Braids, Links, and Knots for Mathematicians
Fall 2022

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Chapter 1

Braids, Links, and Knots

1.1 Braids

Definition 1.1: Braid

A "braid" is a picture drawn in a very particular way:

1. Decide how many strands you want to have in your braid. (Draw n dots on top and bottom)
2. From each top dot, draw an arc to connect one of the bottom dots. This arc can move left and right but never up. 3 strands should never intersect at a point.
3. At each crossing, decide which stand is over, and which is under.

Note:-

Strands can NOT start and finish from the same side (top or bottom).

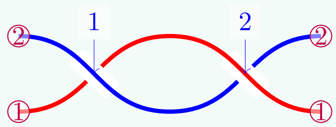
Proposition 1.1 Property of *Braids*

Two braids are the same, so long as you can get from one to the other pulling on strings.

Note:-

You can never pass one strand through another.
Never tear a strand.

Example 1.1 (Example Braid)



Definition 1.2: Braid "Multiplication"

Given braids α and β, $\alpha \cdot \beta$ is to be obtained by stacking the diagrams of α and β.

Example 1.2 (α and β braids)

Stack the braids (where bottom nodes of α match up with top nodes of β)
Then, simplify the resulting braid.
This is braid multiplication.

Definition 1.3: Braid Identity

The braid that has NO crossings is the identity braid.
Each arc is directly connected to the node below/above it
It has the property that it is the identity under braid multiplication.

Definition 1.4: Braid Inversion

Given a braid α, α^{-1} is to be obtained by reversing the direction of each arc in α.

Example 1.3 (α braid inversion)

Every time an arc crossed another in α, flip which arc is on top
This is braid inversion.

$\alpha^{-1} \cdot \alpha \equiv I$

Example 1.4 (σ_i braid)

This braid is formed by taking the identity braid and crossing the ith and $(i+1)$th strands.

Theorem 1.1 σ_i Theorem

If α is any braid, then α can always be written as a product of multiple σ_i and σ_i^{-1} braids.

any braid can therefore be decomposed into a product of σ_i and σ_i^{-1} braids.

Note:-

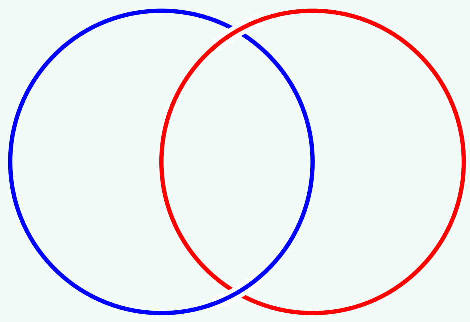
The decomposition of a given braid is NOT unique.
I.E., multiple different braids can have the same decomposition.

1.2 Links

Definition 1.5: Link

A <u>link</u> is what happens when you take a braid and join the top and bottom dots.
Links are NOT braids.

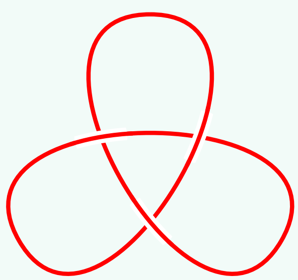
Example 1.5 (Link example)



Definition 1.6: Trefoil

A <u>trefoil</u> is a link that is a braid of 3 strands.

Example 1.6 (Trefoil)
Draw a braid of 3 strands
Join the top and bottom dots
This is a trefoil



Proposition 1.2 Braid Property
<u>Let</u> α, β be braids, then Links of α and β are identical $\iff$ you can transform α into β via a sequence of the following moves

This is Markov's Theorem (Not Markov chain Markov!):

1. $\gamma \cdot \gamma'$ $\gamma' \cdot \gamma$
2. γ