CS 471/571(Fall 2023): Introduction to Artificial Intelligence

Lecture 22: Bayes Nets – Sampling

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Source: http://ai.berkeley.edu/home.html

Reminders

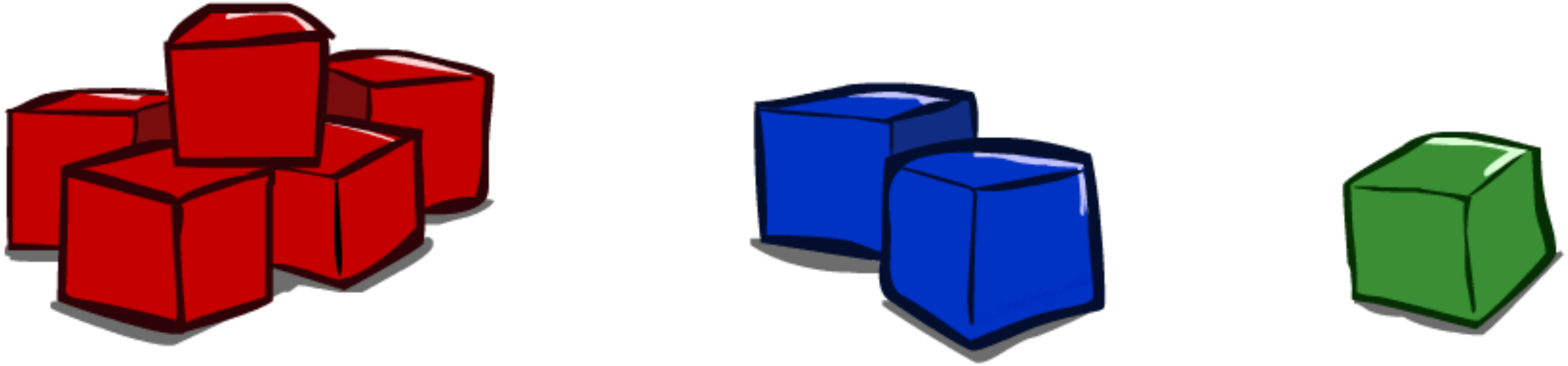
- Programming project 3
  - Deadline: November 20th, 2023

- Written assignment 4:
  - Deadline: November 29th, 2023

Bayes' Nets

- ✓ Representation
- ✓ Conditional Independences
- Probabilistic Inference
  - ✓ Enumeration (exact, exponential complexity)
  - ✓ Variable elimination (exact, worst-case exponential complexity, often better)
  - ✓ Inference is NP-complete
  - Sampling (approximate)
- Learning Bayes' Nets from Data

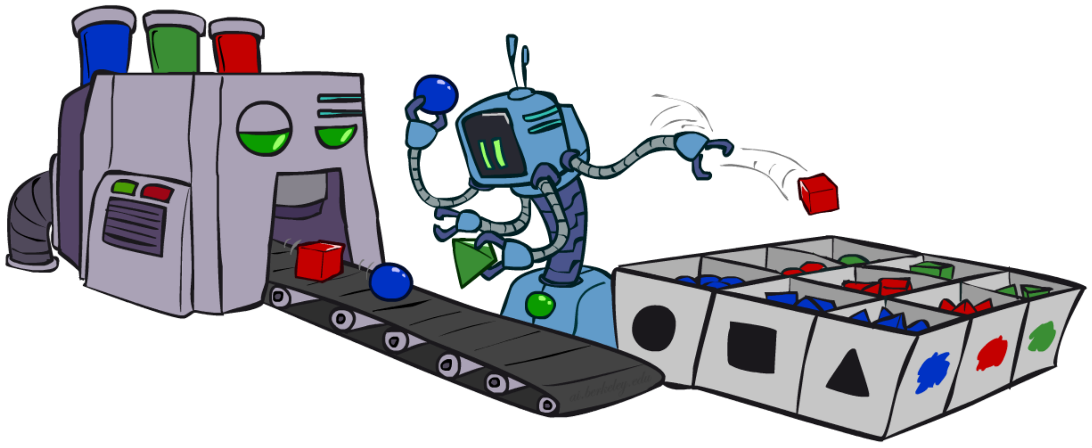
Approximate Inference: Sampling



Sampling

- Sampling is a lot like repeated simulation
  - Predicting the weather, basketball games, …
- Basic idea
  - Draw N samples from a sampling distribution S
  - Compute an approximate posterior probability
  - Show this converges to the true probability P

- Why sample?
  - Learning: get samples from a distribution you don't know
  - Inference: getting a sample is faster than computing the right answer (e.g. with variable elimination)



Sampling

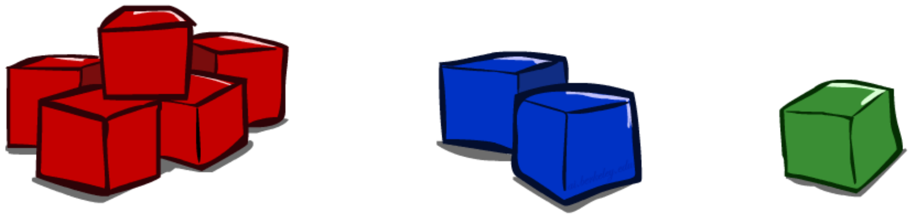
- Sampling from given distribution
  - Step 1: Get sample u from uniform distribution over [0, 1)
    - E.g. random() in python
  - Step 2: Convert this sample u into an outcome for the given distribution
    - Each target outcome is associated with a sub-interval of [0,1)
    - Sub-interval size is equal to probability of the outcome.

- Example

C	P(C)
red	0.6
green	0.1
blue	0.3

$$0 \leq u < 0.6, \rightarrow C = red$$
$$0.6 \leq u < 0.7, \rightarrow C = green$$
$$0.7 \leq u < 1, \rightarrow C = blue$$

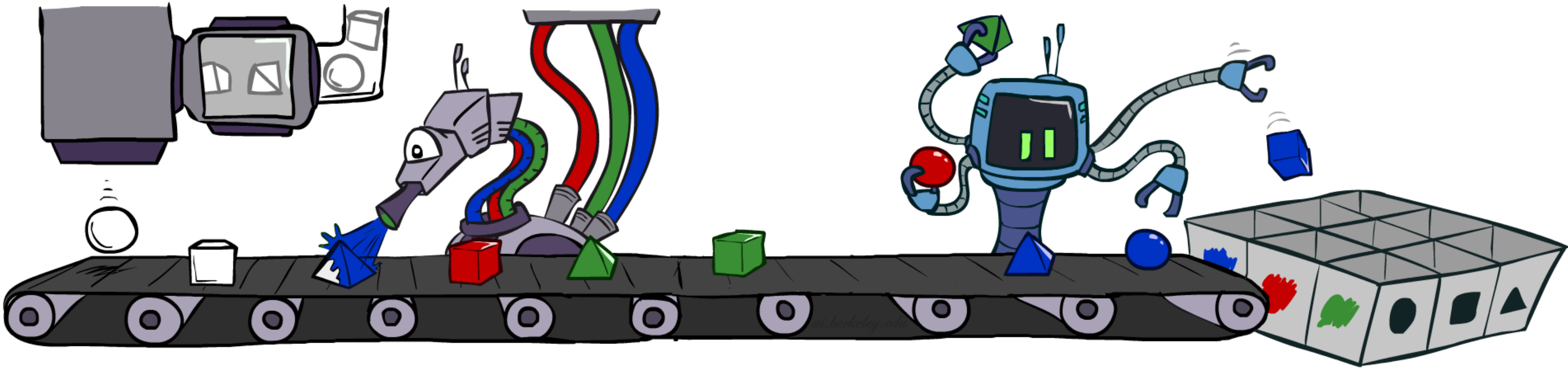
- If random() returns u = 0.83, then our sample is C = blue
- E.g, after sampling 8 times:



Sampling in Bayes' Nets

- Prior Sampling
- Rejection Sampling
- Likelihood Weighting
- Gibbs Sampling

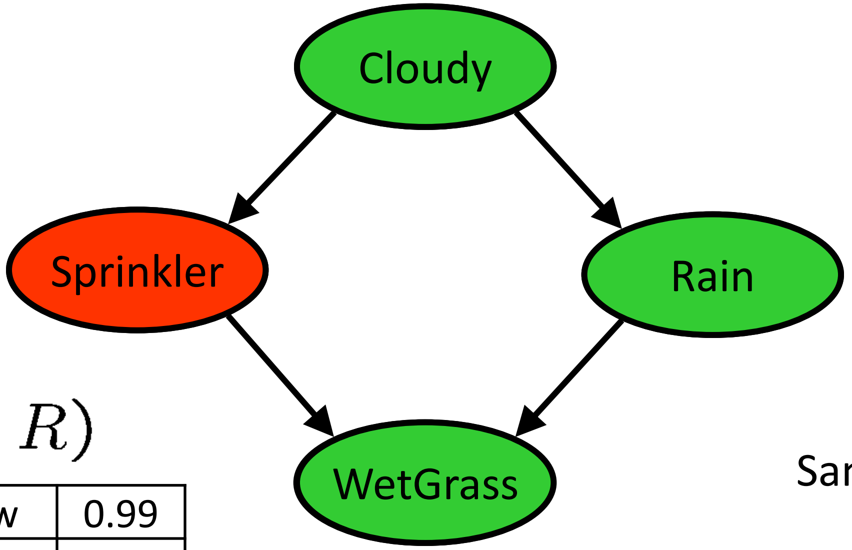
Prior Sampling



Prior Sampling

$P(C)$

+c	0.5
-c	0.5



$P(S|C)$

+c	+s	0.1
	-s	0.9
-c	+s	0.5
	-s	0.5

$P(R|C)$

+c	+r	0.8
	-r	0.2
-c	+r	0.2
	-r	0.8

$P(W|S,R)$

+s	+r	+w	0.99
		-w	0.01
	-r	+w	0.90
		-w	0.10
-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

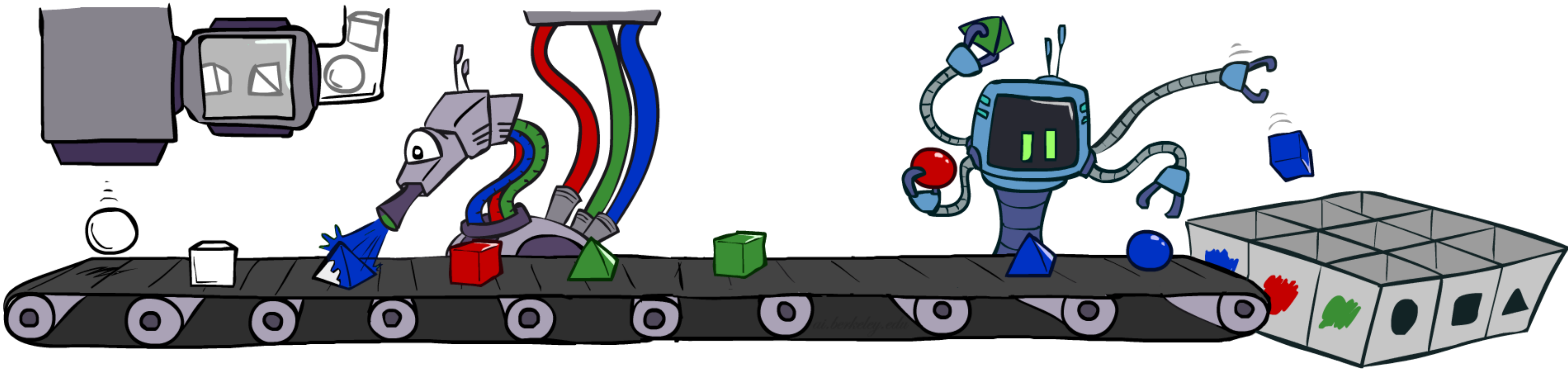
+c, -s, +r, +w

-c, +s, -r, +w

…

Prior Sampling

- For i = 1, 2, …, n
  - Sample x_i from $P(X_i \mid Parents(X_i))$
- Return $(x_1, x_2, \ldots, x_n)$



Prior Sampling

- This process generates samples with probability:

$$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i|\text{Parents}(X_i)) = P(x_1 \ldots x_n)$$

…i.e. the BN's joint probability

- Let the number of samples of an event $N_{PS}(x_1 \ldots x_n)$

- Then
$$\begin{aligned} \lim_{N\to\infty} \widehat{P}(x_1, \ldots, x_n) &= \lim_{N\to\infty} N_{PS}(x_1, \ldots, x_n)/N \\ &= S_{PS}(x_1, \ldots, x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

- I.e., the sampling procedure is consistent

Example

- We'll get a bunch of samples from the BN:

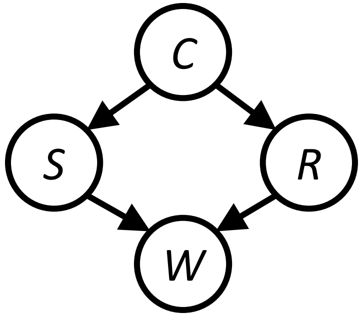
  +c, -s, +r, +w

  +c, +s, +r, +w

  -c, +s, +r, -w

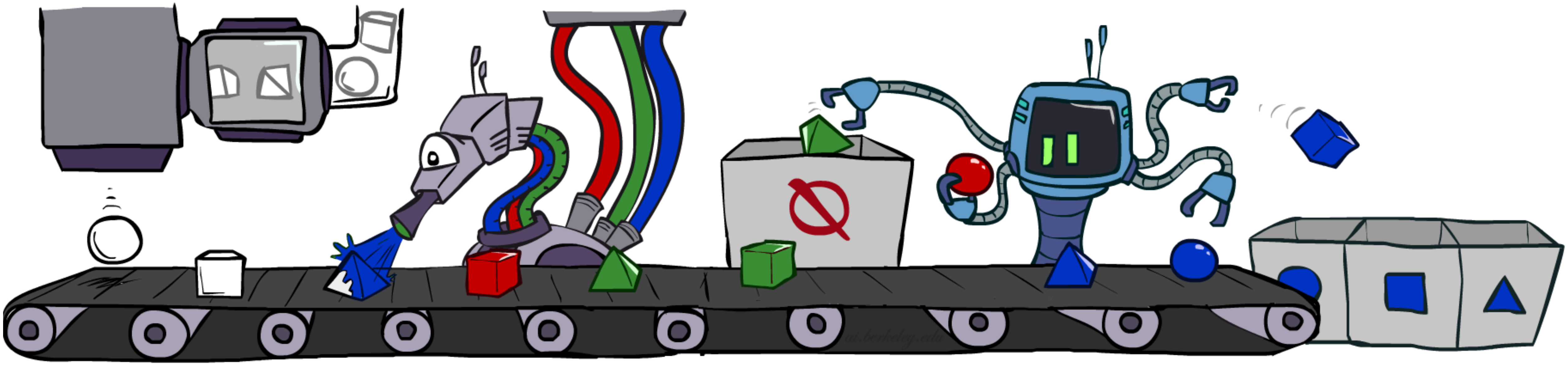
  +c, -s, +r, +w

  -c, -s, -r, +w



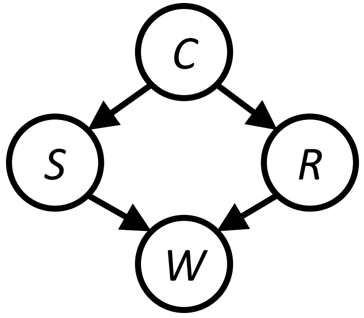
- If we want to know P(W)
  - We have counts <+w:4, -w:1>
  - Normalize to get P(W) = <+w:0.8, -w:0.2>
  - This will get closer to the true distribution with more samples
  - Can estimate anything else, too
  - What about P(C | +w)?   P(C | +r, +w)? P(C | -r, -w)?
  - Fast: can use fewer samples if less time (what's the drawback?)

Rejection Sampling



Rejection Sampling

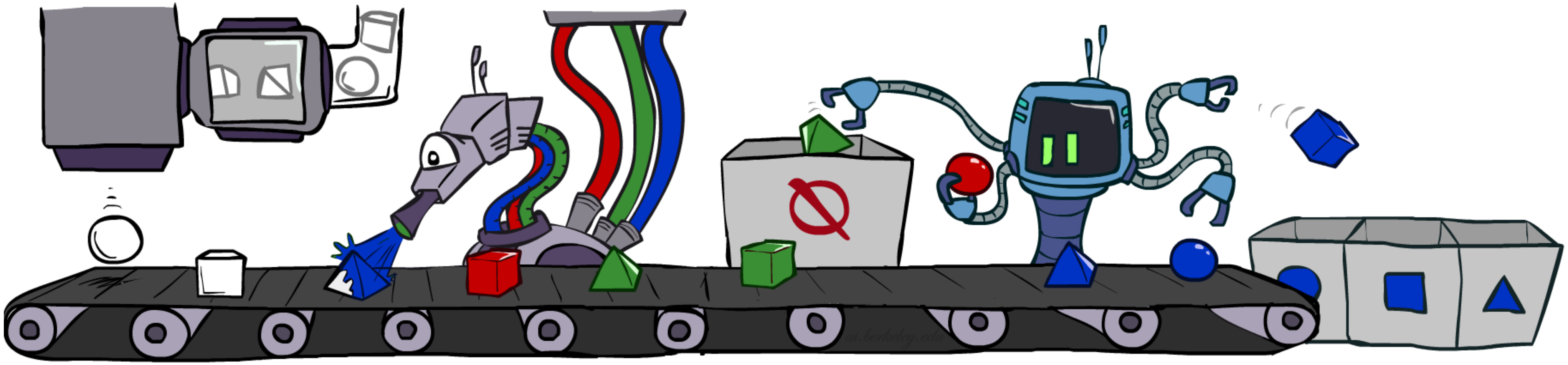
- Let's say we want P(C)
  - No point keeping all samples around
  - Just tally counts of C as we go
- Let's say we want P(C | +s)
  - Same thing: tally C outcomes, but ignore (reject) samples which don't have S=+s
  - This is called rejection sampling
  - It is also consistent for conditional probabilities (i.e., correct in the limit)



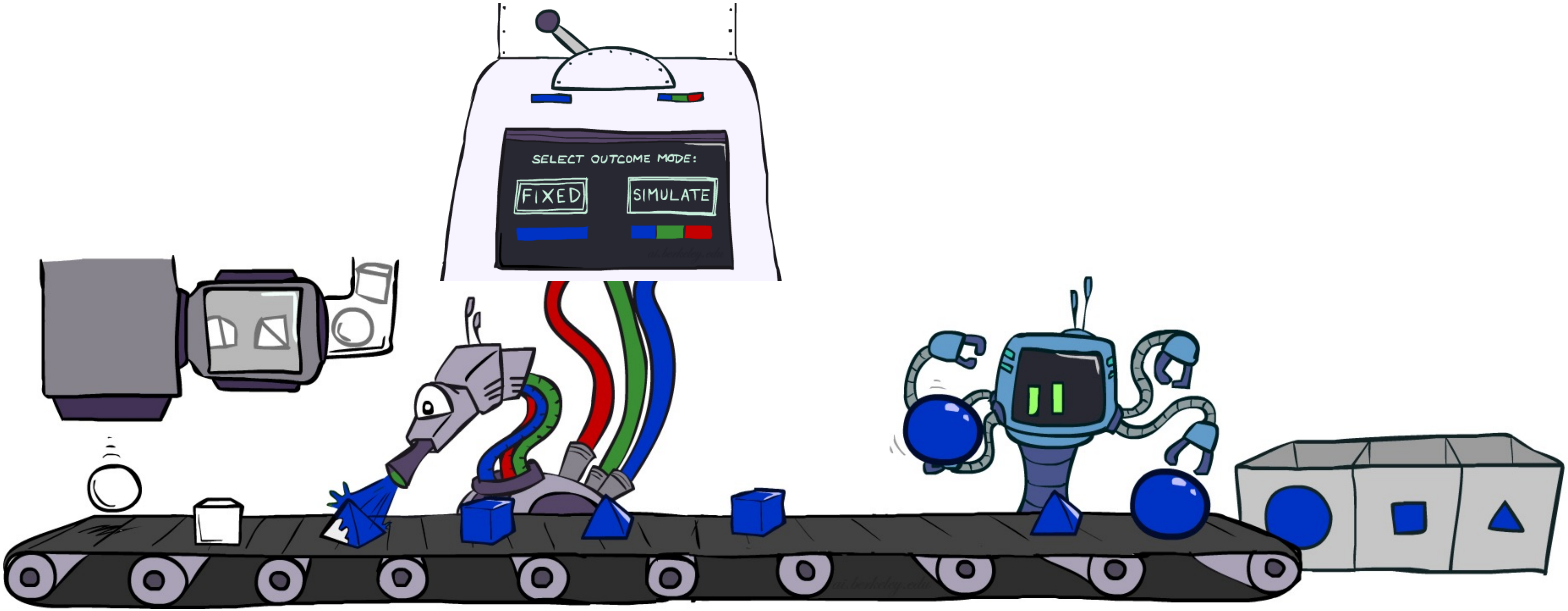
+c, -s, +r, +w
+c, +s, +r, +w
-c, +s, +r, -w
+c, -s, +r, +w
-c, -s, -r, +w

Rejection Sampling

- Input: evidence instantiation
- For i = 1, 2, …, n
  - Sample x_i from $P(X_i \mid Parents(X_i))$
  - If x_i not consistent with evidence
    - Reject: return – no sample is generated in this cycle
- Return $(x_1, x_2, \ldots, x_n)$

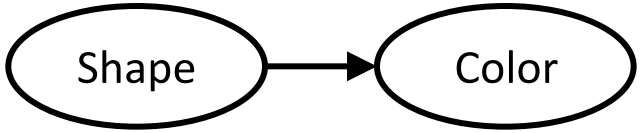


Likelihood Weighting

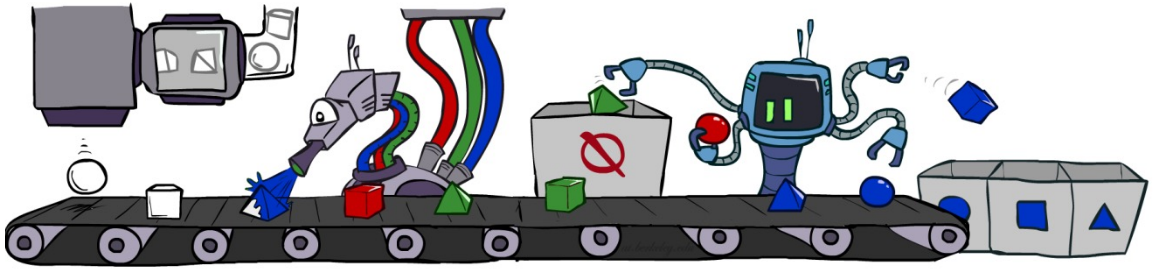


Likelihood Weighting

- Problem with rejection sampling:
  - If evidence is unlikely, rejects lots of samples
  - Evidence not exploited as you sample
  - Consider P(Shape | blue)

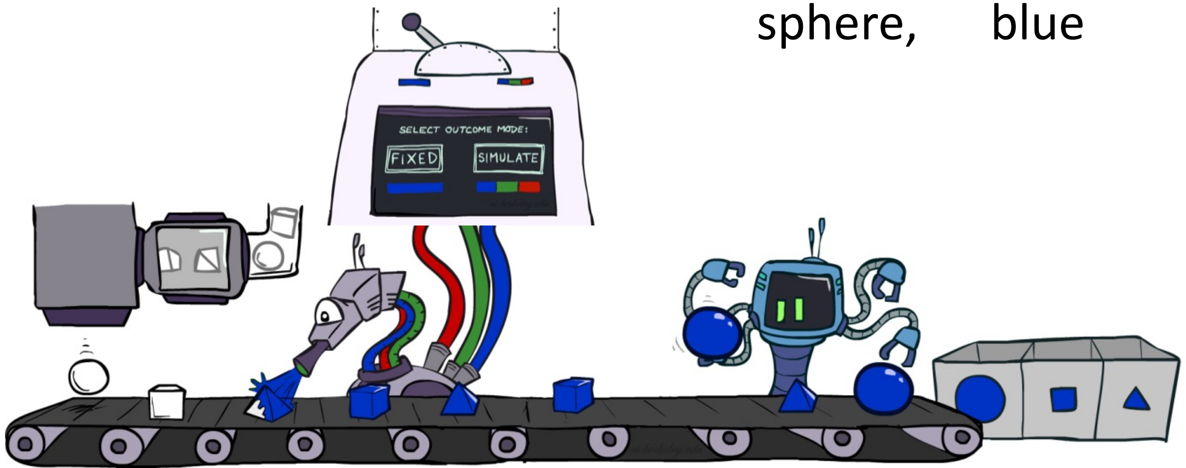


~~pyramid, green~~
~~pyramid, red~~
sphere, blue
~~cube, red~~
~~sphere, green~~



- Idea: fix evidence variables and sample the rest
  - Problem: sample distribution not consistent!
  - Solution: weight by probability of evidence given parents

pyramid, blue
pyramid, blue
sphere, blue
cube, blue
sphere, blue



Likelihood Weighting

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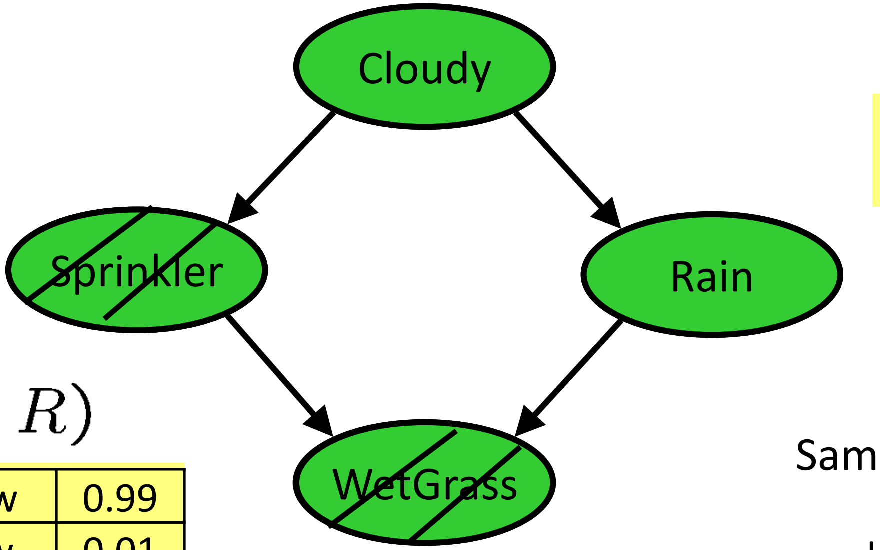

$P(R|C)$

+c	+r	0.8
	-r	0.2
-c	+r	0.2
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$P(W|S,R)$

+s	+r	+w	0.99
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	-r	+w	0.90
		-w	0.10
-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

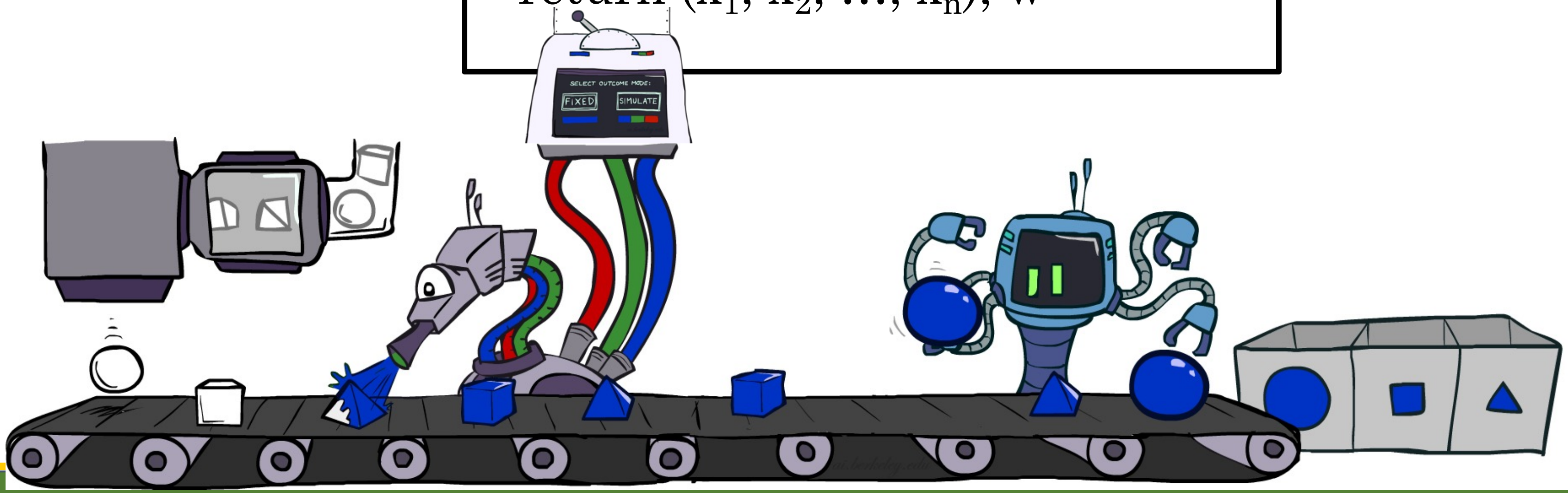
+c, +s, +r, +w

...

$$w = 1.0 \times 0.1 \times 0.99$$

Likelihood Weighting

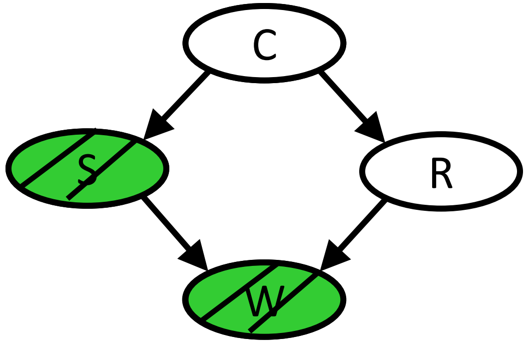
- Input: evidence instantiation
- w = 1.0
- for i = 1, 2, …, n
  - if X_i is an evidence variable
    - X_i = observation x_i for X_i
    - Set w = w * $P(x_i \mid \text{Parents}(X_i))$
  - else
    - Sample x_i from $P(X_i \mid \text{Parents}(X_i))$
- return $(x_1, x_2, \ldots, x_n)$, w



Likelihood Weighting

- Sampling distribution if z sampled and e fixed evidence

$$S_{WS}(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{l} P(z_i|\text{Parents}(Z_i))$$



- Now, samples have weights

$$w(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{m} P(e_i|\text{Parents}(E_i))$$

- Together, weighted sampling distribution is consistent

$$S_{\mathrm{WS}}(z,e) \cdot w(z,e) = \prod_{i=1}^{l} P(z_i|\text{Parents}(z_i)) \prod_{i=1}^{m} P(e_i|\text{Parents}(e_i))$$

$$= P(\mathbf{z},\mathbf{e})$$