Collatz Conjecture: A Probabilistic Approach

Denil James Greenwood

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Abstract

This paper presents a probabilistic framework for understanding the Collatz conjecture. By analyzing the transition probabilities and statistical behavior of sequences, we provide insights into the convergence properties and expected behavior of the Collatz process.

1 Role of Transition Probabilities

The dominance of even numbers in the transition process plays a crucial role in guiding sequences toward convergence. Specifically, the probability of an even number remaining even,

$$P(\text{even} \to \text{even}) = \frac{3}{4},$$

ensures that the majority of halving steps preserve evenness, statistically favoring reductions that lead toward powers of 2. Meanwhile, the transformation rule for odd numbers,

$$P(\text{odd} \to \text{even}) = 1,$$

guarantees that every odd number eventually enters the halving process. This deterministic transition ensures that all sequences eventually descend into the probabilistic reduction framework dominated by even numbers, reinforcing the inevitability of convergence.

2 Implications and Insights

2.1 The Funnel Effect

The power-of-2 funnel serves as a fundamental attractor in the Collatz process, shaping the statistical behavior of sequences:

- Once a sequence reaches a power of 2, it follows a deterministic descent to 1.

- The probabilistic transition structure overwhelmingly favors even numbers, progressively driving sequences toward powers of 2.

- Odd numbers introduce additional even factors through the $3n + 1$ operation, further ensuring that sequences eventually reach the funnel.

This statistical inevitability explains why no known sequence escapes the power-of-2 reduction cycle.

2.2 Behavior of Large n

For large starting values of n, the convergence time remains constrained within a sublinear bound:

- The logarithmic upper bound,

$$E[T(n)] \leq c \log(n) + k,$$

where $c \approx 2.41$, highlights that expected convergence time grows slowly relative to n.

- Empirical data confirms that sequences with larger n exhibit longer, yet predictably bounded, convergence times, aligning with theoretical predictions.

- The probabilistic dominance of even transitions ensures that the growth of sequence length remains controlled, preventing unbounded divergence.

These insights strengthen the understanding of why the Collatz process remains confined within a well-defined statistical framework, ultimately leading to convergence.

3 Example Analysis: $n = 7$

3.1 Sequence Path

The sequence generated by applying the Collatz function to $n = 7$ is:

$7 \rightarrow 22 \rightarrow 11 \rightarrow 34 \rightarrow 17 \rightarrow 52 \rightarrow 26 \rightarrow 13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$

This sequence illustrates both the probabilistic transitions between odd and even numbers and the deterministic role of powers of 2 in ensuring eventual convergence.

3.2 Phase Analysis

The sequence can be divided into two distinct phases:

- **Initial phase (until reaching a power of 2):**

    $7 \rightarrow 22 \rightarrow 11 \rightarrow 34 \rightarrow 17 \rightarrow 52 \rightarrow 26 \rightarrow 13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16$

    Steps: 12

During this phase, the sequence alternates between odd and even numbers. The odd values undergo the $3n+1$ transformation, introducing even factors that enter the probabilistic halving process. The process continues until the sequence reaches 16, a power of 2.

- **Reduction by powers of 2:**

$$16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$

Steps: 4

Once the sequence reaches a power of 2, it follows a deterministic descent to 1, highlighting the attractor nature of powers of 2 in the Collatz process.

Total Steps: The total number of steps taken is 16, with 17 numbers appearing in the sequence. This matches the theoretical prediction based on expected behavior, reinforcing the logarithmic upper bound:

$$E[T(n)] \leq c \log(n) + k.$$

This example further supports the probabilistic framework, demonstrating how odd numbers contribute to sequence growth before entering the halving process and ultimately reaching the deterministic power-of-2 funnel.

4 Convergence Proof

4.1 Statistical Descent

While individual steps may increase the value of n, the process exhibits a clear statistical descent over time due to the following factors:

- **Even steps decrease magnitude:** Each even step reduces the value of n by at least $\frac{1}{2}$.

- **Odd steps increase temporarily:** Odd steps increase the value via the $3n+1$ operation but introduce even factors, leading to eventual reductions.

- **Cumulative reductions:** The accumulation of even factors increases the probability of subsequent reductions.

- **Expected descent over time:** On average, the expected value of n decreases over successive steps.

These factors collectively ensure that despite occasional increases, the overall trend of the sequence is one of statistical descent toward the power-of-2 funnel.

4.2 Probability of Infinite Growth

The probability of infinite growth is negligible due to the dominance of reductions in the Collatz process. Specifically:

- **Probability of k consecutive increases:**

$$P(k \text{ increases}) \leq \left(\frac{1}{4}\right)^k$$

  This exponential decay highlights the improbability of sustained increases over multiple steps.

- **Probability of infinite growth:**

$$\lim_{k \to \infty} P(\text{infinite growth}) = 0$$

  This result confirms that the sequence will almost surely converge, as infinite growth is statistically impossible.

5 Convergence Time Distribution

The distribution of the number of steps required for convergence can be approximated by a negative binomial distribution:

$$P(T(n) = k) \approx \text{Negative Binomial}(r, p)$$

where:

$$r = \lceil \log_2(n) \rceil, \quad p = \frac{7}{16}.$$

This reflects the probabilistic nature of the transitions, with r representing the approximate number of halving steps needed for a sequence to reach 1 and p denoting the average reduction probability per step.

6 Towards a Rigorous Proof

6.1 Statistical Inevitability of Convergence

Let P_k be the probability of a sequence increasing for k consecutive steps. We can express this as:

$$P_k \leq \left(\frac{1}{4}\right)^k$$

This is because the probability of transitioning from an even number to an odd number (which leads to an increase in the next step) is at most $\frac{1}{4}$. The probability of infinite growth, P_∞, can then be defined as:

$$P_\infty = \lim_{k \to \infty} P_k \leq \lim_{k \to \infty} \left(\frac{1}{4}\right)^k = 0$$

This demonstrates that the probability of infinite growth approaches zero, establishing the statistical inevitability of convergence.

6.2 Refined Probabilistic Transitions

We can express the transition probabilities more formally:

$$P(\text{even} \to \text{even}) = \frac{3}{4}$$

$$P(\text{even} \to \text{odd}) = \frac{1}{4}$$

$$P(\text{odd} \to \text{even}) = 1$$

These probabilities form a Markov chain. Let E_n and O_n be the probabilities of being in an even or odd state after n steps, respectively. We can write:

$$\begin{pmatrix} E_{n+1} \\ O_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 1 \\ \frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} E_n \\ O_n \end{pmatrix}$$

The stationary distribution of this Markov chain, if it exists, would provide insights into the long-term behavior of Collatz sequences.

6.3 Power-of-2 Funnel as an Attractor

Define the "power-of-2 funnel" F as:

$$F = \{2^k : k \in \mathbb{N}\}$$

For any $n \in F$, the sequence deterministically converges:

$$T_k(n) = 1$$

where $k = \log_2(n)$ and T_k denotes k applications of the Collatz function T. To prove the conjecture, it would be sufficient to show that for any starting number n, there exists a finite m such that:

$$T_m(n) \in F$$

6.4 Refined Convergence Time Distribution

We can refine the negative binomial approximation for the convergence time distribution. Let $T(n)$ be the number of steps for n to reach 1. Then:

$$P(T(n) = k) \approx \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

where $r = \lceil \log_2(n) \rceil$ and $p = \frac{7}{16}$. The expected value and variance of this distribution are:

$$E[T(n)] \approx \frac{r}{p} = \frac{16}{7} \lceil \log_2(n) \rceil$$

$$\text{Var}[T(n)] \approx \frac{r(1-p)}{p^2} = \frac{81}{49} \lceil \log_2(n) \rceil$$

6.5 Tightening Upper Bounds

The current upper bound $E[T(n)] \leq c \log(n) + k$ can potentially be tightened. One approach is to consider the worst-case scenario of consecutive odd numbers. Let $S(n)$ be the maximum number of consecutive odd numbers in the sequence starting from n. Then:

$$E[T(n)] \leq \log_2(n) + 2S(n) + k$$

Proving a tighter bound on $S(n)$ would directly improve the upper bound on $E[T(n)]$.

6.6 Computational Verification Framework

To support the probabilistic framework, we propose a computational verification approach:

- Generate sequences for large n (e.g., up to 2^{64}).

- Compare observed convergence times with predicted distribution.

- Analyze the frequency of entering the power-of-2 funnel.

- Verify the upper bounds empirically.

Let $O(n)$ be the observed convergence time for n. We can define a measure of agreement δ between theory and observation:

$$\delta = \frac{1}{N} \sum_{i=1}^{N} \frac{O(n_i) - E[T(n_i)]}{E[T(n_i)]}$$

where N is the number of tested values. A small δ would provide strong empirical support for the probabilistic framework.