CS 471/571(Fall 2023): Introduction to Artificial Intelligence

Lecture 4: Constraint Satisfaction Problems (Part 1)

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Source: http://ai.berkeley.edu/home.html

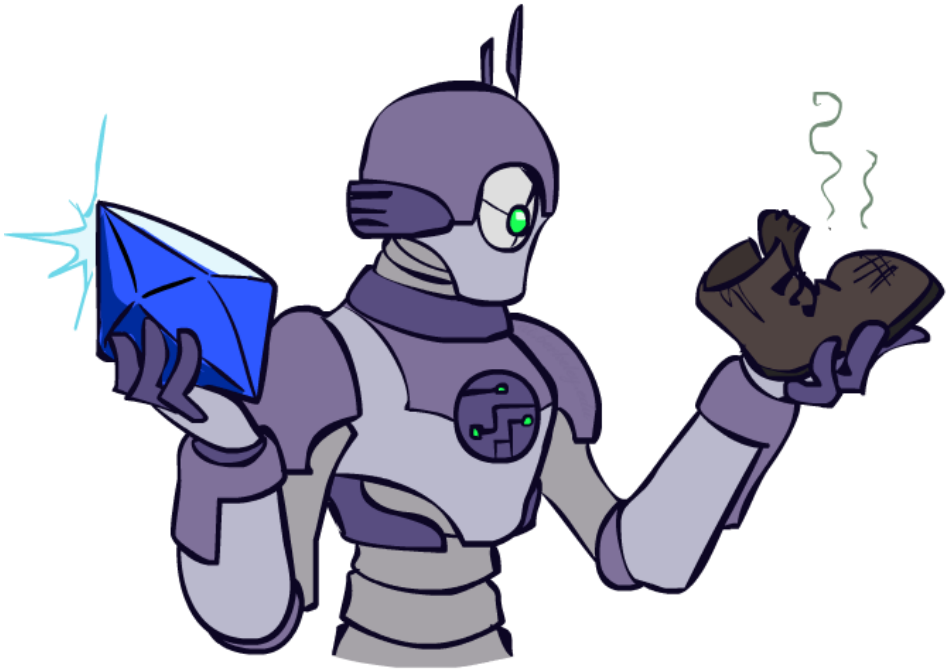
Reminder

- Written assignment 1: Search
  - Deadline: Oct 11th, 2023

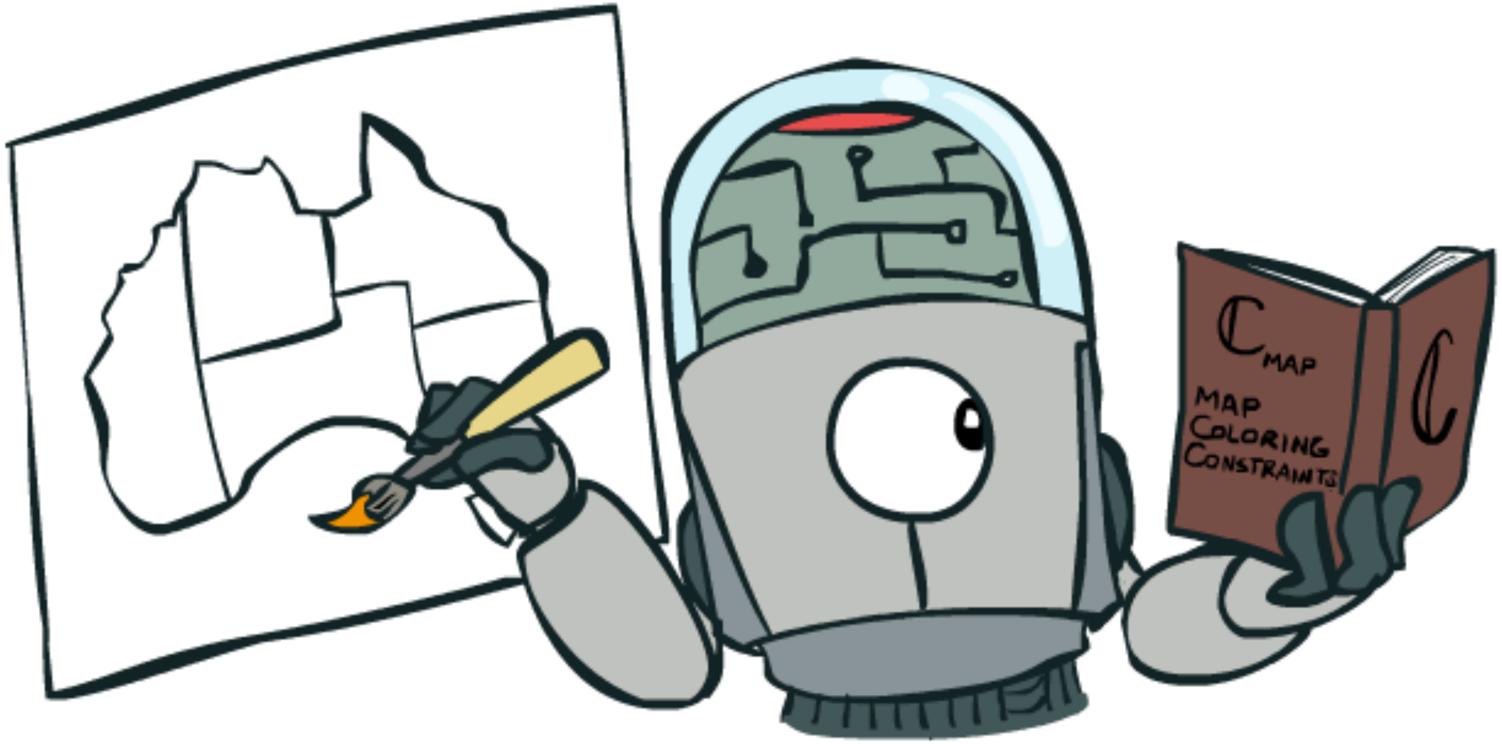
- Project 1: Search
  - Deadline: Oct 16th, 2023

Today

- Constraint Satisfaction Problems
  - Backtracking Search
  - Filtering
  - Ordering



Constraint Satisfaction Problems

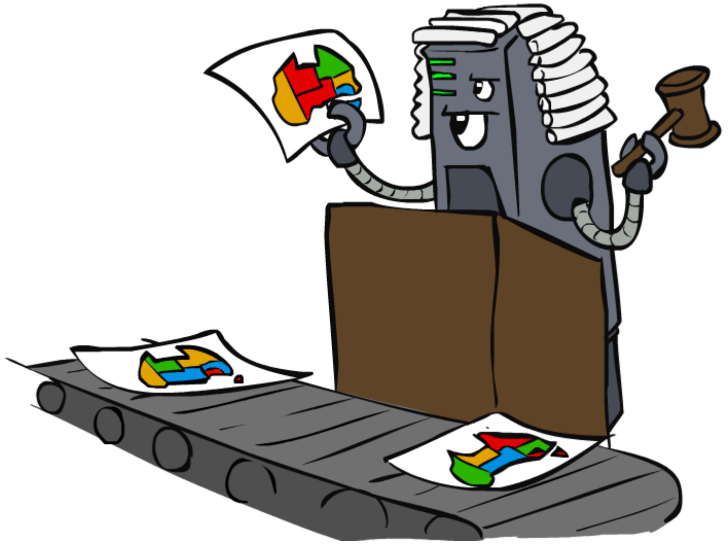


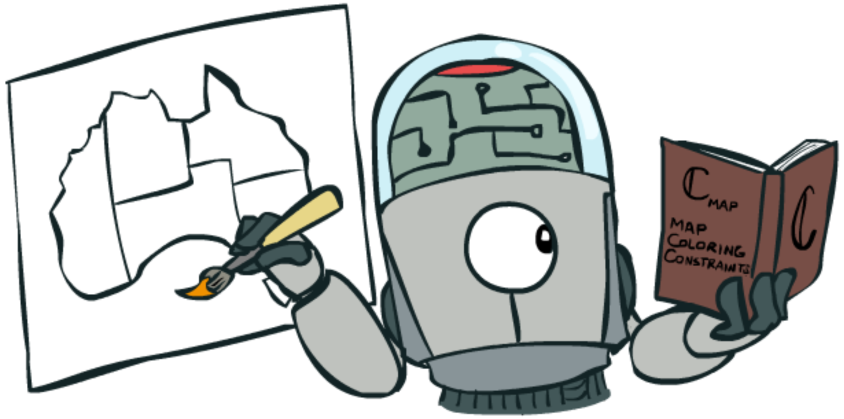
Constraint Satisfaction Problems

- Standard search problems:
  - State is a “black box”: arbitrary data structure
  - Goal test can be any function over states
  - Successor function can also be anything

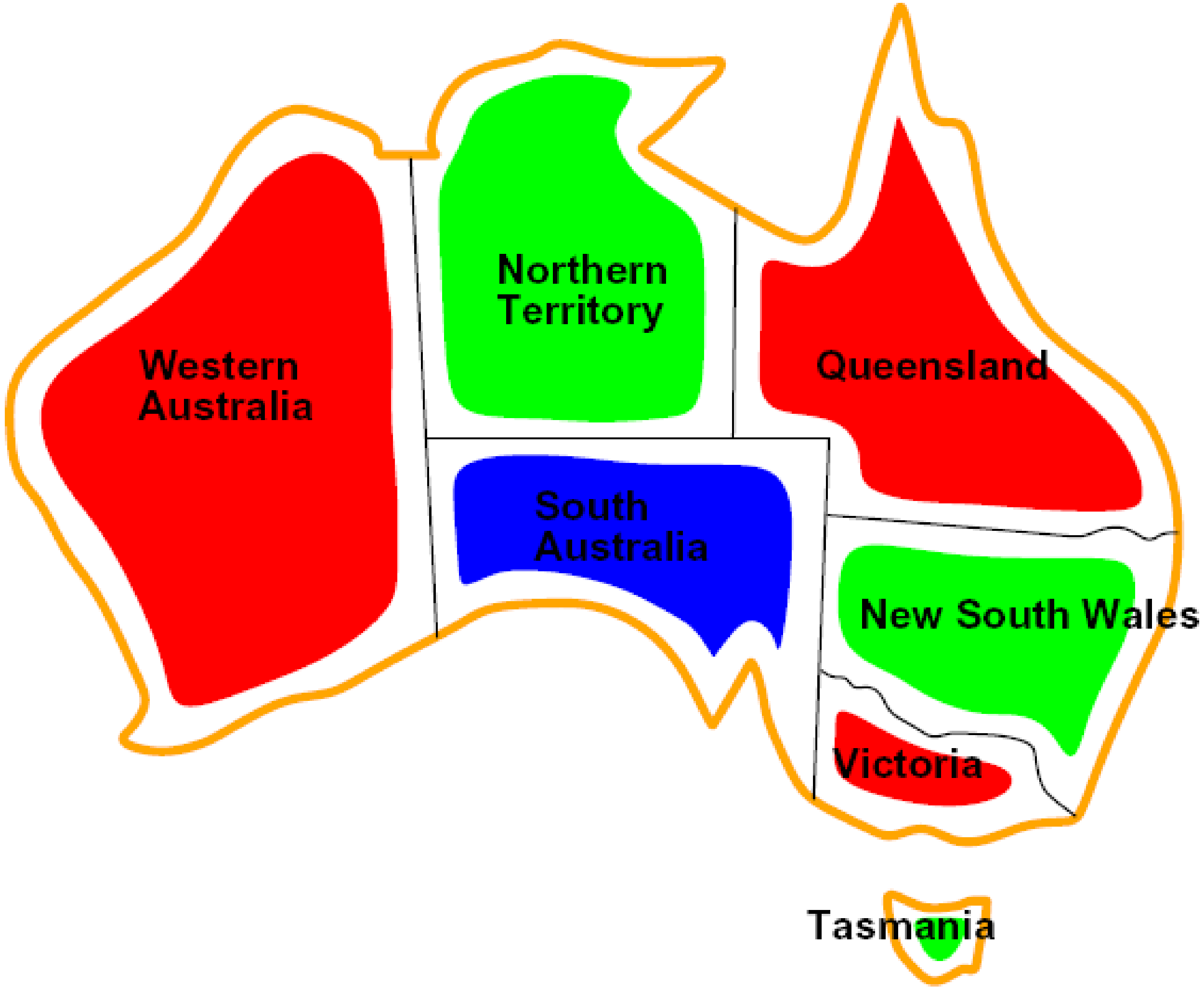
- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - State is defined by variables X_i with values from a domain D (sometimes D depends on i)
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables

- Allows useful general-purpose algorithms with more power than standard search algorithms





CSP Examples



Example: Map Coloring

- Variables: WA, NT, Q, NSW, V, SA, T

- Domains: $D = \{\text{red}, \text{green}, \text{blue}\}$

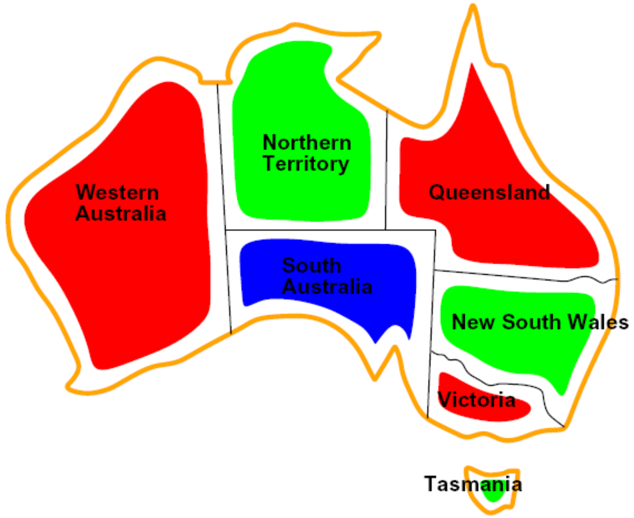
- Constraints: adjacent regions must have different colors

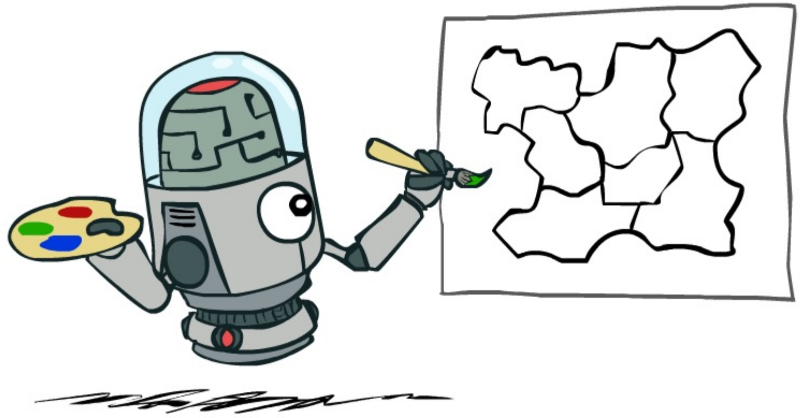
  Implicit: $WA \neq NT$

  Explicit: $(WA, NT) \in \{(\text{red}, \text{green}), (\text{red}, \text{blue}), \ldots\}$

- Solutions are assignments satisfying all constraints, e.g.:

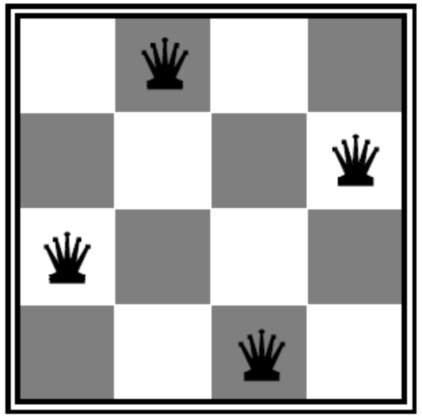
  {WA=red, NT=green, Q=red, NSW=green, V=red, SA=blue, T=green}

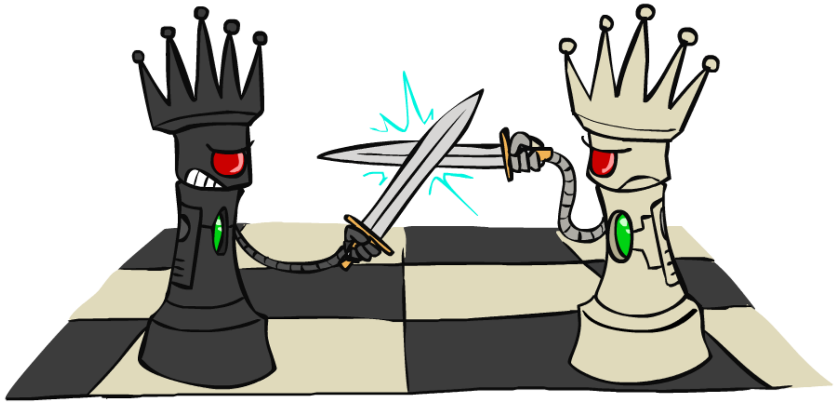




Example: N-Queens

- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0, 1\}$
  - Constraints





$$\forall i,j,k \;\; (X_{ij}, X_{ik}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i,j,k \;\; (X_{ij}, X_{kj}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i,j,k \;\; (X_{ij}, X_{i+k,j+k}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i,j,k \;\; (X_{ij}, X_{i+k,j-k}) \in \{(0,0),(0,1),(1,0)\}$$

$$\sum_{i,j} X_{ij} = N$$

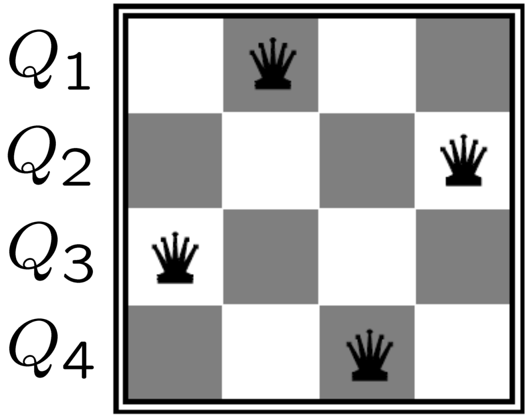
Example: N-Queens

- Formulation 2:
  - Variables: Q_k
  - Domains: $\{1, 2, 3, \ldots N\}$
  - Constraints:

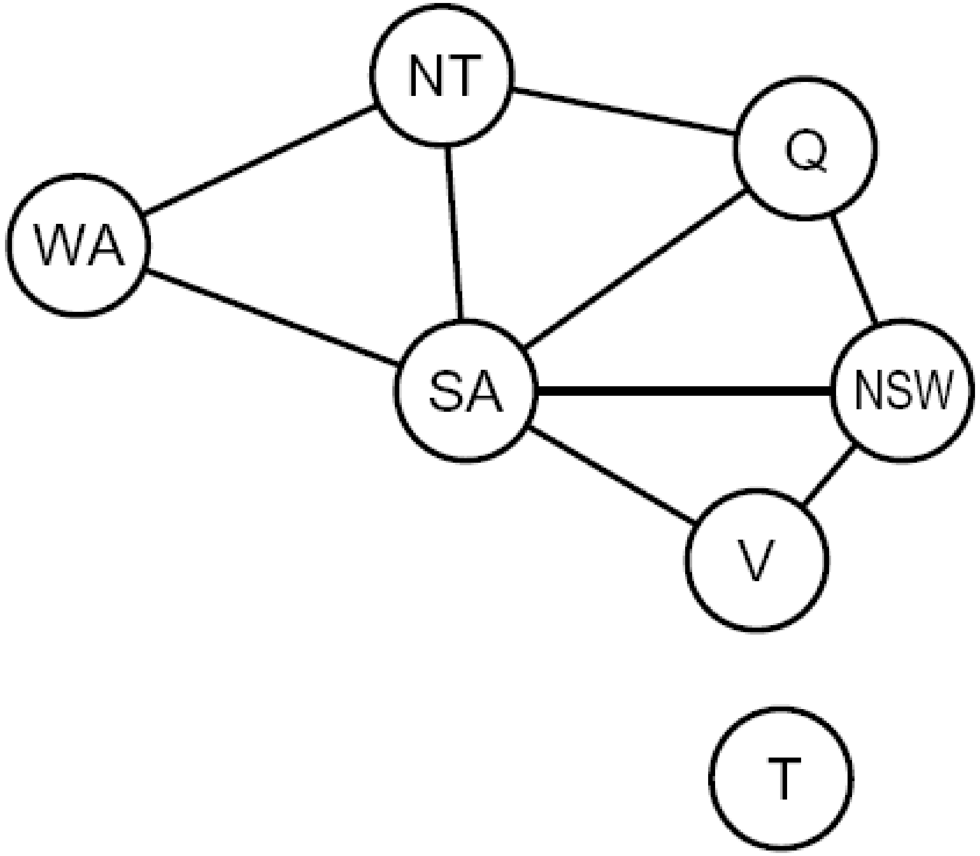
Implicit: $\forall i, j$ non-threatening(Q_i, Q_j)

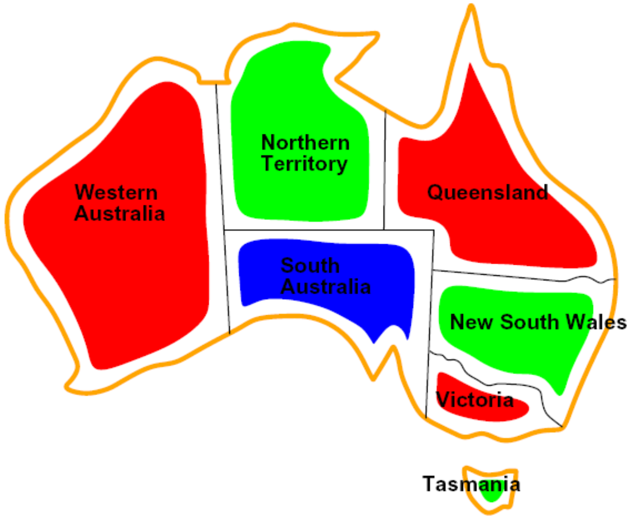
Explicit: $(Q_1, Q_2) \in \{(1,3), (1,4), \ldots\}$

$\ldots$



Constraint Graphs



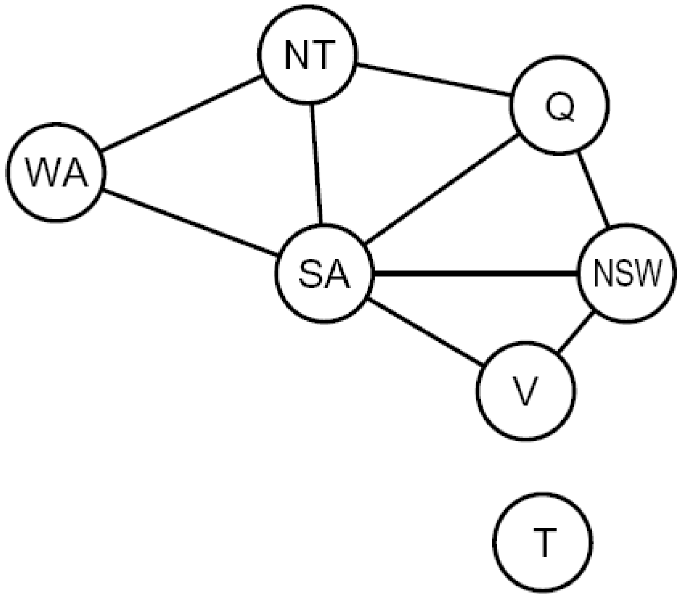


Constraint Graphs

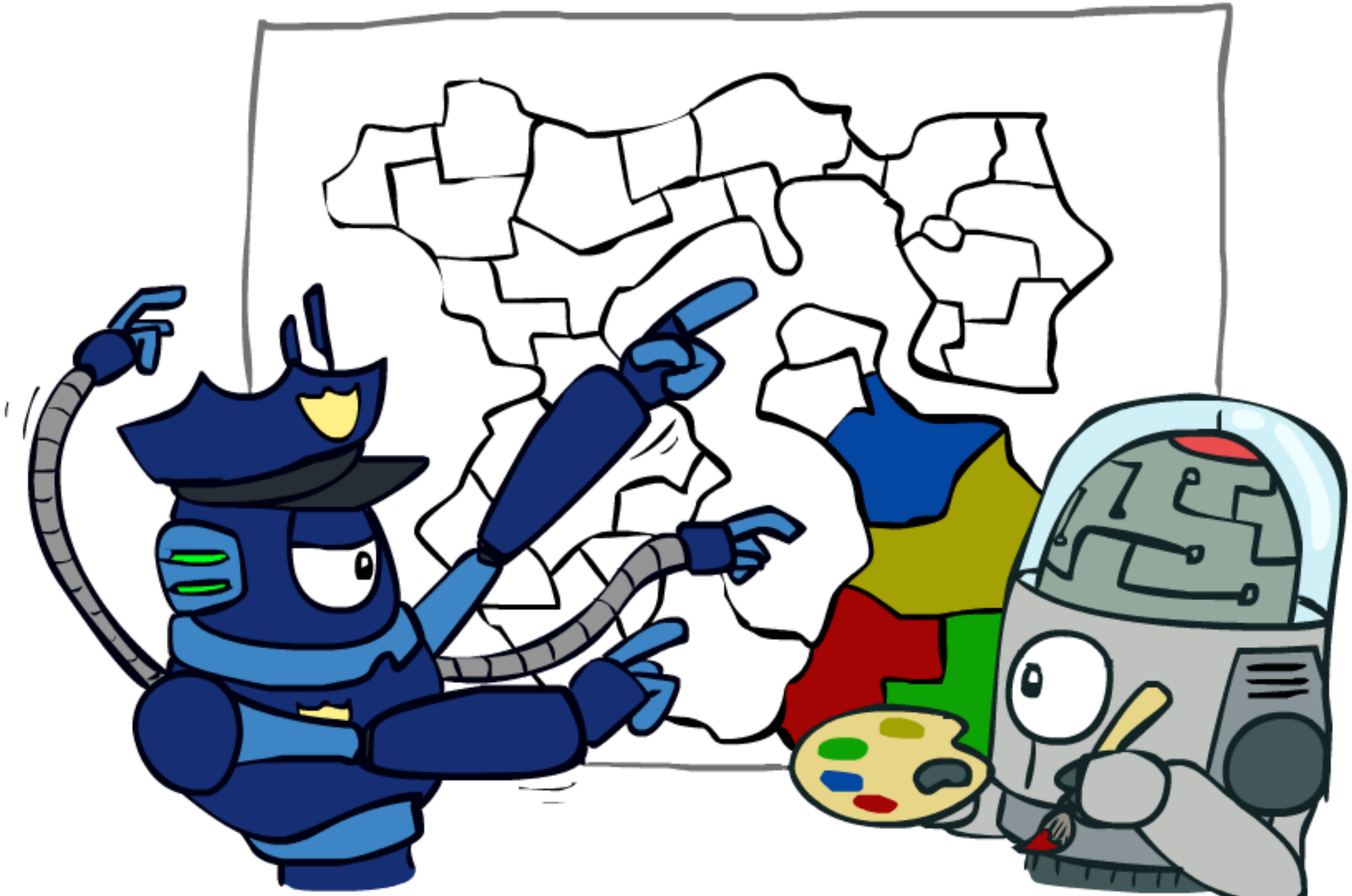
- Binary CSP: each constraint relates (at most) two variables

- Binary constraint graph: nodes are variables, arcs show constraints

- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!



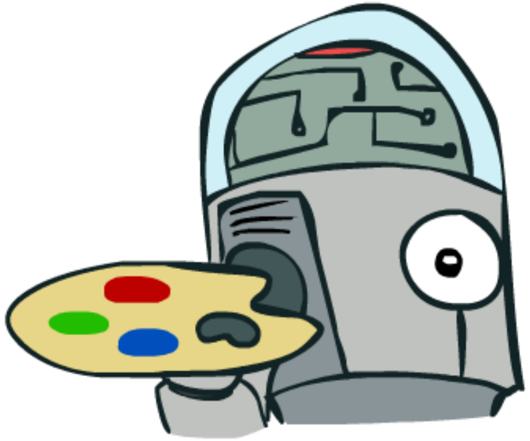
Varieties of CSPs and Constraints



Varieties of CSPs

- Discrete Variables
  - Finite domains
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)

  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end days for each job



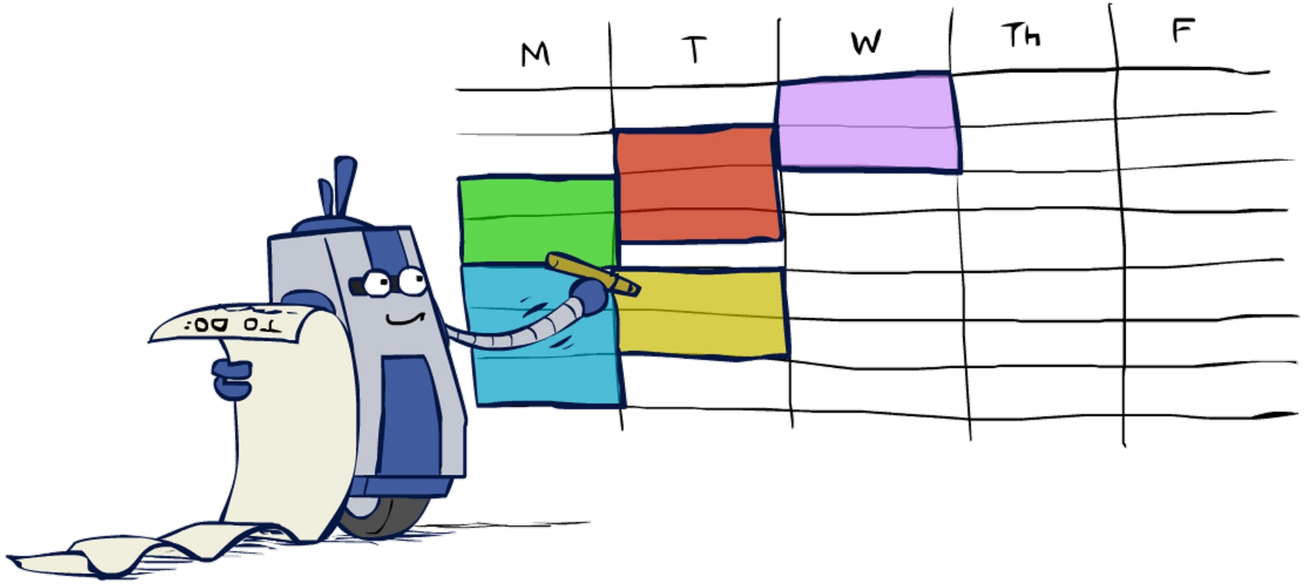
- Continuous variables
  - E.g., start/end times for Hubble Telescope observations

Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equivalent to reducing domains), e.g.: $SA \neq green$
  - Binary constraints involve pairs of variables, e.g.: $SA \neq WA$
  - Higher-order constraints involve 3 or more variables
- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems

Real-World CSPs

- Scheduling problems: e.g., when can we all meet?
- Timetabling problems: e.g., which class is offered when and where?
- Assignment problems: e.g., who teaches what class
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Circuit layout
- Fault diagnosis
- … lots more!
- Many real-world problems involve real-valued variables…



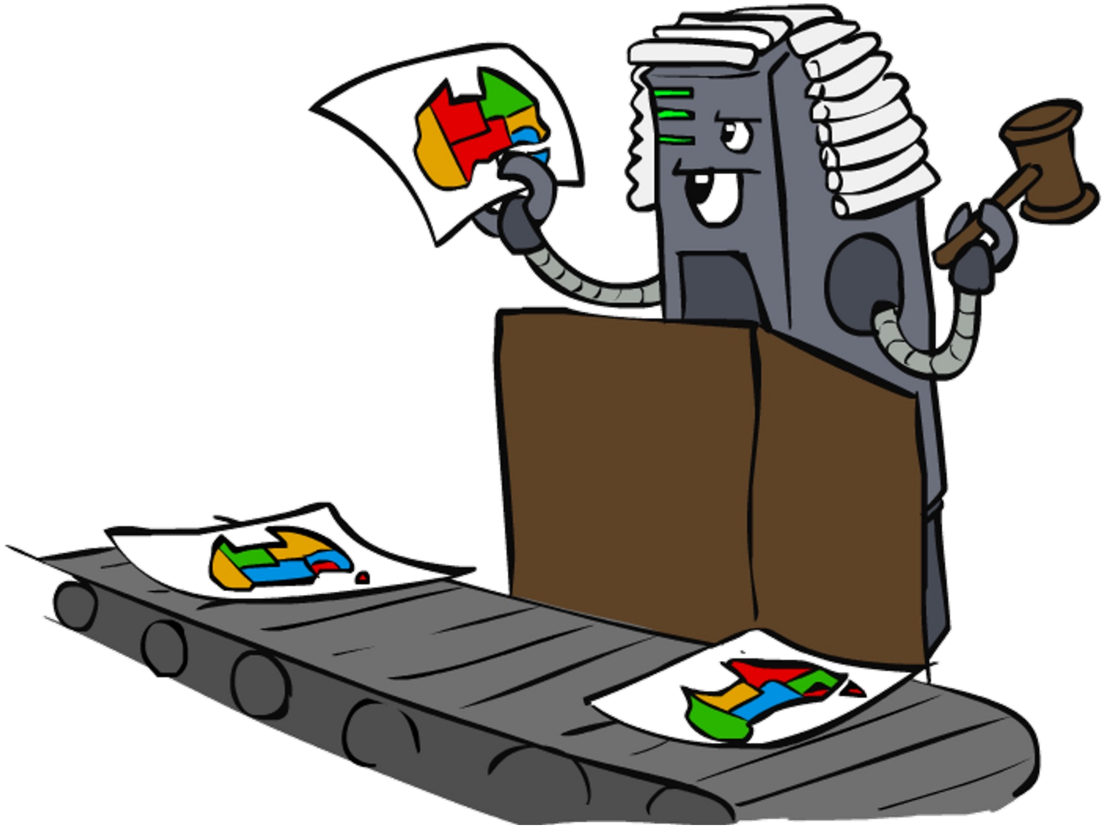
Solving CSPs

Standard Search Formulation

- Standard search formulation of CSPs

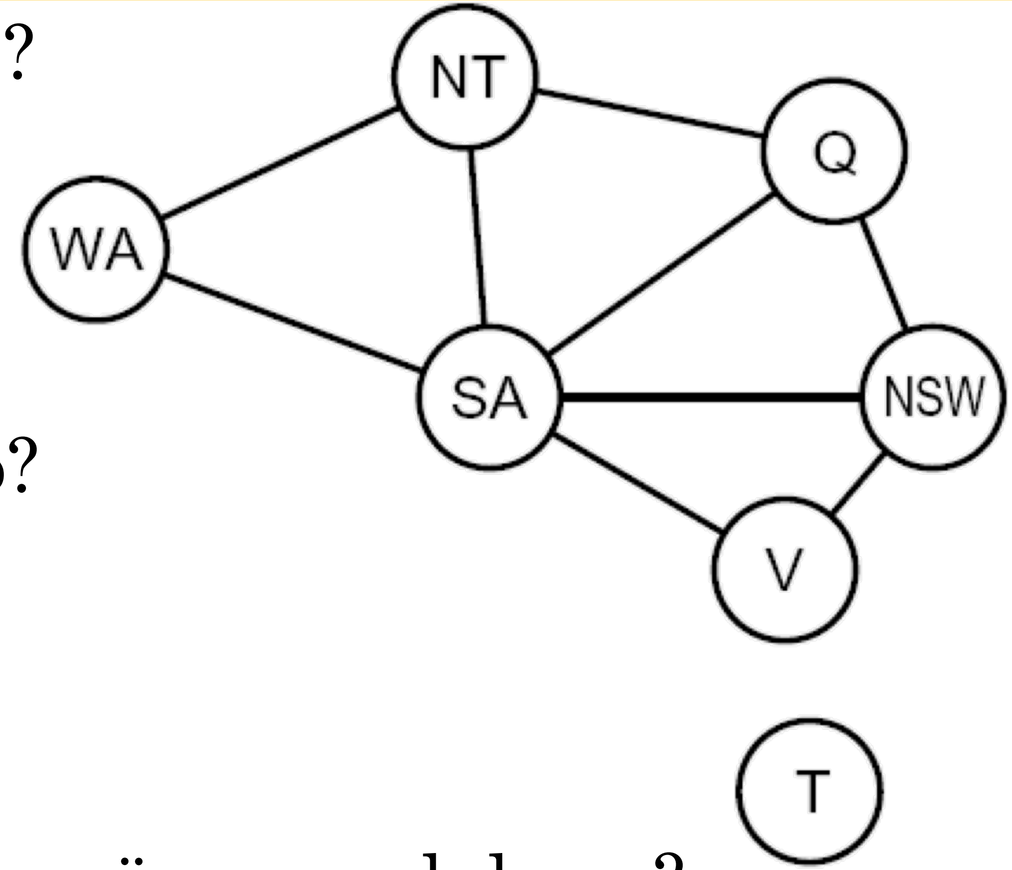
- States defined by the values assigned so far (partial assignments)
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints

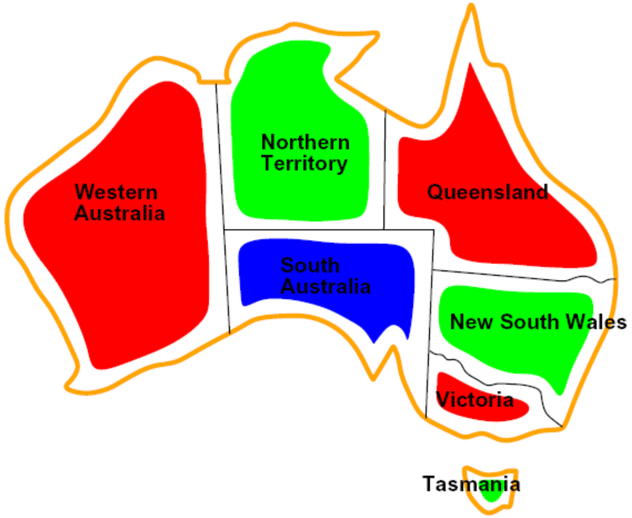
- We'll start with the straightforward, naïve approach, then improve it



Search Methods

- What would BFS do?



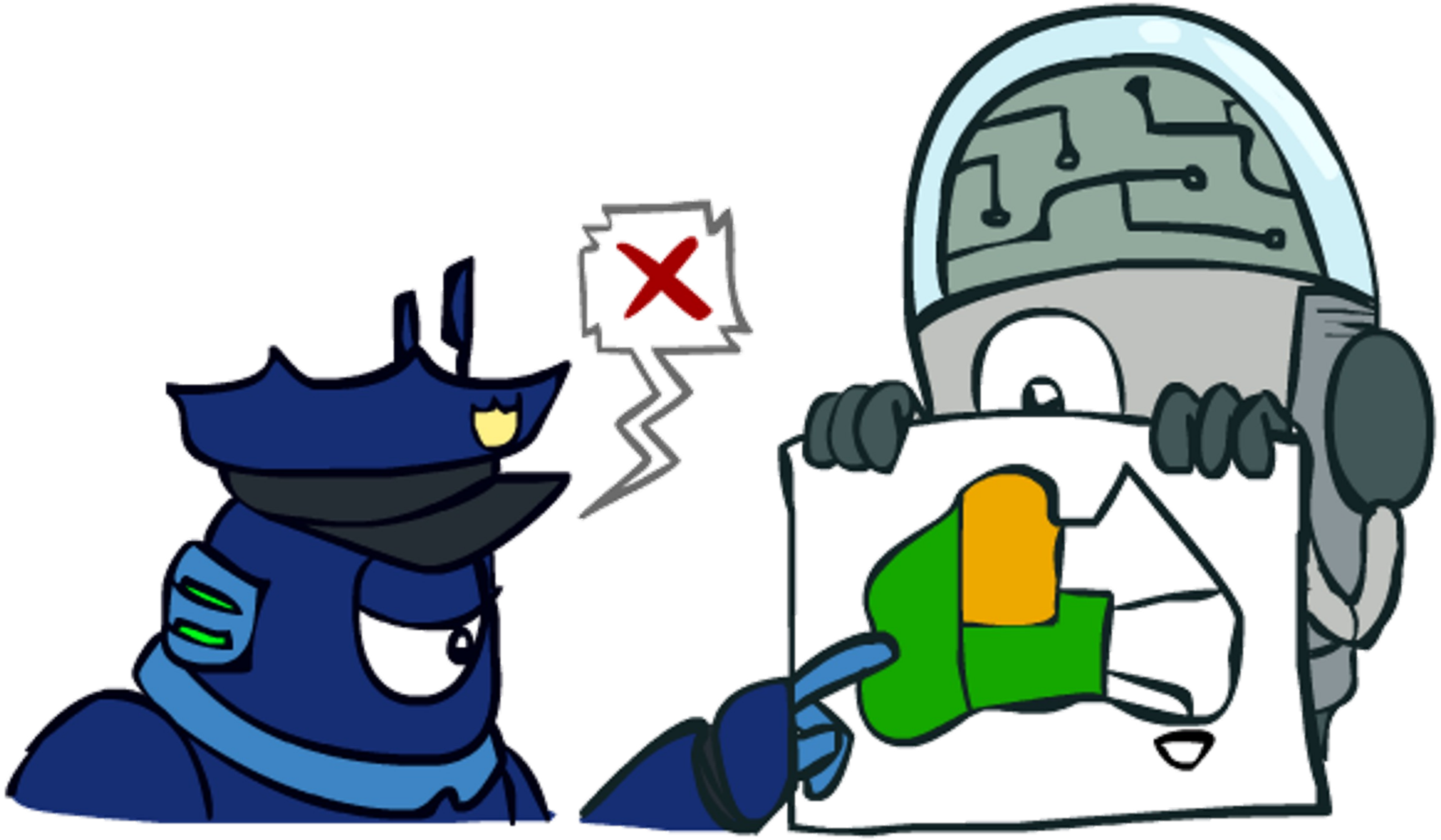


- What would DFS do?

- What problems does naïve search have?

[Demo: coloring -- dfs]

Backtracking Search



Backtracking Search

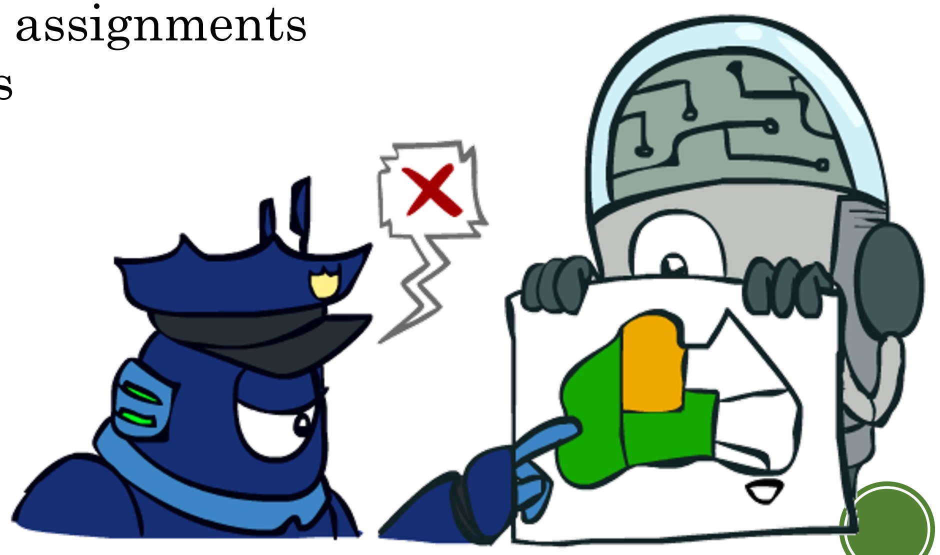
- Backtracking search is the basic uninformed algorithm for solving CSPs

- **Idea 1: One variable at a time**
  - Variable assignments are commutative, so fix ordering
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step

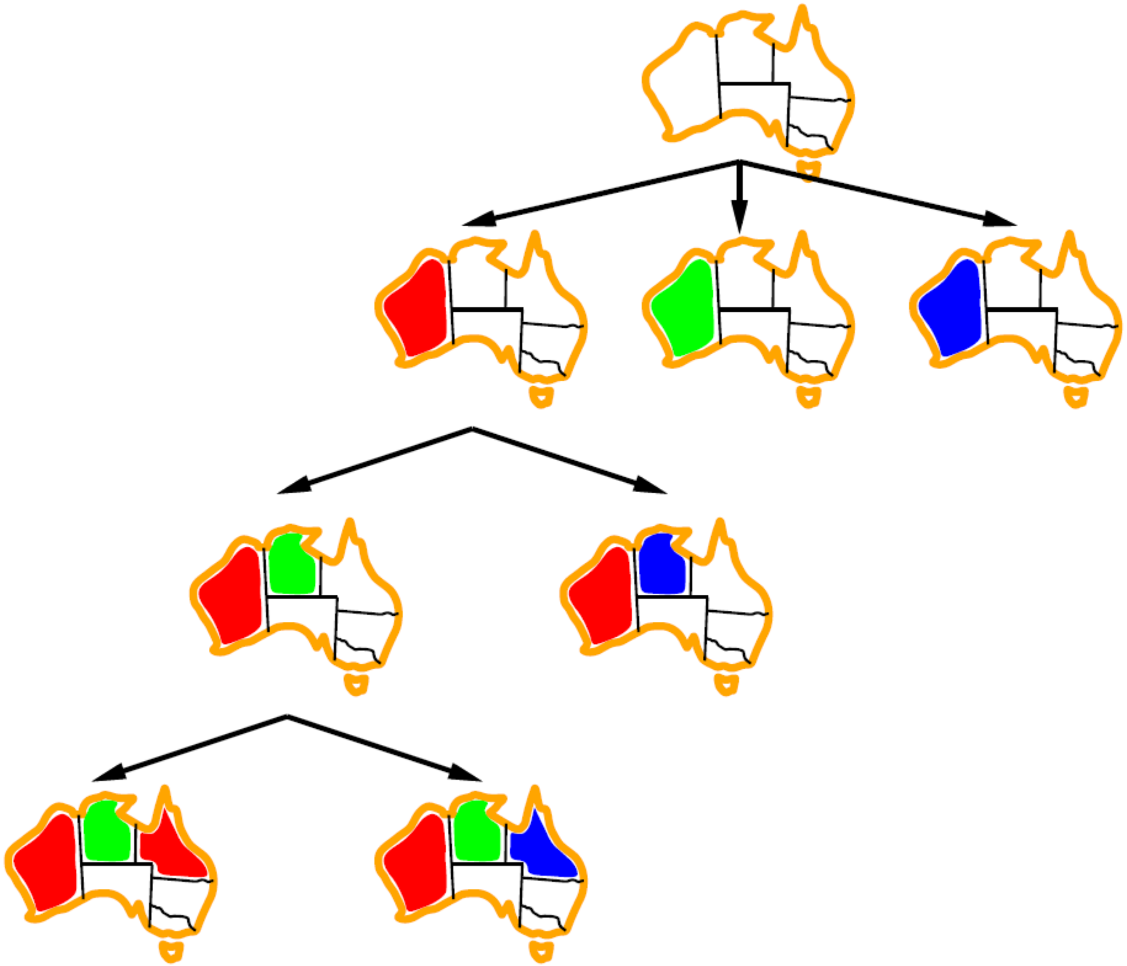
- **Idea 2: Check constraints as you go**
  - I.e. consider only values which do not conflict with previous assignments
  - Might have to do some computation to check the constraints
  - "Incremental goal test"

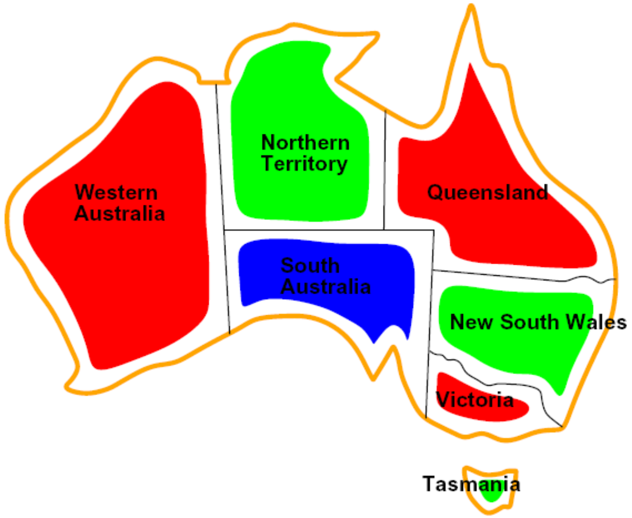
- Depth-first search with these two improvements is called *backtracking search* (not the best name)

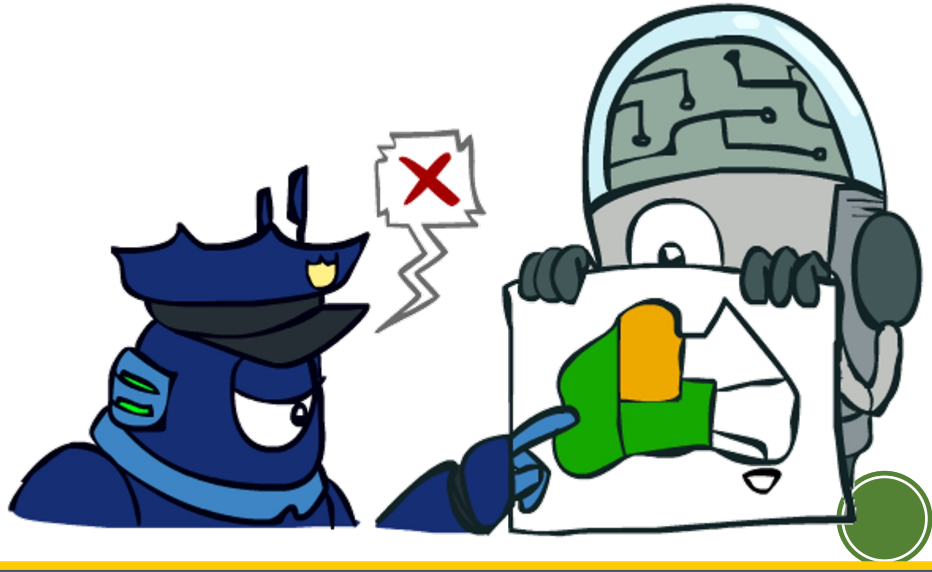
- Can solve n-queens for $n \approx 25$



Backtracking Example







Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
       if value is consistent with assignment given CONSTRAINTS[csp] then
           add {var = value} to assignment
           result ← RECURSIVE-BACKTRACKING(assignment, csp)
           if result ≠ failure then return result
           remove {var = value} from assignment
   return failure
```

- Backtracking = DFS + variable-ordering + fail-on-violation
- What are the choice points?

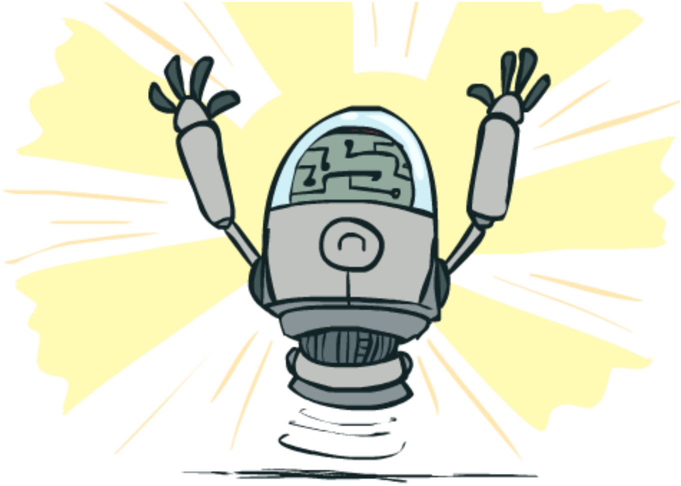
Improving Backtracking

- General-purpose ideas give huge gains in speed

- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?

- Filtering: Can we detect inevitable failure early?

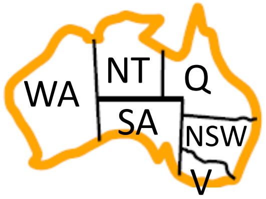
- Structure: Can we exploit the problem structure?

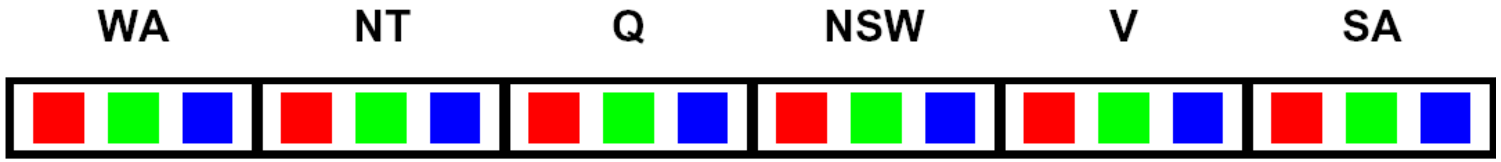


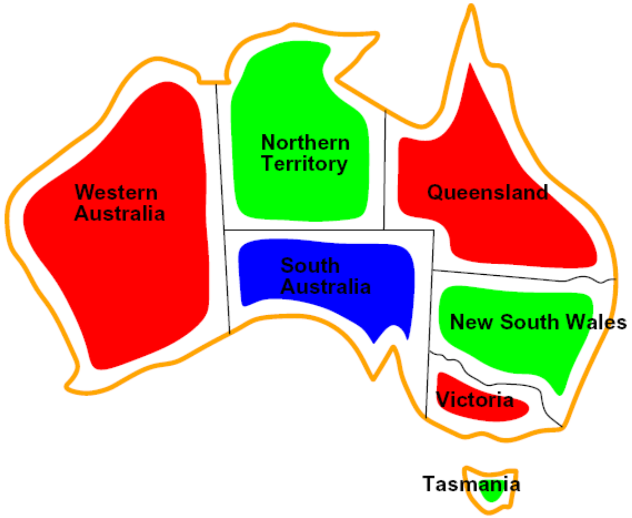
Filtering

Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: Cross off values that violate a constraint when added to the existing assignment

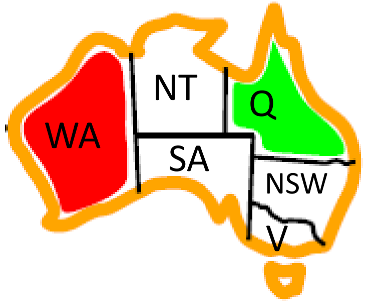


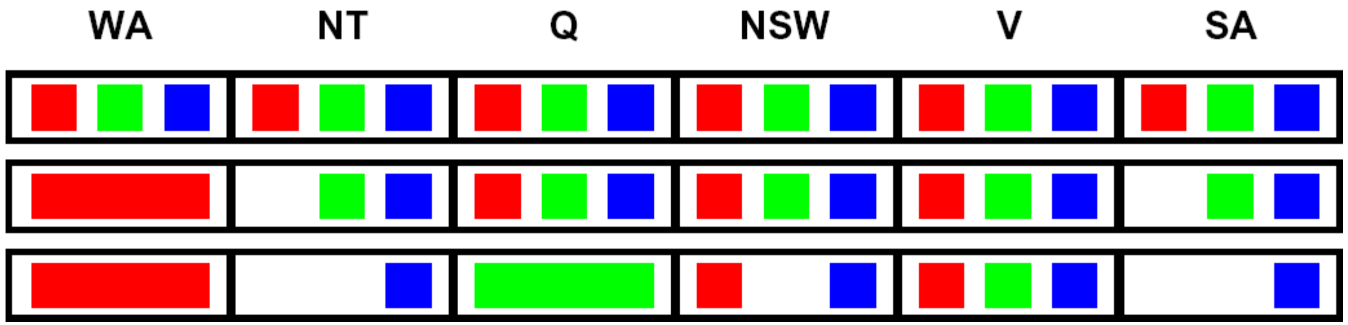




Filtering: Constraint Propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

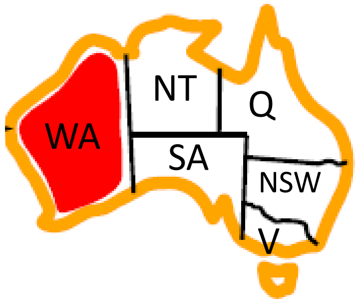


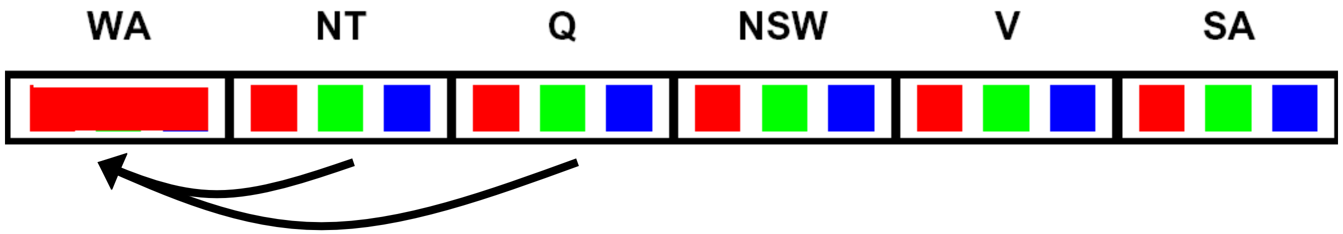


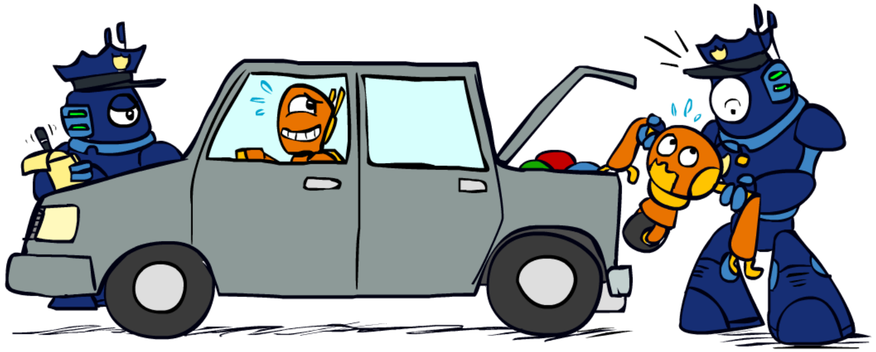
- NT and SA cannot both be blue!
- Why didn't we detect this yet?
- *Constraint propagation:* reason from constraint to constraint

Consistency of A Single Arc

- An arc X → Y is consistent iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint





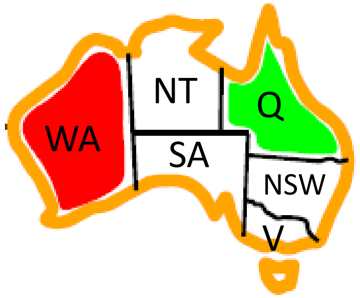


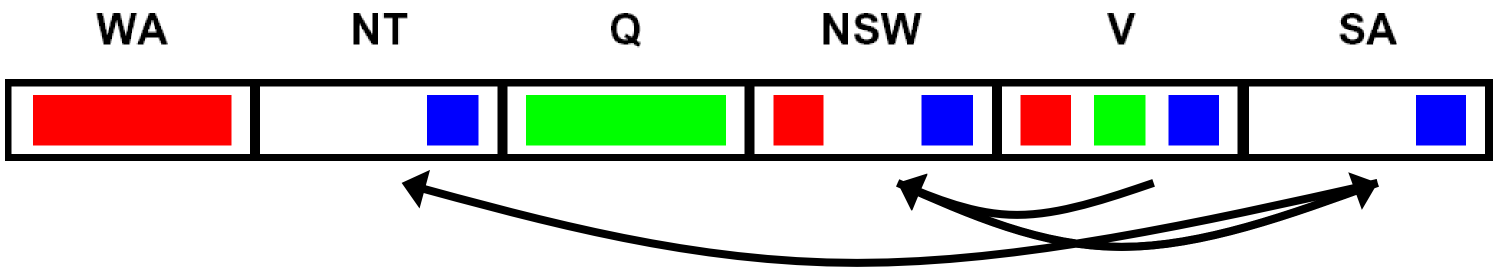
Delete from the tail!

- Forward checking: Enforcing consistency of arcs pointing to each new assignment

Arc Consistency of an Entire CSP

- A simple form of propagation makes sure all arcs are consistent:





- Important: If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- What's the downside of enforcing arc consistency?

Remember: Delete from the tail!