Replacement Models:

The replacement problems are concerned with the situations that arise when some items such as men, machines, electric bikes etc need replacement due to their decreased efficiency failure or breakdown. Such decreased efficiency or complete breakdown may either be gradual or all of a sudden.

Notes: The replacement problem arises because of the following factors.

(i) The old item has become worse or requires expensive maintainance.

(ii) The old item has failed due to accident

(iii) A more efficient design of equipment has become available in market

Thus, the problem of replacement is to decide the best policy to determine the age at which the replacement is most economical. Instead of continuing at cost due to factors like maintainance.

The objective is to find the optimum period of replacement. We shall discuss the following main type of replacement situation.

- Replacement of items that deteriorate with time
- Replacement of items which do not deteriorate

but fail after certain amount of use.

For items which do not detoriate but fail all of a sudden, following are the two types of replacement policies.

* Individual replacement policy:

Under this policy, an item is replaced ~~after~~ immediately after its failure.

* Group replacement policy:

Under this policy, we take decisions as to when all the items must be replaced irrespective of the fact that items have failed or have not failed. with the provision that if any item fails, before the replacement time it may be individually replaced.

The replacement situations may be placed into 4 categories.

(i) replacement of capital equipment that become worse with time.

Eg. Machine tools, planes etc

(ii) Group replacement of items that fail completely. Eg. Radio tubes, light bulbs etc

(iii) Problems of mortality and Staffing

(iv) Miscellaneous problems

Replacement policies for items whose maintainance cost increase with time and money value is not considered.

Proof:

(a) When time 't' is not ~~conside~~ continuous variable

Let R_t = Maintainance cost at time 't'

C = Capital cost of the item

S = The scrap value of the item

Obviously, annual cost of the item at any time

$t = R_t + C - S$

Since the maintainance cost incurred during n years

$$\int_0^n R_t \, dt$$

The total cost incurred on the item will become

$$P(n) = \int_0^n R_t \, dt + C - S$$

hence average total cost is given by

$$F(n) = \frac{P(n)}{n} = \frac{1}{n}\int_0^n R_t \, dt + \frac{C-S}{n}$$

Now to find n for which $f(n)$ is minimum. Therefore differenciating $f(n)$ w.r.t n and equating it to 0

$$\frac{dF(n)}{dn} = 0 = \frac{1}{n} R_n - \frac{1}{n^2} \int_0^n R_t \, d_t + \frac{C-S}{n^2} \rightarrow ①$$

$$\therefore R_n = \frac{1}{n} \int_0^n R_t \, dt + \frac{C-S}{n}$$

$$= \frac{P(n)}{n} \qquad (\text{Using } ①)$$

Hence maintainance cost at the n^{th} year is equal to Average total cost.

Remarks:

We observe that for the total cost to be minimum, the average total cost should be equal to the current maintainance cost provided R_n is not decreasing and $R_0 = 0$. Hence in this type of example replacement should be done when average total cost up to date becomes equals to maintainance cost provided an explicit expression is given for all the cost under consideration.

When t is a random discrete variable, tabular method is used as illustrated in the following example

We write $F(n) = \frac{P(n)}{n} = \frac{\sum_{t=1}^{n} R_t}{n} + \left(\frac{C-S}{n}\right)$

$F(n)$ is minimum if $\Delta F(n-1) < 0 < \Delta F(n)$

$\Delta F(n) = F(n+1) - F(n)$ is satisfied

$$\Rightarrow \left[\frac{\sum_{t=1}^{n+1} R_t}{n+1} + \frac{C-S}{n+1}\right] - \left[\frac{\sum_{t=1}^{n} R_t}{n} + \frac{C-S}{n}\right]$$

$$\Rightarrow \left[\frac{R_{n+1}}{n+1} + \frac{\sum_{t=1}^{n} R_t}{n+1}\right] - \sum_{t=1}^{n} \frac{R_t}{n} + (C-S)\left(\frac{1}{n+1} - \frac{1}{n}\right)$$

$$\Rightarrow \frac{R_{n+1}}{n+1} + \sum_{t=1}^{n} R_t\left(\frac{1}{n+1} - \frac{1}{n}\right) + (C-S)\left(\frac{1}{n+1} - \frac{1}{n}\right)$$

$$\Rightarrow \frac{R_{n+1}}{n+1} - \left(\sum_{t=1}^{n} \frac{R_t}{n(n+1)} + \frac{C-S}{n(n+1)}\right)$$

Since $\Delta F(n) > 0$ for min of $F(n)$

$$\frac{R_{n+1}}{n+1} > \sum_{t=1}^{n} \frac{R_t}{n(n+1)} + \frac{C-S}{n(n+1)}$$

$$R_{n+1} > \sum_{t=1}^{n} \frac{R_t}{n} + \frac{C-S}{n}$$

$$R_{n+1} > \frac{P(n)}{n} \quad \text{(From ①)}$$

Similarly, $R_n \leq \frac{P(n)}{n}$ can be as,

$$\Delta F(n-1) < 0$$

Hence, $R_{n+1} > \frac{P(n)}{n} > R_n$

Remarks,

In ~~as~~ numerical problems, we consider the minimum value of the average annual cost i.e minimum of $\frac{P(n)}{n}$ to determine the optimum replacement period

Money Value, Present Worth Factor (PWF)

Money Value:

Since money has value of time, we often speak money is worth 10% per year. This can be explained in the following two ways.

(i) In one way, spending ruppees 100 today would be equivalent to spending ₹110 in a years time.

(ii) Consequently, ₹1 after a year from now is equivalent to $(1.1)^{-1}$ rupee today.

Present Worth Factor (PWF)

As we have just seen about, ₹1 a year is equivalent to $(1.1)^{-1}$ rupee today at the interest ~~to~~ rate 10% per year. ₹1 spent 2 years from now is equivalent to $(1.1)^{-2}$ today. Similarly, we can say ₹1 spent 'n' years from now is equivalent to $(1.1)^{-n}$ today. The quantity $(1.1)^{-n}$ is called present worth factor or present value of ₹1 spent 'n' years from now.

Dicount Rate (Depriciation Value)

The present worth factor of unit amount to be spent after 1 year is given by

$$V = (1+r)^{-1}$$

where r is the interest rate, then v is called discount rate, technically known as depriciation value.

Problems:

1) A machine owner finds from his past records that the cost per year of maintaining a machine who purchase price is ₹6000 are as given below.

Year	1	2	3	4	5	6
Maintainance cost (₹)	1000	1200	1400	1800	2300	2800
Resale value (₹)	3000	1500	750	375	200	200

Determine at what age is replacement due.

Soln:

$$P(n) = \frac{\Sigma R_n + C - S}{n}$$

ΣR_n - maintainance cost

C - capital

S - resale value

Year	Main cost	ΣR_n	$C-S$	$P(n) = \Sigma R_n + C - S$	Avg
1	1000	1000	3000	4000	4000
2	1200	2200	4500	6700	3350
3	1400	3600	5250	8850	2950
4	1800	5400	5625	11025	2756
5	2300	7700	5800	13500	2700
6	2800	10500	5800	16300	2717

increases.

Optimum replacement year is 5.

Minimum cost is in ~~minimum~~ 5th year

Optimum replacement plan replace the machine at end of 5th year.

② The cost of a machine is ₹6100 and its scrap value is ₹100. The maintainance cost found from experience are as follows.

Year	1	2	3	4	5	6	7	8
Maint. cost (₹)	100	250	400	600	900	1200	1600	2000

When should the machine be replaced?

Soln:

Year	Maint. cost	R_n	C - S	P(n)	Avg cost
1	100	100	6000	6100	6100
2	250	350	6000	6350	3175
3	400	750	6000	6750	2250
4	600	1350	6000	7350	1837
5	900	2250	6000	8250	1650
6	1200	3450	6000	9450	1575
7	1600	5050	6000	11050	1579
8	2000	7050	6000	13050	1631

Optimum Replacement Year : 6

Minimum average cost is in 6th year. So machine should be replaced by 6th year.

A taxi owner estimates from his past records that the cost per for operating a taxi whose purchase when new is ₹ 60,000 are as given below

Age	1	2	3	4	5
operating cost (₹)	10000	12,000	15,000	18,000	20,000

After 5 years the operating cost is ₹ 60,000 k where k = 6, 7, 8, 9, 10. k denoting age in years. If the resale value is decreased by 10% of purchased price each year. What is the best replacement policy?

Soln:

Year	Operating cost	ΣR_n	Resale Value (10% of 60000)	C−S	P(n)
1	10000	10,000	54,000	6000	16,000
2	12,000	22,000	48,000	12,000	34,000
3	15,000	37,000	42,000	18,000	55,000
4	18,000	55,000	36,000	24,000	79,000
5	20,000	75,000	30,000	30,000	1,05,000
6	36,000	1,11,000	24,000	36,000	1,47,000
7	42,000	1,53,000	18,000	42,000	1,95,000
8	48,000	2,01,000	12,000	48,000	2,49,000
9	54,000	2,55,000	6,000	54,000	309,000
10	60,000	3,15,000	0	60,000	3,75,000

Avg cost

1600

7000

18333

19750

21000

24500

27857

31125

34833

37500

optimum replacement Year is 1st year

(4) (i) Machine A cost ₹9000. Annual operating cost is ₹200 for the 1st year and then increases by ₹2000 every year. Determine the optimum replacement policy.

(ii) Machine B cost ₹10,000. Annual operating cost is ₹400 for the 1st year and then increased ₹800 every year. ~~Determine~~ ~~the~~ You Now own a Machine A which is 1 year old. Should you replace it with B if so when?

Money value (change)

$$w(n) = \frac{C + \sum R_n V^{n-1}}{\sum_{r} V^{n-1}} \quad , \quad V = (1+r)^{-1}$$

Problem

Assume that present value of 1 ruppee to be spent in year's time is ₹ 0.9 and C = ₹ 3000. Capital cost of equipment and the running cost are given in the table below. When should the machine be replaced.

Year	1	2	3	4	5	6	7
Running Cost	500	600	800	1000	1300	1600	2000

Year	R_n	V^{n-1}	$R_n V^{n-1}$	$\sum R_n V^{n-1}$	$\sum V^{n-1}$
1	500	1	500	500	1
2	600	0.9	540	1040	1.9
3	800	0.81	648	1688	2.71
4	1000	0.73	730	2418	3.44
5	1300	0.66	858	3276	4.10
6	1600	0.59	944	4220	4.69
7	2000	0.53	1060	5280	5.22

$C+\sum R_n V^{n-1}$	$w(n)$
3500	3500
4040	2126.3
4688	1729.8
5418	1575
6276	1530.73 *
7220	1539.4
8280	1586.2

optimum Replacement Year = 5

Since $W(n)$ is minimum at 5th year, the optimum replacement plan is end of 5th year

② The cost of a new machine is ₹5000. The maintainance cost of n^{th} year is given by $C_n = 500 \times n-1$, $n = 1, 2, 3, \ldots$ Suppose that money is worth 5% per year. After how many years will it be economically to replace the machine by a new one

$$V = (1+r)^{-1} \qquad r = 5\%$$

$$= (1+0.05)^{-1} = 0.9523$$

Year	R_n	V^{n-1}	$R_n V^{n-1}$	$\Sigma R_n V^{n-1}$	ΣV^{n-1}	$C + \Sigma R_n V^{n-1}$	$W(n)$
1	0	1	0	0	1	5000	5000
2	500	0.9523	~~1295.4~~ 476.15	476.15	1.9523	5476.15	2804.9
3	1000	0.9068	906.8	1382.95	2.8591	6382.95	2232.6
4	1500	0.8636	1295.4	2678.35	3.7227	7678.35	2062.57
5	2000	0.8224	1644.8	4323.15	4.5451	9323.15	2051.25
6	2500	0.78319	1957.75	6280.9	5.3282	11280.9	2117.2

The replacement year is 5th year

It is economical to replace at end of 5th year

Incase Salvage value is considered, then

$$W(n) = \frac{C + \Sigma R_n V^{n-1} - S_n V^n}{\sum_1 V^{n-1}}$$

will give desired result i.e Minimum Avg weighted cost.

Q A production machine installed has initial investment of ₹30,000 and its salvage value at the end of i^{th} years of its use is estimated as ₹ $\frac{30000}{i+1}$

The annual operating & maintainence cost in the 1st year is ₹15,000 & increases by ₹1000 in each subsequent years for 5 years and increases by ₹5000 in each year there after. Replacement policy is to be planned over a period of 7 years. During this period cost of capital may be taken as 10% per year. Solve the problem for optimal replacement.

$V = (1+0.10)^{-1} = 0.9090$ $(r = 10\%)$

Year	R_n	V^{n-1}	$R_n V^{n-1}$	ΣV^{n-1}	$\Sigma R_n V^{n-1}$	S_n
1	15,000	1	15000	1	15000	15000
2	16,000	0.9090	14544	1.9090	29544	10000
3	17,000	0.8262	14045.4	2.735	43589.4	7500
4	18,000	0.751	13518	3.486	57107.4	6000
5	19,000	0.6827	12971.3	4.169	70054.4	5000
6	24,000	0.6206	14904	4.7902	84988.4	4285.7
7	29,000	0.5641	16356	5.354	101344.4	3750

V_n	$B_n V_n$	$C + \Sigma R_n V^{n-1} - B_n V_n$	$W(n)$
0.9090	13635	81865	81865
0.8262	8262	801282	26666.81
0.751	56205	67956.9	24847.12
0.683	4098	83009.4	23812.2
0.621	3105	96179.4	23262.03
0.564	2417.13	112571.27	23500.32
0.5127	1922.62	129421.78	24172.31

Optimum replacement year = 5

(4) A manufacturer is offered two machines A and B. A is priced at ₹5000 and running cost are estimated at ₹800 for each of the first 5 years increasing by ₹200 per year in the 6th year and subsequent year.

Machine B which has capacity as A cost ₹2500 but will have a running cost of ₹1200 per year for 6 years increasing by ₹200 per year there after. If mony is worth 10% per year which machine should be purchased.

$V = (1 + 0.10)^{-1} = 0.9090$

Machine A

Year	R_n	V^{n-1}	R_nV^{n-1}	ΣV^{n-1}	ΣR_nV^{n-1}
	800	1	800	1	800
1	800	0.9090	727.2	1.9090	1527.2
2	800	0.8262	660.96	2.7352	2188.16
3	800	0.751	600.8	3.4862	2788.96
4	800	0.6827	546.16	4.1689	3335.12
5	800	0.6206	620.6	4.7895	3955.72
6	1000	0.5641	676.92	5.3536	4632.64
7	1200	0.5127	717.78	5.8663	5350.42
8	1400	0.4661	745.76	6.3324	6096.18
9	1600	0.4237	762.66	6.7561	6858.84
10	1800				

Year	$C+\Sigma R_nV^{n-1}$	$w(n)$
1	5800	5800
2	6527.2	3419.17 ~~2836.57~~
3	7188.96	2628.31
4	7788.96	2234.22
5	8335.12	1999.35
6	8955.72 ~~9632.64~~	1869.86
7	9632.64	1799.28
8	10350.42	1764.38
9	11096.18	1752.28
10	11858.84	1755.27

Machine B

Year	R_n	v^{n-1}	Σv^{n-1}	$\cancel{\Sigma} R_n v^{n-1}$	$\Sigma R_n v^{n-1}$
1	1200	1	1	1200	1200
2	1200	0.9090	1.9090	1090.8	1210.8
3	1200	0.8262	2.7352	991.44	2202.24
4	1200	0.751	3.4862	901.2	3103.44
5	1200	0.6827	4.1689	819.24	3922.68
6	1200	0.6206	4.7895	744.72	4667.4
7	1400	0.5641	5.3536	789.74	5457.14
8	1600	0.5127	5.8663	820.32	6277.46
9	1800	0.4661	6.3324	838.98	7116.44
10	2000	0.4237	6.7561	847.4	7963.84

Year	$C + \Sigma R_n v^{n-1}$	$w(n)$
1	3700	3700
2	3710.8 ~~2410.8~~	1943.8
3	4702.24 ~~3402.24~~	1719.15
4	5603.44	1607.32
5	6422.68	1540.61
6	7167.4	1,496.48
7	7957.14	1,486.3
8	8777.46	1,496.25 *
9	9616.44	1518.60
10	10463.84	1548.77

Machine A should be purchased.

Group Replacement Policy:

Theorem: One should group replace at the end of i^{th} period if the cost of individual replacements for the i^{th} period is greater than the average cost per period through the end of i^{th} period.

One should not group replace at the end of i^{th} period if the cost of individual replacement at the end of $i-1$ less than average cost through the end of i^{th} period.

Q) The following failure rates have been observed for certain items:

End of Month	1	2	3	4	5
Probability of failure to date	0.10	0.30	0.55	0.85	1

The cost of replacing an individual item is ₹1.25. The decision is made to replace all items simultaneously at fixed intervals and also replace individual items as they fail. If the cost of group replacement is 50 paise. What is the best interval for group replacement At what group replacement a item would policy of strictly individually replacement become preferable to the adopted policy.

Soln:

Assume that item failing during a month are replaced at the end of month.

Suppose that there are 1000 items in use, let P_i be the probability that an item which was new when placed in position for use fails during its month of its life.

$P_1 = 0.10$

$P_2 = 0.30 - 0.10 = 0.20$

$P_3 = 0.55 - 0.30 = 0.25$

$P_4 = 0.85 - 0.55 = 0.30$

$P_5 = 1 - 0.85 = 0.15$

N_i denote the no. of replacement at the end of the end.

N_0 = No. of items at beginning (assume)
$= 1000$

$N_1 = N_0 P_1 = 1000 \times 0.10 = 100$

$\cancel{N_2 = N_0 P_2 = 1000 \times 0.20 = 200}$

$\cancel{N_3 = N_0 P_3 =}$

$\cancel{N_2 = N_0 P_1 + N_1 P_1}$

$\cancel{= 100 +}$

$N_2 = N_0P_2 + N_1P_1$

$= 1000 \times 0.20 + 100 \times 0.10$

$= 200 + 10 = 210$

$N_3 = N_0P_3 + N_1P_2 + N_2P_1$

$= 1000 \times 0.25 + 100 \times 0.20 + 210 \times 0.10$

$= 291$

$N_4 = N_0P_4 + N_1P_3 + N_2P_2 + N_3P_1$

$= 1000 \times 0.30 + 100 \times 0.25 + 210 \times 0.20 + 291 \times 0.10$

$= 396$

$N_5 = N_0P_5 + N_1P_4 + N_2P_3 + N_3P_4 + N_4P_1$

$= 1000 \times 0.15 + 100 \times 0.30 + 210 \times 0.25 + 291 \times 0.20 + 396 \times 0.10$

$= \cancel{330}.\ 331$

The expected life of each item $= \sum_{i=1}^{5} i\,p_i$

$= 1 \times P_1 + 2 \times P_2 + 3 \times P_3 + 4 \times P_4 + 5 \times P_5$

$= 3.2$

The cost of individual replacement.

One month failure $= \frac{1000}{3.2} = 312.5$

Individual cost $= 312.5 \times 1.25 = 390.625$

End of Month	Individual Replacement	Total cost I.R + G.R	Average cost
1	100	$100 \times 1.25 + 1000 \times 0.50$ $= 625$	625
2	310	$310 \times 1.25 + 1000 \times 0.50$ $= 887.5$	443.7
3	601	$601 \times 1.25 + 1000 \times 0.50$ $= 1251.25$	417.08
4	997	$997 \times 1.25 + 1000 \times 0.50$ $= 1746.3$	436.56
5			

The average cost is lowest in the 3rd month, it is optimal to have a group replacement after every 3 month.

Further, the avg cost is more than 390.6 for individual replacement, the policy of individual replacement is Preferrable

② Let $p(t)$ be the Probability that a machine in a group 30 machines would break down in period t. The cost of repairing broken machine is ₹200. Preventive maintainance is performd by servicing all the 30 machines at the end T units of time. The preventive maintainance cost would be ₹50 per machine. Find optimal T which will minimizes expected total cost per period of servicing. Given that

$$p(t) = \begin{cases} 0.03 & \text{for } t = 1 \\ p(t-1) + 0.01 & \text{for } t = 2, 3, \ldots 10 \\ 0.13 & \text{for } t = 11, 12, \ldots \end{cases}$$

	$p(t)$	N
1	0.03	
1	0.04	
2	0.05	
3	0.06	
4	0.07	2
5	0.08	3
6	0.09	3
7	0.10	4
8	0.11	4
9	0.12	5
10	0.13	6
11	~~0.13~~	
~~12~~ 12	0	

$N_0 = 30$

$N_1 = N_0 P_1 = 30 \times 0.03 = 0.9 \approx 1$

$N_2 = N_0 P_2 + N_1 P_1$

$= 30 \times 0.04 + \cancel{0.9}\,1 \times 0.03$

$= 1.2 + 0.027 = 1.23$

$N_3 = N_0 P_3 + N_1 P_2 + N_2 P_1$

$= 30 \times 0.05 + 1 \times 0.04 + 1.23 \times 0.03$

$= 1.5 + 0.04 + 0.0369$

$= 1.357$

$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$

$= 30 \times 0.06 + 1 \times 0.05 + 1.23 \times 0.04 + 1.57 \times 0.03$

$= 1.8 +$

The expected life of each machine

$\sum_{i=1}^{11} i \times p_i = 6.38$

Avg no. of machines fail per month $= \frac{30}{6.38}$

$= 4.70 \approx 5$

Cost of individual replacement $= 5 \times 200$

$= 1000$

End of month	Individual replacement	Total cost	Average Cost
1	~~0.03~~ 1	$1 \times 200 + 30 \times 15 = 650$	
2	~~0.07~~ 2	$2 \times 200 + 30 \times 15 = 850$	
3	~~0.11~~ 2	$4 \times 200 + 30 \times 15 =$	
4	~~0.15~~ 4	$6 \times 200 + 30 \times 15$	
5	~~0.25~~ 8	$8 \times 200 + 30 \times 15$	
6	~~0.33~~ 11	$11 \times 200 + 30 \times 15$	

PERT & CPM Networks

Introduction:

A project is defined as a combination of interrelated activities all of which must be executed in a certain order to achieve a set goal.

A large and complex project involves usually no. of interrelated activities requiring men, machines and materials. It is impossible for the management to make and execute an optimum schedule for such a project just by intuition based on organisational capabilities and work pressure.

A system scientific approach has become a necessity for such project. So a no. of methods applying networks scheduling techniques have been developed.

Program Evaluation Review Technique (PERT) & Critical Path Method (CPM) are two of the many ~~methods~~ network technique which are widely used for planning, scheduling & controlling large complex projects.

~~So~~ PERT

The 3 main managerial functions for any projects are

Initial & terminal node are numbered as $i-j$ $(j>i)$. For example, if A is the event who's initial node i & terminal node is j, then it is denoted by diagrammatically by

$$(i) \xrightarrow{A} (j)$$

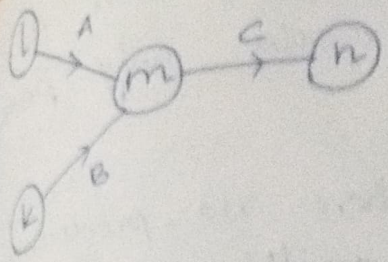
The name of the activity is written over the arrow & not inside the circle. The diagram in which arrow represents activity is called arrow diagram. ~~the initial & te.~~

The initial & terminal nodes of activites are also called tail and head events.
If an activity B can start immediately after an activity A then its is denoted by

$$(i) \xrightarrow{A} (M) \xrightarrow{B} (n)$$

A is called immediate predecessor of B and B is called immediate successor of A.

If C can ~~ea~~ start only after completing activites A and B then it is diagrammatically represented.



ation:

A is predecessor of B is denoted

as A < B

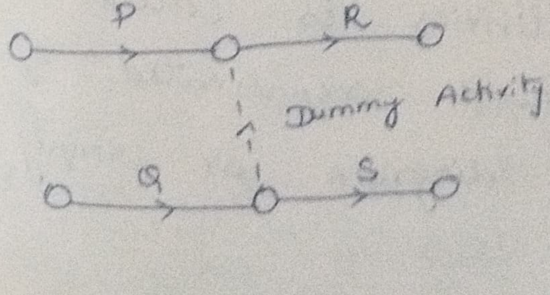
B is successor of A is denoted

as B > A.

If the project contains two or more activities which have some of their immediate predecessors in ~~commo~~ common then there is a need for introducing what is called dummy activity

Dummy activity is a imaginery activities which does not consume any resource and serve the purpore of indicating the predecessor or successor. Relationship clearly in any activity.

The need for any dummy activity is illustrated by the following



Let P, Q be the predecessors of R and Q ~~then~~ be the only predecessor of S.

Activities which have no predecessor are called start activites of the project.

All the start activities can be made to have the same initial node.

Activities which have no successors are called terminal activities of the project.

These can be made to have same terminal node (end node) of the ~~following~~ project.

A project consists of no. of activities to be performed in some technological sequence. For example, while constructing a building, the activity of laying the foundation should be done before the activity of erecting the walls of the building.

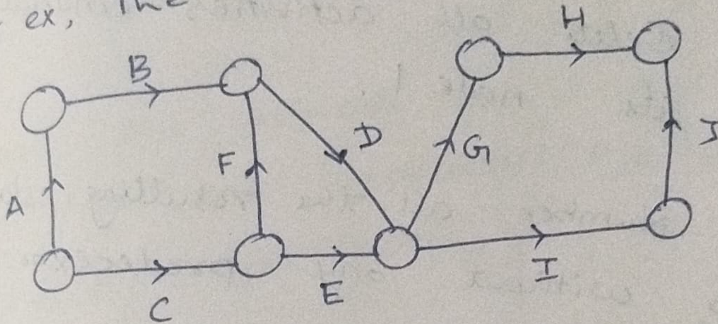
The diagram denoted all the activities of a project by arrows taking into account the technological sequence of the activities is called the project network represented by activity on arrow diagram or simply arrow diagram.

Note: There is another representation of project network representing activites on called AON diagram. To avoid confusion, we use only activity on arrow diagram throughout the text.

Rule for contructing a Project Network:

Rule 1 : There must be no loops.

For ex, The activities F, D, E



obviously form loop, which is obviously not ~~loop~~ possible in any real project network.

Rule 2 : Only one activity should connect any two nodes.

Rule 3 : No dangling should appear in a project network, i.e no node of any activity except the terminal node of the project should be left without any activity emanating from it. Such a node can be joined to the terminal node of the project to avoid.

Nodes may be numbered using the rules below:

(Ford and fulkerson)

Rule 1 : Number the start node which has no predecessor

Rule 2 : Delete all activities emanating from ~~its~~ node 1.

Rule 3 : Number all the resulting start nodes without any predecessor as 2, 3.

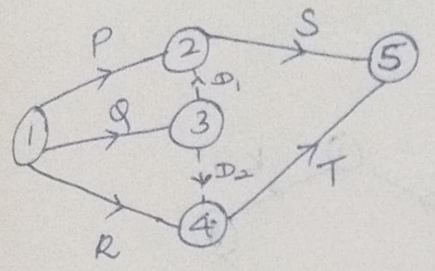
Rule 4 : Delete all the activities originating from the start nodes 2, 3 in step 3.

Rule 5 : Number all the resulting ~~old~~ new start nodes without any predecessor next to last number used in step 3.

Rule 6 : Repeat the process, until the terminal node without any successor activities is reached & Number this ~~terminal~~ nodes suitably.

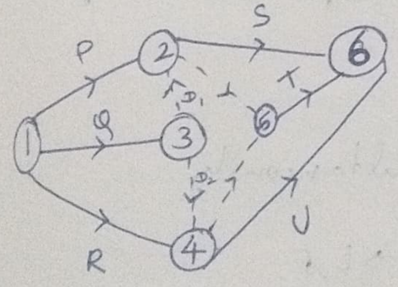
Problem:

If there are 5 activities P, Q, R, S, T such that P, Q, R have no immediate predecessor but S & T have immediate predecessors P, Q and Q, R respectively. Represent this situation in network.



Q. Draw the network for a project whose activities and predecessors are gn below.

Activity :	P	Q	R	S	T	U
Predecessor :	–	–	–	P, Q	P, R	Q, R



~~Q For the project, draw the~~

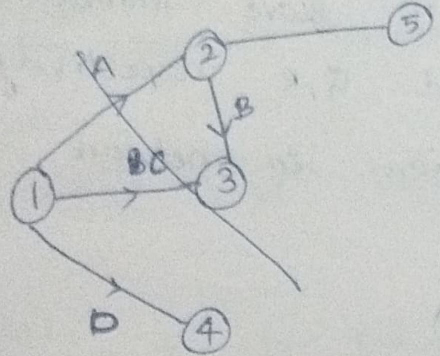
Q. Draw the network for the project whose activities with their predecessor relationship are given below.

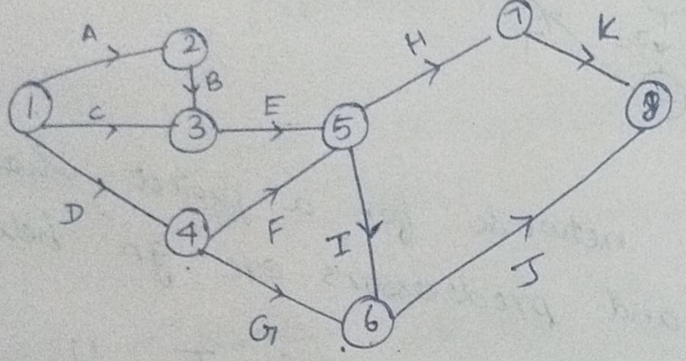
A, C, D can start simultaneously;

E > B, C ;

F, G > D , H, I > E, F

J > I, G , K > H , B > A





4 Construct the network for the problem whose activities and relationships are as given below.

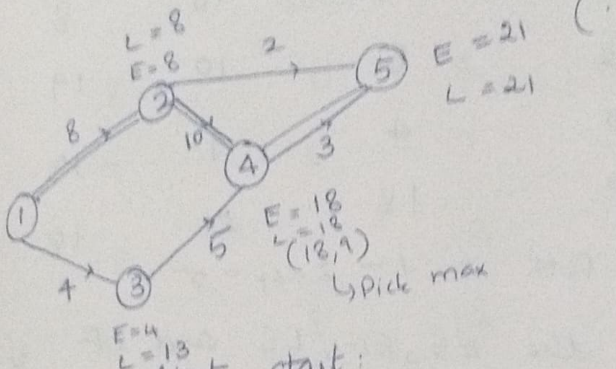
A, B, E can start simultaneously

B, C > A, G, F > D, C H > E, F.

etwork Computations - Earliest Completion Time of a project & Critical Path

Compute the earliest start, earliest finish, latest start, lastest finish of each activity of the project given below.

Activity	1-2	1-3	2-4	2-5	3-4	4-5
Duration	8	4	10	2	5	3



To find earliest start:

$ES_j = Max[ES_i + t_{ij}]$

where ES_i = Earliest start time of all activities from node i to t_{ij} is the estimated duration of the activity i to j

Formula for the latest start time of the activities from the event i of activity i to j:

$LS_i = Min(LS_j - t_{ij})$ for all defined i to j activities,

where t_{ij} is the estimated duration of the activity i to j

Critical Path (8) marks

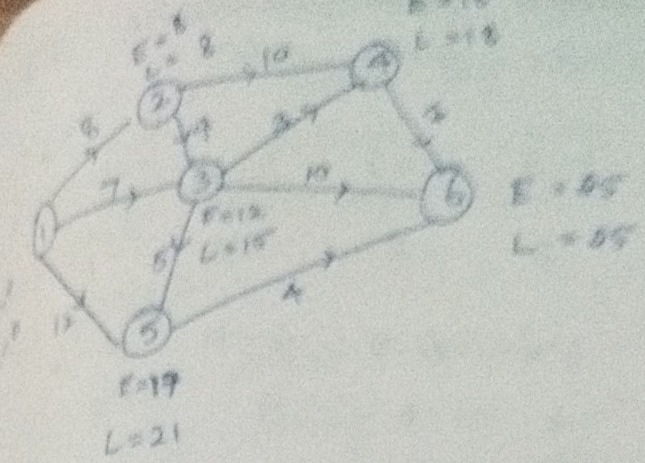
Path connecting the first initial node to the very last terminal node of longest duration in any project network is called critical path.

Activity	Duration	Earliest Start	Finish $EF = ES + t_{ij}$	Latest Start $LS = LF - t_{ij}$	finish L
1-2	8	0	8	0	8
1-3	4	0	4	9	13
2-4	10	8	18	8	18
2-5	2	8	10	19	21
3-4	5	4	9	13	18
4-5	3	18	21	18	21

Critical Path = 1 - 2 - 4 - 5

(2) Calculate the ES, EF, LS and LF of each activity of the project given below. and determine the critical path.

Activity	1-2	1-3	1-5	2-3	2-4	3-4	3-5	3-6	4-6	5-6
Duration (in weeks)	8	7	12	4	10	3	5	10	7	4

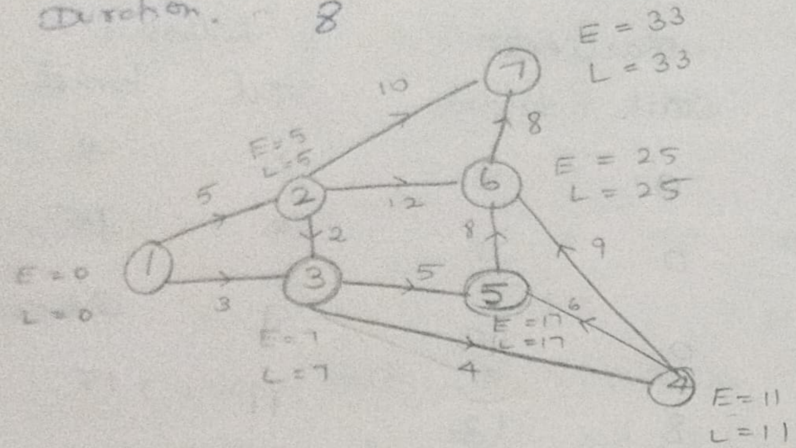


Activity	Duration	Earliest Start	Earliest finish	Latest Start	Latest finish
1-2	8	0	8	0	8
1-3	7	0	7	8	15
1-5	12	0	12	9	21
2-3	4	8	12	11	15
2-4	10	8	18	8	18
3-4	3	12	15	15	18
3-5	5	12	17	16	21
3-6	10	12	22	15	25
4-6	7	18	25	18	25
5-6	4	17	21	21	25

Critical Path → 1-2-4-6

③ Draw the network and determine the critical path for given data

Activity	1-2	1-3	2-3	2-6	2-7
Duration	5	3	2	12	10
Activity	3-5	3-4	4-5	4-6	5-6
Duration	5	4	6	9	8
Activity	6-7				
Duration.	8				



Critical Path = 1-2-3-4-5-6-7

Floats:

Total float of an activity is defined as the difference between the latest finish and the earliest finish of the activity (or) the difference between the latest start and the earliest start of the activity.

Formula:

$$\left.\begin{array}{l}\text{Total float of an} \\ \text{activity } i-j\end{array}\right\} = (LF)_{ij} - (EF)_{ij}$$

$$= (LS)_{ij} - [ES]_{ij}$$

Total float of the activity is the amount of time by which that particular activity may be delayed without affecting the duration of the project.

If the total float is positive then it may be indicate that the resource for the activity is more than adequate.

If the total float of an activity is zero, it may indicate that the resources are just adequate for the activity.

If the total float of an activity is negative, it may indicate that the resources for that activity are inadequate.

There are 3 other types of floats for an activity, namely

* free float
* Independent float
* Interference float

Free Float : (F.F)

Free float of an activity is the portion of the total float, which can be used for rescheduling that activity without affecting the succeeding activity. It can be calculated as follows

Free ~~float of an~~ activity (i to j) } = Total float of an activity (i to j) − (L−E) of

Free float of i−j } = Total float of i−j − (L−E) of Event j

where L = Latest occurrence

E = Earliest Occurrence

Obviously, free float <= total float ~~of~~ for any activity

<u>Independent</u> <u>Float</u>: (I.F)

I.F of an activity is the amount of time by which the activity can be rescheduled without affecting the preceding or succeeding activity of that activity.

<u>Formula</u>:

I.F of activity i−j } = Free float of i−j − (L−E) of event i

I.F <= free float for any activity

Thus, $I.F \leq F.F \leq T.F$

Interference Float:

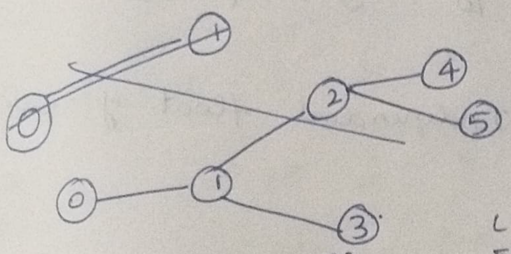
Interference float of an activity i-j is nothing but slack of head event j

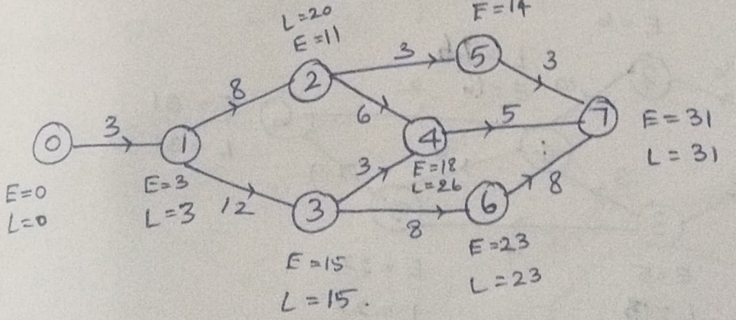
Interference float = Total float (i-j) − Free float (i-j)

Problem

Calculate free float, total float & independent float for the project, whose activities are given below.

Activity	0-1	1-2	1-3	2-4	2-5	3-4	3-6	4-7	5-7	6-7
Duration (weeks)	3	8	12	6	3	3	8	5	3	8





Critical path 0-1-3-6-7

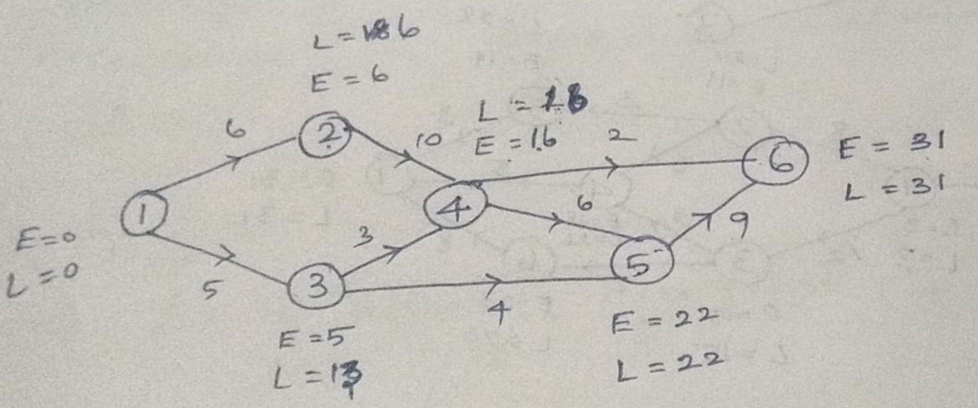
Project duration 31 weeks

Activity	Duration	E.S	E.F	L.S	L.F	T.F	F.F	I.F
0-1	3	0	3	0	3	0	0	0
1-2	8	3	11	12	20	9	0	0
1-3	12	3	15	3	15	0	0	0
2-4	6	11	17	20	26	9	1	-8
2-5	3	11	14	25	28	14	0	9
3-4	3	15	18	23	26	8	0	0
3-6	8	15	23	15	23	0	0	0
4-7	5	18	23	26	31	8	8	0
5-7	3	14	17	28	31	14	14	0
6-7	8	23	31	23	31	0	0	0

(2) Draw the network and determine the CP for the given data.

Jobs	1-2	2-3	2-4	3-4	3-5	4-5	4-6	5-6
Duration (weeks)	6	5	10	3	4	6	2	9

Find T.F, F.F, Independent float of each activity



Critical path. 1 - 2 - 4 - 5 - 6

Project duration 31 weeks.

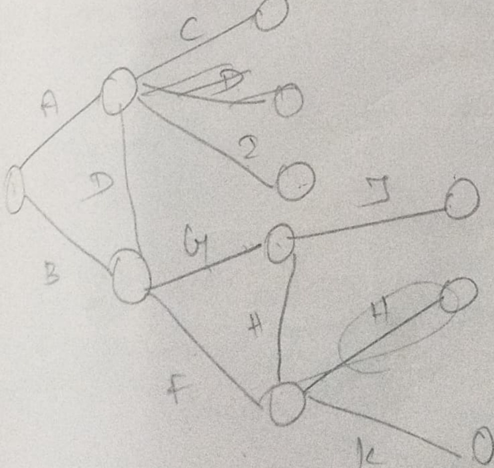
Duration	E.S	E.F	L.S	L.F	T.F	F.F	I.F
6	0	6	0	6	0	0	0
5	0	5	8	~~6~~ 13	8	0	8
10	6	16	6	16	0	0	0
3	5	8	13	16	8	8	0
4	5	9	18	22	13	~~5~~ 13	5
6	16	22	16	~~16~~ 22	0	0	0
2	16	18	29	~~16~~ 31	13	13	13
9	22	31	22	~~22~~ 31	0	~~13~~ 0	0

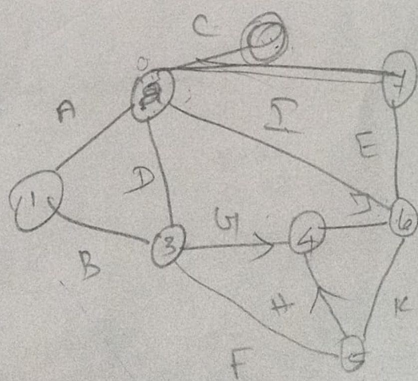
A small CPM project consist of 11 activities A, B ... J, K the precedence relationships are A, B can start immediately. A < C, D, I ; B < G, F ; D < G, F ; G, H < J ; I, J, K < E F < H, K

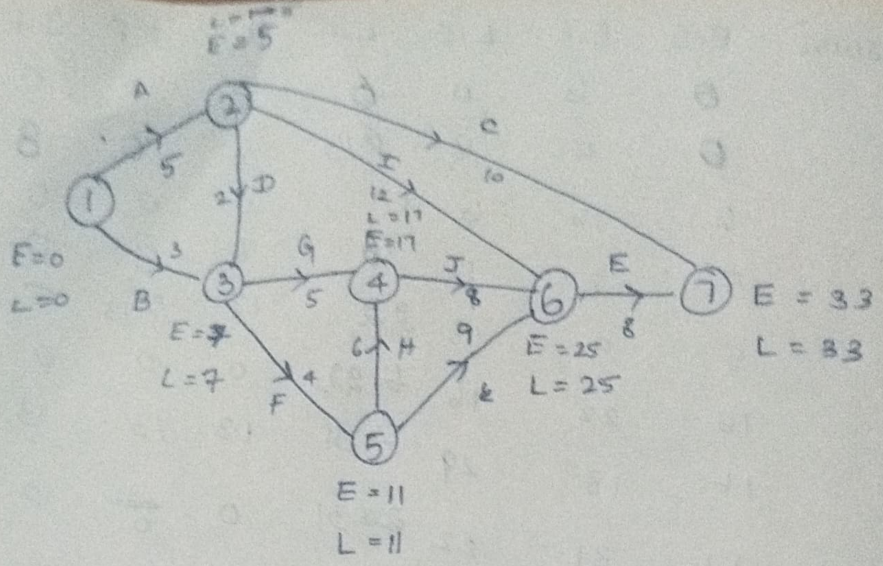
The durations of the activities are as below.

Activity	A	B	C	D	E	F	G	H	I	J	K
Duration (days)	5	3	10	2	8	4	5	6	12	8	9

Determine the critical path, find F.F, T.F, I.F







Critical path : 1-2-7 = 15

1-2-6-7 = 25

1-2-3-4-6-7 = 28

1-2-3-5-6-7 = 28

1-3-5-~~4~~-6-7 = 29

1-~~2~~-3-5-4-6-7 = 33

Project duration = 33 days.

(16 marks) (X)

...ruct the network for the project whose ...ivities and three time estimates of these ...ivities are given below. compute

...) Expected duration of each activity (t_e)

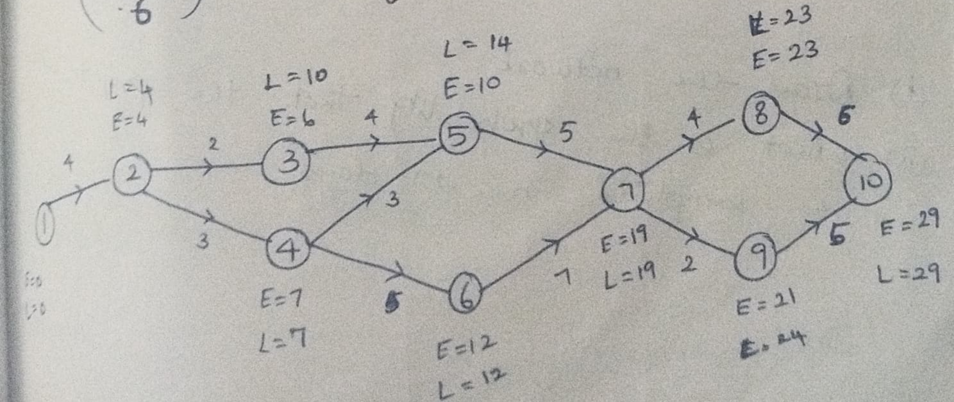
...) Expected variance of each activity (σ^2)

...) Expected variance of project length (σ_c^2) ↳ Critical path variance

...vity	t_o	t_m	t_p	t_e	σ^2
-2	3	4	5	4	1/9
-3	1	2	3	2	1/9
-4	2	3	4	3	1/9
-5	3	4	5	4	1/9
-5	1	3	5	3	4/9
-6	3	5	7	5	4/9
-7	4	5	6	5	1/9
-7	6	7	8	7	1/9
-8	2	4	6	4	4/9
-9	1	2	3	2	1/9
-10	4	5	8	6	4/9
-10	3	6	7	5	4/9

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$ [∵ Expected duration]

$$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$$ [∵ Expected variance]



Critical path : 1-2-4-6-7-8-10

Project duration : 29 weeks

$$\left.\begin{array}{l}\text{Expected variance}\\ \text{of project Length}\end{array}\right\} = \begin{array}{l}\text{Sum of expected variance}\\ \text{of all critical activites}\end{array}$$

$$= 15/9 \quad \left(\overset{1-2}{1/9} + \overset{2-4}{1/9} + \overset{4-6}{4/9} + \underset{6-7}{1/9} + \underset{7-8}{4/9} + \underset{8-10}{4/9} \right)$$

② A project consist of the following activities and time estimates.

Activity	Least time (days)	greatest time (days)	Most likely time (days)
1-2	3	15	6
1-3	2	14	5
1-4	6	30	12
2-5	2	8	5
2-6	5	17	11
3-6	3	15	6
4-7	3	27	9
5-7	1	7	4
6-7	2	8	5

(i) Draw the network

(ii) what is the probability that the project will be kompleted in 27 days?

Sol:

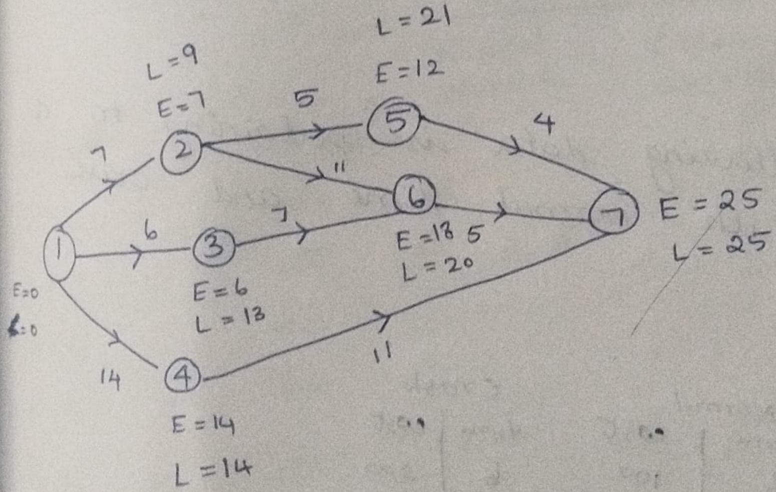
Optimistic time or least time = t_o or a

Pessimistic time or greatest time = t_p or b

Most likely time = t_m or m

$$t_e = \frac{t_o + 4t_m + t_p}{6} \; ; \quad \sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$$

t_e	σ^2
7	4
6	4
14	16
5	1
11	4
7	4
11	16
4	1
5	1



Critical path : 1-4-7

Project duration : 25 days

~~Expected~~

Expected Variance of project length } = ~~25~~ 16+16 = 32

$$Z = \frac{T_E - T_S}{\sigma_c}$$

$\sigma_c = \sqrt{32} = 5.656$

$T_S = 27$ $T_E = 25$

$$Z = \frac{27 - 25}{5.656} = \frac{+2}{5.656} = +0.35$$

(0.3 → 0.05 ↓ in table)

Probability of $T_S <= 27$ = Probability of $Z <= 0.35$

$P(T_S <= 27) = P(Z <= 0.35)$

$= 0.1368$

$= 13.68\%$

~~Hashing~~

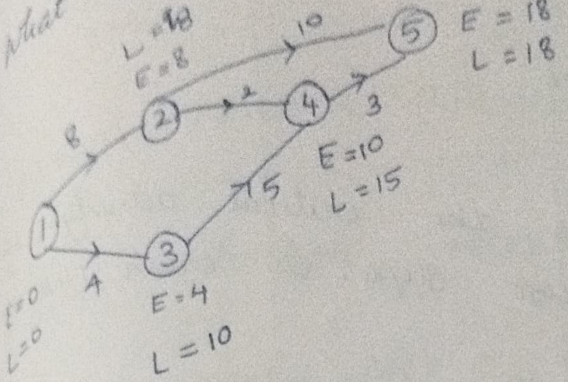
Crashing:

① The following data is pertaining to a project with normal time and crash time.

Activity	Normal time	Normal cost	Crash time	Crash cost
1-2	8	100	6	200
1-3	4	150	2	350
2-4	2	50	1	90
2-5	10	100	5	400
3-4	5	100	1	200
4-5	3	80	1	100
		580		

If the indirect cost is ₹100 per day, find the least cost schedule (optimum duration)

What is the minimum duration?



Critical path = 1-2-5

Normal project duration = 18 days.

Total cost = direct cost + indirect cost

= 580 + 1800 (18×100)

= 2380

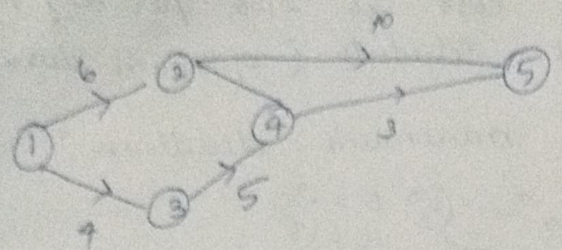
cost slope table :

$$\text{cost slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal duration} - \text{Crash duration}}$$ ~~Normal~~

Activity	cost slope
✓1-2	50 *
1-3	100
2-4	40
✓2-5	60
3-4	25
4-5	10

Check the cost slope of critical path

Take the minimum value (50,60) = 50.



Step 1 : 1-2 is the critical activity of least cost slope. So crash 1-2 by 2 days.

Current critical path is Same (1-2-5).

Current duration 16 days

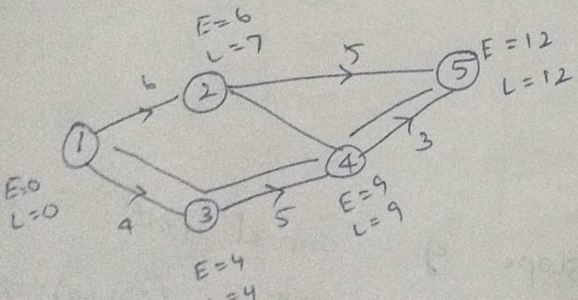
Current total cost = previous total cost + Inc in direct cost − dec in indirect cost

= 2330 + (2×50) − (2× 100)

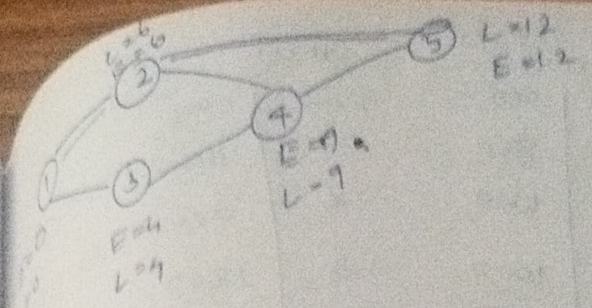
= 2330 + 100 − 200

= 2280

Step 2 : Crash 2-5 by 4 days



Critical Path is changing, So reduce 5 days to 4 days in 2-5 interval



current critical path $\Rightarrow$ 1-2-5

1-3-4-5

current duration $\Rightarrow$ 12 days

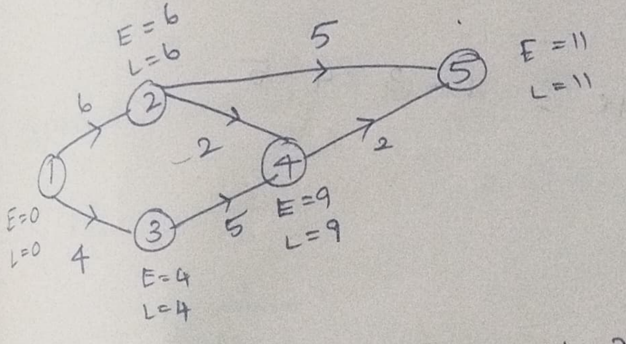
current total cost $= 2280 + (4 \times 60) - (4 \times 100)$

$= 2280 + 240 - 400$

$= 2120$

Step 3 : crash the activities 2-5 & 4-5 by 1 day

if 2 days ↓ critical path changes



current critical paths 1-2-5

1-3-4-5

current total cost = Previous cost + Inc in direct cost − dec in indirect cost

2 activities crashed

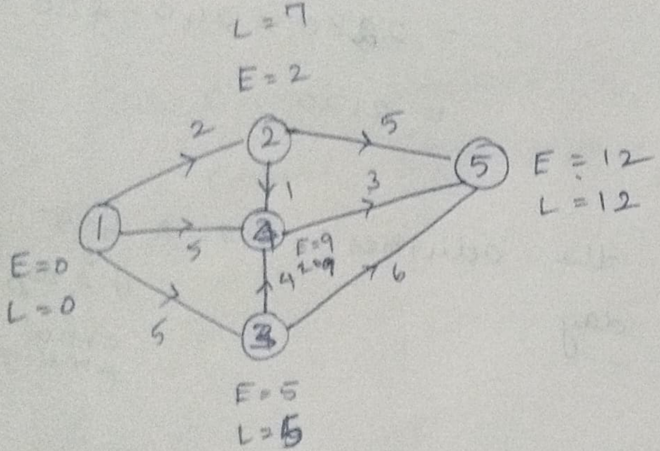
$= 2120 + [(1 \times 60) + (1 \times 10)] - [1 \times 100]$

$= 2120 + 70 - 100$

$= 2090$

②

Activity	Normal		crash	
	time	cost	time	cost
1-2	2	800	1	1400
1-3	5	1000	2	2000
1-4	5	1000	3	1800
2-4	1	500	1	500
2-5	5	1500	3	2100
3-4	4	2000	3	3000
3-5	6	1200	4	1600
4-5	3	900	2	1600



Critical path = 1 - 3 - 4 - 5

Queuing Model

Unit - 4

Definitions ⓧ

Arrival rate - λ

Model I

$(M/M/1) : (\infty / FCFS)$

No. of channel = 1
Capacity - ∞

- To find the average (expected) no. of units units (L_s)

$$L_s = \frac{\rho}{1-\rho} \quad \text{where } \rho = \frac{\lambda}{\mu} < 1$$

μ ↳ Service rate

- To find the average length of queue

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$= \frac{\rho^2}{1-\rho}$$

- Expected waiting time in the system

$$W_s = \frac{L_s}{\lambda}$$

$$= \frac{1}{\mu - \lambda}$$

- Waiting time in the queue

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{\lambda}{\mu(\mu - \lambda)}$$

- Expected waiting time of the customer who has to wait $(W/W>0)$

$$(W/W>0) = \frac{1}{\mu-\lambda}$$

- Expected length of the non-empty queue

$$(L/L>0) = \frac{\mu}{\mu-\lambda}$$

- Probability of queue size $\geq N$ is ρ^N

- Probability of waiting time in the system $\geq t$ $= \int_t^\infty (\mu-\lambda) e^{-(\mu-\lambda)w} \, dw$

- Probability of Waiting time in the queue $\geq t$

$$= \int_t^\infty \rho(\mu-\lambda) e^{-(\mu-\lambda)w} \, dw$$

- Traffic Intensity (or) Utilization factor

$$= \lambda/\mu$$

- Idle time $= 1-\rho$

Problem

The railway marshalling yard goods train arrival at a rate of 30 trains per day. Assuming that inter arrival time follows an exponential distribution and the service time distribution is also exponential with an average of 36 mins. Calculate the following

(i) The mean queue size

(ii) The probability that queue size exceeds 10.

(iii) If the input of the train increases to an average 33 per day, what will be the changes in (i) & (ii)

$$\lambda = \frac{30}{1} \quad ; \quad \mu = \frac{1}{36}$$

$$\lambda = \frac{30}{24 \times 60} = \frac{1}{48} \text{ min}$$

(i) $L_s = \frac{\rho}{1-\rho}$ $\Rightarrow \rho = \lambda/\mu = \cancel{\frac{48}{36}} = \frac{36}{48}$

$= 0.75$

$$= \frac{0.75}{0.25} = 3$$

$$L_q = 3 - 0.75$$

$$= 2.25$$

(ii) The probability that queue size exceeds 10 $\Big\} = \rho^N$

$= \rho^{10}$

$= 0.75^{10}$

$= 0.056$

(iii) $\lambda = \dfrac{33}{24 \times 60} = 0.0229$

$\rho = \dfrac{\lambda}{\mu} = \dfrac{0.0229}{1/36}$

$= 0.8244$

$L_s = \dfrac{\rho}{1-\rho}$

$= \dfrac{0.8244}{1-0.8244}$

$= 4.69 \approx 5$

$L_q = L_s - \rho$

$= 4.69 - 0.8244$

$= 3.87$

The probability the queue size exceeds 10 $= \rho^{10}$

$= (0.8244)^{10}$

$= 0.145$

② In a Supermarket the average arrival rate of customer is 10 in every 30 mins.

$\lambda = \dfrac{10}{30} = 1/3$

Following poisson process the average time taken by the casier to list and calculate the customers purchases is 2.5 mins, following exponential distributions What is the probability that queue length exceeds 6, what is the expected time

...spent by a customer in the system.

$\mu = 1/2.5$

Probability that queue length exceeds 6 $\Big\} = \rho^6 = (0.833)^6$

$= 0.3348$

$\rho = \lambda/\mu = \frac{2.5}{3} = 0.833$

$W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{2.5} - \frac{1}{3}}$ ~~$= 14.99$ min~~

≈ 15 mins.

(3) In a public telephone booth, the arrivals are on the average 15 per hr. A call on the average takes 3 mins. If there is just 1 phone, find

(i) Expected no. of callers in the booth at any time.

(ii) The proportion the time the booth is expected to be idle.

$\lambda = \frac{15}{60} = \frac{1}{4}$ $\qquad \mu = 1/3$

(i) $(L / L > 0) = \frac{\mu}{\mu - \lambda} = \frac{1/3}{1/3 - \frac{1}{4}} = \frac{\frac{1}{3}}{1/12} = 4$

(ii) the booth expected to be idle $= 1 - \rho$

$1 - \lambda/\mu = 1 - \frac{3}{4} = 1/4 = 0.25$

(4) A TV repair man finds that the time spent on his job has an exponential distribution with mean 30 mins. If he repairs sets in the order in which they come in. If the arrival of sets is poisson with an average rate of 10/8 hr per day. What is the expected idle time day?

How many jobs are ahead of average set just brought in?

$$\lambda = \frac{10}{8} \text{ per hour}$$

$$\mu = \cancel{\frac{1}{30 \times 60} = \frac{1}{1800}} \text{ hour} \quad \frac{1}{30} \times 60 = 2 \text{ hrs}$$

Model 2:

$(M|M|1):(N/FCFS)$

Here the capacity of the system is limited say N. Infact arrivals will not exceed N in any case. The various measures of the model are

(1) $P_0 = \dfrac{1-\rho}{1-\rho^{N+1}}$ where $\rho = \dfrac{\lambda}{\mu}$ $\left\{\dfrac{\lambda}{\mu} > 1 \text{ is allowed}\right\}$

(2) $P_n = \dfrac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} \rho^n$ for $n = 0, 1, 2, \ldots N$

$L_s = P_0 \sum_{n=0}^{N} n\rho^n$

$L_q = L_s - \frac{\lambda}{\mu}$

$W_s = \frac{L_s}{\lambda}$

$W_q = \frac{L_q}{\lambda}$

Problem

If a period of two hrs in a day (8 to 10 am) trains arrive at the yard every 20 mins

$\lambda = \frac{1}{20}$, $\mu = \frac{1}{36}$

but the service time continuous to be 36 mins. Then calculate for this period

(i) The probability that the yard is empty

(ii) Average queue length assuming that capacity of the yard is 4 trains only.

$N = 4$ $\qquad \frac{\lambda}{\mu} = \frac{36}{20} = \frac{9}{5} = 1.8$

(i) $P_0 = \frac{1-\rho}{1-\rho^{N+1}}$

$= \frac{1-1.8}{1-(1.8)^5}$

$= \cancel{\frac{-0.8}{1-}} = 0.04$

(ii) $L_s = P_0 \sum_{n=0}^{N} n\rho^n$

$= 0.04(1.8 + 6.48 + \cancel{41.99}\ 17.496 + 41.9904)$

$= 0.04(67.7664)$

$= 2.710656 \approx 2.9 \approx 3$ trains.

$L_q = L_s - \frac{\lambda}{\mu}$

$= \cancel{3+1.8}$ $0.7 - 1.8$

$= 0.9$

≈ 1 train.

② A barber shop has space to accomodate only 10 customers. He can service only 1 person at a time. If a customer comes to his shop and finds it full, he goes to the next shop. Customers randomly arrive at an average rate $\lambda = 10/\text{hrs}$ and the barbers service time is negative exponential with an average of $1/\mu = 5$ mins/customers. Find P_0, P_n

$\lambda = \frac{10}{60}$ mins $= 1/6$

$\rho = \frac{\lambda}{\mu} = \frac{5}{6} = 0.83$

$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$

$= \frac{1-0.83}{1-(0.83)^{11}}$

$= \frac{0.17}{1-0.1345}$

$= \frac{0.17}{0.865}$

$= 0.1965$

$P_n = \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} \rho^n$

$= 0.1965 \times [0.83 + 0.6889 + 0.5\ldots + 0.4745 + 0.3999 + 0.324 + 0.2713 + 0.2252 + 0.1869 + 0.1551]$

$= 0.799$

In a one man barber shop the customers arrive following poisson process at an average rate of 5 per hr and they are served according to exponential distribution with an average service rate of 10 mins assuming that only 5 seats are available for waiting customers. Find the average time a customer, ~~find the average time a customer~~ spends in the system.

$\lambda = 5/60 = 1/12 \quad \mu = 1/10$

$W_s = \frac{L_s}{\lambda} \Rightarrow L_s = P_0 \sum_{n=0}^{N} n \rho^n$

$\rho = \frac{\lambda}{\mu} = \frac{10}{12} = \frac{5}{6} = 0.83$

$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$

$= \frac{1-0.83}{1-(0.83)^6}$

$= \frac{0.17}{1-0.327}$

$= \frac{0.17}{0.673}$

~~$= 0.2805$~~

$= 0.25$

$L_s = 0.25 \times [0.83 + 1.3778 + 1.7154 + 1.898 + 1.969]$

$= 0.25 \times 7.7902$

$= 1.94755$

$W_s = \frac{L_s}{\lambda}$

$= 1.9455 \times 12$

$= 23.37$