



Model based prediction

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Basic idea

1. Assume the data follow a probabilistic model
2. Use Bayes' theorem to identify optimal classifiers

Pros:

- Can take advantage of structure of the data
- May be computationally convenient
- Are reasonably accurate on real problems

Cons:

- Make additional assumptions about the data
- When the model is incorrect you may get reduced accuracy

Model based approach

1. Our goal is to build parametric model for conditional distribution $P(Y = k|X = x)$
2. A typical approach is to apply [Bayes theorem](#):

$$Pr(Y = k|X = x) = \frac{Pr(X = x|Y = k)Pr(Y = k)}{\sum_{\ell=1}^K Pr(X = x|Y = \ell)Pr(Y = \ell)}$$

$$Pr(Y = k|X = x) = \frac{f_k(x)\pi_k}{\sum_{\ell=1}^K f_{\ell}(x)\pi_{\ell}}$$

3. Typically prior probabilities π_k are set in advance.

4. A common choice for $f_k(x) = \frac{1}{\sigma_k \sqrt{2\pi}} e^{-\frac{(x-\mu_k)^2}{\sigma_k^2}}$, a Gaussian distribution

5. Estimate the parameters (μ_k, σ_k^2) from the data.

6. Classify to the class with the highest value of $P(Y = k|X = x)$

Classifying using the model

A range of models use this approach

- Linear discriminant analysis assumes $f_k(x)$ is multivariate Gaussian with same covariances
- Quadratic discriminant analysis assumes $f_k(x)$ is multivariate Gaussian with different covariances
- [Model based prediction](#) assumes more complicated versions for the covariance matrix
- Naive Bayes assumes independence between features for model building

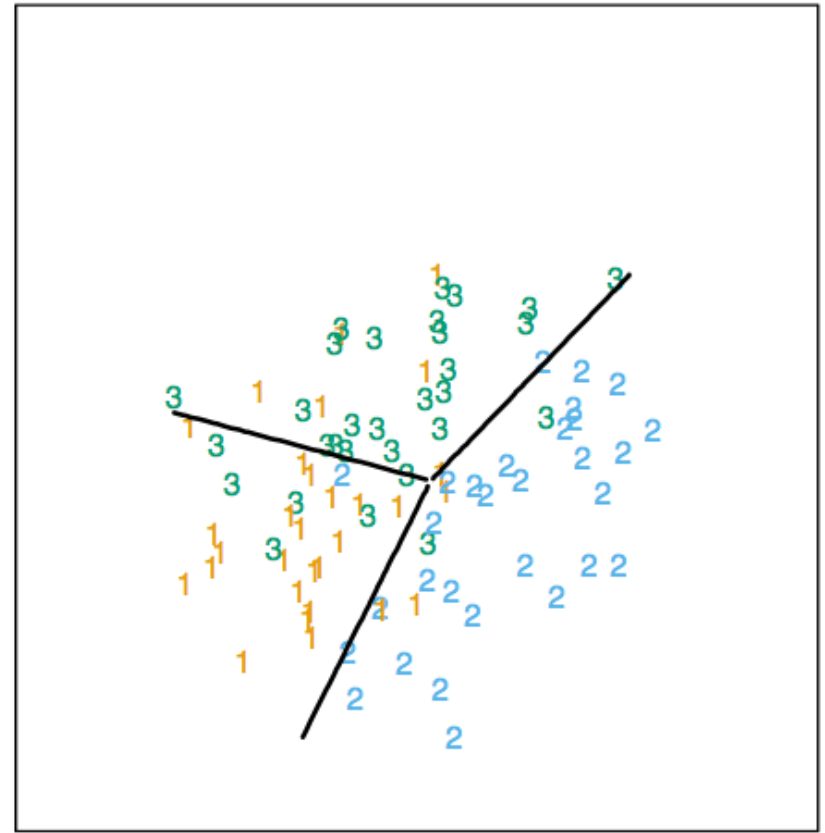
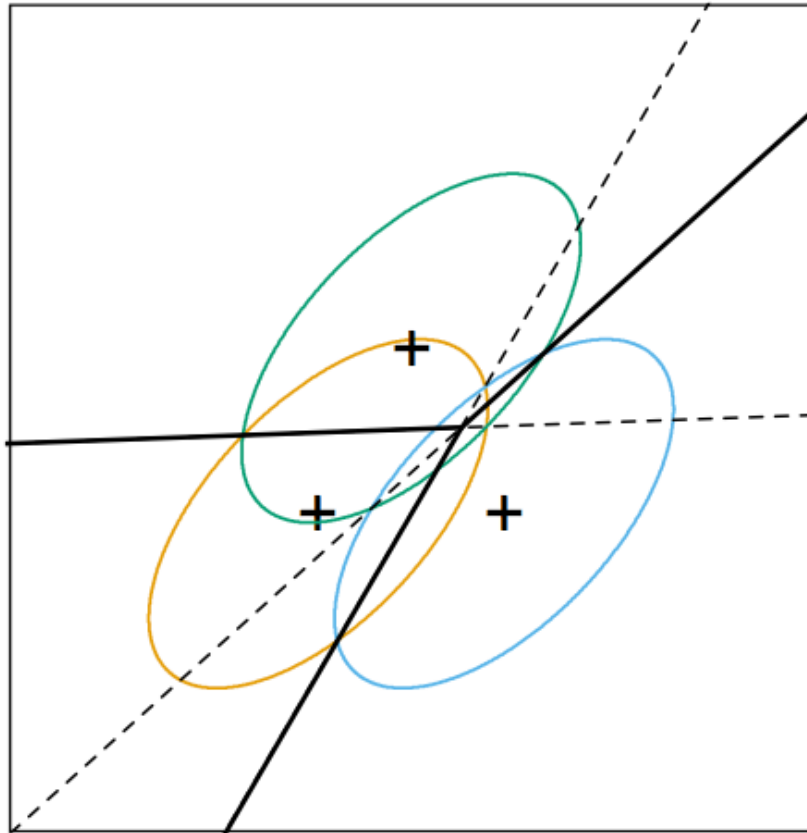
<http://statweb.stanford.edu/~tibs/ElemStatLearn/>

Why linear discriminant analysis?

$$\begin{aligned} & \log \frac{\Pr(Y = k|X = x)}{\Pr(Y = j|X = x)} \\ &= \log \frac{f_k(x)}{f_j(x)} + \log \frac{\pi_k}{\pi_j} \\ &= \log \frac{\pi_k}{\pi_j} - \frac{1}{2} (\mu_k + \mu_j)^T \Sigma^{-1} (\mu_k + \mu_j) \\ & \quad + x^T \Sigma^{-1} (\mu_k - \mu_j) \end{aligned}$$

<http://statweb.stanford.edu/~tibs/ElemStatLearn/>

Decision boundaries



Discriminant function

$$\delta_k(x) = x^T \Sigma^{-1} \mu_k - \frac{1}{2} \mu_k^T \Sigma^{-1} \mu_k + \log(\mu_k)$$

- Decide on class based on $\hat{Y}(x) = \operatorname{argmax}_k \delta_k(x)$
- We usually estimate parameters with maximum likelihood

Naive Bayes

Suppose we have many predictors, we would want to model: $P(Y = k|X_1, \dots, X_m)$

We could use Bayes Theorem to get:

$$\begin{aligned} P(Y = k|X_1, \dots, X_m) &= \frac{\pi_k P(X_1, \dots, X_m|Y = k)}{\sum_{\ell=1}^K P(X_1, \dots, X_m|Y = \ell) \pi_{\ell}} \\ &\propto \pi_k P(X_1, \dots, X_m|Y = k) \end{aligned}$$

This can be written:

$$\begin{aligned} P(X_1, \dots, X_m, Y = k) &= \pi_k P(X_1|Y = k) P(X_2, \dots, X_m|X_1, Y = k) \\ &= \pi_k P(X_1|Y = k) P(X_2|X_1, Y = k) P(X_3, \dots, X_m|X_1, X_2, Y = k) \\ &= \pi_k P(X_1|Y = k) P(X_2|X_1, Y = k) \dots P(X_m|X_1, \dots, X_{m-1}, Y = k) \end{aligned}$$

We could make an assumption to write this:

$$\approx \pi_k P(X_1|Y = k) P(X_2|Y = k) \dots P(X_m|Y = k)$$

Example: Iris Data

```
data(iris); library(ggplot2)
names(iris)
```

```
[1] "Sepal.Length" "Sepal.Width"  "Petal.Length" "Petal.Width"  "Species"
```

```
table(iris$Species)
```

```
setosa versicolor virginica
    50         50         50
```

Create training and test sets

```
inTrain <- createDataPartition(y=iris$Species,  
                                p=0.7, list=FALSE)  
training <- iris[inTrain,]  
testing <- iris[-inTrain,]  
dim(training); dim(testing)
```

```
[1] 45 5
```

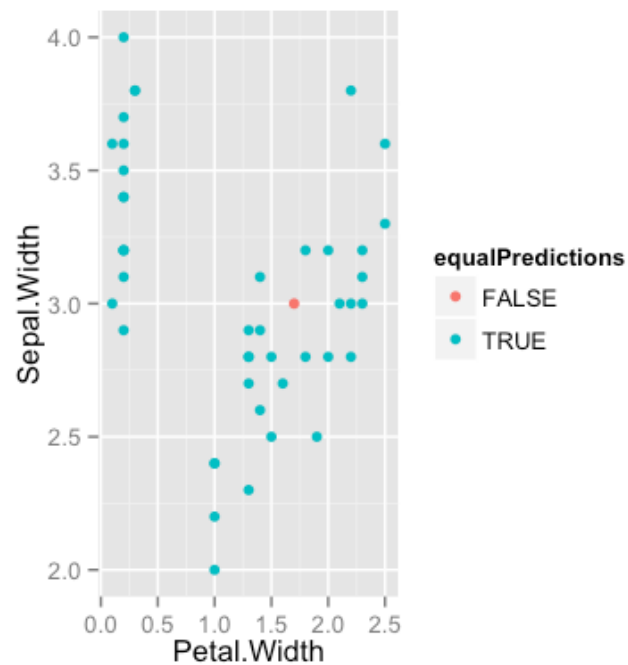
Build predictions

```
modlda = train(Species ~ ., data=training, method="lda")
modnb = train(Species ~ ., data=training, method="nb")
plda = predict(modlda, testing); pnb = predict(modnb, testing)
table(plda, pnb)
```

plda	pnb		
	setosa	versicolor	virginica
setosa	15	0	0
versicolor	0	13	1
virginica	0	0	16

Comparison of results

```
equalPredictions = (plda==pnb)  
qplot(Petal.Width,Sepal.Width,colour=equalPredictions,data=testing)
```



Notes and further reading

- [Introduction to statistical learning](#)
- [Elements of Statistical Learning](#)
- [Model based clustering](#)
- [Linear Discriminant Analysis](#)
- [Quadratic Discriminant Analysis](#)