project 3

$$p(x \mid \alpha, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\alpha}{\sigma}\right)^2}$$ normal distribution

single measurement per experiment

likelihood $$p(\alpha \mid x, \sigma) \sim p(x \mid \alpha, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\alpha}{\sigma}\right)^2}$$

maximizing $p(x \mid \alpha, \sigma)$ also maximizes $p(\alpha \mid x, \sigma)$

Find α that maximizes likelihood

$$\frac{\partial p(x \mid \alpha, \sigma)}{\partial \alpha} = \frac{1}{\sigma\sqrt{2\pi}} \left(e^{-\frac{1}{2}\left(\frac{x-\alpha}{\sigma}\right)^2} \right) \left(-\left(\frac{x-\alpha}{\sigma}\right) \right) = 0$$

$$= -p(x \mid \alpha, \sigma) \frac{x-\alpha}{\sigma} = 0$$

$x_{most-likely} = \alpha$ mean, as expected.

Find σ that maximizes likelihood. $p(x \mid \alpha, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\alpha}{\sigma}\right)^2}$

$$\frac{\partial p(x \mid \alpha, \sigma)}{\partial \sigma} = \frac{1}{\sqrt{2\pi}} \left(\frac{-1}{\sigma^2} e^{-\frac{1}{2}\left(\frac{x-\alpha}{\sigma}\right)^2} + \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{x-\alpha}{\sigma}\right)^2} \left(-\left(\frac{x-\alpha}{\sigma}\right)\right)\left(\frac{-(x-\alpha)}{\sigma^2}\right) \right) = 0$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\alpha}{\sigma}\right)^2} \left(\frac{-1}{\sigma^2} + \frac{(x-\alpha)^2}{\sigma^4} \right) = 0$$

$$= \frac{1}{\sigma} p(x \mid \alpha, \sigma) \left(-1 + \frac{(x-\alpha)^2}{\sigma^2} \right) = 0$$

$$\frac{-\sigma^2 + (x-\alpha)^2}{\sigma^2} = 0$$

$\sigma^2 = (x-\alpha)^2$ as expected. variance multiple measurements / exp.

likelihood $\quad p(\alpha \mid \{x\}, \sigma) \sim \prod_{i=1}^{N} p(x_i \mid \alpha, \sigma)$

log likelihood $\quad \ln\big(p(\alpha \mid \{x\}, \sigma)\big) \sim \sum_{i=1}^{N} \ln\big(p(x_i \mid \alpha, \sigma)\big) = \sum_{i=1}^{N} \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-1}{2\sigma^2}(x_i - \alpha)^2}\right)$

$$= \sum_{i=1}^{N} \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2}(x_i - \alpha)^2$$

$$= \sum_{i=1}^{N} \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2}(x_i - \alpha)^2$$

$$= \sum_{i=1}^{N} -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}(x_i - \alpha)^2$$

$$= -\frac{N}{2}\ln(2\pi) - \frac{N}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i - \alpha)^2$$

find α that maximizes log likelihood.

$$\frac{\partial \ln(p(\alpha \mid x, \sigma))}{\partial \alpha} = \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i - \alpha) = 0$$

$$\sum_{i=1}^{N}(x_i - \alpha) = 0$$

$$\sum_{i=1}^{N} x_i - N\alpha = 0$$

$\alpha = \frac{1}{N}\sum_{i=1}^{N} x_i \qquad$ the mean, as expected.

find σ^2 that maximizes log likelihood.

$$\frac{\partial \ln(p(\alpha \mid x, \sigma))}{\partial \sigma^2} = -\frac{N}{2}\,\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{N}(x_i - \alpha)^2 = 0$$

$$-N\sigma^2 + \sum_{i=1}^{N}(x_i - \alpha)^2 = 0$$

$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \alpha)^2 \qquad$ as expected.

if all $x_i = x$ then $\alpha = x$, $\sigma^2 = (x - \alpha)^2$

st. dev. of α.

$$\frac{\sqrt{N}}{\sigma\sqrt{2\pi}}\, e^{-\frac{N}{2}\left(\frac{x-\alpha}{\sigma}\right)^2} = \frac{1}{\tilde{\sigma}\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{\alpha-\bar{x}}{\tilde{\sigma}}\right)^2} \qquad \tilde{\sigma} = \frac{\sigma}{\sqrt{N}}$$

process

1. Generate Data from Gaussian Distribution. use numpy.random.normal() function.

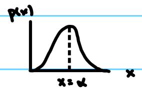
   N_{exp} experiments

   N_{meas} measurements / experiment.

   μ = fixed value

   σ = fixed value

2. plot data.



3. From data, numerically estimate parameter α + error σ_α by maximizing likelihood.

4. compare w/ analytic results $\alpha = \mu$, $\sigma_\alpha^2 = \frac{\sigma^2}{N}$