Mathematik 1 - AI

Blatt 6

A51 a) i) $a_n = 1 - \frac{1}{n}$

$a_1 = 1 - \frac{1}{1} = 0$, $a_2 = 1 - \frac{1}{2} = \frac{1}{2}$, $a_3 = 1 - \frac{1}{3} = \frac{2}{3}$, $a_4 = 1 - \frac{1}{4} = \frac{3}{4}$

$a_5 = 1 - \frac{1}{5} = \frac{4}{5}$

ii) $a_n = n^2 + (-1)^n \cdot n$

$a_1 = 1^2 + (-1)^1 \cdot 1 = 1 - 1 = 0$, $a_2 = 2^2 + (-1)^2 \cdot 2 = 4 + 2 = 6$, $a_3 = 3^2 + (-1)^3 \cdot 3 = 9 - 3 = 6$

$a_4 = 4^2 + (-1)^4 \cdot 4 = 16 + 4 = 20$, $a_5 = 5^2 + (-1)^5 \cdot 5 = 25 - 5 = 20$

iii) $a_n = \frac{n}{n+1} - \frac{n+1}{n}$

$a_1 = \frac{1}{2} - \frac{2}{1} = -\frac{3}{2}$, $a_2 = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}$, $a_3 = \frac{3}{4} - \frac{4}{3} = -\frac{7}{12}$

$a_4 = \frac{4}{5} - \frac{5}{4} = -\frac{9}{20}$, $a_5 = \frac{5}{6} - \frac{6}{5} = -\frac{11}{30}$

iv) $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ mit $a_1 = 1$

$a_2 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2}$, $a_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{17}{12}$

$a_4 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \frac{1}{2}\left(\frac{17}{12} + \frac{24}{17}\right) = \frac{577}{408}$, $a_5 = \frac{1}{2}\left(\frac{577}{408} + \frac{2}{577/408}\right) = \frac{1}{2}\left(\frac{577}{408} + \frac{816}{577}\right) \approx 1{,}4142\ldots$

v) $a_{n+1} = a_n + a_{n-1}$ mit $a_0 = 0$ und $a_1 = 1$

$a_2 = 1 + 0 = 1$, $a_3 = 1 + 1 = 2$, $a_4 = 3$

b) i) $a_n = \frac{2n-1}{3n+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n-1}{3n+1} = \lim_{n\to\infty} \frac{n(2-\frac{1}{n})}{n(3+\frac{1}{n})} = \lim_{n\to\infty} \frac{2 - 1/n}{3 + 1/n} = \underline{\underline{\frac{2}{3}}}$$

ii) $a_n = \left(\frac{2n+1}{n}\right)^{10}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{2n+1}{n}\right)^{10} = \lim_{n\to\infty} \left(2 + \frac{1}{n}\right)^{10} = \left(\lim_{n\to\infty} 2 + \frac{1}{n}\right)^{10} = 2^{10} = \underline{\underline{1024}}$$

wg. Stetigkeit der Potenzfkt.

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zu b)

iii) $a_n = (-1)^n \frac{n}{1-n^2}$

$$a_n = (-1)^n \frac{n}{1-n^2} \underset{\uparrow}{=} (-1)^n \frac{n \cdot \frac{1}{n^2}}{(1-n^2)\cdot\frac{1}{n^2}} = (-1)^n \frac{\overbrace{\frac{1}{n}}^{\to 0}}{\underbrace{\frac{1}{n^2}}_{\to 0}-1}$$

Zähler und Nenner durch die höchste im Bruch auftretende Potenz in n teilen

$$\text{Wg.} \quad -\frac{\frac{1}{n}}{\frac{1}{n^2}-1} \leq a_n \leq \frac{\frac{1}{n}}{\frac{1}{n^2}-1} \quad \text{und}$$

$$\lim_{n\to\infty} -\frac{\frac{1}{n}}{\frac{1}{n^2}-1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n^2}-1} = 0 \quad \text{folgt aus dem}$$

Sandwich-Lemma: $\lim\limits_{n\to\infty} a_n = 0$

iv) $a_n = \frac{-\frac{1}{3}n^3+4n^2-1}{\frac{2}{9}n^3-n^2+1}$

$$a_n = \frac{-\frac{1}{3}n^3+4n^2-1}{\frac{2}{9}n^3-n^2+1} \underset{\uparrow}{=} \frac{(-\frac{1}{3}n^3+4n^2-1)\cdot\frac{1}{n^3}}{(\frac{2}{9}n^3-n^2+1)\cdot\frac{1}{n^3}} =$$

Zähler und Nenner durch n^3 teilen

$$= \frac{-\frac{1}{3}+4\frac{1}{n}-\frac{1}{n^3}}{\frac{2}{9}-\frac{1}{n}+\frac{1}{n^3}} \xrightarrow{(n\to\infty)} \frac{-\frac{1}{3}}{\frac{2}{9}} = -\frac{1}{3}\cdot\frac{9}{2} = \underline{\underline{\frac{3}{2}}}$$

v) $a_n = \sqrt{n+1} - \sqrt{n}$

$$a_n = \underbrace{\sqrt{n+1}}_{\downarrow\,\infty} - \underbrace{\sqrt{n}}_{\downarrow\,\infty} \underset{\uparrow}{\overset{\text{Trick}}{=}} \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})} =$$

mit $(\sqrt{n+1}+\sqrt{n})$ erweitern

$$\underset{\uparrow}{=} \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\underbrace{\sqrt{n+1}}_{\downarrow\,\infty}+\underbrace{\sqrt{n}}_{\downarrow\,\infty}} \xrightarrow{n\to\infty} 0$$

3. bin. Formel

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zu b) vi) $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ mit $a_1 = 1$.

$$\begin{array}{ccccc} a_{n+1} & = & \frac{1}{2}\big(a_n & + & \frac{2}{a_n}\big) \\ \downarrow & & \downarrow & & \downarrow \text{ für } n \to \infty \\ a & & a & & \frac{2}{a} \end{array}$$

Somit folgt für den Grenzwert $\lim\limits_{n\to\infty} a_n = a$: $a = \frac{1}{2}\left(a + \frac{2}{a}\right)$.

$$a = \frac{1}{2}\left(a + \frac{2}{a}\right) \quad | \cdot 2$$

$$\Leftrightarrow \quad 2a = a + \frac{2}{a} \quad | - a$$

$$\Leftrightarrow \quad a = \frac{2}{a} \quad | \cdot a$$

$$\Leftrightarrow \quad a^2 = 2$$

$$\Leftrightarrow \quad a = \pm\sqrt{2}$$

Da $a_n > 0$ für alle $n \in \mathbb{N}$ gilt, kommt nur $a = +\sqrt{2}$ in Frage,

d.h. $\lim\limits_{n\to\infty} a_n = \sqrt{2}$

A52 a) i) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$

Partialsummen: $S_0 = \sum_{n=0}^{0} \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^0 = \underline{\underline{1}}$, $S_1 = \sum_{n=0}^{1} \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 = 1 + \frac{1}{2} = \underline{\underline{\frac{3}{2}}}$

$$S_2 = \sum_{n=0}^{2} \left(\frac{1}{2}\right)^n = \underbrace{\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1}_{S_1} + \left(\frac{1}{2}\right)^2 = S_1 + \frac{1}{4} = \frac{3}{2} + \frac{1}{4} = \underline{\underline{\frac{7}{4}}}$$

$$S_3 = \sum_{n=0}^{3} \left(\frac{1}{2}\right)^n = \underbrace{\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2}_{S_2} + \left(\frac{1}{2}\right)^3 = S_2 + \left(\frac{1}{2}\right)^3 = \frac{7}{4} + \frac{1}{8} = \underline{\underline{\frac{15}{8}}}$$

$$S_4 = \sum_{n=0}^{4} \left(\frac{1}{2}\right)^n = \underbrace{\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3}_{S_3} + \left(\frac{1}{2}\right)^4 = \frac{15}{8} + \frac{1}{16} = \frac{31}{16}$$

Bei den weiteren Aufgaben verwenden wir die Beziehung $\boxed{S_n = S_{n-1} + a_n}$ $(n \geq 1)$

ii) $\sum_{n=0}^{\infty} \frac{1}{n!}$

$$S_0 = \sum_{n=0}^{0} \frac{1}{n!} = \frac{1}{0!} = \frac{1}{1} = \underline{\underline{1}} \; ; \quad S_1 = S_0 + \frac{1}{1!} = 1 + 1 = \underline{\underline{2}}$$

$$S_2 = S_1 + a_2 = 2 + \frac{1}{2!} = 2 + \frac{1}{2} = \underline{\underline{\frac{5}{2}}} \; ; \quad S_3 = S_2 + a_3 = \frac{5}{2} + \frac{1}{3!} = \frac{5}{2} + \frac{1}{6} = \underline{\underline{\frac{8}{3}}}$$

$$S_4 = S_3 + a_4 = \frac{8}{3} + \frac{1}{4!} = \frac{8}{3} + \frac{1}{24} = \frac{65}{24}$$

iii) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

$$S_1 = \sum_{n=1}^{1} (-1)^{n-1} \frac{1}{n} = (-1)^0 \cdot 1 = \underline{\underline{1}} \; ; \quad S_2 = S_1 + a_2 = 1 + (-1)^2 \cdot \frac{1}{2} = 1 + \frac{1}{2} = \underline{\underline{\frac{3}{2}}}$$

$$S_3 = S_2 + a_3 = \frac{1}{2} + (-1)^3 \cdot \frac{1}{3} = \frac{3}{2} - \frac{1}{3} = \underline{\underline{\frac{7}{6}}}$$

$$S_4 = S_3 + a_4 = \frac{7}{6} + (-1)^4 \cdot \frac{1}{4} = \underline{\underline{\frac{17}{12}}}$$

$$S_5 = S_4 + a_5 = \frac{17}{12} + (-1)^5 \cdot \frac{1}{5} = \frac{17}{12} - \frac{1}{5} = \underline{\underline{\frac{73}{60}}}$$

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iv) $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$

$S_0 = \sum_{n=0}^{0} \frac{n^2}{2^n} = \frac{0^2}{2^0} = \underline{\underline{0}}$; $S_1 = S_0 + a_1 = 0 + \frac{1^2}{2^1} = \underline{\underline{\frac{1}{2}}}$

$S_2 = S_1 + a_2 = \frac{1}{2} + \frac{2^2}{2^2} = \frac{1}{2} + 1 = \underline{\underline{\frac{3}{2}}}$; $S_3 = S_1 + a_3 = \frac{3}{2} + \frac{3^2}{2^3} = \frac{3}{2} + \frac{9}{8} = \underline{\underline{\frac{21}{8}}}$

$S_4 = S_3 + a_4 = \frac{21}{8} + \frac{4^2}{2^4} = \frac{21}{8} + \frac{16}{16} = \underline{\underline{\frac{29}{8}}}$

v) $\sum_{n=1}^{\infty} \frac{n!}{(2n-1)!}$

$S_1 = \sum_{n=1}^{1} \frac{n!}{(2n-1)!} = \frac{1!}{(2-1)!} = \underline{\underline{1}}$; $S_2 = S_1 + a_2 = 1 + \frac{2!}{3!} = 1 + \frac{1}{3} = \underline{\underline{\frac{4}{3}}}$

$S_3 = S_2 + a_3 = \frac{4}{3} + \frac{3!}{5!} = \frac{4}{3} + \frac{\cancel{1 \cdot 2 \cdot 3}}{\cancel{1 \cdot 2 \cdot 3} \cdot 4 \cdot 5} = \frac{4}{3} + \frac{1}{20} = \underline{\underline{\frac{83}{60}}}$

$S_4 = S_3 + a_4 = \frac{83}{60} + \frac{4!}{7!} = \frac{83}{60} + \frac{1}{210} = \underline{\underline{\frac{583}{420}}}$;

$S_5 = S_4 + a_5 = \frac{583}{420} + \frac{5!}{9!} = \frac{583}{420} + \frac{1}{3024} = \underline{\underline{\frac{2999}{2160}}}$.

vi) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n!}$

$S_1 = \sum_{n=1}^{1} \frac{2^{n+1}}{n!} = \frac{2^2}{1!} = \underline{\underline{4}}$; $S_2 = S_1 + a_2 = 4 + \frac{2^3}{2!} = 4 + 4 = \underline{\underline{8}}$

$S_3 = S_2 + a_3 = 8 + \frac{2^4}{3!} = 8 + \frac{8}{3} = \underline{\underline{\frac{32}{3}}}$

$S_4 = S_3 + a_4 = \frac{32}{3} + \frac{2^5}{4!} = \frac{32}{3} + \frac{4}{3} = \frac{36}{3} = \underline{\underline{12}}$

$S_5 = S_4 + a_5 = 12 + \frac{2^6}{5!} = 12 + \frac{8}{15} = \underline{\underline{\frac{188}{15}}}$

A52 b) i) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$

Quotientenkrit: $a_n = \left(\frac{1}{2}\right)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)^n} = \frac{1}{2} \xrightarrow{n\to\infty} \frac{1}{2} < 1 \quad \Rightarrow \text{ Reihe ist konvergent}$$

ii) $\sum_{n=0}^{\infty} \frac{1}{n!}$

Quotientenkrit: $a_n = \frac{1}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1/(n+1)!}{1/n!}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \longrightarrow 0 \text{ für } n\to\infty$$

$\Rightarrow$ Reihe ist konvergent

iii) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

Leibnizkrit: $\sum_{k=0}^{\infty} (-1)^k a_k$ ist konvergent, falls $a_0 \geq a_1 \geq a_2 \geq \dots$

mit $\lim_{k\to\infty} a_k = 0$

Nun ist $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ eine monoton fallende Folge mit Grenzwert 0

$\Rightarrow$ Reihe ist konvergent nach Leibnizkrit.

iv) $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$

Quotientenkrit: $a_n = \frac{n^2}{2^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2/2^{n+1}}{n^2/2^n}\right| = \left|\frac{(n+1)^2 \cdot 2^n}{n^2 \cdot 2^{n+1}}\right| = \left(1+\frac{1}{n}\right) \cdot \frac{1}{2} \longrightarrow 0 \text{ für } n\to\infty$$

$\Rightarrow$ Reihe ist konvergent

v) $\sum_{n=1}^{\infty} \frac{n!}{(2n-1)!}$

Quotientenkrit: $a_n = \frac{n!}{(2n-1)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!/(2n+1)!}{n!/(2n-1)!}\right| = \left|\frac{(n+1)!\,(2n-1)!}{n!\,(2n+1)!}\right| =$$

$$= \frac{n}{2n(2n+1)} = \frac{1}{2(2n+1)} \longrightarrow 0 \text{ für } n\to\infty$$

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zu b), vi) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n!}$

Quotientenkrit.: $a_n = \frac{2^{n+1}}{n!}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{2^{n+2}/(n+1)!}{2^{n+1}/n!} \right| = \frac{2^{n+2} \; n!}{(n+1)! \; 2^{n+1}} = \frac{2}{n+1} \longrightarrow 0 \text{ für } n \to \infty$$

=> Reihe ist konvergent nach Quotientenkriterium

A53 a) $\lim\limits_{x\to 2} \dfrac{(x-2)(3x+1)}{4x-8}$

$$\lim_{x\to 2} \frac{\overbrace{(x-2)(3x+1)}^{\to 0}}{\underbrace{4x-8}_{\to 0}} = \lim_{x\to 2} \frac{3x^2-5x-2}{4x-8} \overset{\text{l'H.}}{=} \lim_{x\to 2} \frac{6x-5}{4} = \frac{7}{4};$$

b) $\lim\limits_{x\to 0^-} \arctan \frac{1}{x}$

Wg. $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$ gilt $\lim\limits_{x\to 0^-} \arctan\frac{1}{x} = \lim\limits_{u\to -\infty} \arctan u = -\frac{\pi}{2}$

c) $\lim\limits_{x\to 1} \dfrac{1-x}{1-\sqrt{x}}$

$$\lim_{x\to 1} \frac{\overbrace{1-x}^{\to 0}}{\underbrace{1-\sqrt{x}}_{\to 0}} \overset{\text{l'H}}{=} \lim_{x\to 1} \frac{-1}{-\frac{1}{2\sqrt{x}}} = \lim_{x\to 1} 2\sqrt{x} = 2\sqrt{1} = 2$$

alternativ (ohne Verwendung der Regel von l'Hospital)

$$\lim_{x\to 1} \frac{1-x}{1-\sqrt{x}} = \lim_{x\to 1} \frac{(1-\sqrt{x})(1+\sqrt{x})}{1-\sqrt{x}} = \lim_{x\to 1} 1+\sqrt{x} = 1+\sqrt{1} = 2$$

d) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin x}$

$$\lim_{x\to 0} \frac{\overbrace{\sin 2x}^{\to 0}}{\underbrace{\sin x}_{\to 0}} \overset{\text{l'H}}{=} \lim_{x\to 0} \frac{2\cos 2x}{\cos x} = \frac{2\cdot 1}{1} = 2$$

e) $\lim\limits_{x\to\infty} \sqrt{x}\cdot e^{-x}$

$$\lim_{x\to\infty} \underbrace{\sqrt{x}}_{\to\infty} \cdot \underbrace{e^{-x}}_{\to 0} = \lim_{x\to\infty} \frac{\overbrace{\sqrt{x}}^{\to\infty}}{\underbrace{e^{x}}_{\to\infty}} \overset{\text{l'H.}}{=} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^{x}} = \lim_{x\to\infty} \frac{1}{\underbrace{2\sqrt{x}e^{x}}_{\to\infty}} = 0$$

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f) $\lim\limits_{x \to 0} \frac{1-\cos x}{\sin^2 x}$

$$\lim_{x \to 0} \frac{\overbrace{1-\cos x}^{\to 0}}{\underbrace{\sin^2 x}_{\to 0}} \overset{l'H.}{=} \lim_{x \to 0} \frac{\sin x}{2\sin \cos x} = \lim_{x \to 0} \frac{1}{2\cos x} = \underline{\underline{\frac{1}{2}}};$$

g) $\lim\limits_{x \to \frac{\pi}{2}} (1-\sin x) \tan x$

$$\lim_{x \to \frac{\pi}{2}} \underbrace{(1-\sin x)}_{\downarrow\, 0} \cdot \underbrace{\tan x}_{\downarrow\, \infty} = \lim_{x \to \frac{\pi}{2}} (1-\sin x) \cdot \frac{\sin x}{\cos x} =$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\overbrace{(1-\sin x) \cdot \sin x}^{\to 0}}{\underbrace{\cos x}_{\downarrow\, 0}} = \lim_{x \to \frac{\pi}{2}} \frac{\sin x - \sin^2 x}{\cos x} \overset{l'H.}{=}$$

$$\overset{l'H}{=} \lim_{x \to \frac{\pi}{2}} \frac{\cos x - 2\sin x \cos x}{\sin x} = \frac{0}{1} = \underline{\underline{0}}$$

h) $\lim\limits_{x \to 1} \frac{1}{x-1} - \frac{1}{\ln x}$

$$\lim_{x \to 1} \underbrace{\frac{1}{x-1}}_{\infty} - \underbrace{\frac{1}{\ln x}}_{\infty}$$

Anwendung der Transformation:

$$f(x) - g(x) = \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)\, g(x)}}$$

=>

$$\lim_{x \to 1} \frac{1}{x-1} - \frac{1}{\ln x} = \lim_{x \to 1} \frac{\overbrace{\ln x - (x-1)}^{\to 0}}{\underbrace{(x-1) \cdot \ln x}_{\to 0}} =$$

$$= \lim_{x \to 1} \frac{\ln x - x + 1}{(x-1)\ln x} \overset{l'H.}{=} \lim_{x \to 1} \frac{\frac{1}{x} - 1}{\ln x + (x-1) \cdot \frac{1}{x}} = \lim_{x \to 1} \frac{\overbrace{\frac{1}{x} - 1}^{\to 0}}{\underbrace{\ln x + 1 - \frac{1}{x}}_{\to 0}} =$$

$$\overset{l'H}{=} \lim_{x \to 1} \frac{-\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{-1}{2} = \underline{\underline{-\frac{1}{2}}};$$

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k) $\lim\limits_{x \to 0} \dfrac{\sin 2x - 2\sin x}{2e^x - x^2 - 2x - 2}$

$$\lim_{x \to 0} \frac{\overbrace{\sin 2x - 2\sin x}^{\to 0}}{\underbrace{2e^x - x^2 - 2x - 2}_{\to 0}} \overset{l'H}{=} \lim_{x \to 0} \frac{\overbrace{2\cos 2x - 2\cos x}^{\to 0}}{\underbrace{2e^x - 2x - 2}_{\to 0}} \overset{l'H}{=}$$

$$\overset{l'H}{=} \lim_{x \to 0} \frac{\overbrace{-4\sin 2x + 2\sin x}^{\to 0}}{\underbrace{2e^x - 2}_{\to 0}} \overset{l'H}{=} \lim_{x \to 0} \frac{-8\cos 2x + 2\cos x}{2e^x} = \frac{-8+2}{2} = \underline{\underline{-3}};$$

A54 a) $y = e^{-2x}\cos x$

$y' = -2e^{-2x}\cos x + e^{-2x}(-\sin x) = -e^{-2x}(2\cos x + \sin x);$

b) $y = e^{x}\sin x$

$y' = e^{x}\cdot\sin x + e^{x}\cos x = e^{x}(\sin x + \cos x)$

c) $y = (x^2-1)^2(x+5)^3$

$y' = 2(x^2-1)\cdot 2x(x+5)^3 + (x^2-1)^2\cdot 3(x+5) =$

$= (x^2-1)\cdot(x+5)\left[4x(x+5)^2 + 3(x^2-1)\right] =$

$= (x^2-1)(x+5)\left[4x(x^2+10x+25)+3x^2-3\right] =$

$= (x^2-1)(x+5)(4x^3+43x^2+100x-3)$

d) $y = (4x-1)^2\sin 2x$

$y' = 2(4x-1)\cdot 4\cdot\sin 2x + (4x-1)^2\cdot 2\cos 2x$

$= 2(4x-1)\cdot\left[4\sin 2x + (4x-1)\cdot\cos 2x\right]$

e) $y = x\ln(x+e^x)^2$

$y' = \ln(x+e^x)^2 + x\cdot\dfrac{1}{(x+e^x)^2}\cdot 2(x+e^x)\cdot(1+e^x)$

$= \ln(x+e^x)^2 + \dfrac{2x(1+e^x)}{x+e^x}$

alternativ:

$y = x\cdot\ln(x+e^x)^2 = 2x\ln(x+e^x)$

$y' = 2\ln(x+e^x) + 2x\cdot\dfrac{1}{x+e^x}\cdot(1+e^x) = 2\ln(x+e^x) + \dfrac{2x(1+e^x)}{x+e^x}$

f) $y = x^x$

$$y = x^x = e^{\ln(x^x)} = e^{x \cdot \ln x}$$

$$y' = e^{x \cdot \ln x} \cdot (x \cdot \ln x)' = e^{x \cdot \ln x} \cdot \left(\ln x + x \cdot \frac{1}{x} \right) =$$

$$= e^{x \ln x} (1 + \ln x) = x^x (1 + \ln x)!$$

g) $y = \ln(\tanh x)$

$$(\tanh x)' = \left(\frac{\sinh x}{\cosh x} \right)' = \frac{\cosh^2 - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$\Rightarrow$

$$y' = \frac{1}{\tanh x} \cdot (\tanh x)' = \frac{\cosh x}{\sinh x} \cdot \frac{1}{\cosh^2 x} = \frac{1}{\sinh x \cdot \cosh x}$$

h) $y = \left(\frac{x+1}{x} \right)^n$

$$y = \left(\frac{x+1}{x} \right)^n = \left(1 + \frac{1}{x} \right)^n$$

$$\Rightarrow y' = n \left(1 + \frac{1}{x} \right)^{n-1} \cdot \left(1 + \frac{1}{x} \right)' = n \left(1 + \frac{1}{x} \right)^{n-1} \cdot \left(-\frac{1}{x^2} \right) = -\frac{n}{x^2} \left(1 + \frac{1}{x} \right)^{n-1}$$

i) $y = \sin(x^2+1) \cos(4x)$

$$y' = \cos(x^2+1) \cdot 2x \cdot \cos(4x) + \sin(x^2+1) \cdot (-4) \sin(4x)$$

$$= 2x \cos(x^2+1) \cos(4x) - 4 \sin(x^2+1) \sin(4x)$$

k) $y = 2x \cdot \sqrt{x^2-1}$

$$y' = 2\sqrt{x^2-1} + 2x \cdot \frac{1}{2\sqrt{x^2-1}} \cdot 2x = 2\sqrt{x^2-1} + \frac{2x^2}{\sqrt{x^2-1}}$$

l) $y = \sqrt[3]{(x^2-4x+10)^2} = (x^2-4x+10)^{2/3}$

$$y' = \frac{2}{3} (x^2-4x+10)^{-1/3} \cdot (2x-4) = \frac{4}{3} \cdot \frac{x-2}{\sqrt[3]{x^2-4x+10}}$$

A 54 Fortsetzung

m) $y = \ln|\cos x|$

Beachte, dass $(\ln|x|)' = \frac{1}{x} \quad (x \neq 0)$

$\uparrow\uparrow$
Betrag

$$y' = \frac{1}{\cos x} \cdot (-\sin x) = -\frac{\sin x}{\cos x} = -\tan x$$

n) $y = \arccos\sqrt{x^2-1}$

$$(\arccos x)' = \frac{1}{-\sin(\arccos x)} \qquad (-1 \le x \le 1) \qquad \left| \; (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))} \right.$$

nach dem trig. Pythagoras gilt $\sin^2\alpha = 1-\cos^2\alpha$

und somit $\sin\alpha = \pm\sqrt{1-\cos^2\alpha}$

Da $0 \le \underbrace{\arccos x}_{\alpha} \le \pi$ gilt und $\sin\alpha \ge 0$ für $0 \le \alpha \le \pi$,

folgt

$$\sin(\arccos x) = +\sqrt{1-\cos^2(\arccos x)} = \sqrt{1-x^2}$$

$$\Rightarrow (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

Damit erhält man

$$y' = -\frac{1}{\sqrt{1-(x^2-1)}} \cdot \frac{1}{2\sqrt{x^2-1}} \cdot 2x =$$

$$= -\frac{x}{\sqrt{2-x^2}\,\sqrt{x^2-1}}$$

A54. Fortsetzung

o) $$y = \frac{1+\cos x}{1-\sin x}$$

$$y' = \frac{-\sin x\,(1-\sin x) - (1+\cos x)(-\cos x)}{(1-\sin x)^2} =$$

$$= \frac{-\sin x + \sin^2 x + \cos x + \cos^2 x}{(1-\sin x)^2} =$$

$$= \frac{\overbrace{\cos^2 x + \sin^2 x}^{1} + \cos x - \sin x}{(1-\sin x)^2} =$$

$$= \frac{1+\cos x - \sin x}{(1-\sin x)^2}$$

p) $$y = \frac{\sqrt{x} - x^2}{x^2+1}$$

$$y' = \frac{\left(\frac{1}{2\sqrt{x}} - 2x\right)\cdot(x^2+1) - (\sqrt{x}-x^2)\cdot 2x}{(x^2+1)^2} =$$

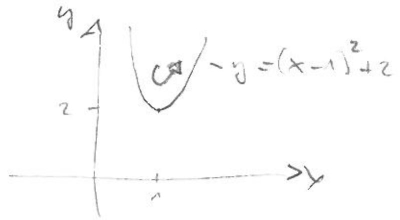
$$= \frac{\dfrac{(1-4x\sqrt{x})\cdot(x^2+1) - (\sqrt{x}-x^2)\cdot 4x\sqrt{x}}{2\sqrt{x}}}{(x^2+1)^2} =$$

$$= \frac{(1-4x\sqrt{x})(x^2+1) - (\sqrt{x}-x^2)4x\sqrt{x}}{2\sqrt{x}\,(x^2+1)^2} =$$

$$= \frac{x^2 - 4x^3\sqrt{x} + 1 - 4x\sqrt{x} - 4x^2 + 4x^3\sqrt{x}}{2\sqrt{x}\,(x^2+1)^2} =$$

$$= \frac{1-3x^2-4x\sqrt{x}}{2\sqrt{x}\,(x^2+1)^2}$$

A55 Krümmung $\kappa = \dfrac{y''(x_0)}{\left(1+y'(x_0)^2\right)^{3/2}}$

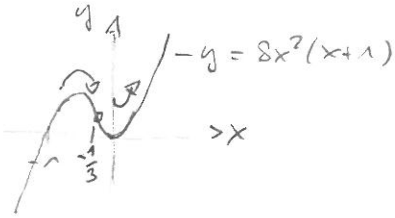


a) $y = (x-1)^2 + 2$

$y' = 2(x-1) = 2x-2$

$y'' = 2 > 0 \Rightarrow y = f(x)$ ist auf ganz $\mathbb{R}$ linksgekrümmt

$P(1;2)$: $\kappa = \dfrac{y''(x_0)}{\left(1+y'(x_0)^2\right)^{3/2}} = \dfrac{2}{(1+0^2)^{3/2}} = \underline{\underline{2}}$;



b) $y = 8x^2(x+1) = 8x^3 + 8x^2$

$y' = 24x^2 + 16x = 8(3x^2+2x)$

$y'' = 48x + 16$

$y''(x) = 0 \Leftrightarrow x = -\frac{1}{3}$ $\quad y''(x) \begin{cases} <0, \text{ falls } x < -\frac{1}{3} & \text{rechtsgekrümmt} \\ >0, \text{ falls } x > -\frac{1}{3} & \text{linksgekrümmt} \end{cases}$

$P(-\frac{1}{2};1)$: $\kappa = \dfrac{y''(x_0)}{\left(1+y'(x_0)^2\right)^{3/2}} = \dfrac{-8}{(1+4)^{3/2}} = -\dfrac{8}{5^{3/2}} = \underline{\underline{-\left(\frac{4}{5}\right)^{3/2}}}$

$y''(-\frac{1}{2}) = -8$ $\qquad = -0{,}8^{3/2} \approx \underline{\underline{-0{,}862}}$

$y'(-\frac{1}{2}) = -2$

c) $y = e^{-x^2}$

$y' = -2x e^{-x^2}$

$y'' = -2e^{-x^2} - 2x(-2x)e^{-x^2} = -2e^{-x} + (4x^2)e^{-x} = 2e^{-x}(2x^2-1)$

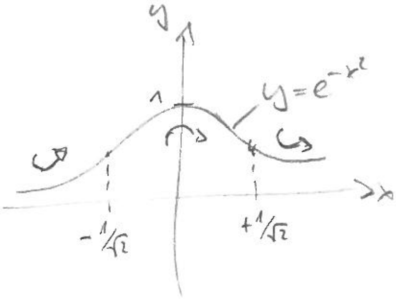
$y''(x) = 0 \Leftrightarrow 2x^2 - 1 = 0 \Leftrightarrow x_{1,2} = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{2}\sqrt{2}$

	$x < -\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}} < x < \sqrt{\frac{1}{2}}$	$x > \sqrt{\frac{1}{2}}$
$y''(x)$	+	−	+
Krümmungsverhalten	linksgekrümmt	rechtsgekrümmt	linksgekrümmt

$y'(0) = 0$

$y''(0) = 2e^{-0}(2\cdot 0^2 - 1) = -2$

$P(0;1)$: $\kappa = \dfrac{y''(x_0)}{\left(1+y'(x_0)^2\right)^{3/2}} = \dfrac{-2}{(1+0)^{3/2}} = \underline{\underline{-2}}$



A56 a) $y = \dfrac{x^2-1}{(x-1)^3} = \dfrac{Z(x)}{N(x)}$

$D_{max} = \mathbb{R} \setminus \{\underbrace{\text{Nst. des Nenners}}_{\text{Def. lücken}}\} = \mathbb{R} \setminus \{+1\}$

Zähler- und Nennerpolynom faktorisieren:

$y = \dfrac{x^2-1}{(x-1)^3} = \dfrac{(x-1)(x+1)}{(x-1)^3} = \dfrac{x+1}{(x-1)^2}$

Nst.: Nst. von $y(x)$ sind die Nst. des Zählers $Z(x)$, die nicht gleichzeitig Nst. des Nenners $N(x)$ sind.

$\Rightarrow$ Nst. von $y(x)$: $x_1 = -1$

Pole: Pol 2. Ordnung (ohne VZW) bei $x = -1$ (VZW = Vorzeichenwechsel)

keine behebbaren Def. lücken, da einzige Def. lücke ein Pol ist.

Asymptoten: senkrechte Asymptote (Polasymptote) $x = 1$

$$\lim_{x \to \pm\infty} \frac{x^2-1}{(x-1)^3} = \lim_{x \to \pm\infty} \frac{x+1}{(x-1)^2} \overset{l'H.}{=} \lim_{x \to \pm\infty} \frac{1}{2(x-1)} = 0$$

$\Rightarrow$ waagrechte Asymptote: $y = 0$

b) $y = \dfrac{x^3-6x^2+12x-8}{x^2-4} = \dfrac{Z(x)}{N(x)}$

$N(x) = x^2-4 = (x-2)(x+2) \Rightarrow$ Def. lücken $x = \pm 2$

$\Rightarrow D_{max} = \mathbb{R} \setminus \{\pm 2\}$

$Z(x)$ faktorisieren: Durch Probieren findet man $Z(2) = 0$.

$\Rightarrow$ Polynomdivision

$$\begin{array}{l}
(x^3-6x^2+12x-8):(x-2) = x^2-4x+4 \\
\underline{-(x^3-2x^2)} \\
\quad -4x^2+12x \\
\quad \underline{-(-4x^2+8x)} \\
\qquad\quad 4x-8 \\
\qquad\quad \underline{4x-8} \\
\qquad\quad\; -\,-
\end{array}$$

A56 b) $\Rightarrow Z(x) = (x-2)\underbrace{(x^2-4x+4)}_{(x-2)^2} = (x-2)^3$

$$\Rightarrow y(x) = \frac{x^3-6x^2+12x-8}{x^2-4} = \frac{(x-2)^3}{(x-2)(x+2)} = \frac{(x-2)^2}{x+2}$$

keine Nst, da Nst. von $Z(x)=(x-2)^2$ nicht in D_{max} liegen.

Pole: Pol 1. Ordnung (mit VZW) bei $x=-2$

hebbare Def.lücke (sind alle Def.lücken, die keine Pole sind):

bei $x=2$

Asymptoten: senkrechte Asymptote (Pol-Asymptote) $x=-2$

Zählergrad > Nennergrad

$\Rightarrow$ Polynomdivision $Z(x):N(x)$

(dabei spielt es keine Rolle, ob man $Z(x)$ und $N(x)$ des ursprünglichen oder des gekürzten Bruchs verwendet – beim gekürzten ist die Rechnung jedoch einfacher)

- ursprüngl. Bruch:

$$\begin{array}{l} (x^3-6x^2+12x-8):(x^2+0x-4) = x-6 \\ -(x^3+0x^2-4x) \\ \hline \quad -6x^2+16x-8 \\ \quad -(-6x^2+0x+24) \\ \hline \quad\quad 16x-32 \leftarrow \text{Rest} \end{array}$$

$$\Rightarrow \frac{x^3-6x^2+12x-8}{x^2-4} = \underbrace{x-6}_{\substack{\text{Asymptote} \\ y=x-6}} + \underbrace{\frac{16x-32}{x^2-4}}_{\to 0 \text{ für } x\to\pm\infty}$$

$$\frac{16x-32}{x^2-4} = \frac{16(x-2)}{(x-2)(x+2)} = \frac{16}{x+2}$$

- gekürzter Bruch:

$$y = \frac{(x-2)^2}{x+2} = \frac{x^2-4x+4}{x+2}$$

$$\begin{array}{l} (x^2-4x+4):(x+2) = x-6 \\ -(x^2+2x) \\ \hline \quad -6x+4 \\ \quad -(-6x-12) \\ \hline \quad\quad +16 \leftarrow \text{Rest} \end{array}$$

$$\Rightarrow \frac{x^2-4x+4}{x+2} = \underbrace{x-6}_{\substack{\text{Asymptote} \\ y=x-6}} + \underbrace{\frac{16}{x+2}}_{\to 0 \text{ für } x\to\infty}$$

A56 In beiden Fällen erhält man $y = x - 6$ als Asymptote für $x \to \pm\infty$.

c) $$y = \frac{2x^3 - 2x}{x^3 + x^2 - x - 1}$$

$Z(x) = 2x^3 - 2x = 2x(x-1)$

$N(x) = x^3 + x^2 - x - 1$

$N(1) = 0 \Rightarrow$ Polynomdivision $N(x) : (x-1)$

$$\begin{array}{l}
(x^3 + x^2 - x - 1) : (x - 1) = x^2 + 2x + 1 \\
\underline{-(x^3 - x^2)} \\
\quad 2x^2 - x \\
\quad \underline{-(2x^2 - 2x)} \\
\qquad x - 1 \\
\qquad \underline{-(x - 1)} \\
\qquad\ \ --
\end{array}$$

$\Rightarrow \quad N(x) = (x-1) \cdot \underbrace{(x^2 + 2x + 1)}_{(x+1)^2} = (x-1)(x+1)^2$

$\Rightarrow$ Def. lücken bei $x = \pm 1$

$\Rightarrow$ $D_{max} = \mathbb{R} \setminus \{\pm 1\}$

Bruch kürzen:

$$y = \frac{2x^3 - 2x}{x^3 + x^2 - x - 1} = \frac{2x(x^2 - 1)}{(x-1)(x+1)^2} = \frac{2x(x-1)(x+1)}{(x-1)(x+1)^2} = \frac{2x}{x+1}$$

Nst.: $x = 0$ (liegt in D_{max}!)

Pole: Pol 1. Ordnung bei $x = -1$ (VZW)

hebbare Deflücke bei $x = 1$

Asymptoten: senkrechte Asymptote (Pol-Asymptote): $x = -1$

A56 Asymptote (Fortsetzung).

Zählergrad ≥ Nennergrad ⇒ Polynomdivision $Z(x) : N(x)$
(3) (3)

verwende dazu den gekürzten Bruch (einfachere Rechnung!)

$$y = \frac{2x}{x+1} \Rightarrow \begin{array}{l} (2x+0):(x+1) = 2 \\ -(2x+2) \\ \hline \quad -2 \ \text{(Rest)} \end{array}$$

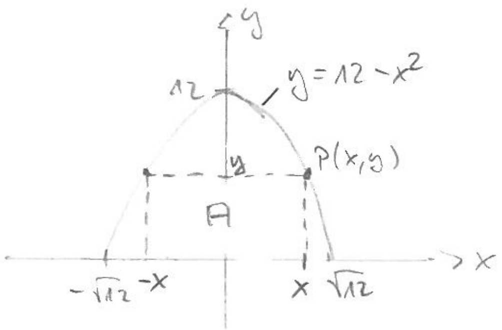
$$\Rightarrow y = \frac{2x}{x+1} = \underbrace{2}_{\substack{\text{Asymptote} \\ y=2}} - \frac{2}{x+1}$$

$\rightarrow 0$ für $x \to \pm\infty$

⇒
Asymptote für $x \to \pm\infty$: $y = 2$

A57

A58 a)



Länge $l = 2x$

Breite $b = y$

Flächeninhalt des Rechtecks:

$$A = x\cdot y$$

Da die Ecke $P(x,y)$ auf dem Graphen G_f von $f(x) = 12 - x^2$ liegt, gilt die Nebenbedingung (NB):

$$y = f(x) = 12 - x^2$$

NB in die Flächenformel eingesetzt, ergibt

$$A = 2x\cdot y = 2x\cdot \underbrace{(12 - x^2)}_{y} = 24x - 2x^3$$

Ges. ist das Maximum der Funktion $A(x) = 12x - x^3$.

$A'(x) = 24 - 6x^2$, $A''(x) = -12x$

$A'(x) = 0 \iff 24 - 6x^2 = 0 \iff x^2 = 4 \iff x = \pm 2$

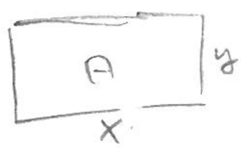
$\overset{x>0}{\Longrightarrow} x = 2$

Wg. $A''(2) = -24 < 0$ hat $A(x)$ bei $x = 2$ lokales Maximum.

$\Rightarrow$ gesuchte Abmessungen: $\underset{(x=2)}{l = 2x} = \underline{\underline{4}}$, $b = y = 12 - x^2 = \underline{\underline{8}}$

b) Zaun mit vorgegebener Länge U_0

i)



A maximal mit Nebenbedingung $U = U_0 = \text{const.}$

$$A = x\cdot y$$

NB: $U_0 = 2x + 2y$

A58 b) i) (Fortsetzung)

NB $U_0 = 2x + 2y$ nach y umstellen

$$y = \frac{1}{2}U_0 - x$$

und in die Flächenformel einsetzen

$$A = x \cdot y = x \cdot \left(\frac{1}{2}U_0 - x\right) = \frac{U_0}{2}x - x^2$$

=> Flächenfunktion $A(x) = \frac{U_0}{2}x - x^2$ maximieren:

$A'(x) = \frac{U_0}{2} - 2x$, $A''(x) = -2 < 0$ (=> Maximum)

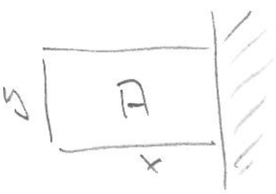
$A'(x) = 0 \iff \frac{U_0}{2} - 2x = 0 \iff x = \frac{U_0}{4}$

$\Rightarrow y = \frac{1}{2}U_0 - x = \frac{U_0}{2} - \frac{U_0}{4} = \frac{U_0}{4}$

$\Rightarrow A_{max} = \left(\frac{U_0}{4}\right)^2 = \frac{U_0^2}{16}$

Seitenverhältnis $x : y = \frac{x}{y} = \frac{U_0/4}{U_0/4} = \frac{1}{1} = 1:1$ (=> Quadrat)

ii) eine Seite (z.B. y) grenzt an eine Mauer



$A = x \cdot y$

$U_0 = 2x + y$ (NB)

$U_0 = 2x + y \iff y = U_0 - 2x$

in die Flächenformel eingesetzt: $A = x \cdot y = x \cdot (U_0 - 2x)$
$= U_0 x - 2x^2$

$A(x) = U_0 x - 2x^2$ maximieren

$A'(x) = U_0 - 4x$, $A'' = -2 < 0$ (=> Maximum)

$A'(x) = 0 \iff U_0 - 4x = 0 \iff x = \frac{U_0}{4}$

$\Rightarrow y = U_0 - 2x = U_0 - 2 \cdot \frac{U_0}{4} = \frac{U_0}{2}$

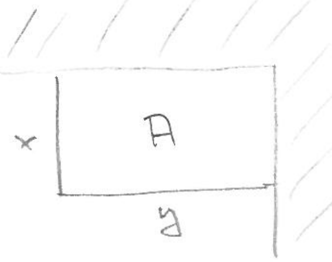
$\Rightarrow A_{max} = \frac{U_0}{4} \cdot \frac{U_0}{2} = \frac{U_0^2}{8}$

Seitenverhältnis $x : y = \frac{x}{y} = \frac{U_0/4}{U_0/2} = \frac{U_0}{4} \cdot \frac{2}{U_0} = \frac{2}{4} = \frac{1}{2} = 1:2$

d.h. y ist doppelt so lang wie x (=> Rechteck)

A58 b) (Fortsetzung)

iii) zwei benachbarte Seiten grenzen an eine Mauer



$A = x \cdot y$

$U_0 = x + y$ (NB)

$U_0 = x + y \Leftrightarrow y = U_0 - x$

in die Flächenformel eingesetzt: $A = x \cdot y = x \cdot (U_0 - x) =$
$= U_0 x - x^2$

$\Rightarrow$
$A(x) = U_0 x - x^2$ maximieren

$A'(x) = U_0 - 2x$, $A''(x) = -2 < 0$ ($\Rightarrow$ Maximum)

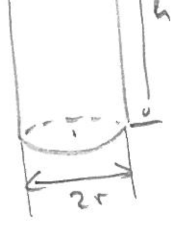
$A'(x) = 0 \Leftrightarrow U_0 - 2x = 0 \Leftrightarrow x = \frac{U_0}{2}$

$\Rightarrow y = U_0 - x = U_0 - \frac{U_0}{2} = \frac{U_0}{2}$

$\Rightarrow A_{max} = \frac{U_0}{2} \cdot \frac{U_0}{2} = \frac{U_0^2}{4}$

Seitenverhältnis $x : y = \frac{x}{y} = \frac{U_0/2}{U_0/2} = \frac{1}{1} = 1:1$ ($\Rightarrow$ Quadrat)

c) Ges.: Längenverhältnis h/r eines geraden Kreiszylinders mit vorgegebenem Volumen V_0 und minimaler Oberfläche A_0



Oberfläche $A_0 = \underbrace{2r^2\pi}_{\text{Grund- und Deckfläche}} + \underbrace{2\pi r h}_{\text{Mantelfläche}}$

Volumen $V_0 = r^2 \pi h = \text{const.}$ (NB)

NB $V_0 = r^2 \pi h$ nach h aufgelöst

$h = \frac{V_0}{r^2 \pi}$

A58 c) (Fortsetzung)

und in die Oberflächenformel eingesetzt:

$$A_0 = 2r^2\pi + 2\pi r h = 2r^2\pi + 2\pi r \cdot \frac{V_0}{r^2\pi}$$

$$= 2\pi r^2 + \frac{2V_0}{r}$$

$\Rightarrow$ Oberflächenfunktion $A_0(r) = 2\pi r^2 + \frac{2V_0}{r}$ minimieren

$$A_0'(r) = 4\pi r - \frac{2V_0}{r^2} \quad , \quad A_0''(r) = 4\pi + \frac{4V_0}{r^3}$$

$$A_0'(r) = 0 \Leftrightarrow$$

$$\Leftrightarrow 4\pi r - \frac{2V_0}{r^2} = 0 \quad | \cdot r^2 \ (r \neq 0)$$

$$\Leftrightarrow 4\pi r^3 - 2V_0 = 0 \quad | + 2V_0 \ | : 4\pi$$

$$\Leftrightarrow r^3 = \frac{2V_0}{4\pi} \left(= \frac{V_0}{2\pi} \right) \quad | \sqrt[3]{\ldots}$$

$$\Leftrightarrow r = \sqrt[3]{\frac{V_0}{2\pi}} = \left(\frac{V_0}{2\pi}\right)^{1/3}$$

in $A_0''(r)$ eingesetzt: $\quad A''(r) = 4\pi + 4V_0 \cdot \frac{1}{r^3}$

$$A_0''\left(\left(\frac{V_0}{2\pi}\right)^{1/3}\right) = 4\pi + 4V_0 \cdot \left[\left(\frac{2\pi}{V_0}\right)^{1/3}\right]^3 = 4\pi + 4V_0 \cdot \frac{2\pi}{V_0} = 4\pi + 8\pi = 12\pi > 0$$

$\Rightarrow$ Minimum

$$h = \frac{V_0}{r^2\pi} = \frac{V_0}{\pi} \cdot \frac{1}{r^2} = \frac{V_0}{\pi} \cdot \left[\left(\frac{2\pi}{V_0}\right)^{1/3}\right]^2 \quad \left(r = \left(\frac{V_0}{2\pi}\right)^{1/3} \text{ eingesetzt!}\right)$$

$$= \frac{V_0}{\pi} \cdot \left(\frac{2\pi}{V_0}\right)^{2/3}$$

$\Rightarrow$ Verhältnis h/r:

$$h/r = h \cdot \frac{1}{r} = \frac{V_0}{\pi} \underbrace{\left(\frac{2\pi}{V_0}\right)^{2/3} \cdot \left(\frac{2\pi}{V_0}\right)^{1/3}}_{\left(\frac{2\pi}{V_0}\right)^{3/3}} = \frac{V_0}{\pi} \cdot \frac{2\pi}{V_0} = \underline{\underline{2}}$$

A51 a) $I_1 = \int_0^{\ln 2} \sqrt{e^x - 1}\, dx$

Subst.: $u = \sqrt{e^x - 1}$

$\Rightarrow du = u'(x)\, dx = \frac{1}{2\sqrt{e^x-1}} \cdot e^x\, dx$

$\Leftrightarrow dx = \frac{2\sqrt{e^x-1}}{e^x} \cdot du = \frac{2u}{e^x} du$

$u = \sqrt{e^x - 1} \Leftrightarrow u^2 = e^x - 1 \Leftrightarrow e^x = u^2 + 1$

$\Rightarrow dx = \frac{2u}{u^2+1} du$

$\Rightarrow I_1 = \int_0^{\ln 2} \sqrt{e^x - 1}\, dx = \int_{u(0)}^{u(\ln 2)} u \cdot \frac{2u}{u^2+1} du$

$u(0) = \sqrt{e^0 - 1} = 0$
$u(\ln 2) = \sqrt{e^{\ln 2} - 1} = \sqrt{2-1} = \sqrt{1} = 1$

$\Rightarrow I_1 = 2\int_0^1 \frac{u^2}{u^2+1} du$ Zählergrad ≥ Nennergrad ⇒ Polynomdivision

$$(u^2 + 0u + 0) : (u^2 + 0u + 1) = 1$$
$$-(u^2 + 0u + 1)$$
$$-1$$

$\Rightarrow \frac{u^2}{u^2+1} = 1 - \frac{1}{u^2+1}$

$\Rightarrow I_1 = 2\int_0^1 1 - \frac{1}{u^2+1} du = 2[u]_0^1 - 2[\arctan(u)]_0^1 =$

$= 2[1-0] - 2[\underbrace{\arctan(1)}_{\pi/4} - \underbrace{\arctan(0)}_{0}]$

$= 2 - 2\frac{\pi}{4} = 2(1 - \pi/4)$

A59 b) $I_2 = \int_1^2 \frac{2x^2+3x-2}{-x^3+x^2}\,dx$

Zählergrad < Nennergrad $\Rightarrow$ Partialbruchzerlegung (PBZ)

Faktorisierung des Nenners: $N(x) = -x^3+x^2 = x^2(-x+1) = x^2(1-x)$

$\Rightarrow$ $x=0$ 2-fache Nst. $\Rightarrow$ Teilansatz $\frac{A_1}{x} + \frac{A_2}{x^2}$

$x=1$ 1-fache Nst. $\Rightarrow$ Teilansatz $\frac{B}{1-x}$

$\Rightarrow$ Gesamtansatz

$$\frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B}{1-x} = \frac{2x^2+3x-2}{x^2(1-x)} \qquad HN = x^2(1-x)$$

$$\frac{A_1x(1-x) + A_2(1-x) + Bx^2}{x^2(1-x)} = \frac{2x^2+3x-2}{x^2(1-x)} \quad | \cdot HN$$

$$A_1x - A_1x^2 + A_2 - A_2x + Bx^2 = 2x^2+3x-2$$

$$(B-A_1)x^2 + (A_1-A_2)x + A_2 = 2x^2+3x-2$$

Koeff.vgl. $\Rightarrow$
(I) $B - A_1 = 2$ $\Rightarrow B = 3$
(II) $A_1 - A_2 = 3$ $\Rightarrow A_1 = 1$
(III) $A_2 = -2$

$$\Rightarrow \frac{2x^2+3x-2}{x^2(1-x)} = \frac{1}{x} - \frac{2}{x^2} + \frac{3}{1-x} = \frac{1}{x} - \frac{2}{x^2} - \frac{3}{x-1}$$

$\Rightarrow$
$$I_2 = \int \frac{1}{x}\,dx - 2\int \frac{1}{x^2}\,dx - 3\int \frac{1}{x-1}\,dx$$

$$= \ln|x| + C_1 - 2(-x^{-1}) + C_2 - 3\ln|x-1| + C_3$$

$$= \ln|x| + \frac{2}{x} - 3\ln|x-1| + C$$

A55 c) $\int x e^{-x} dx$

$$\underbrace{\int x e^{-x} dx}_{u \cdot v'} = \underbrace{-x e^{-x}}_{u \cdot v} - \underbrace{\int 1 \cdot (-e^{-x}) dx}_{u' \cdot v} =$$

$$\begin{bmatrix} u = x & u' = 1 \\ v = -e^{-x} & v' = e^{-x} \end{bmatrix}$$

$$= -x e^{-x} + \int e^{-x} dx = -x e^{-x} + (-e^{-x}) + C$$

$$= e^{-x}(-x-1) + C = -e^{-x}(x+1) + C$$

A 60 a) Fläche zwischen $f(x) = x^2 - 4$, $g(x) = \frac{1}{2}x + 1$

für $x \geq 0$

Gemeinsame Schnittpunkte:

$$f(x) = g(x)$$
$$x^2 - 4 = \frac{1}{2}x + 1$$
$$x^2 - \frac{1}{2}x - 5 = 0$$

$$\Rightarrow x_{1/2} = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} + 20}}{2} = \frac{\frac{1}{2} \pm \frac{9}{2}}{2}$$

$$\Rightarrow x_1 = \frac{5}{2}, \quad x_2 = -2$$

$$\Rightarrow A = \left| \int_0^{5/2} f(x) - g(x)\, dx \right|$$

$$\int_0^{5/2} f(x) - g(x)\, dx = \int_0^{5/2} x^2 - \frac{1}{2}x - 5\, dx = \left[\frac{x^3}{3} - \frac{x^2}{4} - 5x \right]_0^{5/2} =$$

$$= \frac{1}{3} \cdot \frac{125}{8} - \frac{1}{4} \cdot \frac{25}{4} - \frac{25}{2} = -\frac{425}{48} \quad \Rightarrow A = \frac{425}{48} = 8\frac{41}{48} \approx 8{,}854 \text{ FE}$$

b) $f: [-1, 1] \to \mathbb{R}$, $x \mapsto \cosh x$, Kurve $\Gamma = G_f$

Bogenlänge $l_\Gamma = \int_{-1}^{1} \sqrt{1 + (f'(x))^2}\, dx$

$$f'(x) = (\cosh x)' = \sinh x \quad \Rightarrow 1 + f'(x)^2 = 1 + \sinh^2 x \underset{\substack{\uparrow \\ \text{hyperbolischer} \\ \text{Pythagoras}}}{=} \cosh^2 x$$

$$\Rightarrow l_\Gamma = \int_{-1}^{1} \sqrt{\cosh^2 x}\, dx = \int_{-1}^{1} |\cosh x|\, dx$$

$$\underset{\substack{\uparrow \\ \cosh x > 0}}{=} \int_{-1}^{1} \cosh x\, dx = [\sinh x]_{-1}^{1} = \sinh(1) - \underbrace{\sinh(-1)}_{= -\sinh(1)} = \sinh(1) + \sinh(1)$$

$$= 2\sinh(1) \approx \underline{\underline{2{,}35}} \text{ LE};$$

c) $f: [0; a] \to \mathbb{R}$, $x \mapsto \sqrt{x} + 1$

i) lineares Mittel: $\bar{f} = \frac{1}{a} \int_0^a f(x)\, dx = \frac{1}{a} \int_0^a \sqrt{x} + 1\, dx = \frac{1}{a} \int_0^a x^{1/2} + 1\, dx$

$$= \frac{1}{a}\left(\left[\frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} \right]_0^a + [x]_0^a \right) =$$

$$= \frac{1}{a}\left(\frac{2}{3} a^{3/2} + a \right) = \frac{2}{3} a^{3/2 - 1} + 1 = \frac{2}{3} a^{1/2} + 1 = \underline{\underline{\frac{2}{3}\sqrt{a} + 1}}$$

A60 c) Fortsetzung

ii) quadratisches Mittel: $\bar{f}^{a} = \sqrt{\frac{1}{a}\int_0^a f^2(x)\,dx}$

$f(x) = \sqrt{x} + 1 \Rightarrow f^2(x) = (\sqrt{x}+1)^2 = x + 2\sqrt{x} + 1$

$\Rightarrow \int_0^a f^2(x)\,dx = \int_0^a x + 2x^{1/2} + 1 \, dx = \left[\frac{x^2}{2} + 2 \cdot \frac{2}{3} x^{3/2} + x \right]_0^a$

$= \frac{a^2}{2} + \frac{4}{3} a^{3/2} + a$

$\Rightarrow \bar{f}^{a} = \sqrt{\frac{1}{a}\left(\frac{a^2}{2} + \frac{4}{3}a^{3/2} + a\right)} = \sqrt{\frac{a}{2} + \frac{4}{3}\sqrt{a} + 1}$

d) $f : [0,1] \to \mathbb{R}, \; x \mapsto -4x(x-1) \; (= f(x))$

Rotationsvolumen V

$V = \pi \int_0^1 f^2(x)\,dx = \pi \int_0^1 16x^2(x-1)^2\,dx = 16\pi \int_0^1 x^2(x^2 - 2x + 1)\,dx$

$= 16\pi \int_0^1 x^4 - 2x^3 + x^2\,dx = 16\pi \left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3} \right]_0^1 =$

$= 16\pi \left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3} - 0 \right) = \frac{16}{30}\pi = \frac{8}{15}\pi \approx 1{,}68 \text{ (VE)}$