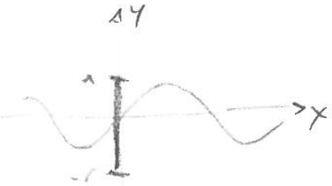
Mathematik 1 - AI

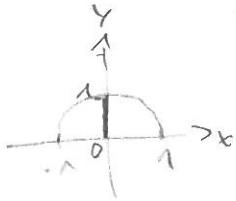
Blatt 2

A11 $f: \mathbb{R} \to \mathbb{R},\ x \mapsto \sin x,\quad g: [-1,1] \to \mathbb{R},\ x \mapsto 1-x^2,$

$h: \mathbb{R}_0^+ \to \mathbb{R},\ x \mapsto \sqrt{x}$



$D(f) = \mathbb{R}$
$W(f) = [-1,1]$



$D(g) = [-1,1]$
$W(g) = [0;1]$

$D(h) = \mathbb{R}_0^+$
$W(h) = \mathbb{R}_0^+$

a) $f \circ g$ existiert wg. $\underset{[-1,1] \subseteq \mathbb{R}}{W(g) \subseteq D(f)}$; $(f \circ g)(x) = f(g(x)) = \sin(1-x^2)$

$W(f \circ g) = [0; \sin(1)]$

b) $f \circ h$ ex. wg. $\underset{\mathbb{R}_0^+ \subseteq \mathbb{R}}{W(h) \subseteq D(f)}$, $(f \circ h)(x) = f(h(x)) = \sin(\sqrt{x})$

c) $g \circ h$ ex. nicht wg. $\underset{\mathbb{R}_0^+ \quad [-1,1]}{W(h) \not\subseteq D(g)}$

d) $g \circ f$ ex. wg. $\underset{[-1;1] \quad [-1,1]}{W(f) \subseteq D(g)}$, $(g \circ f)(x) = g(f(x)) = 1 - \sin^2 x$

e) $h \circ f$ ex. nicht wg. $\underset{[-1,1] \quad \mathbb{R}_0^+}{W(f) \not\subseteq D(h)}$

f) $h \circ g$ ex. wg. $\underset{[0;1] \quad \mathbb{R}_0^+}{W(g) \subseteq D(h)}$, $(h \circ g)(x) = h(g(x)) = \sqrt{1-x^2}$

g) $f \circ (g \circ h)$ ex. nicht, da $g \circ h$ nicht ex.

h) $f \circ (h \circ g)$ ex. wg. $W(h \circ g) \subseteq W(h) \subseteq D(f)$; $(f \circ (h \circ g))(x) = f(h(g(x))) =$
$= \sin\sqrt{1-x^2}$

i) $g \circ (h \circ f)$ ex. nicht, da $h \circ f$ nicht ex.

k) $g\circ(f\circ h)$ ex. da $W(f\circ h) \subseteq W(f) \subseteq D(g)$,

$$g(f(h(x)) = 1 - \sin^2\sqrt{x}$$

l) $h\circ(f\circ g)$ ex., da $\underset{[0;\sin(1)]}{W(f\circ g)} \subseteq \mathbb{R}_0^+ = D(h)$

$$h(f(g(x)) = \sqrt{\sin(\underbrace{1-x^2}_{0\leq 1-x^2\leq 1})} \quad (-1\leq x\leq 1)$$

m) $h\circ(g\circ f)$ ex. da $W(g\circ f) \subseteq \underset{[0,1]}{W(g)} \subseteq \underset{\mathbb{R}_0^+}{D(h)}$

$$h(g(f(x)) = \sqrt{1-\sin^2 x}$$

A12 Beh.: $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$

$f: \{x\geq 1\} \to \mathbb{R}_0^+,\ x\mapsto 2x-2$, $\quad g: \mathbb{R}_0^+ \to \mathbb{R}_0^+,\ x\mapsto\sqrt{x}$

Bestimmung von f^{-1}:

$$y = 2x-2$$
$$y+2 = 2x$$
$$x = \tfrac{1}{2}y+1$$
$$\overset{x\leftrightarrow y}{\Rightarrow} f^{-1}(x) = \tfrac{1}{2}x+1$$

Bestimmung von g^{-1}:

$$y = \sqrt{x}$$
$$y^2 = x$$
$$x = y^2$$
$$\overset{x\leftrightarrow y}{\Rightarrow} g^{-1}(x) = x^2$$

Bestimmung von $g\circ f$:

$$(g\circ f)(x) = g(f(x)) = \sqrt{2x-2}$$

$$g\circ f: \{x\geq 1\} \to \mathbb{R}_0^+$$

$$W(g\circ f) = \mathbb{R}_0^+$$

Bestimmung von $(g\circ f)^{-1}$:

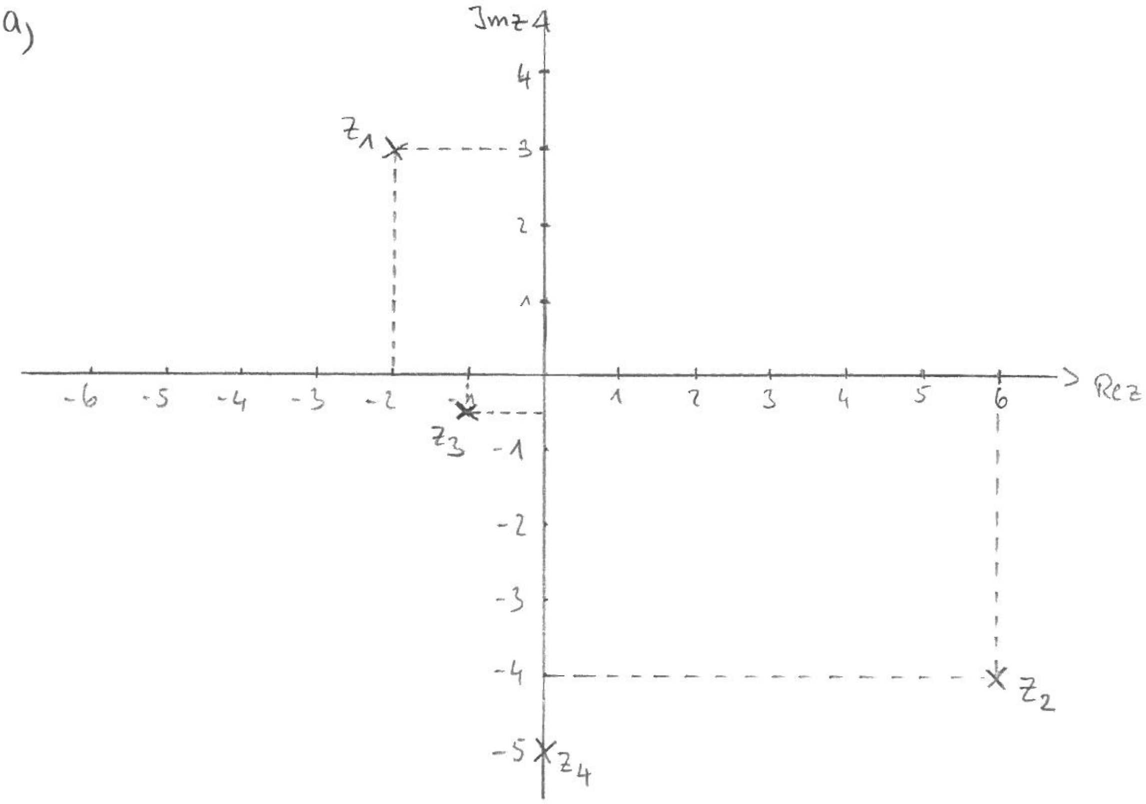
$$y = \sqrt{2x-2} \quad |\,(\,)^2$$
$$y^2 = 2x-2$$
$$y^2+2 = 2x$$
$$x = \tfrac{1}{2}y^2+1 \quad (\geq 1)$$
$$\overset{x\leftrightarrow y}{\Rightarrow} (g\circ f)^{-1}(x) = \tfrac{1}{2}x^2+1$$

$$(g\circ f)^{-1}: \mathbb{R}_0^+ \to \{x\geq 1\}$$

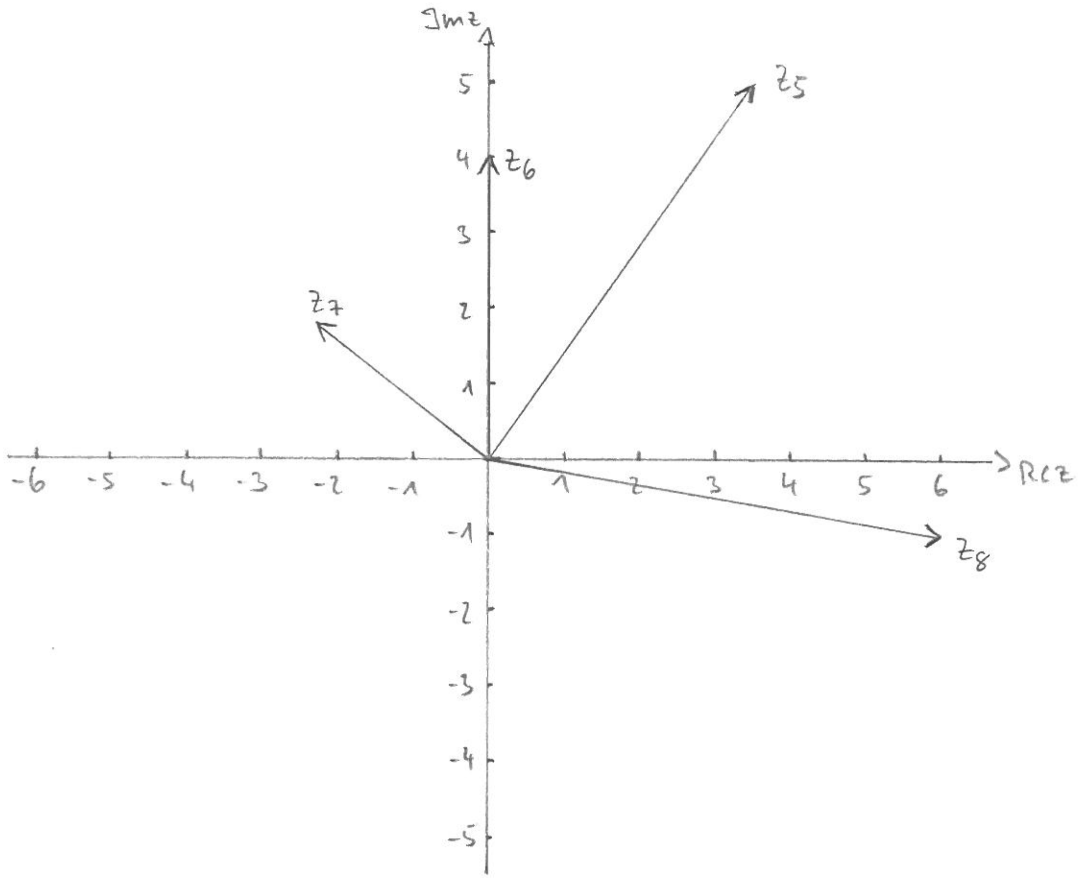
$$(g \circ f)^{-1}(x) = \frac{1}{2}x^2 + 1$$

$$(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x)) = \frac{1}{2}x^2 + 1$$

A13 a)



b)



A14 a) $3(-1+4i)-2(7-i) = -3+12i-14+2i = \underline{\underline{-17+14i}};$

b) $(3+2i)(2-i) = 6-3i+4i+2 = \underline{\underline{8+i}};$

c) $(i-2)\left[2(1+i)-3(i-1)\right] = (i-2)\cdot\left[2+2i-3i+3\right] =$

$= (i-2)(5-i) = 5i+1-10+2i = \underline{\underline{-9+7i}};$

d) $\dfrac{2-3i}{4-i} = \dfrac{(2-3i)(4+i)}{(4-i)(4+i)} = \dfrac{8+2i-12i+3}{16+1} = \dfrac{11-10i}{17} = \underline{\underline{\dfrac{11}{17}-\dfrac{10}{17}i}};$

e) $(4+i)(3+2i)(1-i) = (12+8i+3i-2)(1-i) = (10+11i)(1-i) =$

$= 10+11i-10i+11 = \underline{\underline{21+i}};$

f) $\dfrac{(2+i)(3-2i)(1+2i)}{(1-i)^2} = \dfrac{(6-4i+3i+2)(1+2i)}{(1-i)^2} = \dfrac{(8-i)(1+2i)}{(1-i)^2} =$

$= \dfrac{8+16i-i+2}{(1-i)^2} = \dfrac{10+15i}{(1-i)^2} = \dfrac{10+15i}{1-2i-1} = \dfrac{10+15i}{-2i}\cdot\dfrac{i}{i} = \dfrac{-15+10i}{2}$

$= \underline{\underline{-\dfrac{15}{2}+5i}}$

g) $(2i-1)^2\cdot\left(\dfrac{4}{1-i}+\dfrac{2-i}{1+i}\right) = (-4-4i+1)\cdot\dfrac{4(1+i)+(2-i)(1-i)}{(1-i)(1+i)}$

$= (-3-4i)\cdot\dfrac{4+4i+2-2i-i-1}{2} = (-3-4i)\dfrac{5+i}{2} =$

$= \dfrac{1}{2}(-3-4i)(5+i) = \dfrac{1}{2}(-15-3i-20i+4) = \dfrac{1}{2}(-11-23i)$

$= \underline{\underline{-\dfrac{11}{2}-\dfrac{23}{2}i}};$

h) $\dfrac{i^4+i^9+i^{16}}{2-i^5+i^{10}-i^{15}} = \dfrac{(i^2)^2+i(i^2)^4+(i^2)^8}{2-i(i^2)^2+(i^2)^5-i(i^2)^7} =$

$= \dfrac{(-1)^2+i(-1)^4+(-1)^8}{2-i(-1)^2+(-1)^5-i(-1)} = \dfrac{1+i+1}{2-i-1+i} = \underline{\underline{2+i}};$

i) $3\left(\dfrac{1+i}{1-i}\right)^2-2\left(\dfrac{1-i}{1+i}\right)^3 =$

$(1+i)^2 = 1+2i-1 = 2i$ $\quad (1+i)^3 = 2i(1+i) = -2+2i$

$(1-i)^2 = 1-2i-1 = -2i$ $\quad (1-i)^3 = (-2i)(1-i) = -2-2i$

$= 3\dfrac{(1+i)^2}{(1-i)^2}-2\dfrac{(1-i)^3}{(1+i)^3} = 3\cdot\dfrac{2i}{-2i}-2\dfrac{-2-2i}{-2+2i} =$

$= -3-2\dfrac{2+2i}{2-2i} = -3-2\dfrac{(2+2i)^2}{(2-2i)(2+2i)} = -3-2\dfrac{4+8i-4}{8} = \underline{\underline{-3-2i}}$

A15 Prüfe $\overline{z^{-1}} = \bar{z}^{-1}$ für $z_1 = 2-3i$ und $z_2 = -4+i$

$\underline{z = 2-3i}$:

$$\left.\begin{aligned} z^{-1} &= \frac{1}{2-3i} = \frac{2+3i}{2^2+3^2} = \frac{2}{13} + \frac{3}{13}i \\ \bar{z}^{-1} &= \frac{1}{2+3i} = \frac{2-3i}{2^2+3^2} = \frac{2}{13} - \frac{3}{13}i \end{aligned}\right\} \Rightarrow \overline{z^{-1}} = \bar{z}^{-1}$$

$\underline{z = -4+i}$:

$$\left.\begin{aligned} z^{-1} &= \frac{1}{-4+i} = \frac{-4-i}{4^2+1^2} = \frac{-4-i}{17} = -\frac{4}{17} - \frac{1}{17}i \\ \bar{z}^{-1} &= \frac{1}{-4-i} = \frac{-4+i}{4^2+1^2} = \frac{-4+i}{17} = -\frac{4}{17} + \frac{1}{17}i \end{aligned}\right\} \Rightarrow \overline{z^{-1}} = \bar{z}^{-1}$$

A16 $z_1 = 1-i$, $z_2 = -2+4i$, $z_3 = \sqrt{3} - 2i$

a) $z_1^2 + 2z_1 - 3 = (1-i)^2 + 2(1-i) - 3 = \cancel{1} - 2i - \cancel{1} + 2 - 2i - 3 =$
$= \underline{\underline{-1-4i}}$;

b) $|2z_2 - 3z_1|^2 = |2(-2+4i) - 3(1-i)|^2 = |-4+8i-3+3i|^2 =$
$= |-7+11i|^2 = (-7)^2 + 11^2 = 49 + 121 = \underline{\underline{170}}$;

c) $|z_1\bar{z}_2 + z_2\bar{z}_1| = |z_1\bar{z}_2 + \overline{z_1\bar{z}_2}| = |2\,\mathrm{Re}(z_1\bar{z}_2)| = (*)$

$z_1 \cdot \bar{z}_2 = (1-i)(-2-4i) = -2-4i+2i-4 = -6-2i$

$(*) = |2\cdot(-6)| = |-12| = \underline{\underline{12}}$;

oder direkt berechnet:

$|z_1\bar{z}_2 + z_2\bar{z}_1| = |(1-i)(-2-4i) + (-2+4i)(1+i)| =$

$= |-2-\cancel{4i}+\cancel{2i}-4-2-\cancel{2i}+\cancel{4i}-4| = |-12| = \underline{\underline{12}}$;

A16 $z_1 = 1 - i$, $z_2 = -2 + 4i$, $z_3 = \sqrt{3} - 2i$

d) $\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + i} \right| = \left| \frac{1 - i - 2 + 4i + 1}{1 - i + 2 - 4i + i} \right| = \left| \frac{3i}{3 - 4i} \right| = \frac{|3i|}{|3 - 4i|} =$

$= \frac{3}{\sqrt{3^2 + 4^2}} = \frac{3}{\sqrt{25}} = \underline{\underline{\frac{3}{5}}}$;

bzw. $\left| \frac{3i}{3-4i} \right| = \left| \frac{3i(3+4i)}{3^2+4^2} \right| = \left| \frac{9i - 12}{25} \right| = \sqrt{\left(\frac{12}{25}\right)^2 + \left(\frac{9}{25}\right)^2} =$

$= \frac{1}{25} \sqrt{12^2 + 9^2} = \frac{1}{25} \cdot \sqrt{144 + 81} = \frac{1}{25} \sqrt{225} = \frac{15}{25} = \underline{\underline{\frac{3}{5}}}$

e) $\frac{1}{2} \left(\frac{z_3}{\bar{z}_3} + \frac{\bar{z}_3}{z_3} \right) = \frac{1}{2} \left(\frac{z_3}{\bar{z}_3} + \overline{\left(\frac{z_3}{\bar{z}_3} \right)} \right) = \operatorname{Re}\left(\frac{z_3}{\bar{z}_3} \right)$ $\left[\operatorname{Re}(z) = \frac{1}{2}(z + \bar{z}) \right]$

$= \operatorname{Re}\left(\frac{\sqrt{3} - 2i}{\sqrt{3} + 2i} \right) = \operatorname{Re}\left(\frac{(\sqrt{3} - 2i)(\sqrt{3} - 2i)}{(\sqrt{3} + 2i)(\sqrt{3} - 2i)} \right) = \operatorname{Re}\left(\frac{3 - 4\sqrt{3}i - 4}{3 + 4} \right)$

$= \operatorname{Re}\left(\frac{-1 - 4\sqrt{3}i}{7} \right) = \underline{\underline{-\frac{1}{7}}}$

bzw. direkt gerechnet:

$\frac{1}{2} \left(\frac{z_3}{\bar{z}_3} + \frac{\bar{z}_3}{z_3} \right) = \frac{1}{2} \frac{z_3^2 + \bar{z}_3^2}{\bar{z}_3 \cdot z_3} = \frac{1}{2} \frac{(\sqrt{3} - 2i)^2 + (\sqrt{3} + 2i)^2}{(\sqrt{3} + 2i)(\sqrt{3} - 2i)} =$

$= \frac{1}{2} \frac{3 - 4\sqrt{3}i - 4 + 3 + 4\sqrt{3}i - 4}{3 + 4} = \frac{1}{2} \frac{-2}{7} = \underline{\underline{-\frac{1}{7}}}$;

f) $\overline{(z_2 + z_3)(z_1 - z_3)} = (\bar{z}_2 + \bar{z}_3)(\bar{z}_1 - \bar{z}_3) =$

$= (-2 - 4i + \sqrt{3} + 2i)(1 + i - \sqrt{3} - 2i) =$

$= (-2 + \sqrt{3} - 2i)(1 - \sqrt{3} - i) =$

$= -2 + 2\sqrt{3} + 2i + \sqrt{3} - 3 - \sqrt{3}i - 2i + 2\sqrt{3}i - 2$

$= \underline{\underline{-7 + 3\sqrt{3} + i\sqrt{3}}}$;

g) $|z_1^2+\bar{z}_2^2|^2+|\bar{z}_3^2-z_2^2|^2 = (*)$

$$\left[\begin{array}{l} z_1^2=(1-i)^2=1-2i+i^2=-2i \\ z_2^2=(-2+4i)^2=4-16i+(4i)^2=-12-16i \\ \Rightarrow \bar{z}_2^2=\overline{z_2^2}=-12+16i \\ \bar{z}_3^2=(\sqrt{3}+2i)^2=3+4\sqrt{3}i-4=-1+4\sqrt{3}i \end{array}\right] \text{ N.R.}$$

$$(*)=|-2i-12+16i|^2+|-1+4\sqrt{3}i+12+16i|^2$$

$$=|-12+14i|^2+|11+i(16+4\sqrt{3})|^2$$

$$=12^2+14^2+|11+i4(4+\sqrt{3})|^2=$$

$$=340+11^2+(4(4+\sqrt{3}))^2=$$

$$=461+16(4+\sqrt{3})^2=461+16(16+8\sqrt{3}+3)=$$

$$=461+16(19+8\sqrt{3})=461+304+128\sqrt{3}=\underline{\underline{765+128\sqrt{3}}};$$

h) $\mathrm{Re}(2z_1^3+3z_2^2-5z_3^2)$

$$\left[\begin{array}{l} z_1^3=z_1^2\cdot z_1=(-2i)(1-i)=-2i-2=-2-2i \\ z_2^2=-12-16i \\ z_3^2=-1-4\sqrt{3}i \quad ; \quad z_3^2=\overline{(\bar{z}_3^2)}=\overline{(-1+4\sqrt{3}i)}=-1-4\sqrt{3}i \end{array}\right]$$

$$\Rightarrow \mathrm{Re}(2z_1^3+3z_2^2-5z_3^2)=2\,\mathrm{Re}(z_1^3)+3\,\mathrm{Re}(z_2^2)-5\,\mathrm{Re}(z_3^2)=$$

$$=2\cdot(-2)+3\cdot(-12)-5\cdot(-1)=\underline{-35}$$

i) $\dfrac{z_2}{z_3}=\dfrac{-2+4i}{\sqrt{3}-2i}=\dfrac{(-2+4i)(\sqrt{3}+2i)}{3+4}=\dfrac{-2\sqrt{3}-4i+4\sqrt{3}i-8}{7}$

$$=\frac{1}{7}\left(-8-2\sqrt{3}+i(4\sqrt{3}-4)\right) \Rightarrow z_1\cdot\frac{z_2}{z_3}=(1-i)\cdot\frac{1}{7}(-8-2\sqrt{3}+i(4\sqrt{3}-4))$$

$$=\frac{1}{7}(1-i)\cdot 2(-4-\sqrt{3}+i2(\sqrt{3}-1))=\frac{2}{7}(-4-\sqrt{3}+i2(\sqrt{3}-1)+4i+\sqrt{3}i+$$

$$+2(\sqrt{3}-1))=\frac{2}{7}(-4-\sqrt{3}+2\sqrt{3}-2+i(2\sqrt{3}-2+4+\sqrt{3}))=$$

A16 i) $= \frac{2}{7}\left(-6+\sqrt{3}+i\left(2+3\sqrt{3}\right)\right)$

$$\Rightarrow \operatorname{Im}\left(z_1 z_2 / z_3\right) = \underline{\underline{\frac{2}{7}\left(2+3\sqrt{3}\right)}};$$

A17 a) $z = 2-2i$; $r = |z| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

$$\varphi = \operatorname{Arg}(z) = -\arccos\left(\frac{x}{r}\right) = -\arccos\left(\frac{2}{2\sqrt{2}}\right)$$
$$= -\arccos\left(\frac{1}{\sqrt{2}}\right) = -\pi/4$$

$$\Rightarrow z = 2\sqrt{2}\underbrace{\cos(-\tfrac{\pi}{4})}_{=\cos(\pi/4)} + i\,2\sqrt{2}\underbrace{\sin(-\pi/4)}_{=-\sin(\pi/4)} = \underline{\underline{2\sqrt{2}e^{-i\pi/4}}}$$

b) $z = -1+\sqrt{3}i$;

$$r = |z| = \sqrt{1^2+\sqrt{3}^2} = \sqrt{1+3} = 2$$

$$\varphi = \operatorname{Arg}(z) = \arccos\left(\frac{x}{r}\right) = \arccos\left(-\frac{1}{2}\right) = \frac{2}{3}\pi$$

$$\Rightarrow z = 2\cos\left(\frac{2\pi}{3}\right) + i2\sin\left(\frac{2\pi}{3}\right) = \underline{\underline{2e^{i\frac{2\pi}{3}}}}$$

c) $z = 2\sqrt{2}+2\sqrt{2}i$

$$r = |z| = \sqrt{(2\sqrt{2})^2+(2\sqrt{2})^2} = \sqrt{8+8} = 4$$

$$\varphi = \operatorname{Arg}(z) = \arccos\left(\frac{x}{r}\right) = \arccos\left(\frac{2\sqrt{2}}{4}\right) = \arccos\left(\frac{\sqrt{2}}{2}\right)$$
$$= \pi/4$$

$$\Rightarrow z = 4\cos(\pi/4) + i4\sin\left(\frac{\pi}{4}\right) = \underline{\underline{4e^{i\pi/4}}}$$

d) $z = -i$

$$r = |z| = \sqrt{0^2+1^2} = 1$$

$$\varphi = \operatorname{Arg}(z) = -\arccos\left(\frac{x}{r}\right) = -\arccos(0) = -\pi/2$$

$$\Rightarrow z = \cos(-\pi/2) + i\sin(-\pi/2) = e^{-i\pi/2}$$

e) $z = -4$

$r = |z| = \sqrt{4^2 + 0^2} = 4$

$\varphi = \mathrm{Arg}(z) = \arccos\left(\frac{x}{r}\right) = \arccos\left(\frac{-4}{4}\right) = \arccos(-1) = \pi$

$\Rightarrow z = 4\cos(\pi) + i4\sin(\pi) = \underline{\underline{4e^{i\pi}}}$

f) $z = -2\sqrt{3} - 2i$

$r = |z| = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{12+4} = 4$

$\varphi = \mathrm{Arg}(z) = -\arccos\left(\frac{x}{r}\right) = -\arccos\left(-\frac{2\sqrt{3}}{4}\right) =$

$= -\arccos\left(-\frac{\sqrt{3}}{2}\right) = \underline{\underline{-\frac{5}{6}\pi}};$

$\Rightarrow z = 4\cos\left(-\frac{5}{6}\pi\right) + i4\sin\left(-\frac{5}{6}\pi\right) = \underline{\underline{4e^{-i\frac{5}{6}\pi}}}$

g) $z = \sqrt{2}i$

$r = |z| = \sqrt{0^2 + \sqrt{2}^2} = \sqrt{2}$

$\varphi = \mathrm{Arg}(z) = \arccos\left(\frac{x}{r}\right) = \arccos(0) = \frac{\pi}{2}$

$\Rightarrow z = \sqrt{2}\cos(\pi/2) + i\sqrt{2}\sin\left(\frac{\pi}{2}\right) = \underline{\underline{\sqrt{2}e^{i\pi/2}}}$

h) $z = \frac{\sqrt{3}}{2} - \frac{3}{2}i$

$r = |z| = \sqrt{(\sqrt{3}/2)^2 + (3/2)^2} = \sqrt{3/4 + 9/4} = \sqrt{3}$

$\varphi = \mathrm{Arg}(z) = -\arccos\left(\frac{x}{r}\right) = -\arccos\left(\frac{\sqrt{3}/2}{\sqrt{3}}\right) =$

$= -\arccos(1/2) = -\frac{\pi}{3}$

$\Rightarrow z = \sqrt{3}\cos(-\pi/3) + i\sqrt{3}\sin(-\pi/3) = \sqrt{3}e^{-i\pi/3}$

$\boxed{= \sqrt{3}\cos(\pi/3) - i\sqrt{3}\sin(\pi/3)}$

A18 a) $z_1 = 6(\cos 135° + i \sin 135°) =$

$= 6\left(-\frac{1}{2}\sqrt{2} + i\,\frac{1}{2}\sqrt{2}\right) = \underline{\underline{-3\sqrt{2} + i\,3\sqrt{2}}}$

b) $z_2 = 3\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = 3(0 + i\cdot 1) = \underline{\underline{3i}}$ $\left(\frac{\pi}{2} \mathrel{\hat=} 90°\right)$

c) $z_3 = 4\left(\cos\frac{7}{4}\pi + i\sin\frac{7}{4}\pi\right) = 4\left(\frac{1}{2}\sqrt{2} - i\,\frac{1}{2}\sqrt{2}\right) = \underline{\underline{2\sqrt{2} - 2\sqrt{2}\,i}}$ $\left(\frac{7}{4}\pi \mathrel{\hat=} 315°\right)$

d) $z_4 = 2\,e^{i\frac{5}{4}\pi} = 2\cos\frac{5}{4}\pi + i\,2\sin\frac{5}{4}\pi =$ $\left(\frac{5}{4}\pi \mathrel{\hat=} 225°\right)$

$= 2\left(-\frac{1}{2}\sqrt{2}\right) + i\,2\left(-\frac{1}{2}\sqrt{2}\right) =$

$= \underline{\underline{-\sqrt{2} - \sqrt{2}\,i}}$

$\left(\frac{7}{6}\pi = 210°\right)$

e) $z_5 = 5\,e^{i\frac{7\pi}{6}} = 5\cos\frac{7}{6}\pi + i\,5\sin\frac{7}{6}\pi =$

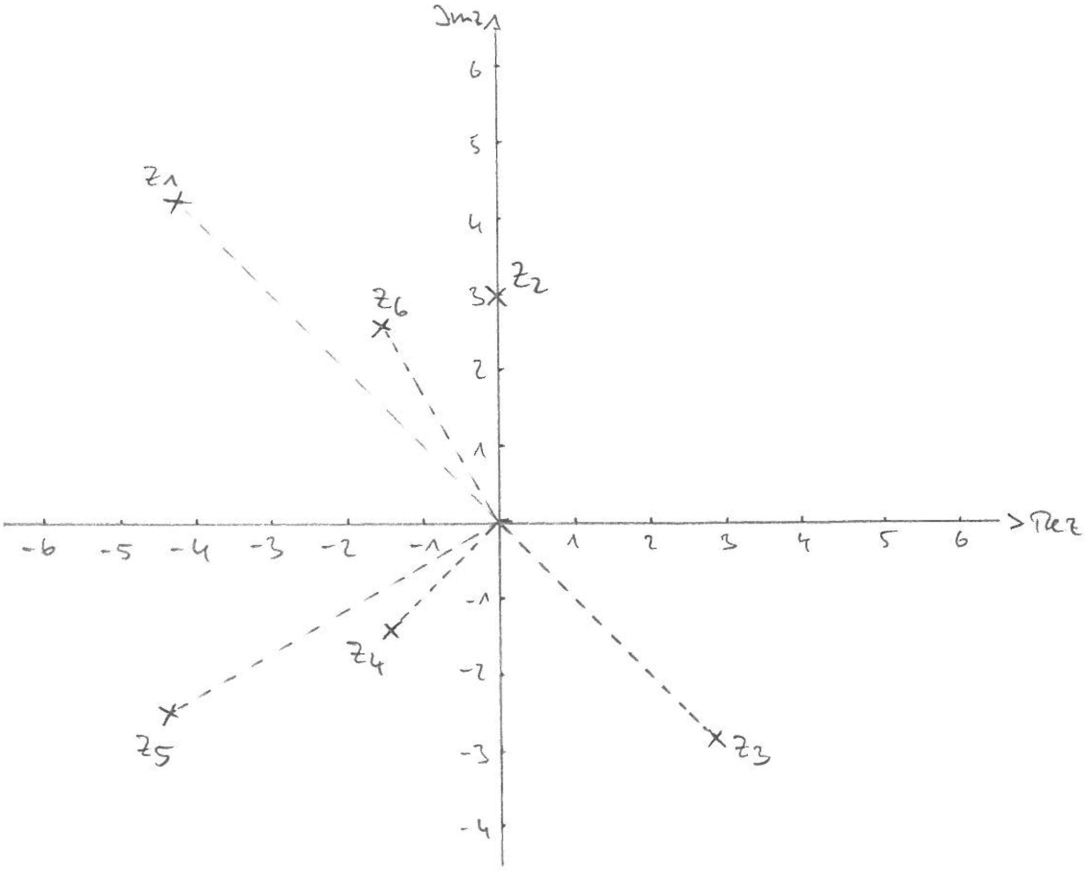
$= 5\left(-\frac{1}{2}\sqrt{3}\right) - i\,5\cdot\frac{1}{2} =$

$= \underline{\underline{-\frac{5}{2}\sqrt{3} - \frac{5}{2}i}}$

$\left(\frac{2\pi}{3} \mathrel{\hat=} 120°\right)$

f) $z_6 = 3\,e^{i\frac{2\pi}{3}} = 3\cdot\cos\frac{2}{3}\pi + i\,3\sin\frac{2}{3}\pi$

$= 3\cdot\left(-\frac{1}{2}\right) + i\cdot 3\cdot\frac{1}{2}\sqrt{3} = \underline{\underline{-\frac{3}{2} + \frac{3}{2}\sqrt{3}\,i}}$



A19 a) <u>Beh.:</u> $\sin(3\alpha) = 3\sin\alpha - 4\sin^3\alpha$

$$z = \cos(3\alpha) + i\sin(3\alpha) = (\cos\alpha + i\sin(\alpha))^3 =$$

$$= \cos^3\alpha + 3\cos^2\alpha \cdot i\sin\alpha + 3\cos\alpha \cdot (i\sin\alpha)^2 + (i\sin(\alpha))^3 =$$

$$= \cos^3\alpha + i\,3\cos^2\alpha \cdot \sin\alpha - 3\cos\alpha \cdot \sin^2\alpha - i\sin^3(\alpha)$$

$$= \cos^3\alpha - 3\cos\alpha \cdot \sin^2\alpha + i(3\cos^2\alpha \cdot \sin\alpha - \sin^3\alpha)$$

vgl. Im(...)

$$\Rightarrow \sin(3\alpha) = 3\cos^2\alpha \cdot \sin\alpha - \sin^3\alpha =$$

$$= 3(1-\sin^2\alpha) \cdot \sin\alpha - \sin^3\alpha =$$

$$= 3\sin\alpha - 3\sin^3\alpha - \sin^3\alpha =$$

$$= \underline{\underline{3\sin\alpha - 4\sin^3\alpha}}$$

b) <u>Beh.:</u> $\cos(4\alpha) = 8\sin^4\alpha - 8\sin^2\alpha + 1$

$$z = \cos(4\alpha) + i\sin(4\alpha) = (\cos\alpha + i\sin\alpha)^4 =$$

$$= \cos^4\alpha + 4\cos^3\alpha \cdot i\sin\alpha + 6\cos^2\alpha \cdot (i\sin\alpha)^2 + 4\cos\alpha \cdot (i\sin\alpha)^3 + (i\sin\alpha)^4$$

$$= \cos^4\alpha + i\,4\cos^3\alpha \cdot \sin\alpha - 6\cos^2\alpha\sin^2\alpha - i\,4\cos\alpha \cdot \sin^3\alpha + \sin^4\alpha$$

$$= \cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha + i(4\cos^3\alpha \cdot \sin\alpha - 4\cos\alpha \cdot \sin^3\alpha)$$

vgl. Re(...)

$$\Rightarrow$$

$$\cos(4\alpha) = \cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha$$

$$= (1-\sin^2\alpha)^2 - 6(1-\sin^2\alpha)\sin^2\alpha + \sin^4\alpha =$$

$$= 1 - 2\sin^2\alpha + \sin^4\alpha - 6\sin^2\alpha + 6\sin^4\alpha + \sin^4\alpha =$$

$$= \underline{\underline{8\sin^4\alpha - 8\sin^2\alpha + 1}}$$

A20 a) $w = 2\sqrt{3} - 2i$, $n = 2$

$r = |w| = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{12+4} = 4$

$\varphi = \mathrm{Arg}(w) = -\arccos\left(\frac{x}{r}\right) = -\arccos\left(\frac{2\sqrt{3}}{4}\right) =$

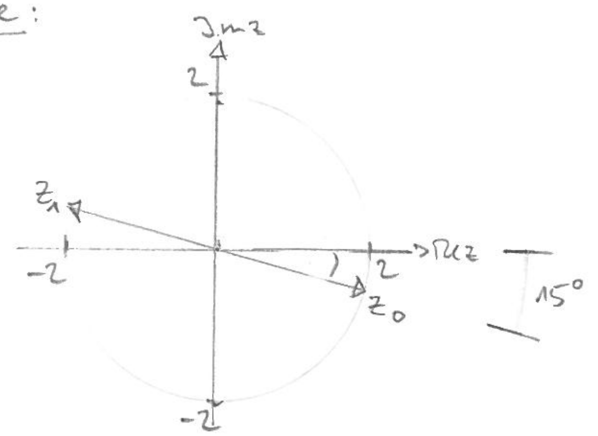
$= -\arccos\left(\frac{1}{2}\sqrt{3}\right) = -\frac{\pi}{6}$

$\Rightarrow w = 4 \cdot e^{-i\pi/6}$

$\Rightarrow$ Quadratwurzeln $z_0 = 2e^{-i\pi/12}$ $\qquad -\frac{\pi}{12} \mathrel{\hat{=}} -15^\circ$

$z_1 = 2 \cdot e^{-i\frac{\pi}{12}} \cdot e^{i\pi} = 2e^{i\frac{11}{12}\pi}$ $\qquad \frac{11}{12}\pi \mathrel{\hat{=}} 165^\circ$

Skizze:



b) $w = -4+4i$, $n = 5$

$r = |w| = \sqrt{4^2+4^2} = \sqrt{32} = 4\sqrt{2}$

$\varphi = \mathrm{Arg}(w) = \arccos\left(\frac{x}{r}\right) = \arccos\left(\frac{-4}{4\sqrt{2}}\right) = \arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{3}{4}\pi$

$\Rightarrow w = \sqrt{32} \cdot e^{i\frac{3}{4}\pi}$

5-te Wurzeln:

$\left(\frac{3\pi}{20} \mathrel{\hat{=}} 27^\circ\right)$ $z_0 = \sqrt{32}^{1/5} \cdot e^{i\frac{3}{20}\pi} = \sqrt{2}e^{i\frac{3}{20}\pi}$

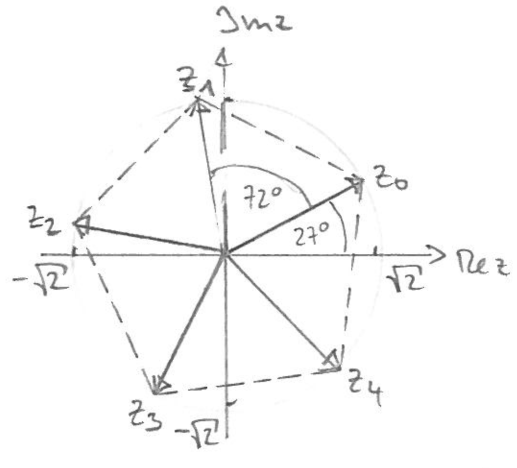
$\left(\frac{2\pi}{5} \mathrel{\hat{=}} 72^\circ\right)$ $z_1 = \sqrt{2}e^{i\left(\frac{3}{20}\pi + \frac{2}{5}\pi\right)} = \sqrt{2}e^{i\frac{11}{20}\pi}$

$z_2 = \sqrt{2}e^{i\frac{19}{20}\pi}$

$z_3 = \sqrt{2}e^{i\frac{27}{20}\pi}$

$z_4 = \sqrt{2}e^{i\frac{35}{20}\pi}$

Skizze:



A20 c) $w = 2+2\sqrt{3}i \quad , n=3$

$$r = |w| = \sqrt{2^2+(2\sqrt{3})^2} = \sqrt{4+12} = 4$$

$$\varphi = +\arccos\left(\frac{x}{r}\right) = \arccos\left(\frac{2}{4}\right) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\Rightarrow w = 4\cdot e^{i\frac{\pi}{3}}$$

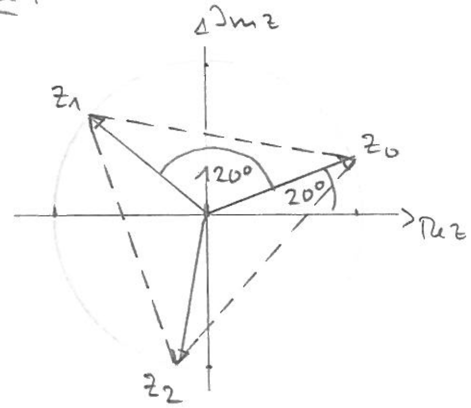
3-te Wurzeln: $\left(\frac{2\pi}{n} = \frac{2}{3}\pi \mathrel{\hat{=}} 120°\right)$

$\left(\frac{\pi}{9} \mathrel{\hat{=}} 20°\right)$ $z_0 = 4^{1/3} e^{i\frac{\pi}{9}} = \sqrt[3]{4}\, e^{i\pi/9}$

$z_1 = \sqrt[3]{4}\, e^{i\frac{\pi}{9}} \cdot e^{i\frac{2\pi}{3}} = \sqrt[3]{4}\, e^{i\frac{7\pi}{9}}$

$z_2 = \sqrt[3]{4}\, e^{i\frac{13}{9}\pi}$

Skizze:



d) $w = -16i \quad , n=4$

$$r = |w| = 16$$

$$\varphi = -\arccos\left(\frac{x}{r}\right) = -\arccos(0) = -\frac{\pi}{2}$$

$$\Rightarrow w = 16\cdot e^{-i\pi/2}$$

4-te Wurzeln: $\left(\frac{2\pi}{4} \mathrel{\hat{=}} 90°\right)$

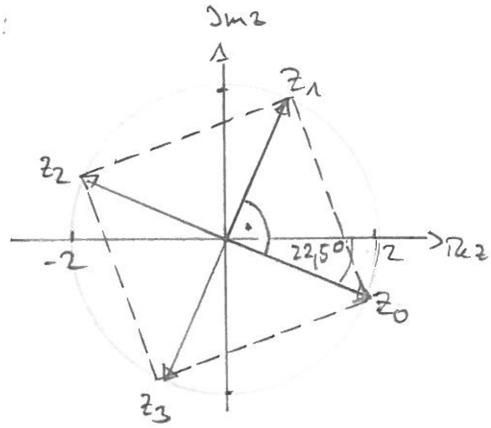
$\left(\frac{\pi}{8} \mathrel{\hat{=}} 22{,}5°\right)$ $z_0 = 16^{1/4}\cdot e^{-i\frac{\pi}{8}} = 2e^{-i\pi/8}$

$z_1 = 2\cdot e^{-i\pi/8}\, e^{i\pi/2} = 2e^{i\frac{3}{8}\pi}$

$z_2 = 2e^{i\frac{7}{8}\pi}$

$z_3 = 2e^{i\frac{11}{8}\pi}$

Skizze:



e) $w = 64, \quad n = 6$

$r = |w| = 64 \; ; \; \varphi = \mathrm{Arg}(w) = 0$

$\Rightarrow \; w = 64 \cdot e^{i0}$

6-te Wurzeln: $\quad \frac{2\pi}{6} = \frac{\pi}{3} \mathrel{\hat{=}} 60°$

$z_0 = 64^{1/6} \cdot e^{i\frac{0}{6}} = 2 \cdot e^0 = 2$

$z_1 = 2 \cdot e^{i\frac{\pi}{3}}$

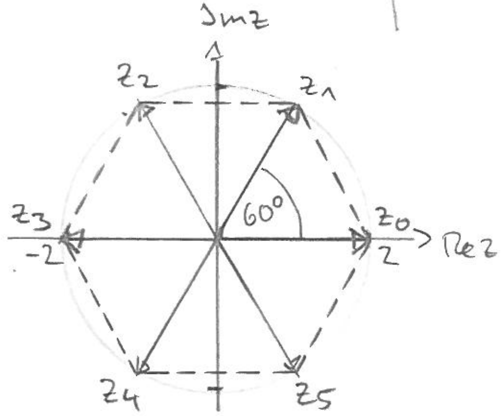
$z_2 = 2 \cdot e^{i\frac{2\pi}{3}}$

$z_3 = 2 \cdot e^{i\pi}$

$z_4 = 2 \cdot e^{i\frac{4\pi}{3}}$

$z_5 = 2 \cdot e^{i\frac{5\pi}{3}}$

Skizze:



f) $w = -1, \quad n = 3$

$r = |w| = 1$

$\varphi = \mathrm{Arg}(w) = +\arccos\left(\frac{x}{r}\right) = \arccos\left(\frac{-1}{1}\right) = \pi$

$\Rightarrow \; w = 1 \cdot e^{i\pi}$

3-te Wurzeln: $\quad \left(\frac{2\pi}{3} \mathrel{\hat{=}} 120°\right)$

$\left(\frac{\pi}{3} \mathrel{\hat{=}} 60°\right) \quad z_0 = 1^{1/3} e^{i\pi/3} = e^{i\pi/3} \quad \left(\frac{\pi}{3} \mathrel{\hat{=}} 60°\right)$

$z_1 = e^{i\frac{\pi}{3}} e^{i\frac{2\pi}{3}} = e^{i\pi}$

$z_2 = e^{i\frac{5}{3}\pi}$

Skizze: