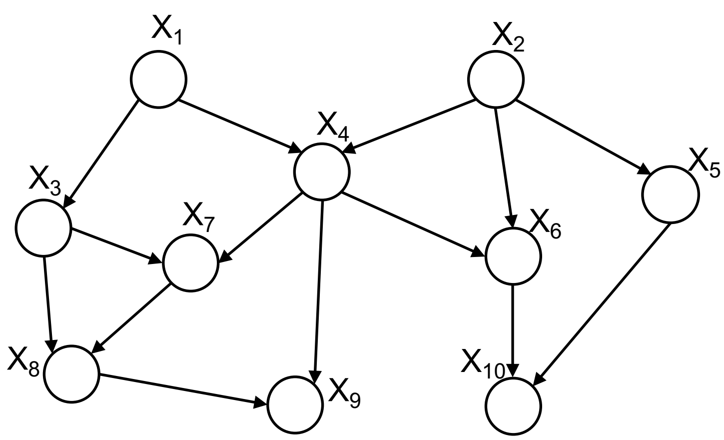
Written Assignment 4: Solution

Deadline: November 29th, 2023

Instruction: You may discuss these problems with classmates, but please complete the write-ups individually. (This applies to BOTH undergraduates and graduate students.) Remember the collaboration guidelines set forth in class: you may meet to discuss problems with classmates, but you may not take any written notes (or electronic notes, or photos, etc.) away from the meeting. Your answers must be **typewritten**, except for figures or diagrams, which may be hand-drawn. Please submit your answers (pdf format only) on **Canvas**.

Q1. Bayes Nets: Independence (20 points)

Consider the following Bayesian network with 10 variables $\{X_1, X_2, \ldots, X_{10}\}$.



Which of the following statements are true:

1. $X_6 \perp\!\!\!\perp X_1 \mid X_2, X_4$
   Answer. True

2. $X_6 \perp\!\!\!\perp X_9 \mid X_4$
   Answer. False

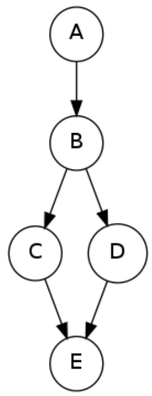
3. $X_3 \perp\!\!\!\perp X_9 \mid X_8$
   Answer. False

4. $X_1 \perp\!\!\!\perp X_2 \mid X_6$
   Answer. False

5. $X_4 \perp\!\!\!\perp X_8 \mid X_3, X_7$
   Answer. True

Q2. Bayes Nets: Inference (50 points)

Assume the following Bayes Net and corresponding CPTs.



A	$P(A)$
0	0.200
1	0.800

B	A	$P(B\|A)$
0	0	0.400
1	0	0.600
0	1	0.200
1	1	0.800

C	B	$P(C\|B)$
0	0	0.600
1	0	0.400
0	1	0.600
1	1	0.400

D	B	$P(D\|B)$
0	0	0.800
1	0	0.200
0	1	0.600
1	1	0.400

E	C	D	$P(E\|C,D)$
0	0	0	0.200
1	0	0	0.800
0	1	0	0.600
1	1	0	0.400
0	0	1	0.800
1	0	1	0.200
0	1	1	0.800
1	1	1	0.200

We are going to compute two conditional probabilities using the variable elimination method:

1. Compute $P(B = 1 \mid E = 1)$. In this question, we first keep table entries that are consistent with the evidence $E = 1$. We then perform variable elimination on hidden variables A, C, D.

   - (8 points) First, we eliminate hidden variable A, which results in a new factor. Provide the table of the new factor.
     Answer. We have: $f_1(B) = \sum_a P(a)P(B \mid a)$, resulting in:

B	$f_1(B)$
0	0.24
1	0.76

   - (8 points) Second, we eliminate hidden variable C, which results in a new factor. Provide the table of the new factor.
     Answer. We have: $f_2(B, D, E = 1) = \sum_c P(c \mid B)P(E = 1 \mid C, D)$, resulting in:

B	D	E = 1	$f_2(B, D, E = 1)$
0	0	1	0.64
0	1	1	0.2
1	0	1	0.64
1	1	1	0.2

- (8 points) Third, we eliminate hidden variable D, which results in a new factor. Provide the table of the new factor.
  Answer. We have: $f_3(B, E = 1) = \sum_d P(d \mid B) f_2(B, d, E = 1)$, resulting in:

B	E = 1	$f_3(B, E = 1)$
0	1	0.552
1	1	0.464

- (5 points) Finally, we join all remaining tables and normalize. Provide the value of $P(B = 1 \mid E = 1)$.
  Answer. We join: $f_4(B, E = 1) = f_1(B) \times f_3(B, E = 1)$, which results in:

B	E = 1	$f_4(B, E = 1)$
0	1	0.13248
1	1	0.35264

  Finally, we perform normalization: $P(B = 1 \mid E = 1) = \frac{f_4(B=1,E=1)}{f_4(B=0,E=1)+f_4(B=1,E=1)}$.
  Therefore, $P(B = 1 \mid E = 1) = 0.727$

2. Compute $P(A = 1 \mid C = 0, E = 0)$. Similarly, in this question, we first keep table entries that are consistent with the evidence $C = 0, E = 0$. We then perform variable elimination on hidden variables B, D.

- (8 points) First, we eliminate hidden variable B, which results in a new factor. Provide the table of the new factor.

  Answer. We have: $f_1(A, C = 0, D) = \sum_b P(b \mid A) P(C = 0 \mid b) P(D \mid b)$, resulting in

A	C = 0	D	$f_1(A, C = 0, D)$
0	0	0	0.408
0	0	1	0.192
1	0	0	0.384
1	0	1	0.216

- (8 points) Second, we eliminate hidden variable D, which results in a new factor. Provide the table of the new factor.

A	C = 0	E= 0	$f_2(A, C = 0, E = 0)$
0	0	0	0.2352
1	0	0	0.2496

Answer. We have: $f_2(A, C = 0, E = 0) = \sum_d f_1(A, C = 0, d)P(E = 0 \mid C = 0, d)$, which results in:

- (5 points) Finally, we join all remaining tables and normalize. Provide the value of $P(A = 1 \mid C = 0, E = 0)$.
  Answer. We join: $f_3(A, C = 0, E = 0) = P(A)f_2(A, C = 0, E = 0)$, which results in:

A	C = 0	E= 0	$f_3(A, C = 0, E = 0)$
0	0	0	0.04704
1	0	0	0.19968

Finally, we join $P(A = 1 \mid C = 0, E = 0) = \frac{f_3(A=1,C=0,E=0)}{f_3(A=0,C=0,E=0)+f_3(A=1,C=0,E=0)}$, resulting in: $P(A = 1 \mid C = 0, E = 0) = 0.809$

Q3. Bayes Nets: Sampling (30 points)

In this question, we will work with the same Bayes net and CPTs as **Q2**.

Q3.1. Rejection Sampling (7 points) In this question, we will perform rejection sampling to estimate $P(C = 1 \mid B = 1, E = 1)$. Perform one round of rejection sampling, using the random samples given in the table below. Variables are sampled in the order A, B, C, D, E.

0.320	0.037	0.303	0.318	0.032	0.969	0.018	0.058	0.908	0.249

Note that the sampling attempt should stop as soon as you discover that the sample will be rejected. In that case mark the assignment of that variable and write "none" for the rest of the variables. When generating random samples, use as many values as needed from the table below, which we generated independently and uniformly at random from $[0, 1)$. Use numbers from left to right. To sample a binary variable W with probability $P(W = 0) = p$ and $P(W = 1) = 1 - p$ using a value a from the table, choose $W = 0$ if $a < p$ and $W = 1$ if $a \geq p$.

Choose the value (0 or 1) that each variable gets assigned to:

- A: 1 B: 0 C: none D: none E: none
- Which variable will get rejected? B.
  In rejection sampling, you reject any sample for which the variables' values do not match the values of the evidence variables in what you are trying to estimate. In this case, any sample where $B \neq 1$ or $E \neq 1$ is rejected. Only B and E can ever be rejected, in this case B was rejected because its sampled value was 0.

Q3.2. Likelihood Weighting (7 points) In this question, we will perform likelihood weighting to estimate $P(C = 1 \mid B = 1, E = 1)$. Generate a sample and its weight, using the random samples given in the table below. Variables are sampled in the order A, B, C, D, E.

When generating random samples, use as many values as needed from the table below, which we generated independently and uniformly at random from $[0, 1)$. Use numbers from left to right. To sample a binary variable W with probability $P(W = 0) = p$ and $P(W = 1) = 1 - p$ using a value a from the table, choose $W = 0$ if $a < p$ and $W = 1$ if $a \geq p$.

0.249	0.052	0.299	0.773	0.715	0.550	0.703	0.105	0.236	0.153

Select the assignments to the variables you sampled.

- A: 1   B: 1   C: 0   D: 0   E: 1
- What is the weight for the sample you obtained above? 0.64
  For likelihood weighting, the evidence variables, in this case B and E, are fixed and the sample is given weight equal to the product of the probabilities of the evidence variables taking on those values given the sampled values for their parents.

  In this case, the weight is equal to: $P(B = 1 \mid A = 1)P(E = 1 \mid C = 0, D = 0) = 0.64$.

Q3.3. Gibbs Sampling (16 points). We observe the value of the variable $E = 1$. In this question, we will perform Gibbs sampling, using the random samples given in the table below.

0.320	0.037	0.303	0.318	0.032	0.969	0.018	0.058	0.908	0.249

When generating random samples, use as many values as needed from the table below, which we generated independently and uniformly at random from $[0, 1)$. Use numbers from left to right. To sample a binary variable W with probability $P(W = 0) = p$ and $P(W = 1) = 1 - p$ using a value a from the table, choose $W = 0$ if $a < p$ and $W = 1$ if $a \geq p$.

Our current sample is $(A = 1, B = 0, C = 1, D = 1, E = 1)$. We are going to generate 2 new samples using Gibbs sampling as follows:

1. To generate the first new sample, the non-evidence variable B is chosen. What would be the value of each variable in the new sample?

   Answer. We first need to compute:

$$
\begin{aligned}
&P(B = 0 \mid A = 1, C = 1, D = 1, E = 1) \\
&\propto P(B = 0 \mid A = 1)P(C = 1 \mid B = 0)P(D = 1 \mid B = 0) = 0.2 \times 0.4 \times 0.2 \\
&P(B = 1 \mid A = 1, C = 1, D = 1, E = 1) \\
&\propto P(B = 1 \mid A = 1)P(C = 1 \mid B = 1)P(D = 1 \mid B = 1) = 0.8 \times 0.4 \times 0.4
\end{aligned}
$$

Therefore,

$$P(B = 0 \mid A = 1, C = 1, D = 1, E = 1) = 0.11$$
$$P(B = 1 \mid A = 1, C = 1, D = 1, E = 1) = 0.89$$

Since $0.11 < 0.32$, the new sample is $(A = 1, B = 1, C = 1, D = 1, E = 1)$

2. To generate the second new sample, the non-evidence variable D is chosen. What would be the value of each variable in the new sample?

Answer. Based on the new sample we just generated, we need to compute:

$$P(D = 0 \mid A = 1, B = 1, C = 1, E = 1) \propto P(D = 0 \mid B = 1)P(E = 1 \mid C = 1, D = 0) = 0.6 \times 0.4$$
$$P(D = 1 \mid A = 1, B = 1, C = 1, E = 1) \propto P(D = 1 \mid B = 1)P(E = 1 \mid C = 1, D = 1) = 0.4 \times 0.2$$

Therefore,

$$P(D = 0 \mid A = 1, B = 1, C = 1, E = 1) = 0.75$$
$$P(D = 1 \mid A = 1, B = 1, C = 1, E = 1) = 0.25$$

Since $0.75 > 0.037$, the new sample is $(A = 1, B = 1, C = 1, D = 0, E = 1)$.