - 2Lx^3 + L^3x)$$

Now, for the easiness of comparison, we shift the origin from left end to the middle of the beam, i.e., we change x into (x- L/2). After simplifying, we get,

$$w = \frac{q_0 L^4}{24EI}\left[\left(\frac{x}{L}\right)^4 - \frac{3}{2}\left(\frac{x}{L}\right)^2 + \frac{5}{16}\right]$$

3.3 Comparison between Euler-Bernoulli and Timoshenko beam theories

First, we plot the deflection of the beam according to both theories. For the sake of comparison, we set all the other parameters as unity.

For that, following python code is implemented.

```
import numpy as np
from matplotlib import pyplot as plt
```

```
x=np.linspace(-0.5,0.5,50)
w1=0.5*(0.25-x**2)+(np.power(x,4)/24)-(x**2/16)+(5/384)
w2=(1/24)*(np.power(x,4) - 1.5*x**2 + (5/16))
```

```
plt.plot(x,-w1,label="Timoshenko beam")
plt.plot(x,-w2,label="Euler beam")
plt.legend()
```

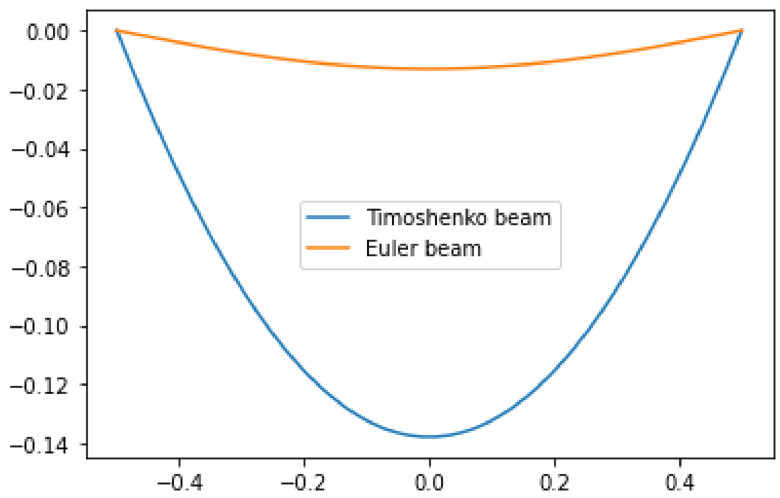

Fig4. Deflection of Beam for both theories

From the graph, it is clearly visible that Timoshenko beam theory predicts more deflection than that of Euler-Bernoulli beam theory. This is due to the fact that the Timoshenko theory accommodates the deflection due to the shearing of the beam in addition to that of simple bending. Even though they differ in values, both of them agrees with the symmetry of the deflection of the beam about its centre.

4. Buckling of Timoshenko Beam

4.1 Introduction

Buckling is a type of instability which occurs when the beams or columns are subjected to compressive axial loads. Earlier, Euler Beam Theory was used to predict the critical buckling load. The new beam theory given Timoshenko gained popularity in design of structures and can be used to study the buckling instability also.

4.2 Mathematical Derivation

The kinematical hypothesis assumed for the beam in Timoshenko Theory is mentioned below:

$$\mathrm{u} = -\mathrm{z}\phi(\mathrm{x}) + \mathrm{u_s}(\mathrm{x})$$

⍰=0

⍰=⍰(⍰)

The strain-displacement relationship includes the non-linear terms in the normal strain.

$$\epsilon_{\mathrm{xx}} = \frac{\mathrm{du}}{\mathrm{dx}} + \frac{1}{2}\left(\left(\frac{\mathrm{du}}{\mathrm{dx}}\right)^2 + \left(\frac{\mathrm{dv}}{\mathrm{dx}}\right)^2 + \left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2\right) = \frac{\mathrm{du}}{\mathrm{dx}} + \frac{1}{2}\left(\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2\right)$$

$$= -\mathrm{z}\frac{\mathrm{d}\psi}{\mathrm{dx}} + \frac{\mathrm{du_s}}{\mathrm{dx}} + \frac{1}{2}\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2$$

$$\epsilon_{xz} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] = \frac{1}{2}\left[-\phi + \frac{dw}{dx}\right]$$

Above we have assumed that $\frac{\mathrm{dw}}{\mathrm{dx}} \gg \frac{\mathrm{du}}{\mathrm{dx}}$

Now according to the principle of virtual work, **LHS** is given by:

$$\int_0^L \int_{-h/2}^{h/2} \left[E\left[\left(\frac{du_s}{dx}\right) + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 - z\frac{d\Psi}{dx}\right]\left[\left(\frac{d\delta u_s}{dx}\right) + \frac{dw}{dx}\frac{d\delta w}{dx} - z\frac{d\delta\phi}{dx}\right] + kG\beta\delta\beta\right] bdxdz$$

Where $\psi(x)$ is the rotation of line elements along the centreline due to bending only. $\beta(x)$ accounts for the shear deformation.

$$\int_0^L \int_{-h/2}^{h/2} E\left[\frac{du_s}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\left[\frac{d\delta u_s}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}\right] bdxdz + 0$$
$$+ \int_0^L E\frac{d\phi}{dx}\frac{d\delta\phi}{dx}\left(\int_{-h/2}^{h/2} z^2\,dz\right)dx + \int_0^L kGA\left(-\phi + \frac{dw}{dx}\right)\delta\phi dx$$
$$+ \int_0^L kGA\left(-\phi + \frac{dw}{dx}\right)\frac{d\delta w}{dx}dx$$

After expansion of previous expression, some terms include only ‘z’ which is an odd function and as we know that integrating and odd function between similar limits will results in zero.

$$\int_0^L \int_{-h/2}^{h/2} E\left[\frac{du_s}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\left[\frac{d\delta u_s}{dx}+\frac{dw}{dx}\frac{d\delta w}{dx}\right] bdxdz + 0 + \int_0^L E\frac{d\phi}{dx}\frac{d\delta\phi}{dx}\left(\int_{-h/2}^{h/2} z^2\,dz\right)dx + \int_0^L kGA\left(-\phi+\frac{dw}{dx}\right)\delta\phi dx + \int_0^L kGA\left(-\phi+\frac{dw}{dx}\right)\frac{d\delta w}{dx}dx$$

To simplify the above expression, let $\left[\left(\frac{du_s}{dx}\right)+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] = (\#)$ and $\left(-\phi\,\frac{dw}{dx}\right) = (\$)$

$$\int_0^L EA(\#)\frac{d\delta u_s}{dx}dx + \int_0^L EA(\#)\frac{dw}{dx}\frac{d\delta w}{dx}dx + \int_0^L EI\frac{d\phi}{dx}\frac{d\delta\Psi\phi}{dx}dx - \int_0^L kGA\,(\$)\delta\phi dx + \int_0^L kGA\,(\$)\frac{d\delta w}{dx}d$$

Now applying integrating by parts: -

$$[EA(\#)\delta u_s]_0^L - \int_0^L \frac{(EA(\#))}{dx}\delta u_s\,\mathrm{dx} + \left[EA(\#)\frac{dw}{dx}\delta\mathrm{w}\right]_0^L - \int_0^L \frac{d\left(EA(\#)\frac{dw}{dx}\right)}{dx}\delta w dx$$

$$+\left[EI\frac{d\phi}{dx}\delta\phi\right]_0^L - \int_0^L \frac{\mathrm{d}\left[EI\frac{d\phi}{dx}\right]}{\mathrm{dx}}\delta\phi dx - \int_0^L kGA(\$)\delta\phi \mathrm{dx} + [kGA(\$)\delta w]_0^L - \int_0^L kGA\frac{d(\$)}{dx}\delta w dx$$

RHS of the principal of virtual work:

$$\int t_i\delta u_i dS = -[P\delta u_s]_0^L + \int_0^L q\delta w dx$$

Now equating LHS = RHS and rearranging similar terms we arrive on the following expression-

$$[(EA(\#)+P)\delta u_s]_0^L - \int_0^L d\frac{(\mathrm{EA}(\#))}{\mathrm{dx}}\delta\mathrm{u_s dx} + \left[\mathrm{EA}(\#)\frac{\mathrm{dw}}{\mathrm{dx}}\delta\mathrm{w}\right]_0^L - \int_0^L\left(\frac{\mathrm{d}\left(\mathrm{EA}(\#)\frac{\mathrm{dw}}{\mathrm{dx}}\right)}{\mathrm{dx}} - \mathrm{q} + \mathrm{kGA}\frac{\mathrm{d}(\$)}{\mathrm{dx}}\right)\delta\mathrm{w\,dx} + \left[\mathrm{EI}\frac{\mathrm{d}\phi}{\mathrm{dx}}\delta\phi\right]_0^L + [kGA(\$)\delta w]_0^L - \int_0^L\left[d\frac{EI\frac{d\phi}{dx}}{dx} + kGA(\$)\right]\delta\phi dx = 0$$

From the above equation, Governing Differential Equations and Boundary Conditions can be obtained.

The governing equations are as followed:

GDE 1: $d\frac{(\mathrm{EA}(\#))}{\mathrm{dx}} = 0 \Rightarrow \mathrm{EA}(\#) = const.$

GDE 2: $\frac{\mathrm{d}\left(\mathrm{EA}(\#)\frac{\mathrm{dw}}{\mathrm{dx}}\right)}{\mathrm{dx}} - \mathrm{q} + \mathrm{kGA}\frac{\mathrm{d}(\$)}{\mathrm{dx}} = 0 \Rightarrow \frac{\mathrm{d}\left(\mathrm{EA}(\#)\frac{\mathrm{dw}}{\mathrm{dx}}\right)}{\mathrm{dx}} + \mathrm{kGA}\frac{\mathrm{d}(\$)}{\mathrm{dx}} = q$

GDE 3: $d\frac{EI\frac{d\phi}{dx}}{dx} + kGA(\$) = 0$

The boundary conditions at $x = 0$ and $x = L$, are as followed:

BC 1: Either $EA(\#) = -P$ or u_s $is\ specified$

BC 2: Either $\mathrm{EA}(\#)\frac{\mathrm{dw}}{\mathrm{dx}} = 0$ or w is specified

BC 3: Either $\mathrm{EI}\frac{\mathrm{d}\phi}{\mathrm{dx}} = 0$ or Ψ is specified

From GDE 1 and BC 1, it can be observed that LHS expression is same. Therefore, it can be confirmed that $\mathrm{EA}(\#) = -P$.

Now modifying the GDE 2, the new equation will be –

$$-P\frac{d^2w}{dx^2} + kGA\frac{d(\$)}{dx} = q$$

$$\Rightarrow -P\frac{d^2w}{dx^2} + kGA\frac{d\left(-\phi + \frac{dw}{dx}\right)}{dx} = q$$

$$\Rightarrow -P\frac{d^2w}{dx^2} + kGA\left(\frac{d^2w}{dx^2} - \frac{\mathrm{d}\phi}{\mathrm{dx}}\right) = q$$

$$\Rightarrow \frac{\mathrm{d}\phi}{\mathrm{dx}} = \left(1 - \frac{P}{kGA}\right)\frac{d^2w}{dx^2} - \frac{q}{kGA}$$

Using the expression of $\frac{\mathrm{d}\phi}{\mathrm{dx}}$ $in\ above\ line\ can\ be\ used\ in\ GDE\ 3\ to\ obtain\ an\ expression\ for\ \Psi.$

$$EI\frac{d^2\phi}{dx^2} + kGA\left(-\phi + \frac{dw}{dx}\right) = 0$$

$$\Rightarrow EI\left(\left(1 - \frac{P}{kGA}\right)\frac{d^3w}{dx^3} - \frac{1}{kGA}\frac{dq}{dx}\right) + kGA\left(-\phi + \frac{dw}{dx}\right) = 0$$

$$\Rightarrow \frac{EI}{kGA}\left(1 - \frac{P}{kGA}\right)\frac{d^3w}{dx^3} - \frac{EI}{(kGA)^2}\frac{dq}{dx} + \frac{dw}{dx} = \Psi$$

Now using expression for ϕ again in the equation of $\frac{\mathrm{d}\phi}{\mathrm{dx}}$ which generated an equation i.e.

$$\boldsymbol{\frac{EI}{kGA}\left(1-\frac{P}{kGA}\right)\frac{d^4w}{dx^4}-\frac{EI}{(kGA)^2}\frac{d^2q}{dx^2}=-\frac{P}{kGA}\frac{d^2w}{dx^2}-\frac{q}{kGA}}$$

If it is assumed that the external loading q = 0, then the more simplified expression is –

$$-\frac{P}{kGA}\frac{d^2w}{dx^2}=\frac{EI}{kGA}\left(1-\frac{P}{kGA}\right)\frac{d^4w}{dx^4}$$

Let $k^2 = \frac{\frac{P}{EI}}{1-\frac{P}{kGA}}$, the final differential equation for deflection of the beam is –

$$\boldsymbol{\frac{d^4w}{dx^4}+k^2\frac{d^2w}{dx^2}=0}$$

NOTE: It can be observed that the equation resembles with that of the Euler- Buckling Beam Theory with change in expression for “k”.

References

[1] Dym, C., Shames, I (2013). Solid mechanics: A variational approach. Springer

[2] Timoshenko, S., Gere, J. (2009). Theory of Elastic Stability. Dover Publications, INC.,Mineola,NewYork

[3]http://www.facweb.iitkgp.ac.in/~jeevanjyoti/teaching/advmechsolids/2022/notes/beam_theory_recorded_notes.pdf

Work Distribution

Vaibhav Tyagi(21ME63R37), will try to obtain G.E. for Buckling using Timoshenko Beam Theory

Dipannoy Dhar(21ME63R32), will introduce Classical Beam Theory and show how Timoshenko Theory is different than the former.

Faudal Fairusy(21ME63R29), will compare the results between these theories using an example such as simply supported beam.