Combinatorics and Graph Theory Homework
Spring 2023

Daniel Grant

Contents

Homework 1 — Page 2

Homework 2 — Page 11

Homework 3 — Page 19

Homework 4 — Page 29

Homework 5 — Page 38

Homework 6 — Page 47

Homework 7 — Page 58

Homework 8 — Page 69

Homework 9 — Page 77

Homework 10 — Page 85

Homework 1

Question 1

In each of the following problems you are given the vertex and edge set of a graph. Determine the values of $\delta(G), \Delta(G)$, and $\kappa(G)$ in each case.

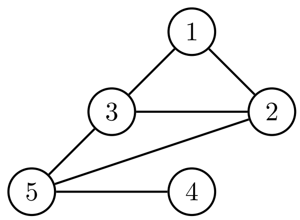
(i) $V = \{1, 2, 3, 4, 5\}, E = \{\{1, 2\}, \{3, 1\}, \{3, 2\}, \{5, 2\}, \{4, 5\}, \{3, 5\}\}$.

(ii) $V = \{1, 2, 3, 4\}, E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\}$.

(iii) $V = \{a, b, c, d\}, E = \{\{a, b\}, \{a, d\}, \{b, d\}, \{b, c\}\}$.

Solution

(i) $V = \{1, 2, 3, 4, 5\}, E = \{\{1, 2\}, \{3, 1\}, \{3, 2\}, \{5, 2\}, \{4, 5\}, \{3, 5\}\}$.



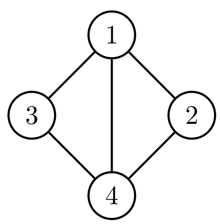
Example Graph with Parameters as Listed Above

(a) $\delta(G) = \mathbf{1}$ (the minimum degree, at node 4)

(b) $\Delta(G) = \mathbf{3}$ (the maximum degree, at node $3, 2, \ldots$)

(c) $\kappa(G) = \mathbf{1}$ (the number of vertex removal needed to disconnect the graph, in this case the removal of 5 disconnects the graph)

(ii) $V = \{1, 2, 3, 4\}, E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\}$.



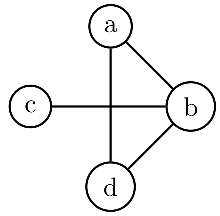
Example Graph with Parameters as Listed Above

(a) $\delta(G) = \mathbf{2}$ (at node 2 and 3)

(b) $\Delta(G) = \mathbf{3}$ (at node 1 and 4)

(c) $\kappa(G) = \mathbf{2}$ (removing 1 and 4 disconnects the graph)

(iii) $V = \{a, b, c, d\}, E = \{\{a, b\}, \{a, d\}, \{b, d\}, \{b, c\}\}$.



Example Graph with Parameters as Listed Above

(a) $\delta(G) = \mathbf{1}$ (at node c)

(b) $\Delta(G) = \mathbf{3}$ (at node b)

(c) $\kappa(G) = \mathbf{1}$ (removing d disconnects the graph)

Question 2

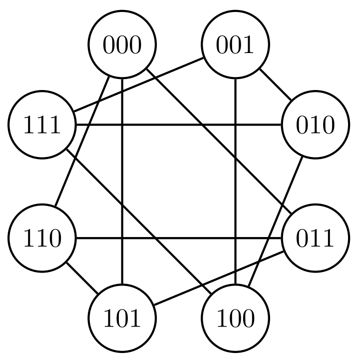
A binary string of length n is just a sequence of n 0's and 1's. For instance, 01110 is a binary string of length 5, whereas 111 is a binary string of length 3 . We are going to look at graphs whose vertex sets are comprised of binary strings.

(i) Let $G = (V, E)$ be the graph whose vertex set is all binary string of length 3 , and whose edges connect vertices that differ in two positions of the string. For instance, $\{010, 100\}$ is an edge in this graph, as these two vertices differ precisely in positions one and two. Draw the graph G. How many components does G have?

(ii) In the previous part you should have found two components. Call these components G_1 and G_2. Do you notice any particular pattern among the strings appearing in G_1 ? what about G_2 ?

(iii) Now let's see if this pattern holds up as you make the strings longer! For this problem, let $G = (V, E)$ be the graph whose vertex set is the set of binary strings with length 9 and whose edge set once again connects strings with exactly two different positions. Prove that G has exactly two components.

(i) Let $G = (V, E)$ be the graph whose vertex set is all binary string of length 3 , and whose edges connect vertices that differ in two positions of the string. For instance, $\{010, 100\}$ is an edge in this graph, as these two vertices differ precisely in positions one and two. Draw the graph G. How many components does G have?



Example Graph with Parameters as Listed Above

The graph above has two connected components, G_1 and G_2. Where G_1 and G_2 are defined as follows:

- G_1 is the graph with ...
  (a) $V(G_1) = \{000, 110, 101, 011\}$
  (b) $E(G_1) = \{\{000, 110\}, \{000, 101\}, \{000, 011\}, \{110, 101\}, \{110, 011\}, \{101, 011\}\}$.
- G_2 is the graph with ...
  (a) $V(G_2) = \{001, 010, 100, 111\}$
  (b) $E(G_2) = \{\{001, 010\}, \{001, 100\}, \{001, 111\}, \{010, 100\}, \{010, 111\}, \{100, 111\}\}$.

(ii) In the previous part you should have found two components. Call these components G_1 and G_2. Do you notice any particular pattern among the strings appearing in G_1 ? what about G_2 ?

(1) The graph G_1 has all of its vertices with an even number of 1's, while the graph G_2 has all of its vertices with an odd number of 1's.

(iii) Now let's see if this pattern holds up as you make the strings longer! For this problem, let $G = (V, E)$ be the graph whose vertex set is the set of binary strings with length 9 and whose edge set once again connects strings with exactly two different positions. Prove that G has exactly two components.

Proof 1 Proof that G has exactly two components

Define sets S_1 and S_2 to be two sets of binary strings of length 9 such that each set contains binary strings that differ by two in terms of their digits.

Let S_1 be the set of binary strings with an even number of 1's, and let S_2 be the set of binary strings with an odd number of 1's.

Then, we want to show that S_1 and S_2 are the only two components of G.

To do this, we will show that any binary string with an even number of 1's is only connected to other binary strings with even numbers of 1's.

Example 1.1 (Two different positions)

Let x be a binary string with an even number of 1's, and let y be a binary string with an odd number of 1's.

Then, we want to show that x and y are not connected in G.

To find a string connected to x, we can simply "flip" two of the bits in the string x. So, by definition this will either:

1. Flip two 0's to 1's (net increase of 2 1's), or
2. Flip two 1's to 0's (net decrease of 2 1's).
3. Flip one 0 to a 1 and one 1 to a 0 (net change of 0 1's).

Therefore, the string x will be connected to a string with an even number of 1's, and cannot be connected to a string with an odd number of 1's. So, string x cannot be connected to string y.

Proof by Contradiction:

Assume that S_1 and S_2 are not the only two components of G.

Then, there must be a third component S_3 that contains some binary string z

z must either have an even or odd number of 1's, since z is a binary string.

S So, z must be connected to either S_1 or S_2, and by definition, since a string can not both have an even and odd number of 1's, z cannot be connected to both.

Therefore, z must be connected to either S_1 or S_2, and not a set S_3

This is a contradiction.

Question 3

Let G be a graph, and assume that exactly two vertices a and b of G have odd degree. Prove that there exists a path connecting a to b.

Proof 2 Proof

Let G be a graph, and assume that exactly two vertices a and b of G have odd degree.

Then, two cases can occur:

1. G is connected. In this case, by the definition of a connected graph, there exists a path connecting a to b. (proof done day 2 in class)
2. G is not connected. In this case, can represent each connected component of G as a subgraph G_i. Then, by the definition of a connected graph, as shown in class:

$$\sum_{v \in V(G_i)} d(v) = 2E(G_i)$$

   So, the sum of the degrees of all vertices in G_i is equal to twice the number of edges in G_i, and is therefore an even number.

   Thus, there must be an <u>even</u> number of vertices in G_i with odd degree, as otherwise the sum of the degrees of all vertices would be odd. So:

$$a \in V(G_i) \iff b \in V(G_i)$$

   So, a and b are in the same connected component. This is case 1, and we are done.

Question 4

For a graph G we will write $\delta(G)$ for the smallest degree of a vertex in G.

(i) Prove that $\kappa(G) \leqslant \delta(G)$. (HINT: Remember that, by definition, there must exist a vertex with exactly $\delta(G)$ neighbors.)

(ii) Assume that G has n vertices, and that $\delta(G) \geqslant n-2$. We will prove that $\kappa(G) = \delta(G)$. To begin, notice that there are only two possible cases $\delta(G) = n-2$ and $\delta(G) = n-1$. If $\delta(G) = n-1$ then your graph must be a complete graph. Prove the desired equality from here.

(iii) Otherwise, our graph has $\delta(G) = n-2$. Remember that what we are trying to prove is that if we remove $n-3$ vertices, then our graph cannot possibly be disconnected. Well, no matter what if you remove $n-3$ vertices the resulting graph has exactly 3 vertices. Draw every single possible graph with three vertices.

(iv) To conclude, what is the smallest possible degree after having removed your $n-3$ vertices? Remember the smallest possible degree to start was $n-2$, what is the worst case scenario after removing $n-3$ vertices? Use this to prove that the removal of these vertices could not disconnect the graph.

(i) Prove that $\kappa(G) \leqslant \delta(G)$. (HINT: Remember that, by definition, there must exist a vertex with exactly $\delta(G)$ neighbors.)

Proof 3 Proof

Let G be a graph such that $\sigma(G) = x$

Then, by definition, there must exist a vertex v with degree x, and no other vertex p such that $d(p) < d(v)$.

Therefore, $\kappa(G) \leqslant \delta(G)$, as to achieve the fewest vertex-removals to disconnect G, you would be remove all vertices connected to v.

(ii) Assume that G has n vertices, and that $\delta(G) \geqslant n-2$. We will prove that $\kappa(G) = \delta(G)$. To begin, notice that there are only two possible cases $\delta(G) = n-2$ and $\delta(G) = n-1$. If $\delta(G) = n-1$ then your graph must be a complete graph. Prove the desired equality from here.

Proof 4 Proof

(a) Case 1: $\delta(G) = n-1$

In this case, G is a complete graph. By definition, this means that every vertex in G is connected to every other vertex. This means that there is no way to create two separate components, as in every set of possible vertex-removals the remaining vertices are in a connected component.

This means that $\kappa(G) = \delta(G)$, as to disconnect G, you would have to remove all vertices but one.

So, $\kappa(G) = n-1$

(b) Case 2: $\delta(G) = n-2$

In this case, G is not a complete graph.

By definition, there must exist a vertex v with degree $n-2$, and no other vertex p such that $d(p) < d(v)$.

So, each vertex is either connected to every other vertex, or is connected to all but one vertices.

So, given two vertices x and y, they are either connected to each other, or not connected to each other but are connected to every other $v_i \in G$

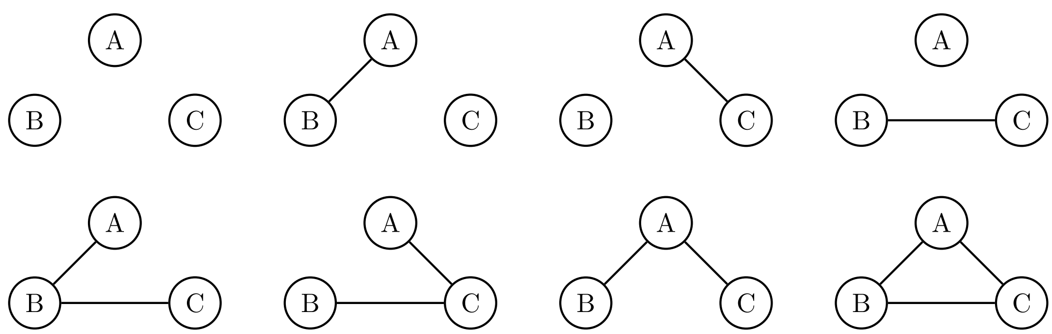
Therefore, the only way to disconnect G is to remove all vertices connected to v and x, leaving v and x as their own separate components with $n-2$ vertex-removals.

Thus $n-2$ vertex-removals are required to disconnect G.

$$\kappa(G) \leqslant n-2$$

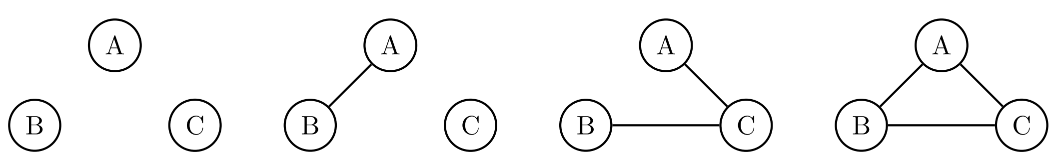
So, in the cases where G is a complete graph, or $\delta(G) = n-2$, $\kappa(G) = \delta(G)$.

(iii) Otherwise, our graph has $\delta(G) = n-2$. Remember that what we are trying to prove is that if we remove $n-3$ vertices, then our graph cannot possibly be disconnected. Well, no matter what if you remove $n-3$ vertices the resulting graph has exactly 3 vertices. Draw every single possible graph with three vertices.



(iv) To conclude, what is the smallest possible degree after having removed your $n-3$ vertices? Remember the smallest possible degree to start was $n-2$, what is the worst case scenario after removing $n-3$ vertices? Use this to prove that the removal of these vertices could not disconnect the graph.

From left to right G_1, G_2, G_3, G_3



(Case 1) $n-3$ removals results in graph G_1

This is inherently not possible, as the degree of each vertex is 0, which directly contradicts the fact that the smallest possible degree is $n-2$, as if a node has a degree of 0 with three nodes remaining, it could have a degree of at **most** $n-3$.

(Case 2) $n-3$ removals results in graph G_2

Again, this is not possible, as their must be at least one vertex with a degree of 0 in this graph, which contradicts the fact that the smallest possible degree is $n-2$.

(Case 3) $n-3$ removals results in graph G_3

This is possible, as the degree of each vertex is 1, which is the smallest possible degree after removing $n-3$ vertices.

(Case 4) $n-3$ removals results in graph G_4

This is also possible, as the degree of each vertex is 2, which could occur after removing $n-3$ vertices from a complete graph, for example, as the degree of each vertex in a complete graph is $n-1$, meaning each vertex is connected to every other vertex.

The worst case scenario is that the degree of each vertex is 2 after $n-3$ vertices are removed (*Graph* G_4), as this would require two additional removals to disconnect the graph, whereas the case of G_3 would only require one additional vertex removal to disconnect the graph.

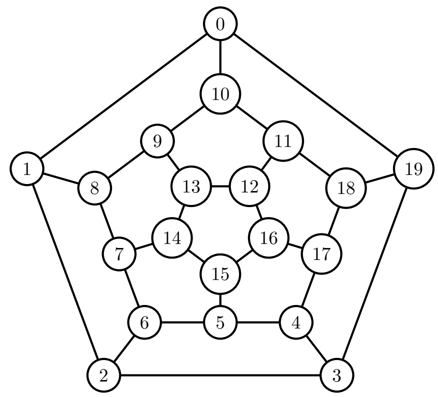
Proof 5 Proof

Since the only possible configurations of a graph after removing $n-3$ vertices with a minimum degree of $n-2$ are G_3, G_4 (*shown above*), and G_1, G_2 are not possible, the resulting graph after $n-3$ removals is necessarily connected.

Homework 2

Question 5: Problem 1 (25 points)

Let G be the graph drawn below:



Graph G

(i) What is the eccentricity of the vertices 1,10 , and 16 ?

(ii) What are the radius and diameter of G ?

Solution:

(i) *What is the eccentricity of the vertices 1,10 , and 16 ?*

(a) The eccentricity of vertex 1 is 5, since the shortest path from $1 \longrightarrow 16$ is 5 edges long, and there is not a longer shortest path to any other vertex v_i in G from vertex 1.

(b) The eccentricity of vertex 10 is 5, since the shortest path from $10 \longrightarrow 5$ is 5 edges long, and there is not a longer shortest path to any other vertex v_i in G from vertex 10.

(c) The eccentricity of vertex 16 is 5, since the shortest path from $16 \longrightarrow 1$ is 5 edges long, and there is not a longer shortest path to any other vertex v_i in G from vertex 16.

(ii) *What are the radius and diameter of G ?*

(a) The radius of G is 5, since the shortest path one vertex can take to get to any other vertex is 5 edges long.

$$\forall v_i \in V(G), Ecc(v_i) \geqslant 5$$

(b) The diameter of G is also 5, since the longest path one vertex can take to get to any other vertex is 5 edges long.

$$\forall v_i \in V(G), Ecc(v_i) \leqslant 5$$

Question 6: Problem 2 (25 points)

In this problem we let Q_n denote the graph whose vertices are binary strings of length n, and whose edges indicate the corresponding string differ in exactly one position.

(i) Draw Q_2 and Q_3.

(ii) Unlike the previous week's homework, show that Q_n is always connected.

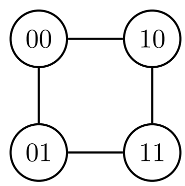
(iii) Prove that Q_n is always bipartite.

(iv) Let R_n denote the graph whose vertices are binary strings of length n with an even number of ones, and whose edges indicate that the two strings differ in exactly two places. This was one of the two connected components from the previous homework's problem on binary strings. Is R_n always bipartite like Q_n ?

Solution:

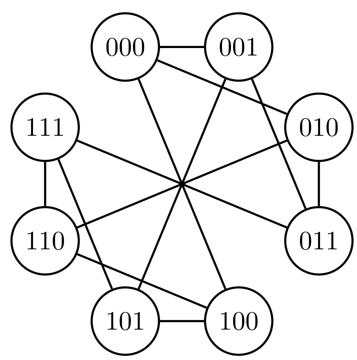
(i) *Draw Q_2 and Q_3.*

(a) Q_2 is a graph with 4 vertices and 6 edges.



Graph Q_2

(b) Q_3 is a graph with 8 vertices and 12 edges.



Graph Q_3

(ii) *Unline the previous week's homework, show that Q_n is always connected.*

Proof 6 Proof that Q_n is always connected:

Let x and y be two vertices in Q_n.

(1) If x and y are the same vertex, then they are connected by definition.

(2) If x and y differ in exactly one position, then they are connected by definition.

(3) If x and y differ in more than one position:

In this case, there must be a binary string z_1 such that x and z_1 differ in exactly one position, and a binary string z_2 such that z_1 and z_2 differ in exactly one position ...

Then, there must be a binary string z_n such that z_{n-1} and z_n differ in exactly one position, and z_n and y differ in exactly one position.

So, x and y are connected by definition.

Therefore, Q_n is always connected.

😄

(iii) *Prove that Q_n is always bipartite.*

Proof 7 Proof that Q_n is always bipartite:

Let S_1 and S_2 be two sets of vertices in Q_n.

Let S_1 be the set of vertices with an even number of ones, and S_2 be the set of vertices with an odd number of ones.

Then, S_1 and S_2 are disjoint, and $S_1 \cup S_2 = V(Q_n)$.

Want to show that S_1 and S_2 are independent.

Let $x \in S_i$ and $y \in S_i$.

Then, x and y differ in an even number of positions, and therefore, x and y do not differ by 1 digit. So, x and y cannot be connect by an edge in Q_n.

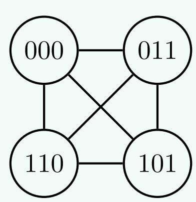
Therefore, S_1 and S_2 are independent.

Since we already know Q_n is connected, then by definition of independent sets, Q_n is bipartite.

😄

(iv) *Let R_n denote the graph whose vertices are binary strings of length n with an even number of ones, and whose edges indicate that the two strings differ in exactly two places. This was one of the two connected components from the previous homework's problem on binary strings. Is R_n always bipartite like Q_n ?*

Example 2.1 (Graph of R_3)



Graph R_3

As the graph above shows, R_3 is not bipartite; the bipartite property does not exist for graphs whose vertices are binary strings of length n with an even number of ones, and whose edges indicate that the two strings differ in exactly two places.

In fact, R_3 is a complete graph (although not all R_n are complete graphs), so is by definition not bipartite.

Question 7: Problem 3 (25 points)

Let G be a connected graph such that exactly one subgraph of G is a cycle. Prove that $|V| = |E|$.

Solution:

Proof 8 Proof that $|V| = |E|$ for a connected graph G such that exactly one subgraph of G is a cycle

Want to show that:

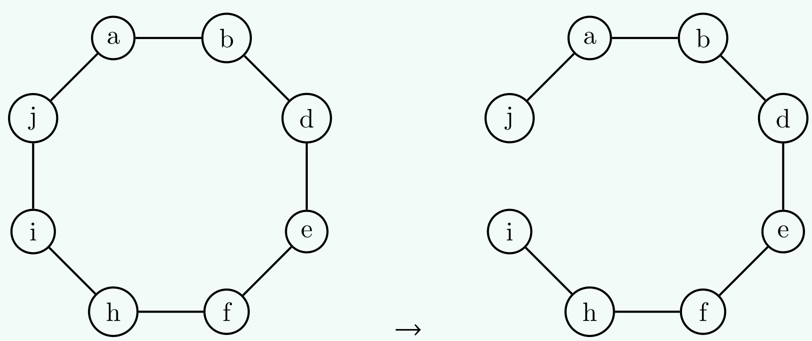
$$G \text{ is a connected graph with exactly one subgraph that is a cycle} \implies |V| = |E|$$

Let G be a connected graph such that exactly one subgraph of G is a cycle.

Then, there exists a path such that $v_1 \to v_2 \to \ldots \to v_{k-1} \to v_k$ and $v_k \to v_1$

Now, say we remove an edge from the cycle. Then every vertex can still reach every other vertex, as vertices that were connected in the cycle can just follow the path along where the cycle was to reach every other vertex.

Example 2.2 (Cycle Edge Removal)



Every vertex can clearly still reach every other vertex.

But now, since there was only one subgraph with a cycle and we just removed an edge from that cycle, then there are no more cycles in the graph.

But, the graph is still connected.

So, by the definition of a tree being a graph with no cycles and is connected, then G with an edge from the cycle removed, G', is a tree.

By the definition of a tree, $|E| = |V - 1|$.

So, for the graph G that has one extra edge than G', $|E| = |V|$.

So, if G is a connected graph such that exactly one subgraph of G is a cycle, then $|V| = |E|$.

Question 8: Problem 4 (25 points)

We will prove two other characterizations of trees.

(i) Prove that a graph G is a tree if and only if it is connected and there is a unique subgraph of G that is both a tree and uses every vertex of G.

(ii) Prove that a graph G is a tree if and only if it has no cycles, but by ADDING a new edge, you always create exactly one cycle. (HINT: In the reverse direction, you are trying to prove that G is connected. Assume for contradiction it is not, and let G_1 and G_2 be two of its components. Can you think of an edge whose addition cannot possibly create a cycle? In the other direction, why does the addition of an edge to a tree only create one cycle? Draw a picture of a single edge being part of two cycles, what is "un-tree-like" about this picture?)

Solution:

(i) *Prove that a graph G is a tree if and only if it is connected and there is a unique subgraph of G that is both a tree and uses every vertex of G.*

Proof 9 Proof that a graph G is a tree if and only if it is connected and there is a unique subgraph of G that is both a tree and uses every vertex of G:

($\Rightarrow$) Want to show that:

$$\text{A graph } G \text{ is a tree} \implies \text{it is connected and there is a unique subgraph of } G \text{ that is both a tree and uses every vertex of } G.$$

If a graph G is a tree, then it is connected by definition of a tree.

Also, by definition of a tree, every vertex in G only has one path to every other vertex. Therefore, if you were to remove an edge, you would necessarily disconnect the tree, as all nodes on one side of that edge by the definition of a tree were only connected to the other side by that single edge.

So, if removing a single edge disconnects the graph, then there can only exist one subgraph that is connected and still uses every vertex; again, by the definition of a tree, this new subgraph would have to be connected, and therefore would have to be the original graph G itself, as all vertices would be the same and all edges would have to remain or the graph would be disconnected.

Therefore, if a graph G is a tree, there is a unique subgraph of G that is both a tree and uses every vertex of G.

($\Leftarrow$) Want to show that:

$$\text{A graph } G \text{ is connected and there is a unique subgraph of } G \text{ that is both a tree and uses every vertex of } G \implies G \text{ is a tree}$$

If the graph G contained a cycle, and it was connected, then there would be multiple subgraphs that are trees, as there would be by definition of a cycle in a graph be multiple edges you could remove from the cycle and still allow for all vertices to reach all others (shown in example above).

So, the graph G cannot contain a cycle, as if it did, then there would not be a unique subgraph that is both a tree and uses every vertex of G—there would be multiple.

Therefore, since G does not contain a cycle but is connected, by the definition of a tree, G is a tree.

😄

(ii) *Prove that a graph G is a tree if and only if it has no cycles, but by ADDING a new edge, you always create exactly one cycle. (HINT: In the reverse direction, you are trying to prove that G is connected. Assume for contradiction it is not, and let G_1 and G_2 be two of its components. Can you think of an edge whose addition cannot possibly create a cycle? In the other direction, why does the addition of an edge to a tree only create one cycle? Draw a picture of a single edge being part of two cycles, what is "un-tree-like" about this picture?)*

Proof 10 Prove that a graph G is a tree if and only if it has no cycles, but by ADDING a new edge, you always create exactly one cycle

($\Rightarrow$) Want to show that:

A graph G is a tree $\implies$ it has no cycles, but by ADDING a new edge, you always create exactly one cycle.

If a graph G is a tree, then it is connected by definition of a tree.

Also, by definition of a tree, every vertex in G only has one path to every other vertex.

So, for example, for a vertex v and u such that $(v, e) \notin E(G)$, there exists a path $v \to \ldots \to u$.

Then, by creating a new edge (v, e), you would create a cycle, as there would now be a path $v \to \ldots \to u \to v$.

This could not create multiple cycles as by the definition of a tree, there is only one path prior to adding an edge from v to u, so the addition of a new edge creates the singular cycle including the path from $v \to u$ and $e(v, u)$.

Therefore, if a graph G is a tree, then by adding a new edge, you always create exactly one cycle.

($\Leftarrow$) Want to show that:

A graph G has no cycles, but by ADDING a new edge, you always create exactly one cycle $\implies$ G is a tree.

Assume for contradiction that G is not connected.
Let G_1 and G_2 be two of its components.

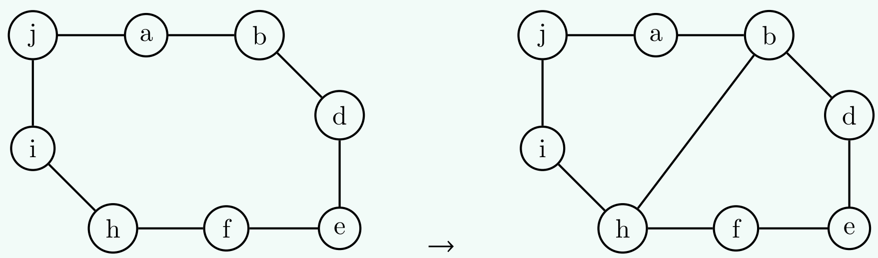
In this case, adding an edge from a vertex from G_1 to a vertex from G_2 could not possibly create a new cycle, as the only path from any vertex in G_1 to G_2 is now that new edge.

So, G must be connected.

Now, want to show that adding a graph that is not a tree could create multiple cycles—this would imply that G must be a tree.

Say you have a graph with a cycle, G'. In this case, the addition of a new edge could create multiple new cycles.

Example 2.3 (Multiple New Cycles after Edge Insertion)



Here, two new cycles are created by adding the edge (b, h):

- $b \to d \to e \to f \to h \to b$
- $h \to i \to j \to a \to b \to h$

So, by adding a new edge to a graph that already has a cycle, you could create multiple cycles.

Therefore, as the graph must be connected, and must not have a cycle, by the definition of a tree, G is a tree.

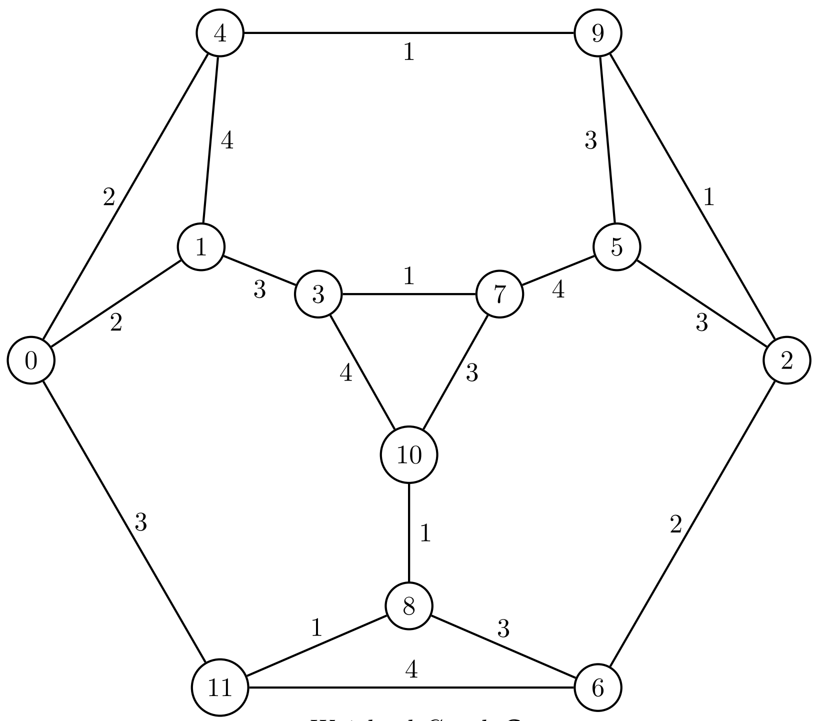
Homework 3

Question 9: (25 points)

(i) Compute a spanning tree of G with minimal cost.

> **Note:-**
> There may well be several such trees. What is the cost of this tree?

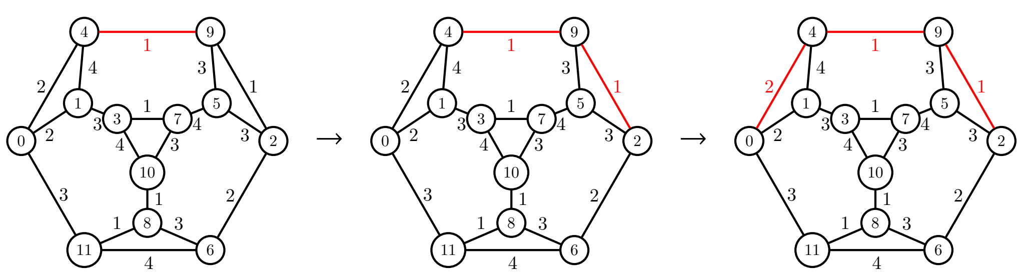
(ii) Compute a spanning tree of G with MAXIMAL cost, and justify why the cost is maximal. (HINT: Kruskal's Algorithm can be directly used here! The thing you need to justify is why you can just replace "least" with "largest" at every step.)

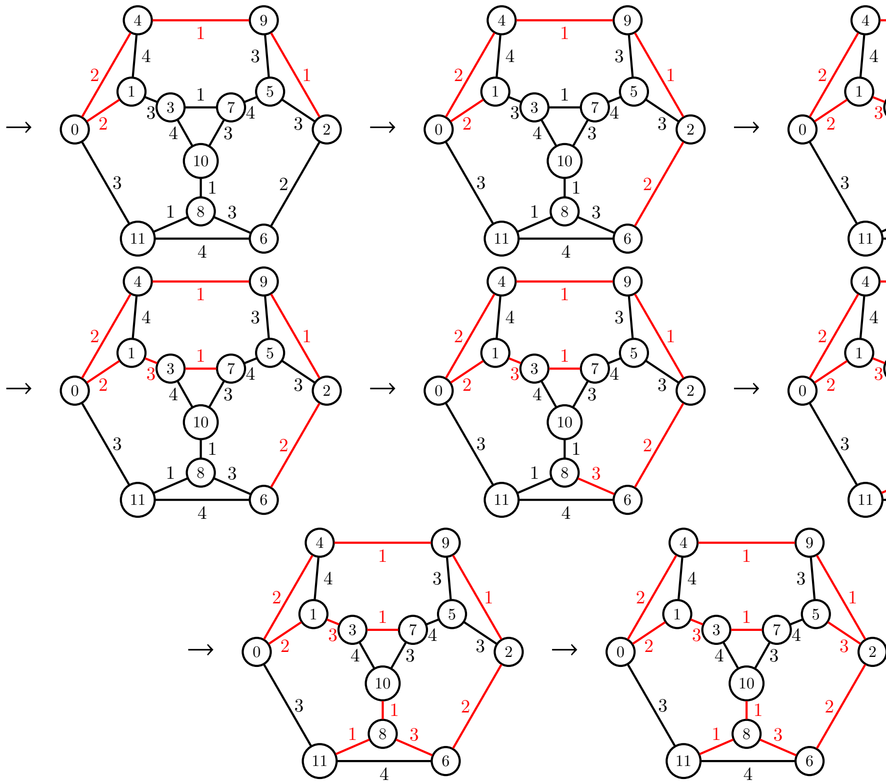


Weighted Graph G

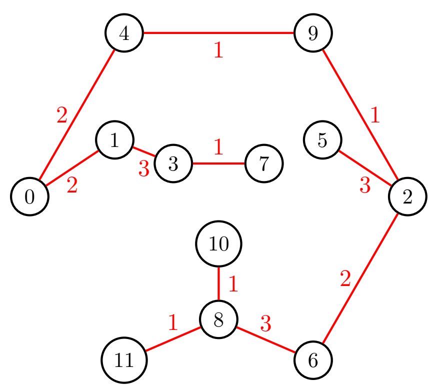
Solution:

- Compute a spanning tree of G with minimal cost.
  - Do do this, will will start from the lowest-cost edge, and then greedily choose the least cost edge from any already visited edge (if a tie, choose one at random) at each step that does not add an edge that connects two vertices already in the spanning tree so far, until every vertex is part of the tree.



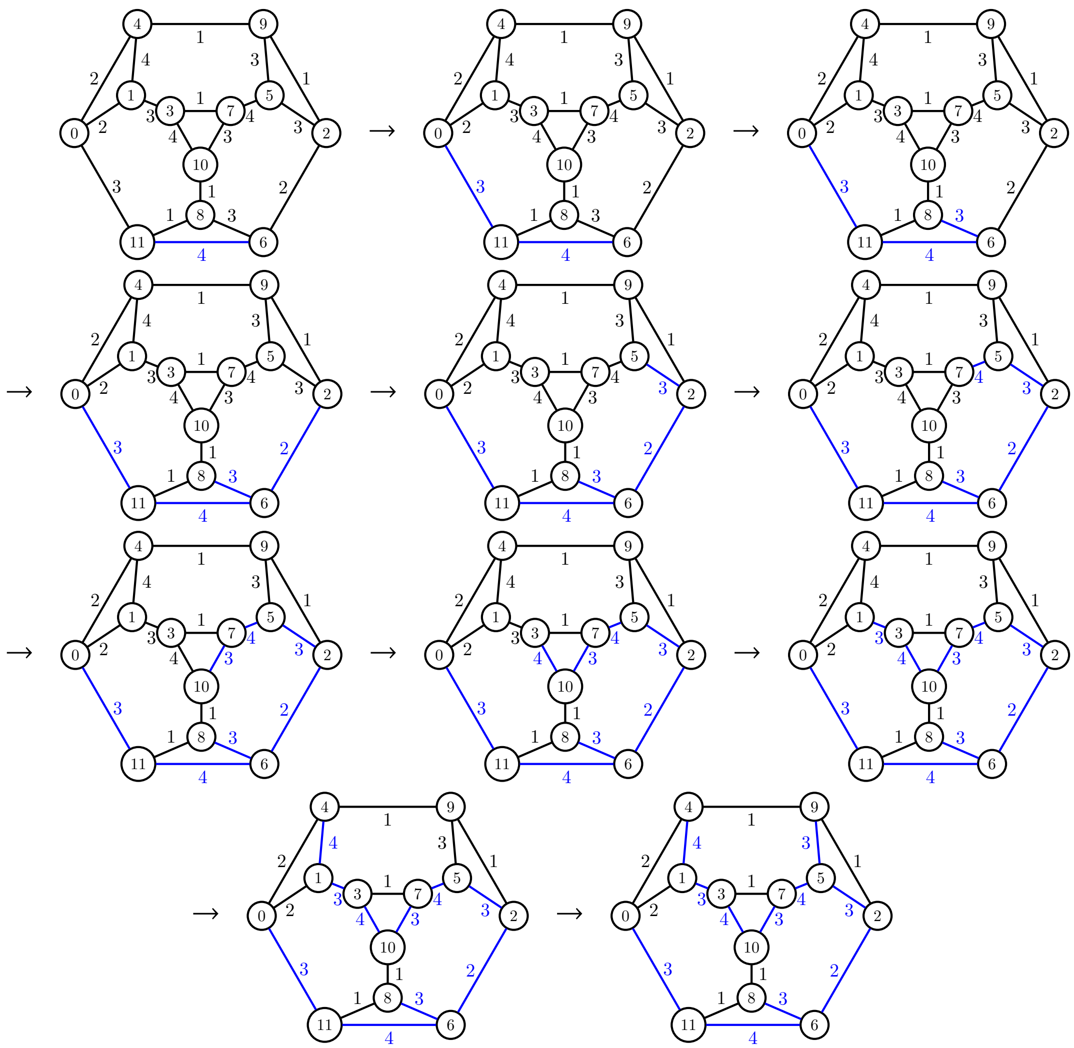


After having ran Kruskal's Algorithm, we get the following spanning tree:

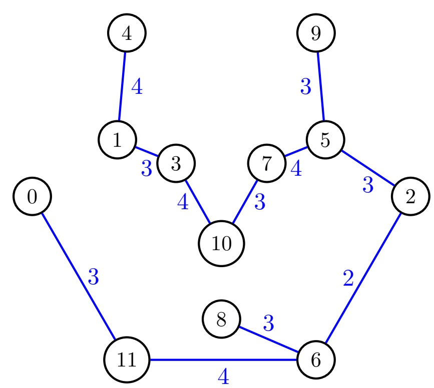


This tree has a total weight of 20, which is the minimum possible weight for a spanning tree of this graph. There are multiple variations of this tree, for example swapping the edge $e(5, 2)$ with $e(5, 9)$, but the total cost of the tree would not decrease.

- Compute a spanning tree of G with MAXIMAL cost, and justify why the cost is maximal.
  - For finding the maximal cost, we can use the same algorithm as before, but instead of finding the minimum cost edge at every step, we find the maximum value step. This works, for example, because if you flipped the signs of all of the edges is the graph, such that they were negative, you would expect Kruskal's Alforithm to find the cheapest spanning tree, which would be the furthest magnitude from 0. Therefore, by definition, Kruskal's algorithm must work to find the maximal spanning tree as well as the minimum spanning tree.



After having ran Kruskal's Algorithm for finding the maximal spanning tree, we get the following graph:



The total weight of the maximal spanning tree is 36. The edges in blue are the ones that are part of the tree. Again, this tree has multiple different variations in terms of which edges were chosen when, and what the final tree looks like, but the overall weight of the maximal spanning tree of this graph will not change in any of the other variations of the tree—the overall weight will still be 36.

Question 10: (25 points)

The following is an alternative algorithm for computing an Euler circuit in a connected graph whose every vertex degree is even. We begin by calling all of the edges of G unmarked. Throughout the course of the algorithm, we will be marking edges as we construct the circuit.

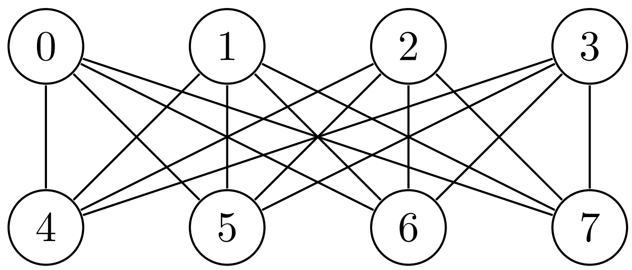
Step 1: Choose a vertex x of your graph. This is our first lead vertex.

Step 2: If all edges of G are marked, then the algorithm ends, otherwise proceed to Step 3.

Step 3: Among all unmarked edges connected to the current lead vertex, choose, if possible, an edge whose removal would not disconnect the subgraph of unmarked edges. If this is not possible, then choose any unmarked edge connected to the lead. Mark this edge, and change the lead vertex to its other endpoint.

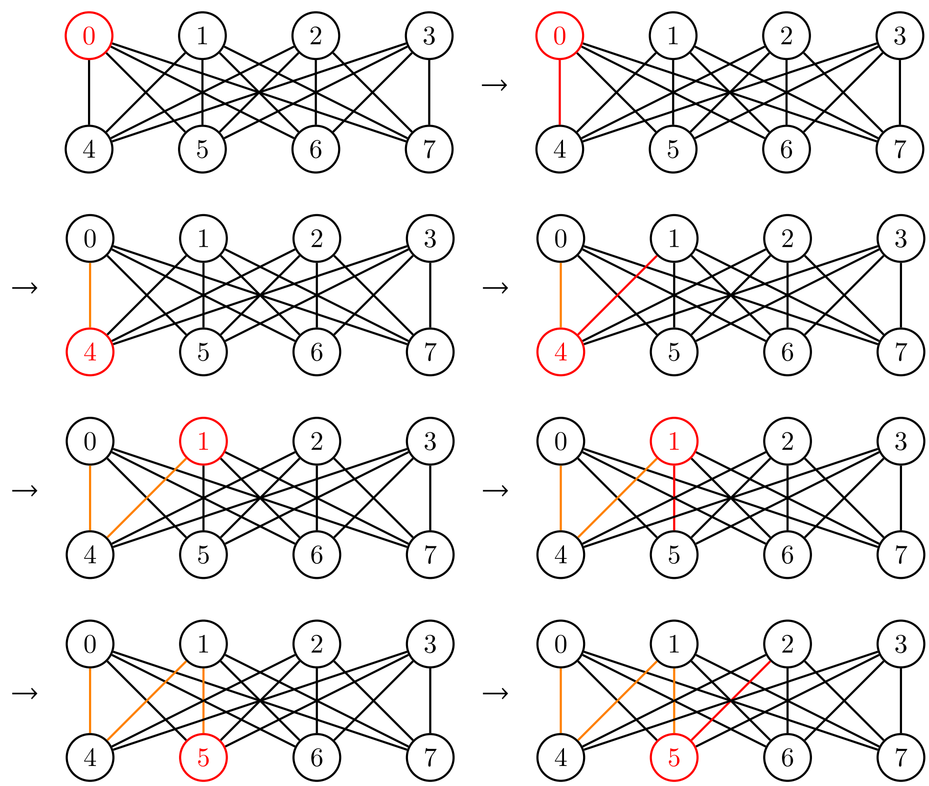
Step 4: Repeat Step 2.

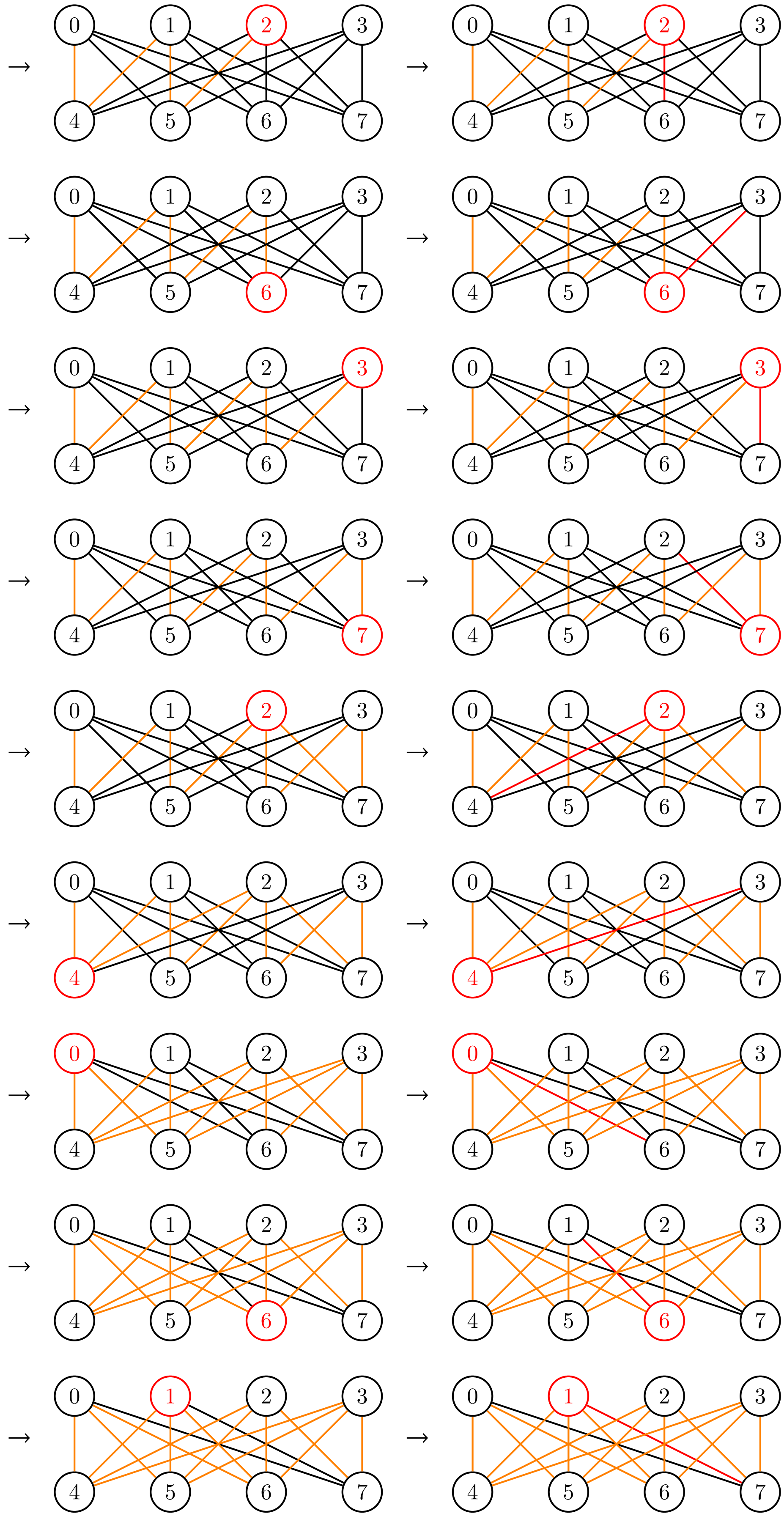
Show, step-by-step, the Euler circuit created by applying this algorithm to the following graph. You may start at any vertex.

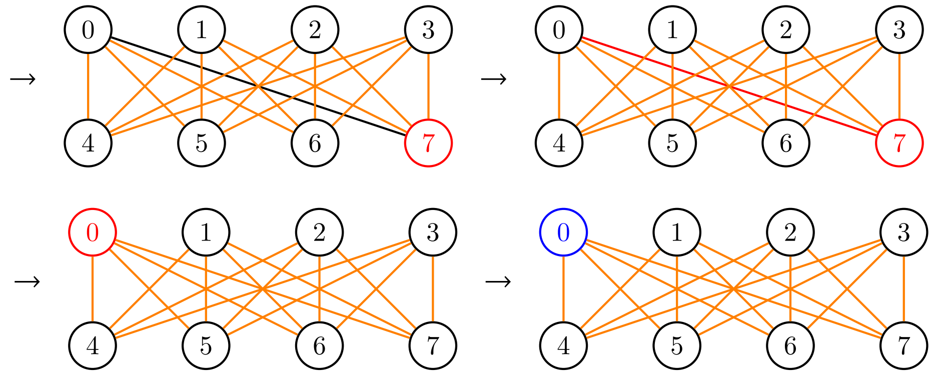


Euler Graph G

Solution:




→ 0 1 2 3 4 5 6 7 → 0 1 2 3 4 5 6 7
→ 0 1 2 3 4 5 6 7 → 0 1 2 3 4 5 6 7
→ 0 1 2 3 4 5 6 7 → 0 1 2 3 4 5 6 7
→ 0 1 2 3 4 5 6 7 → 0 1 2 3 4 5 6 7
→ 0 1 2 3 4 5 6 7 → 0 1 2 3 4 5 6 7
→ 0 1 2 3 4 5 6 7 → 0 1 2 3 4 5 6 7
→ 0 1 2 3 4 5 6 7 → 0 1 2 3 4 5 6 7
→ 0 1 2 3 4 5 6 7 → 0 1 2 3 4 5 6 7
→ 0 1 2 3 4 5 6 7 → 0 1 2 3 4 5 6 7



Note:-

Every edge has been traversed, and we have ended up at the same node as we started at, and thus have created a Euler Circuit.

Question 11: (25 points)

An Eulerian trail is a trail that hits every edge (i.e. it does not necessarily end where it started). Prove that any graph G with at most two vertices of odd degree must have an Eulerian trail. HINT: If the graph has no odd degree vertices then it must have an Eulerian trail (why?). We may therefore assume that G has two vertices a, b of odd degree (why can't it only be one?). If $\{a, b\}$ is not an edge of G, consider the graph G with the edge $\{a, b\}$ added. How can you adapt this argument if the edge $\{a, b\}$ already exists?

Proof 11 A graph G has at most two vertices of odd degree $\implies$ G has an Eulerian trail

(Case 1) if G has no odd degree vertices, then G has an Eulerian trail by the theorem shown in class. In essence, this is because:

i. Since every vertex has an even degree, you can create a cycle that involves every vertex.
ii. With the remaining edges, you can create cycles with those edges.
iii. Then, you can add in those cycles to the original cycle, and you will have a Eulerian trail, since you have now traversed every edge.

So, there is no need to prove this case, as there exists an Euler tour (which is also a Euler trail).

(Case 2) G has one vertex of odd degree

This is inherently not possible as:

$$\sum_{v \in V} d(v) = 2m$$

where m is the number of edges in G. Thus, if G has one vertex of odd degree, then $d(v)$ would be odd, which is not possible.

(Case 3) G has two vertices of odd degree In this case, we can assume that G has two vertices of odd degree, lets say vertices u, v. Then, lets add a new vertex w to the graph G, and new edges from $u \to w$ and $v \to w$. Now, w has a degree of 2, and $d(v)$ and $d(u)$ are even. Therefore, this new graph G' with w only has even degree vertices.

Now, since G' has only even degree vertices, we can apply the first case, and create a Eulerian tour. So, in this Eulerian tour, we can start from the w vertex, traverse all vertices, and return back to w.

Applying this logic to G, without the vertex w and edges $u \to w$ and $v \to w$, we can create a Eulerian trail. This is because instead of starting at w, we can start at u or v, and instead of finishing back at w, we can finish one edge away at u if we started at v, or vice versa.

So, in this case, the graph G has an Eulerian trail.

Therefore, if G has at most two vertices of odd degree, then G has an Eulerian trail.

😄

Question 12: (25 points)

Let G be a connected graph. We say a collection of cycles $\{C_1, \ldots, C_g\}$ is independent if each of the cycles has at least one edge that is not shared with the others. For instance, if G is the graph that is two squares sharing a side then G has precisely three cycles, the two inner squares as well as the outer rectangle. Any pair of these is independent, whereas the trio is not independent, as the outer rectangle in this case would not have an edge unique to it. Prove that you can always find an independent collection of cycles that has $|E| - |V| + 1$ members. HINT: This is actually a question about spanning trees. How does the quantity $|E| - |V| + 1 = |E| - (|V| - 1)$ relate with a spanning tree of G ?

Proof 12 For any graph, there exists an independent collection of cycles that has $|E| - |V| + 1$ members

(Case 1) G is a tree

In this case, since $|E| = |V| - 1$, then $|E| - |V| + 1 = 0$. Thus, there are no cycles, as is expected of a tree.

(Case 2) G is not a tree, and therefore has a non-zero number of cycles

Let G_S be a spanning tree of G. Then, $|E| = |V| - 1$, and $|E| - |V| + 1 = 0$.

Now for every additional edge in G, we can add a new cycle to G_S, as that new edge will make the new cycle independent to every prior cycle added to G_S. So, for every edge beyond the initial $|V| - 1$ edges a new independent cycle is added to the graph.

So, simplifying this expression to solve for the number of cycles, we get:

$$\begin{aligned}
(\#\text{ of independent cycles}) &= (\text{total } \#\text{ of edges}) - (\#\text{ of edges in a spanning tree of that graph}) \\
&= |E| - (|V| - 1) \\
&= |E| - |V| + 1
\end{aligned}$$

Notice this is what we wanted to show.

So, for any graph, there exists an independent collection of cycles that has $|E| - |V| + 1$ members.

😄

Homework 4

Question 13: Problem 1 (25 points)

Let $n \geqslant 1$ be an integer, and let G be the following graph: G is bipartite with one partition having n vertices, and the other having $n+1$ vertices, and every possible edge is included in the graph (that maintains the bipartition).

(1) Compute $\delta(G)$ and $|V|$.

(2) Prove that G cannot possibly have a Hamiltonian cycle. Along with the previous part, this shows that the primary assumption of Dirac's theorem is the best possible.

(1) Compute $\delta(G)$ and $|V|$.

- Compute $\delta(G)$.

  Since G is bipartite, we know that each vertex in the first partition, P_1 could at most be connected to every vertex in P_2 in accordance with the bipartite definition. Therefore, the lowest degree of a vertex would be n, as a vertex from the partition with $n+1$ vertices must be connected to every vertex in the partition of n vertices. Using the stipulation that every possible edge is included in the graph (that maintains the bipartition), we know that this case must be true. Therefore:

$$\delta(G) = n$$

- Compute $|V|$.

  Since there are two partitions, one with $n+1$ vertices and one with n vertices, we know that the total number of vertices is:

$$|V| = n + n + 1 = 2n + 1$$

(2) Prove that G cannot possibly have a Hamiltonian cycle. Along with the previous part, this shows that the primary assumption of Dirac's theorem is the best possible.

- Prove that G cannot possibly have a Hamiltonian cycle.

Proof 13 Proof of no Hamiltonian cycle

Since every vertex in P_1 (the partition with n vertices) has edges to only vertices in P_2 (the partition with $n+1$ vertices), then two cases can occur:

(Case 1) Start the cycle from a vertex in P_1

Then, the cycle must go to a vertex in P_2, then back to a vertex in P_1, then back to a vertex in P_2, and so on.

Since there are n vertices in P_1 and $n+1$ vertices in P_2, we will run out of unique vertices to visit in P_1 before visiting all vertices in P_2.

Therefore, there is no Hamiltonian cycle in G that starts from a vertex in P_1.

(Case 2) Start the cycle from a vertex in P_2

Then, the cycle must go to a vertex in P_1, then back to a vertex in P_2, then back to a vertex in P_1, and so on.

Since there are $n + 1$ vertices in P_2 and n vertices in P_1, we will run out of unique vertices to visit in P_1 before we are able to revisit the starting node in P_2, and thus a Hamiltonian path is possible, but a cycle is not, as we cannot end at the same node we started at.

Therefore, there is no Hamiltonian cycle in G that start from a vertex in P_2.

So, we have shown that G cannot possibly have a Hamiltonian cycle.

😄

Note:-

You can also prove this by showing that since G is bipartite, it cannot have a cycle of odd length, but G has an odd number of vertices, so a Hamiltonian cycle on G would have an odd length (This is a contradiction).

Question 14: Problem 2 (25 points)

Let G be a connected planar graph whose every vertex has degree 4 and assume that every face of G is a triangle, and every edge of G is on the boundary of two faces.

(1) Prove that G has twice as many edges as vertices. HINT: Planarity and connectivity have nothing to do with this.

(2) Prove that G must have 6 vertices. HINT: Planarity and connectivity have everything to do with this.

(3) Draw the (unique) graph G that satisfies all of these properties.

(1) Prove that G has twice as many edges as vertices. HINT: Planarity and connectivity have nothing to do with this.

As shown in class, the following relationship relates the number of edges and the degree of vertices in a graph G:

$$\frac{1}{2}\sum_{v\in V} d(v) = |E|$$

Lets say that the number of vertices in G is n. Then, we know that the number of edges in G is:

$$|E| = \frac{1}{2}\sum_{v\in V} d(v) = \frac{1}{2}\sum_{v\in V} 4 = 2n$$

Since there are necessarily $2n$ edges in G, and we assumed that there are n vertices in G, we can say that there are twice as many edges as vertices in G.

(2) Prove that G must have 6 vertices. HINT: Planarity and connectivity have everything to do with this.

Proof 14 Proof of 6 vertices

Since G is planar, we know that it must have at least 3 vertices.

Now, to show that G has 6 vertices, we can use the following theorem, since $n \geq 3$ and G is a fully-triangulated planar graph:

> **Theorem 4.1** Euler's Formula
>
> Let G be a planar graph with n vertices. Then, G has at most $3n - 6$ edges.

Since we know that each vertex in G has degree 4, and we just found that because of this there are twice as many edges as vertices, we can write the following equation:

$$\begin{aligned} |E| &\leqslant 3n - 6 \\ 2n &\leqslant 3n - 6 \end{aligned}$$

Then, simplifying the equation, we get:

$$\begin{aligned} -n &\leqslant -6 \\ n &\geqslant 6 \end{aligned}$$

So, we know that G has at least 6 vertices.

Now, to show that G has exactly 6 vertices, we can use the following corollary since every face of G

is a triangle:

$$f \leq \frac{2}{3}|E|$$

To show the upper bound for the number of vertices.

Using Euler's Formula:

$$\begin{aligned} n - |E| + f &= 2 \\ n - \frac{1}{3}|E| &= 2 \\ n &= \frac{1}{3}|E| + 2 \end{aligned}$$

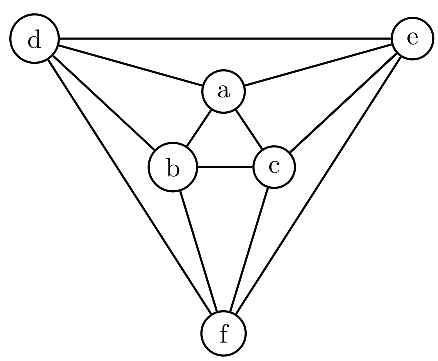
Now, upperbounding the number of edges in G:

$$\begin{aligned} 3n &\leq n + 12 \\ 2n &\leq 12 \\ n &\leq 6. \end{aligned}$$

Since we now know that $n \geq 6$, and $n \leq 6$, we can say that $n = 6$.

😄

(3) Draw the (unique) graph G that satisfies all of these properties.



Graph G

Question 15: Problem 3 (25 points)

Let G be a graph such that every edge appears on an odd number of cycles. We will prove that G has an Eulerian Circuit.

(1) The idea of this proof is similar to that of the handshake lemma. For any edge e, let $c(e)$ be the number of cycles that e appears on, and for any vertex a let $c(a)$ be the number of cycles that a appears on. Fix a vertex a in G. Prove that

$$\sum_{e \text{ is connected to } a} c(e) = 2c(a)$$

(2) Use the formula from the previous step to show that a has even degree, and conclude that G has an Eulerian Circuit.

(1) The idea of this proof is similar to that of the handshake lemma. For any edge e, let $c(e)$ be the number of cycles that e appears on, and for any vertex a let $c(a)$ be the number of cycles that a appears on. Fix a vertex a in G. Prove that

$$\sum_{e \text{ is connected to } a} c(e) = 2c(a)$$

Proof 15 Proof of formula

if the vertex a is part of a cycle, then there is necessarily two edges that are connected to a that are also part of that cycle, as a cycle must enter into a and leave via a different edge, as by the definition of a cycle edges mus not repeat. So, for every cycle that a is a part of, two of the edges connected to a must also be a part of that cycle.

And, since cycles can not repeat vertices, there can not be more than two edges connected to a that are a part of a cycle involving the vertex a.

So, for each cycle that a is a part of, there are two edges connected to a that are also a part of that cycle. Therefore:

$$\sum_{e \text{ is connected to } a} c(e) = 2c(a)$$

Notice this is what we wanted to show.

☺

(2) Use the formula from the previous step to show that a has even degree, and conclude that G has an Eulerian Circuit.

Proof 16 Proof of Eulerian Circuit

Since we know that any vertex v must appear on an odd number of cycles, we can write the following equation:

$$c(v) = 2k + 1 \text{ for some } k \in \mathbb{N}$$

Then, using the formula we just proved, we can write the following equation:

$$\sum_{e \text{ is connected to } v} c(e) = 2c(v)$$
$$\sum_{e \text{ is connected to } v} c(e) = 2(2k+1)$$
$$\sum_{e \text{ is connected to } v} c(e) = 4k+2$$

Now, since we know that the number of edges connected to a vertex must be even, we can recall:

$$\deg(v) = \# \text{ of edges connected to } v$$

Therefore, we know that the degree of any vertex in G is even.

So, now that we know that the degree of every vertex in G is even and nonzero sinze every vertex must be a part of at least 1 cycle, if we use the fact that G is connected, then by the definition of a Euler circuit:

$$\text{An undirected graph has an Eulerian cycle} \iff \begin{cases} \text{Every vertex has even degree} \\ \text{The graph is connected} \end{cases}$$

So, we know that G has an Eulerian Circuit.

😄

Question 16: Problem 4 (25 points)

Let G be a graph with n vertices and e edges, and let $\bar{G}$ denote the graph that has the exact same vertex set as G, while having the exact complement of the edges of G. For instance, if G is the graph that looks like the letter Y, then $\bar{G}$ is a triangle with an isolated vertex in the middle.

(1) Recall from class that in any connected planar graph $G, e \leqslant 3n - 6$. Prove that this "connected" assumption is actually unnecessary here. HINT: The picture you should start with here is a bunch of disjoint blobs floating in space. Can you add edges to this picture so that it becomes connected while not breaking planarity?

(2) It is a fact (that you do not need to prove) that $\bar{G}$ has $\frac{n(n-1)}{2} - e$ edges. If we assume that $n \geqslant 11$, prove that it is impossible for both G and $\bar{G}$ to be planar. HINT: Assume that both are planar for contradiction, and prove that it is impossible for BOTH G and $\bar{G}$ to satisfy all of the necessary inequalities.

(1) Recall from class that in any connected planar graph $G, e \leqslant 3n - 6$. Prove that this "connected" assumption is actually unnecessary here. HINT: The picture you should start with here is a bunch of disjoint blobs floating in space. Can you add edges to this picture so that it becomes connected while not breaking planarity?

Proof 17 Proof of connected assumption

If G is connected, this is already true as proved in class.

If G is not connected, then we can break it up into its connected components.

Say we have k connected components in G. Call these components $G_1, G_2, \ldots, G_k$.

Connect the k components together with $k - 1$ edges to create G'.

Now, we have a connected graph with the k connected components that is still planar.

Note:-

This is because any two planar components can be connected with an edge to create a planar graph as the components can just be redrawn such that the new edge is just crossing empty space between the disconnected graphs

$$\implies |E_{G'}| \leq 3|V_{G'}| - 6$$

Since G' is connected, we know that $|V_{G'}| = n$ and $|E_{G'}| = e + k - 1$, so $|E_{G'}| > |E_G|$

$$\implies |E_G| \leq 3|V_G| - 6$$

So, we can conclude that the connected assumption is not necessary.

☺

(2) It is a fact (that you do not need to prove) that $\bar{G}$ has $\frac{n(n-1)}{2} - e$ edges. If we assume that $n \geqslant 11$, prove that it is impossible for both G and $\bar{G}$ to be planar. HINT: Assume that both are planar for contradiction,

and prove that it is impossible for BOTH G and $\bar{G}$ to satisfy all of the necessary inequalities.

Proof 18 Proof of Planarity

$$|E_{\overline{G}}| = \frac{n(n-1)}{2} - |E|$$

Assume for the sake of contradiction that both G and $\overline{G}$ are planar.

Then:

$$\implies |E_G| \leq 3|V_G| - 6$$
$$\implies |E_{\overline{G}}| = \frac{n(n-1)}{2} - |E| \leq 3|V| - 6$$

Plugging in, we get:

$$\implies \frac{n(n-1)}{2} - (3|V| - 6) \leq 3|V| - 6$$
$$\implies \frac{n(n-1)}{2} \leq 6|V| - 12$$
$$\implies n^2 \leq 13|V|^2 - 24$$

Now, plugging in n as 11:

$$\implies 11^2 \leq 13(11) - 24$$
$$\implies 121 \leq 143 - 24$$
$$\implies 121 \leq 119$$

Contradiction!

Since the n^2 term grows much faster than the n term, we can conclude that n must be less than 11 for both G and $\overline{G}$ to be planar, as any value larger than 11 for n would also be invalid.

Therefore, it is impossible for both G and $\overline{G}$ to be planar for $|V| \geq 11$

😄

Homework 5

Question 17: Problem 1 (25 points)

In this problem we define the average degree of a graph,

$$ad(G) = \frac{\sum_{v \in V} d(v)}{|V|}$$

Prove that the average degree of a connected planar graph is strictly less than 6 .

Proof 19 Proof that the average degree of a connected planar graph is strictly less than 6

So, to begin, first lets assume for contradiction that the average degree is above 6. In this case, we have the following equation:

$$\frac{\sum_{v \in V} \deg v}{|V|} \geq 6$$

Now, by the Handshaking Lemma:

$$\sum_{v \in V} \deg v = 2|E|$$

We can simplify the previous equation to:

$$\begin{aligned} \frac{2|E|}{|V|} &\geq 6 \\ 2|E| &\geq 6|V| \\ |E| &\geq 3|V| \end{aligned}$$

But, using a formula shown in class, for a planar graph we have the following equation (derived from Euler's Formula)

$$|E| \leq 3|V| - 6$$

But notice this is a contradiction. The following equations can not both be true:

$$\begin{gathered} |E| \leq 3|V| - 6 \\ \text{and} \\ |E| \geq 3|V| \end{gathered}$$

So, the average degree of a connected planar graph must be strictly less than 6

😄

Question 18: Problem 2 (25 points)

We say that a graph with $n \geqslant 3$ vertices is maximally planar if it is planar, but adding any single edge creates a graph that is not planar.

(i) Prove that every face in a maximally planar graph must be a triangle. HINT: Your instinct might be to take any face bigger than a triangle and just draw a chord through the region. This isn't so clear cut! It might be possible that all vertices on the boundary of the face are adjacent, the edges just happen to be outside the face. You must show that this cannot be the case.

(ii) Prove that every edge in a maximally planar graph must be on the boundary of two faces. You may use, without proof, the fact that any edge that does not bound any face always disconnects the graph when removed. HINT: The picture you should draw here is two "blobs" connected by a single edge. Why is this picture accurate to the situation, and why does it show that the graph cannot possibly be maximally planar?

(iii) Write down the number of edges and faces of a maximally planar graph as a function of n.

(i) Prove that every face in a maximally planar graph must be a triangle. HINT: Your instinct might be to take any face bigger than a triangle and just draw a chord through the region. This isn't so clear cut! It might be possible that all vertices on the boundary of the face are adjacent, the edges just happen to be outside the face. You must show that this cannot be the case.

Proof 20 Every face in a maximally planar graph must be a triangle

First, we know that the graph is connected, as if it wasn't we could add an edge to connect the components while still retaining planarity. Similarly, every boundary of a region in the graph must be a cycle, as otherwise we could add an edge to connect the disparate vertices in the graph, while again retaining planarity.

So, now that we know the boundary of every region will be a cycle, we can consider a few cases.

The boundary can not consist of one or two vertices, as there can not be a face made up of less than 3 vertices.

In the case of the boundary being made up of 3 vertices, this is a triangular face, and we do not need to consider this further as it is what we are looking for.

Finally, in a boundary that contains more that 3 vertices, we know that there must be a walk of vertices V_1, V_2, V_3 that is along the border of the region. We know that vertices V_1 and V_2 are not connected, as in that case the region would not be bounded by V_2 as well.

Since $e(V_1, V_3) \notin |E|$, this is a contradiction—we can safely add that edge without breaking planarity, as the new edge would bisect the existing face with a boundary of more than 3 edges, which contradicts the definition of a maximally planar graph.

So, a maximally planar graph must have only triangular faces.

😄

(ii) Prove that every edge in a maximally planar graph must be on the boundary of two faces. You may use, without proof, the fact that any edge that does not bound any face always disconnects the graph when removed. HINT: The picture you should draw here is two "blobs" connected by a single edge. Why is this picture accurate to the situation, and why does it show that the graph cannot possibly be maximally planar?

Proof 21 Proof that every edge in a maximally planar graph must be on the boundary of two faces

Assuming that the number of vertices is greater than two (as in the case of a maximally planar graph with $|V| = 2$, an edge can in fact not be on the boundary of two faces), then let e_i be an edge that does not bound two faces.

Let V_1 and V_2 be the two vertices that are connected by e_i. Since there are more than 2 vertices, either V_1 or V_2 must be connected to another vertex.

Lets call this vertex V_3, and lets assume that V_3 is connected to V_2. Since e_i is not on the boundary of two faces, we know that V_1 must not be connected to V_3, or any other vertices adjacent to V_2, as that would create a second face that would contradict the supposition that e_i only bounded one face.

So, while preserving planarity, we can safely add an edge between V_1 and some edge that neighbors V_2, such as V_3. This contradicts the definition of a maximally planar graph.

So, every edge in a maximally planar graph must be on the boundary of two faces.

😊

(iii) Write down the number of edges and faces of a maximally planar graph as a function of n.

Solution

Since a maximally planar graph assumes that no edges can be added while retaining planarity, we can use an upper bound of the number of edges in a planar graph derived from Euler's Formula:

$$|E| \leq 3|V| - 6$$

We can say that a maximally planar graph must have $3|V| - 6$ edges.
Now, directly applying Eulers formula, we can show the number of faces in a maximally planar graph:

$$|F| = |E| - |V| + 2$$

Now, plugging the number of edges in a maximally planar graph into the equation we get the following:

$$|F| = (3|V| - 6) - |V| + 2$$
$$|F| = 2|V| - 4$$

So, given n vertices, for a maximally planar graph we can say:

$$|E| = 3|V| - 6$$
$$|F| = 2|V| - 4$$

Question 19: Problem 3 (25 points)

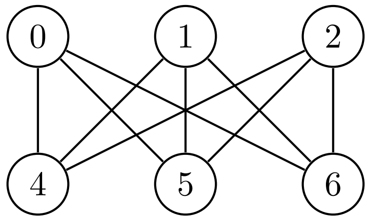
Compute the Chromatic Number $\chi(G)$ for the following graphs:

(i) $K_{3,3}$

(ii) The Petersen Graph

(iii) The Dodecahedron Graph

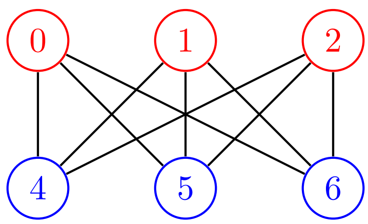
(i)



$K_{3,3}$

Solution

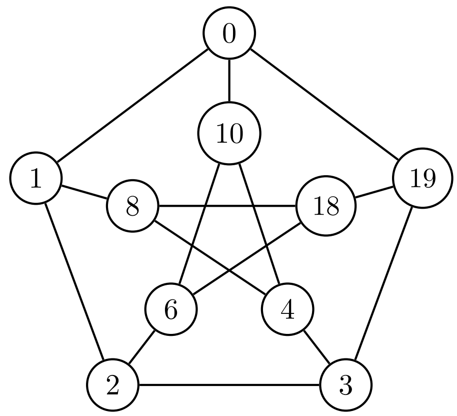
The following graph minimized the number of colorings in the graph $K_{3,3}$:



Colored $K_{3,3}$

Since $K_{3,3}$ is bipartite (as shown in class, but is also fairly clear), there is a possible two-coloring of the graph.

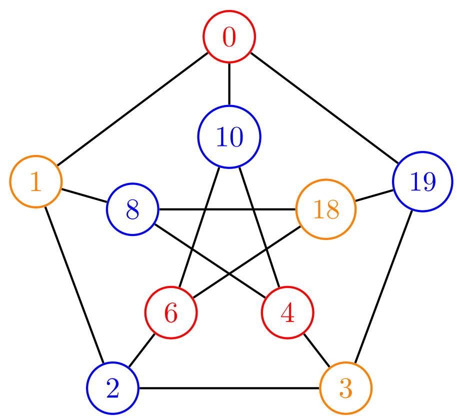
(ii)



Petersen Graph

Solution

The following graph minimizes the number of colorings in the Petersen Graph:

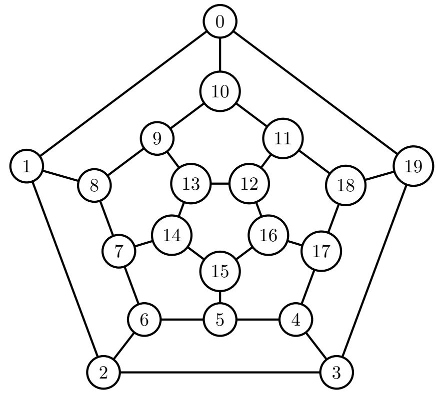


colored Petersen Graph

For this graph, it is clear that $\chi(G) = 3$. Since all vertices in the graph have a degree of 3, the lower bound for χ is 3.

Since I have shown there exists 3-coloring graph of the Petersen Graph, $\chi(G) = 3$.

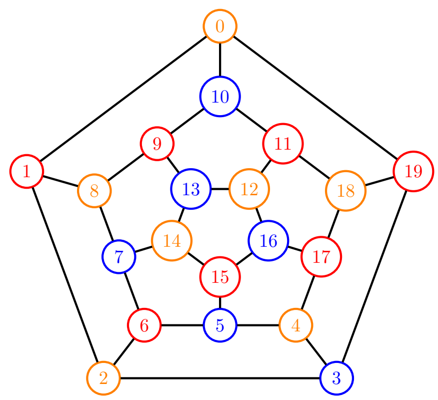
(iii)



dodecahedron graph

Solution

For the Dodecahedron graph, we can achieve the following coloring:



dodecahedron graph

Since every vertex in the dodecahedron graph has a degree of exactly 3, we know that the lower bound of $\chi(G)$ is also 3.

In this case, I have shown a 3-coloring of the graph, so $\chi(G)$ must be 3.

Question 20: Problem 4 (25 points)

Let G be a connected graph, and write $\tau(G)$ for the length of the longest possible path in G. We will prove that $\chi(G) \leqslant \tau(G) + 1$

(i) To begin, prove that if G is any connected graph with minimum degree $\delta(G)$, then G has a path of length $\delta(G)$. (HINT: Given any path whose length is shorter than $\delta(G)$, prove that this path can be extended.)

(ii) Now let G be a connected graph with chromatic number $\chi(G)$. Prove that there exists a subgraph H of G such that $\chi(H) = \chi(G)$, but for any vertex v the graph $H - v$ has $\chi(H - v) < \chi(H)$.

(iii) If H is the subgraph from the last part, show that the minimal degree of H is at least $\chi(G) - 1$, and conclude that there exists a path in H of length $\chi(G) - 1$.

(iv) Put the previous steps together to finish the proof!

(i) To begin, prove that if G is any connected graph with minimum degree $\delta(G)$, then G has a path of length $\delta(G)$. (HINT: Given any path whose length is shorter than $\delta(G)$, prove that this path can be extended.)

Proof 22 A connected graph G with minimum degree $\delta(G) \implies G$ has a path of length $\delta(G)$

Since G has a minimum degree of $\delta(G)$, then it must have atleast $\delta(G)$ vertices, that are connected by the definition of the degree of a vertex.

So, since there are atleast $\delta(G) + 1$ vertices in the graph G that are connected, there must be a path of length G that exists between those verices.

By contradiction:

Lets assume the path P_1 of length $< \delta(G)$ is the longest path on G.

Since every vertex in G must have a degree of atleast $\delta(G)$, then the endpoints of the path P_1 would have to be connected to at least $\delta(G)$ other vertices.

Since the path is only of length $\delta(G) - k$ where k is a non-zero positive integer, there must be atleast $\delta(G) - k$ vertices connected to a starting or ending point of P_1 that is not in the path already.

Therefore, this is a contradiction, and P_1 must not be the longest path in G.

😊

(ii) Now let G be a connected graph with chromatic number $\chi(G)$. Prove that there exists a subgraph H of G such that $\chi(H) = \chi(G)$, but for any vertex v the graph $H - v$ has $\chi(H - v) < \chi(H)$.

Proof 23 There exists a subgraph H of G such that any vertex in the graph $H - v$ has $\chi(H - v) < \chi(H)$

Start with a graph G. If removing a vertex of G lowers the value of $\chi(G)$, we are done.

If not, then remove that edge of G, and repeat the process.

This will eventually lead to a graph H that has a chromatic number of $\chi(H) = \chi(G)$, and removing any vertex from H will lower the value of $\chi(H)$.

☺

(iii) If H is the subgraph from the last part, show that the minimal degree of H is at least $\chi(G)-1$, and conclude that there exists a path in H of length $\chi(G)-1$.

Proof 24 The minimal degree of H is at least $\chi(G)-1$, so $\exists$ a path in H of length $\chi(G)-1$

Since removing any vertex from H would reduce the value of $\chi(H)$, want to use this to prove that the minimal degree of H is at least $\chi(G)$.

This is true because if there exists a node in H, say v_1, such that $\delta(v_1) < \chi(G)-1$, then removing that node would not decrease the value of $\chi(H)$, because as shown in class when removing a node that has a degree less than the value for $\chi-1$, it can not decrease the overall value of $\chi(H)$ for the graph.

So, using this fact, we know that the minimal degree of H is at least $\chi(G)-1$. Then, want to use this to show that there exists a path in H of length $\chi(G)-1$.

Using our proof from part (i) of this problem, this is true, and there must exist a path in H such that the length of that path is greater than $\chi(G)-1$, given that any vertex in H has a degree of $\chi(G)-1$ or greater.

☺

(iv) Put the previous steps together to finish the proof!

Proof 25 Finishing the proof that for a connected graph G, $\tau(G)+1 \geq \chi(G)$

We know that for a graph G, from the previous parts of this problem, there exists a subgraph of G such that any vertex reduces the value of $\chi(G)$.

Again, from the previous parts of this problem, that subgraph must have a path length greater than $\chi(G)-1$. So, we can rewrite this as:

$$\tau(G) \geq \chi(G)-1\chi(G)-1 \leq \tau(G)\chi(G) \leq \tau(G)+1$$

Since this path exists on a subgraph of G, by the definition of a subgraph, it must also be a valid path on G.

So, for a given connected graph G, $\exists$ a path of length $\tau(G)$ such that $\chi(G) \leq \tau(G)+1$.

☺

Homework 6

Question 21: Problem 1 (25 points)

Here are two specific kinds of graphs that must always have a perfect matching.

(i) Let G be the graph whose vertices are all binary strings of length n, and whose edges connect two strings that agree in at least one position. Prove that G has a perfect matching (HINT: begin by computing vertex degrees).

(ii) Let G be a bipartite graph whose every vertex has the same degree. Prove that G must have a perfect matching.

Solution:

(i) Let G be the graph whose vertices are all binary strings of length n, and whose edges connect two strings that agree in at least one position. Prove that G has a perfect matching (HINT: begin by computing vertex degrees).

Proof 26 G has a perfect matching

For a given binary string s of length n, every other binary string of length n will be part of an edge including s, with the exception of the binary string that differs in every spot from s.

For a binary string of n characters, there are 2^n possible variations. So, for a given binary string s, there are:

$$2^n - 2 \text{ edges.}$$

The minus two accounts for the duplication of s as well as the string that differs in every spot from s.

So:

$$\forall v \in V(G), d(v) = 2^n - 2$$

Using Hall's Theorem, which states the following:

Theorem 6.1 Hall's Theorem

For a bipartite graph G on the parts X and Y, the following conditions are equivalent.

(1) There is a perfect matching of X into Y.

(2) For each $T \subseteq X$, the inequality $|T| \leqslant |N_G(T)|$ holds.

Since all vertices have the same degree, simply let one half of the vertices be in a set X, and the other half in Y (since there are exactly 2^n vertices for a given value of n, there are an even number of vertices, so $|X| = |Y|$).

For each $T \subseteq X$, $|T| \leq N_G(T)$ since each vertex has a degree of $2^n - 2$, which for any value of $n > 1$, is greater than one.

Therefore, by Hall's theorem, this graph has a perfect matching.

☺

(ii) Let G be a bipartite graph whose every vertex has the same degree. Prove that G must have a perfect matching.

Proof 27 G must have a perfect matching

Assuming that G is connected (as otherwise this claim does not hold), then $\forall v \in G, d(v) \geq 1$. Call the constant degree of all vertices to be d–d must be greater than 1.

Furthermore, since the graph is bipartite, we can call the two sets X and Y, such that $\forall v_1, v_2 \in A, e(v_1, v_2) \notin E(G)$ and $\forall v_1, v_2 \in B, e(v_1, v_2) \notin E(G)$ by the definition of a bipartite graph.

Now, let T be a subset of X. Want to show that $\forall T$, $|T| \leq N_G(T)$.

First, we can say:

$$\sum_{i=0}^{|T|} d(v_{\text{ in T }}) = d \cdot |T|$$

Since every vertex has a degree of d, even though vertices in T might be adjacent to the same vertices in Y, there must be **at least** $|T|$ vertices that the vertices in T connect to (as the vertices that are adjacent to those in T must also at least have a summed degree of $d \cdot |T|$).

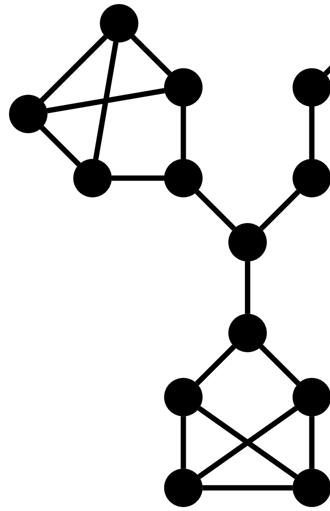
Therefore, $|T| \leq N_G(T)$.

Therefore, by Hall's theorem, this graph has a perfect matching.

😄

Question 22: Problem 2 (25 points)

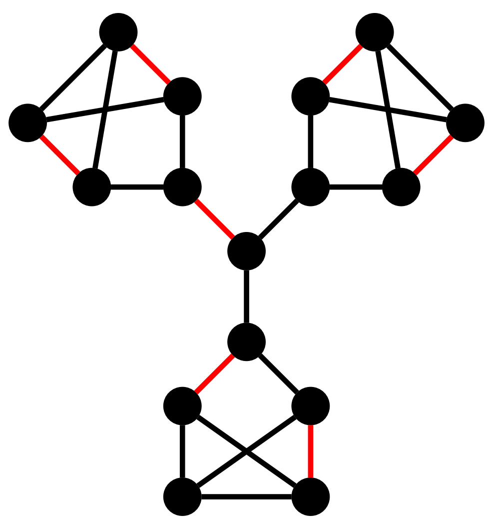
(i) For the graph drawn below, find a maximum matching, then argue why your matching is maximum.



(ii) Recall the hypercube graph Q_4. This is the graph whose vertices are binary strings of length 4 , and whose edges indicate the two strings differ in exactly one position. In a previous homework, you proved that this graph was bipartite $Q_4 = A \sqcup B$, where A is the set of strings with an even number of 1 's, and B is the set of strings with an odd number of 1 's. Prove that Q_4 has a perfect matching.

Solution:

(i) For the graph drawn below, find a maximum matching, then argue why your matching is maximum.



Graph G, with a maximum matching

Solution

This graph gives a possible maximum matching of G since all but two vertices are adjacent to one chosen edge at most. All vertices in the two remaining vertices' neighborhoods are connected to another edge chosen in the matching, so no edge could be added while still adhering to the properties of a maximal matching. Therefore, this graph is a maximal matching.

However, to show that the graph is a maximum matching, must also show that no other possible maximum matching has more edges than this matching. This can be shown using Berge's Theorem, which states:

Theorem 6.2 Berge's Theorem

A matching μ is maximum if and only if there are no augmenting paths.

Looking at the graph, by the definition of an augmenting path, we know that the endpoints must be unsaturated vertices, of which there are only two. The possible paths between these vertices however cannot satisfy the property of augmenting paths that edges alternate between being selected (red) and not selected (black) in the matching, as can be seen above. So, the matching is a maximum matching.

(ii) Recall the hypercube graph Q_4. This is the graph whose vertices are binary strings of length 4 , and whose edges indicate the two strings differ in exactly one position. In a previous homework, you proved that this graph was bipartite $Q_4 = A \sqcup B$, where A is the set of strings with an even number of 1 's, and B is the set of strings with an odd number of 1 's. Prove that Q_4 has a perfect matching.

Proof 28 Q_4 has a perfect matching

Since we know Q_4 is bipartite, and we also know that $|V| = 2^4 = 16$, we can say that $|A| = |B| = 8$ since the number of binary strings with an even number of 1s and the number of binary strings with an odd number of 1s are equal.

So, if we find that one set (say, A) matches into the other, then this graph has a perfect matching.

Lets choose the matching such that only the last digit varies from each vertex of the edge pair. Then, we would have the following edges:

$$\begin{gathered}
0000 \longrightarrow 0001 \\
0010 \longrightarrow 0011 \\
0100 \longrightarrow 0101 \\
0110 \longrightarrow 0111 \\
1000 \longrightarrow 1001 \\
1010 \longrightarrow 1011 \\
1100 \longrightarrow 1101 \\
1110 \longrightarrow 1111
\end{gathered}$$

Since each vertex in A is matched to a vertex in B, $|A| = |B|$, we have a perfect matching.

😊

Question 23: Problem 3 (25 points)

In this course we have discussed proper colorings of vertices, but there is also a rich theory of proper colorings of edges, which is made even more interesting by its relation to Hall's theorem. An edge coloring of a graph G with n colors is a function $f : E \to \{1, \dots, n\}$. we say that an edge coloring is proper if any two edges which share a common endpoint are different colors. Just as with vertex colorings, we can define the edge chromatic number $\chi'(G)$ to be the smallest number of colors needed to properly color the edges of G.
Now let G be a bipartite graph whose every vertex has the same degree, d. In this problem we will prove $\chi'(G) = d$

(i) Prove that G has a proper edge coloring with d colors. This shows $\chi'(G) \leqslant d$.

(ii) Conclude the proof by arguing it is impossible to properly edge color G with less than d colors.

Solution:

(i) Prove that G has a proper edge coloring with d colors. This shows $\chi'(G) \leqslant d$.

Proof 29 G has a proper edge coloring with d colors

Let G be a bipartite graph with bipartition X and Y such that every vertex in G has degree d. We want to show that G can be properly edge colored using d colors.

To do this, we first prove that there exists a matching of size d in G. Take any subset S of X of size d. Since every vertex in S is adjacent to exactly d vertices in Y, there must be at least d vertices in Y that are adjacent to some vertex in S. Let Y' be the set of d vertices in Y that are adjacent to vertices in S. Then, the subgraph $G[S \cup Y']$ induced by S and Y' has minimum degree d. By Hall's theorem, there exists a matching of size d in $G[S \cup Y']$.

This matching corresponds to a proper edge coloring of G using d colors. To see why, note that every edge in G has one endpoint in X and the other endpoint in Y. Therefore, every edge has one endpoint in S or $X \setminus S$, and the matching ensures that no two edges with a common endpoint are assigned the same color. Therefore, G can be properly edge colored using d colors.

☺

(ii) Conclude the proof by arguing it is impossible to properly edge color G with less than d colors.

Proof 30 Proper edge coloring must use ≥ d colors

To prove that it is impossible to properly edge color G with less than d colors, we will use a proof by contradiction. Suppose that G can be properly edge colored using only $k < d$ colors, and let f be such a proper edge coloring.

Consider the set E_x of edges incident with a vertex x in X. Since every vertex in X is adjacent to exactly d vertices in Y, we have $|E_x| = d$ for every x in X. Since E_x contains d edges, and $d > k$, there must be at least two edges in E_x that are assigned the same color by f. Let vw and $v'w'$ be two such edges, both incident with x.

Assume that v is in X and w and v' are in Y. Since G is bipartite, v' cannot be connected to w, since that would create an odd cycle. Therefore, v' must be connected to some other vertex w' in Y. Since $d > k$, there must be a color available to color the edge $v'w'$ that is different from the color assigned to vw. But this contradicts the fact that f is a proper edge coloring, since vw and $v'w'$

share a common endpoint x and cannot be assigned the same color.

Therefore, we have a contradiction, and it is impossible to properly edge color G with less than d colors.

Since d is a lower bound for $\chi'(G)$ as shown in part (i), and we now know that $\chi'(G) \nless d$, this completes the proof:

$$\chi'(G) = d \text{ for a bipartite graph } G \text{ whose every vertex has degree } d.$$

😄

Question 24: Problem 4 (25 points)

An $n \times n$ latin square is an $n \times n$ arrangement of the numbers $\{1, \ldots, n\}$, such that in every row and every column, each number appears exactly once. In this problem, we will describe an algorithm which one can use to construct $n \times n$ latin squares.

Step 1 Begin by filling in the first row of the square with the numbers $\{1, \ldots, n\}$, in any desired order.

Step 2 If the entire square has been filled in, the algorithm terminates. Otherwise, assume you have filled in the first r rows, where $1 \leqslant r \leqslant n-1$.

Step 3 For each $i \in \{1, \ldots, n\}$, let C_i be the set of numbers which do NOT appear in the i-th column of your partially constructed square. Form a bipartite graph G whose vertex set is $\{1, \ldots, n\} \sqcup \{C_1, \ldots, C_n\}$. There will be an edge in this graph from i to C_j if and only if i is an element of C_j.

Step 4 Construct a matching of $\{C_1, \ldots, C_n\}$ into $\{1, \ldots, n\}$. Call this matching $\mathcal{M}$.

Step 5 We fill in the $(r+1)$-st row of our partially constructed square by inserting the number i into the column C_j whenever $\left\{i, C_j\right\} \in \mathcal{M}$.

Step 6 Repeat Step 2.

(i) Prove that Step 4 in this algorithm makes sense. That is, prove that The bipartite graph G always has such a matching.

(ii) Argue that the output of this algorithm is correct. In other words, what is ulimtately produced is indeed a latin square.

(iii) Use this algorithm to construct a 4×4 latin square. Show each step.

Solution:

(i) Prove that Step 4 in this algorithm makes sense. That is, prove that The bipartite graph G always has such a matching.

Solution

To show that Step 4 in this algorithm makes sense, we need to prove that the bipartite graph G always has a matching.

We have a bipartite graph G, where $U = 1, \ldots, n$ is the set of rows, $V = C_1, \ldots, C_n$ is the set of columns, and $(i, C_j) \in E$ if and only if i does not appear in column C_j in the partially constructed square. We need to show that G always has a matching from V to U while the latin square is not filled out.

By Hall's Theorem, it suffices to show that for any subset $S \subseteq V$, the number of rows adjacent to S is at least $|S|$. Let $S \subseteq V$ be arbitrary.

Initially, every vertex in S is connected to all but one column-that is, the column that started with that value of i. As the algorithm continues, the degree of the vertices in S decrease, until by the final loop they each have a degree of one.

Therefore, for each loop, there will be atleast one edge connected to each of the rows in U.

This implies that there are at least $|S|$ rows adjacent to S, as required by halls theorem to garentee the fact that there will be a perfect matching such that each row in U can be matched to a unique column in V.

(ii) Argue that the output of this algorithm is correct. In other words, what is ulimtately produced is indeed a latin square.

Solution

To show that the output of this algorithm is correct, we need to prove that the partially constructed square is a latin square at the end of each iteration. We proceed by induction on the number of rows constructed. The base case when $r = 1$ is trivially true.

Assume that we have constructed r rows, and that the partially constructed square is a latin square. Then, we need to show that after Step 4, the $(r + 1)$-st row of the partially constructed square is also a latin square. By the construction of G, any two columns C_i and C_j have at most one common element (namely, the $(r+1)$-st element of the square). Thus, we can fill in the $(r+1)$-st row with distinct elements, and it suffices to show that each element appears exactly once in each column.

Let C_j be a column. If C_j contains the $(r + 1)$-st element, then it is already filled in, and we need to show that the other elements appear exactly once in C_j. Otherwise, C_j is matched to a row i in Step 4, and we fill in the $(r + 1)$-st element with i.

By the induction hypothesis, each of the first r elements in C_j appears exactly once in the first r rows. Since i does not appear in column j and the first r rows contain each number at most once in every column, it follows that there are at least $n - r$ numbers in C_j. Therefore, we have

$$\deg(j) = C_j \leq n - r.$$

By Hall's Theorem, there exists a matching of $C_1, \ldots, C_n$ into $1, \ldots, n$. Let M be such a matching. To fill in the $(r + 1)$-th row, we insert i in the j-th column if and only if i and C_j are matched by M. This completes Step 5.

We can repeat Steps 2-5 until the entire square is filled in. It is clear from the construction that every number appears exactly once in every row and every column. Therefore, the output of this algorithm is a valid Latin square.

(iii) Use this algorithm to construct a 4×4 latin square. Show each step.

Solution

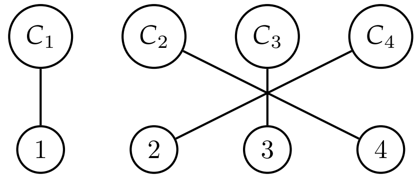
First, will begin by choosing values from 1 to 4 and putting them in random indices for the first row as described by **Step 1**.

3	2	1	4

Now, $C_1, \ldots, C_4$ are as following:

$$C_1 = 1, 2, 4$$
$$C_2 = 1, 3, 4$$
$$C_3 = 2, 3, 4$$
$$C_4 = 1, 2, 3$$

Next, we will construct a matching as described in **Step 3** and **Step 4**:



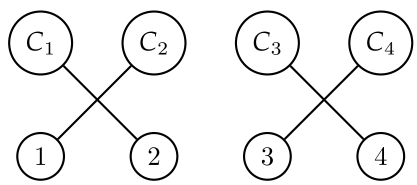
Now, we can update the table to the following:

3	2	1	4
1	4	3	2

And we can update $C_1, \ldots, C_4$ to:

$$C_1 = 2, 4$$
$$C_2 = 1, 3$$
$$C_3 = 2, 4$$
$$C_4 = 1, 3$$

We repeat the process for the third row by constructing another matching:



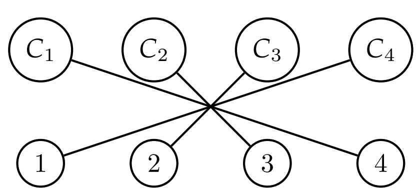
Now, we can update the table to the following:

3	2	1	4
1	4	3	2
2	1	4	3

Now we can update $C_1, \ldots, C_4$ to:

$$C_1 = 4$$
$$C_2 = 3$$
$$C_3 = 2$$
$$C_4 = 1$$

We can constructing one last matching for the final row:



Now, we can finish the table:

3	2	1	4
1	4	3	2
2	1	4	3
4	3	2	1

It is quite clear that this table is indeed a latin square; each column contains all 4 unique values from $1 \ldots 4$.

Homework 7

Question 25: Problem 1 (25 points)

For the purposes of this problem we work with a deck of cards, where each card has one of 15 ranks paired with one of four suits. When discussing cards, a "hand" refers to a collection of cards with no particular order prescribed to them. Words like "double" or "triple" always refer to collections of cards where a rank is repeated twice (or three times, respectively).

(i) How many 5 card hands have exactly one double and one triple.

(ii) How many 6 card hands have at least one triple?

(iii) How many 7 card hands have exactly three doubles, but no triples or quads?

Solution:

Note:-

There are 15 ranks, and 4 suits, so 60 cards rather than the usual 52.

(i) How many 5 card hands have exactly one double and one triple.

Solution

To count the number of such hands, we can first choose the rank for the double and the triple. This can be done in $\binom{15}{1}$ and $\binom{14}{1}$ ways.

For the triple, we need to choose which 3 of the 4 suits will be used. This can be done in $\binom{4}{3}$ ways. Finally, we need to choose which 2 of the 4 suits will be used for the double. This can be done in $\binom{4}{2}$ ways.

To obtain the total number of 5 card hands with exactly one double and one triple, we multiply the number of ways to choose the double and triple ranks, and then assign suits:

$$\binom{15}{1} \cdot \binom{4}{3} \cdot \binom{14}{1} \cdot \binom{4}{2}$$

(ii) How many 6 card hands have at least one triple?

Solution

To count the number of such hands, we can first choose the rank for the triple. This can be done in $\binom{15}{1}$ ways.

For the triple, we need to choose which 3 of the 4 suits will be used. This can be done in $\binom{4}{3}$ ways. Then, we need to choose the 3 other cards remaining from the deck. There are 57 cards remaining (assuming that having 4 cards of a rank still counts as a triple with just an extra card), so we can any 3 of those 57:

$$\binom{57}{3}$$

To obtain the total number of 6 card hands with at least one triple, we multiply the number of ways to choose the triple rank, and then assign suits and remaining cards:

$$\binom{15}{1}\binom{4}{3}\binom{57}{3}$$

But, also need to subtract the cases where we might be double-counting for a case like having 3 8s and 3 7s (counted as different than having 3 7s then 3 8s).

So, final counting is:

$$\binom{15}{1}\binom{4}{3}\binom{57}{3} - \binom{15}{2}\binom{4}{3}^2$$

(iii) How many 7 card hands have exactly three doubles, but no triples or quads?

Solution

To count the number of such hands, we can first choose the rank all of the doubles. This can be done in $\binom{15}{3}$ ways.

For the first double, we need to choose which 2 of the 4 suits will be used. This can be done in $\binom{4}{2}$ ways. Then, we can do the same thing with the second and third double:

$$\binom{15}{3} \cdot \binom{4}{2}^3$$

Now, to add the final card, we have to make sure it is not one of the 3 ranks already chosen (it can be any suit):

$$\binom{12}{1}\binom{4}{1}$$

To obtain the total number of 7 card hands with exactly three doubles, but no triples or quads, we multiply the number of ways to choose the double ranks with their suits, then multiply it by the additional card remaining:

$$\binom{15}{3}\binom{4}{2}^3\binom{12}{1}\binom{4}{1}$$

Question 26: Problem 2 (25 points)

For each of the following graphs, compute a formula for $\chi_G(n)$.

(i) The graph that looks like the letter Y.

(ii) More generally, a tree G.

(iii) The cycle of length r.

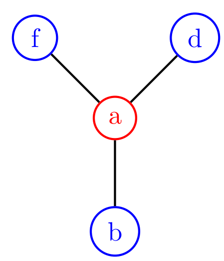
(iv) The graph obtained from the complete graph K_n by removing a single edge.

Solution:

(i) The graph that looks like the letter Y.

Solution

No matter how many vertices are added to the tree that looks like the letter Y, you will need exactly 2 colors, as there is only one vertex with degree 3, which can be the first color, and then along each of the sides of the Y shape the colors can alternate.



Example Y-shaped graph G

Starting from the middle vertex, you have n number of colors to choose from. Then, for each of the other vertices, you have $n-1$ colors to choose from.

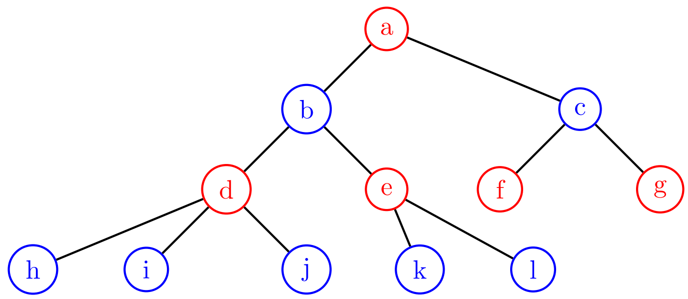
So, the equation for $\chi_G(n)$ is:

$$\chi_G(n) = n \cdot (n-1)$$

(ii) More generally, a tree G.

Solution

A tree is a connected graph with no cycles. Therefore, we can start with any vertex of the tree and color it with the first color. Then we move on to the neighbors of this vertex and color them with the second color. We continue this way, coloring vertices layer by layer, until all vertices are colored. Since the tree has no cycles, we will never need to use more than 2 colors.



Example tree G

For the first vertex, you have n possible selections for the color of that vertex. Then, for each neighbooring vertex, there is a total of $n-1$ choices for the color of that vertex, as it simply cannot match the color of the vertex you came from.

This pattern continues as you traverse the tree, since there is no cycle, so you never reach a point where there are less than $n-1$ choices for a vertex.
So, the equation for $\chi_G(n)$ is:

$$\chi_G(n) = n \cdot (n-1)^{v-1}$$

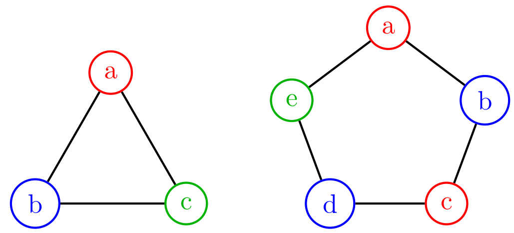
Note:-

The $(n-1)$ term is to the power of $(v-1)$ because you do not want to recount the vertex you started at.

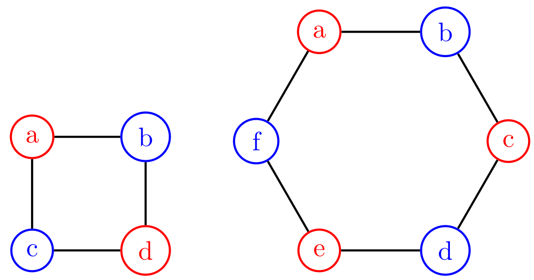
(iii) The cycle of length r.

Solution

This equation will finally need to use the number of vertices, n, as the length of the cycle matters–if the cycle is odd, you need 3 colors, whereas if the cycle is of an even length, you only need 2.



Graphs with $n = 3$ and $n = 5$ (odd)



Graphs with $n = 4$ and $n = 6$ (even)

First, to solve this problem, we can find the chromatic number of a line graph. We can find this by thinking of an example graph of length r with n possible colors. Then, we know for the first vertex, we can choose any of the n possible colors. However, for each subsequent $r - 1$ vertices, there are only $n - 1$ choices for that vertex. So, the chromatic polynomial for a line graph is as follows:

$$\chi_G(k) = n(n-1)^{r-1}$$

Now, we can use the deletion-contraction formula:

$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k)$$

Knowing this formula, we can begin to solve for the chromatic polynomial of a cycle.

With this equation in mind, the first thing we can notice is when we remove an edge from any cycle G of length r, it simply becomes a line-graph of length r of which we already know the chromatic polynomial for.

Then for the second term of the deletion-contraction formula, we know that this is just a cycle of length $r - 1$, which we know also can expand using the deletion-contraction formula. So, we end up getting an alternating function, shown below:

$$\begin{aligned}
\chi_G(k) &= \chi_{G-e}(k) - \chi_{G/e}(k) \\
&= n(n-1)^{r-1} - \chi_{G/e}(k) \\
&= n(n-1)^{r-1} - (\chi_{G-e}(k) - \chi_{G/e}(k)) \\
&= n(n-1)^{r-1} - (n(n-1)^{r-2} - \chi_{G/e}(k)) \\
&= n(n-1)^{r-1} - (n(n-1)^{r-2} - (n(n-1)^{r-3} - \chi_{G/e}(k))) \\
&= n(n-1)^{r-1} - n(n-1)^{r-2} + n(n-1)^{r-3} - \dots
\end{aligned}$$

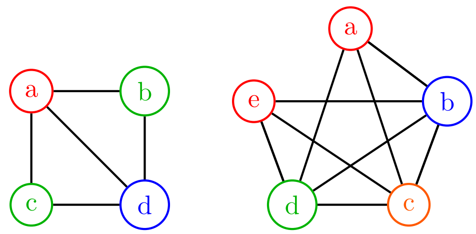
This converges to the following equation for the chromatic polynomial of a cycle:

$$\chi_G(k) = \sum_{i=0}^{r-1} (-1)^i n(n-1)^{r-1-i}$$

(iv) The graph obtained from the complete graph K_n by removing a single edge.

Solution

When removing one edge from a complete graph, the two vertices that were disconnected can be the same color, as they are no longer adjacent to each other; however, all of the other vertices, as they are all connected to each other, must be different colors.



Graph of $K_4 - e$ and $K_5 - e$

So, begining at the two nodes that can be the same color, we can choose any of the n colors for those two vertices.

So, lets break this into cases. In the first case, lets say we choose the two vertices not connected by an edge to be the same color. In this case, each following vertex has $(n-1), (n-2), \ldots n(n-(k-2))$ choices for their color. We can represent this in the following way:

$$n \cdot \prod_{i=1}^{k-2}(n-i)$$

In the second case, lets say the first two vertices are colored different colors. In this case:

$$n \cdot \prod_{i=1}^{k-1}(n-i)$$

Adding these together, we get the equation for the chromatic polynomial for complete graphs with one edge removed:

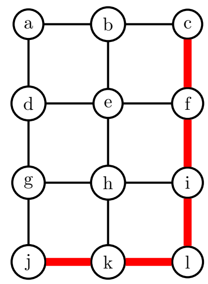
$$n \cdot \left(\cdot \prod_{i=1}^{k-2}(n-i) + n \cdot \prod_{i=1}^{k-1}(n-i) \right)$$

Question 27: Problem 3 (25 points)

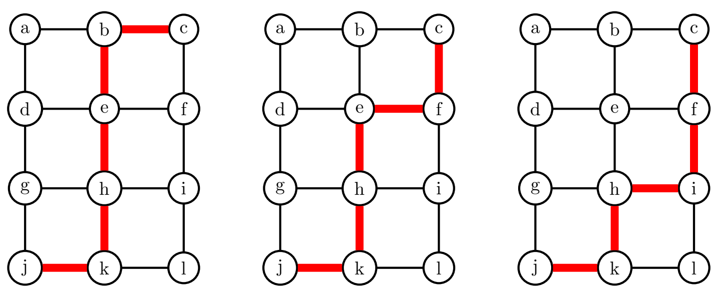
For any $n, m \geqslant 2$ the $(n \times m)$-lattice is the graph G which can be drawn as an n vertex by m vertex grid. For instance, the (2×2)-lattice is a square, whereas the (3×2)-lattice is comprised of two squares sharing an edge. How many walks exist in the $(n \times m)$-lattice which start at the bottom left corner and end the top right corner while only taking steps to the right and upwards?

Solution:

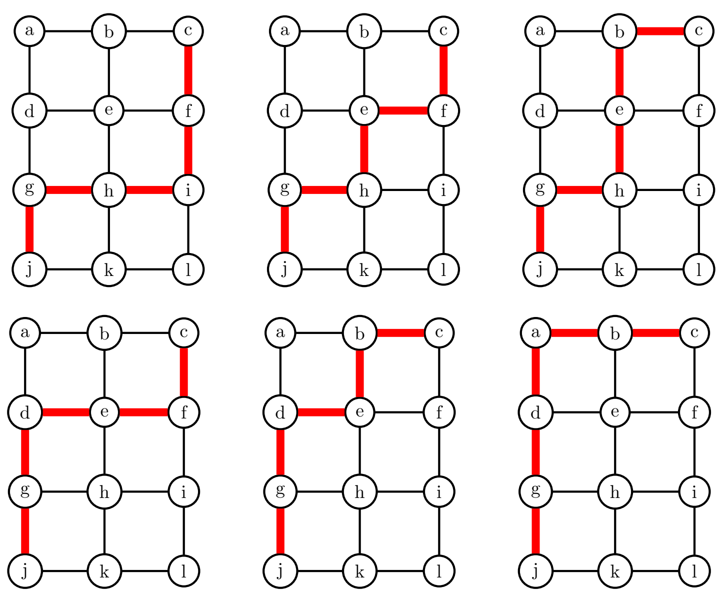
I will use this example going forward, showing all possible traversals of the $(3x4)$-lattice graph, G:



All possible traversals of G starting with two "right" moves



All possible traversals of G starting with one "right" move



All possible traversals of G starting with zero "right" moves

Solution

To count the number of walks from the bottom left corner to the top right corner of an $(n \times m)$-lattice, taking only steps to the right and upwards, we need to count the number of ways to choose a sequence of $n-1$ rightward steps and $m-1$ upward steps.

One way to approach this is to label each step in the sequence as either R" for rightward or U" for upward. For example, a possible sequence for the (3×4)-lattice shown above is "RUURU". We can see that this sequence has two rightward steps and three upward steps.

Notice that every possible sequence of $n-1$ rightward steps and $m-1$ upward steps will correspond to a unique path from the bottom left corner to the top right corner of the lattice. Conversely, every such path will correspond to a unique sequence of steps.

Therefore, the problem reduces to counting the number of possible sequences of $n-1$ R's and $m-1$ U's. This can be done using the "choose" formula.
We want to use this formula to count the number of ways to choose a subset of $n-1$ items from a set of $n+m-2$ items. We can think of the set of $n+m-2$ items as consisting of $n-1$ R's and $m-1$ U's, in some order. Then, we can choose a subset of $n-1$ items by selecting which of the $n-1$ positions in the sequence will contain an r, and filling in the remaining positions with u's. the number of ways to do this is precisely $\binom{n+m-2}{n-1}$.

Therefore, the total number of walks from the bottom left corner to the top right corner of an $(n \times m)$-lattice, taking only steps to the right and upwards, is:

$$\binom{n+m-2}{n-1}$$

Note:-

By the **Fundamental Theorem of Combinatorics** (The Law of Double Counting), the following equation must hold for all n:

$$\binom{n+m-2}{n-1} = \binom{n+m-2}{m-1}$$

Question 28: Problem 4 (25 points)

Prove the following identity for all non-negative integers m, n by counting in two ways:

$$\binom{n}{m} 2^{n-m} = \sum_{k=m}^{n} \binom{n}{k} \binom{k}{m}$$

Solution:

Proof 31 Proof by Counting

Suppose you are a teacher and you want to form a committee of n students. You want to choose a subset of m students to be on the committee, and then choose a subset of the remaining $n - m$ students to help with organizing an event. You also want to split the committee into two smaller subcommittees: subcommittee A and subcommittee B. You want to count the total number of ways to form the committee using two different methods.

(Method 1) First, choose a subset of m students to be on the committee in $\binom{n}{m}$ ways. Then, for each of these choices, you can choose a subset of the remaining $n-m$ students to help with organizing the event in 2^{n-m} ways, since each of the remaining students can either be in subcommittee A or subcommittee B. Therefore, the total number of ways to form the committee is:

$$\binom{n}{m} 2^{n-m}$$

(Method 2) Additionally, you can also count the total number of ways to form the committee by first choosing a subset of k students to be on both the committee and the organizing team, with $m \leqslant k \leqslant n$. There are $\binom{n}{k}$ ways to do this. Then, from this subset of k students, you can choose a subset of m students to be on the committee in $\binom{k}{m}$ ways. Since $m \leqslant k \leqslant n$, we can sum over all possible values of k to get:

$$\sum_{k=m}^{n} \binom{n}{k} \binom{k}{m}$$

Since we are counting the same thing in two different ways, by the **Fundamental Theorem of Combinatorics** we know that the two expressions are equal. Therefore, we have shown that:

$$\binom{n}{m} 2^{n-m} = \sum_{k=m}^{n} \binom{n}{k} \binom{k}{m}.$$

😄

Homework 8

Question 29: Problem 1 (25 points)

Assume that a positive integer cannot have 0 as its first digit.

(i) How many 10 digit positive integers have at least two pairs of repeated digits?

(ii) How many seven digit positive integers have a run of exactly four consecutive repeated digits?

(iii) How many five digit positive integers have no consecutive repeated digits?

Solution:

(i) How many 10 digit positive integers have at least two pairs of repeated digits?

Solution

To use the Principle of Inclusion/Exclusion, we can first find the number of possible 10 digit positive integers. To do this, we have 9 choices for the first digit, since it cannot be 0. Then, for each subsequent digit, we have 10 choices. Therefore, the number of 10 digit positive integers is:

$$9 \cdot 10^9$$

Now, we want to find the number of 10 digit positive integers that do not have any pairs of repeated digits. For this counting problem, we can choose any of the digits 1-9 for the first digit, then subsequent digits can choose from the remaining 9, remaining 8, etc. So, we can represent this as:

$$9 * 9!$$

Finally, we want to find the number of 10 digit positive integers that have exactly one pair of repeated digits. To find this, we do the following.

We can first find the different single-pair integers where the pair is not a 0, and then when the pair is a zero.

When the pair is not a zero:

$$\binom{9}{1} \cdot \binom{10}{2} \cdot 9!$$

When the pair is a zero:

$$\binom{9}{2} \cdot 9!$$

Thus, to find the total number of 10 digit possible integers that have at least one repeated pair, we can take the total number of all the possible integers, and subtract the cases in which there are 0 or 1 pairs of repeated numbers. Thus, we get:

$$(9 \cdot 10^9) - (9 * 9!) - \left(\binom{9}{1} \cdot \binom{10}{2} \cdot 9! + \binom{9}{2} \cdot 9!\right)$$

(ii) How many seven digit positive integers have a run of exactly four consecutive repeated digits?

Solution

First, we can start with the case where the first digit is part of the run; In this case, there are 9 choices for the value of the first 4 digits, and then 9 choices for the value following that set of 4 numbers (since it must be exactly four consecutive, not 5). For the last two digits, each have 10 choices. Thus:

$$9^2 \cdot 10^2$$

Now, we can find the cases in which the second digit is the start of the run. The set of 4 consecutive numbers has a total of 10 choices for their value. If the value they chose is a 0, then both the first and 6th digit have 9 choices, and the last value has 10. If they did not choose 0, then the first digit has 8 choices, the 6th has 9, and the 7th has 10. Thus:

$$9^2 \cdot 10 + 8 \cdot 9^2 \cdot 10$$

Now, we can find the case when the third digit starts the run. In this case, the first digit has 9 choices, as does the second, as does the final digit. Thus:

$$9^3 \cdot 10$$

Finally, for the case when the 4th digit starts the run, the first digit has 9 choices, the second has 10, the third has 9, and the last four have 10.

$$10^2 \cdot 9^2$$

Adding this all together, we get the following:

$$9^2 \cdot 10^2 + 9^2 \cdot 10 + 8 \cdot 9^2 \cdot 10 + 9^3 \cdot 10 + 10^2 \cdot 9^2$$

(iii) How many five digit positive integers have no consecutive repeated digits?

Solution

The first digit can choose any of the 9 digits from 1-9. Then, the second digit can be any number but the first number–thus, it has 9 choices. This logic follows for the rest of the digits. Thus:

$$9^5$$

Question 30: Problem 2 (25 points)

The following two problems involve a lottery game.

(i) In this game, each ticket consists of a selection of eight different numbers from 1 to 76. At the end of each week, seven distinct "winning" numbers are selected by the state. How many possible tickets will match at least three of the winning numbers?

(ii) How many lottery tickets will have exactly four even numbers, and exactly four odd? What proportion of all possible tickets is this?

Solution:

Note:-

The tickets are a collection of numbers, and thus the order of the numbers does not matter.

(i) In this game, each ticket consists of a selection of eight different numbers from 1 to 76. At the end of each week, seven distinct "winning" numbers are selected by the state. How many possible tickets will match at least three of the winning numbers?

Solution

To find the number of tickets that have at least 3 winning numbers as part of the 8 digits that are chosen, we can first choose 3 of the possible 7 winning numbers in the following way:

$$\binom{7}{3}$$

Then, to find the remaining 5 digits, we just need to choose any 5 of the remaining 69 numbers:

$$\binom{69}{5}$$

Therefore, combining those answers, we find the number of tickets that have at least 3 winning numbers:

$$\binom{7}{3} \cdot \binom{69}{5}$$

(ii) How many lottery tickets will have exactly four even numbers, and exactly four odd? What proportion of all possible tickets is this?

Solution

Out of the possible 1 to 76 digits, exactly half of them are odd, and half are even–38 in both cases. Thus, to find the number of tickets that have 4 even numbers and 4 odd numbers, we can multiply the number of ways to choose 4 even numbers by the number of ways to choose 4 odd numbers:

$$\binom{38}{4} \cdot \binom{38}{4} = \binom{38}{4}^2$$

To find the proportion of all possible tickets, we can divide the number of tickets that have 4 even and 4 odd numbers by the total number of possible tickets:

$$\frac{\binom{38}{4}^2}{\binom{76}{8}}$$

Question 31: Problem 3 (25 points)

How many five card hands from a standard deck of playing cards contain at least one card from each of the four suits?

Solution:

To have a five card hand with at least one card from each of the four suits, you nknow that 2 of the cards must be of the same suit. Therefore, there are:

$$\binom{13}{2} \text{ ways of choosing those cards, given a suit.}$$

Now, for the rest of the cards, you have 13 choices each, as you can chose any of the cards from the 13 cards of that suit.

$$13^3 \text{ ways of choosing the remaining 3 cards.}$$

Now, we need to choose which suit is given the double, so we can multiply this result by 4 to represent the fact that the double could be clubs, diamonds, hearts, or spades.

So, the final answer is:

$$\binom{13}{2} \cdot 13^3 \cdot 4 = 685464$$

Question 32: Problem 4 (25 points)

Let n, k be positive integers. We write $S(n,k)$ to denote the Stirling numbers of the second kind. More specifically $S(n,k)$ counts the total number of ways to partition $\{1,2,\ldots,n\}$ into k disjoint non-empty subsets. For instance, $S(4,2) = 7$ because the only partitions with 2 non-empty parts are given by

$$[\{1\},\{2,3,4\}],[\{2\},\{1,3,4\}],[\{3\},\{1,2,4\}],[\{4\},\{1,2,3\}],[\{1,2\},\{3,4\}],[\{1,3\},\{2,4\}],[\{1,4\},\{2,3\}]$$

(i) Compute $S(4,3)$, and $S(3,2)$ by writing down all possible partitions.

(ii) Prove that $S(n,n-1) = \left(\begin{array}{c} n \\ 2 \end{array}\right)$ by counting in two ways.

(iii) Prove that $S(n,2) = 2^{n-1} - 1$ by counting in two ways.

Solution:

(i) Compute $S(4,3)$, and $S(3,2)$ by writing down all possible partitions.

Solution

Here are all possible partitions for $S(4,3)$, and $S(3,2)$, and their Stirling numbers of the second kind:

$$\begin{aligned} S(4,3) = {} & [\{1\},\{2\},\{3,4\}],[\{1\},\{3\},\{2,4\}],[\{1\},\{4\},\{2,3\}], \\ & [\{2\},\{3\},\{1,4\}],[\{2\},\{4\},\{1,3\}],[\{3\},\{4\},\{1,2\}] \\ & = 6 \end{aligned}$$

$$\begin{aligned} S(3,2) = {} & [\{1\},\{2,3\}],[\{2\},\{1,3\}],[\{3\},\{1,2\}] \\ & = 3 \end{aligned}$$

(ii) Prove that $S(n,n-1) = \left(\begin{array}{c} n \\ 2 \end{array}\right)$ by counting in two ways.

Proof 32 $S(n,n-1) = \left(\begin{array}{c} n \\ 2 \end{array}\right)$

We can prove that $S(n,n-1) = \binom{n}{2}$ by counting the number of ways to partition n distinct objects into $n-1$ non-empty subsets in two ways.

By definition, the left side of the equality is the number of ways to partition n items into $n-1$ subsets.

Now, for the RHS of the equation. First, consider that we have n distinct objects and we want to partition them into $n-1$ non-empty subsets. We can start by selecting any two objects from the n objects and placing them into the same subset. We now have $n-2$ objects left to partition into $n-2$ non-empty subsets. We can do this in only one way since we have reduced the problem to partitioning $n-2$ objects into $n-2$ non-empty subsets. Therefore, this counting problem can be reduced to simply choosing 2 of the n items to be in the non-singleton set.

Therefore:

$$\Rightarrow S(n, n-1) = \binom{n}{2}.$$

Notice this is what we wanted to show.

😊

(iii) Prove that $S(n, 2) = 2^{n-1} - 1$ by counting in two ways.

Proof 33 $S(n, 2) = 2^{n-1} - 1$

Say we have a list of n people who need to be split between two teams. We know then that no singular person can be in either of the two teams. So, the two teams are disjoint sets. Therefore, by the definition of Stirling numbers of the second kind, $S(n, 2)$ counts the number of ways to partition n people into the two different teams, such that neither team is empty.

Now, to show that the right side is equal to the left, lets consider the following scenario. For each of the n people, we make a choice: to put them on the first team, or to put them on the second team. So, for each of the n people, we have 2 choices for which team to put them in. This gives us:

$$2^n \text{ choices for which team to put each person on.}$$

However, there are some added complexities here; since we are not differentiating between the two team, i.e. there is no "A" team and "B" team, we are double counting the case where one partition, with sets A_1 and B_1, is the "flipped" partition with sets A_2 and B_2 such that $A_1 = B_2$ and $B_1 = A_2$. To account for this, we want to remove a factor of 2, since which "team" the first person is partitioned to does not matter.

$$2^{n-1} \text{ choices, accounting for the fact that the sets are not distinct.}$$

Finally, since we know that neither team can be empty, we want to acount for the case where the first $n-1$ people were partitioned to the same team, in which case the last person would have to go onto the other team. This takes away exactly one case from our counting problem.

$$2^{n-1} - 1 \text{ choices, also accounting for the fact that neither team can be empty. .}$$

So:

$$S(n, 2) = 2^{n-1} - 1$$

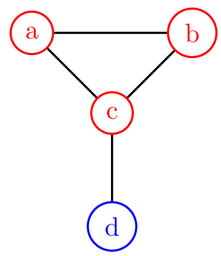
Notice this is what we wanted to show.

😊

Homework 9

Question 33: Problem 1 (25 points)

Use PIE to compute the chromatic polynomial of the lollipop graph. That is, the graph obtained from the tree that looks like the letter Y by adding a single edge.



lollipop graph G

Solution: To use the Principle of Inclusion/Exclusion, we can first define a few sets–notably, sets that define pairs of vertices being the same color, as that cannot occur for the graph to have a valid coloring.

Thus, we define the following sets:

$$S_1 = \{ \text{ a and b are colored the same } \}$$
$$S_2 = \{ \text{ a and c are colored the same } \}$$
$$S_3 = \{ \text{ b and c are colored the same } \}$$
$$S_4 = \{ \text{ c and d are colored the same } \}$$

Therefore, we want to find the intersection of the complement of these sets–that is, the possible graphs where a is not the same color as b, a is not the same color as c, b is not the same color as c, and c is not the same color as d.

We can then use PIE (II) to solve this problem.

First, lets find the size of all of the sets combined. For each set, we know that there are n choices for the pair of 2 vertices, and then the other 2 vertices can be any color. Therefore, each of the sets $S_1, \ldots, S_4$ has size n^3. So:

$$\mid S_1 \mid + \mid S_2 \mid + \mid S_3 \mid + \mid S_4 \mid = 4 \cdot n^3$$

Since we are over counting, we now want to find the number of double intersections between sets. For each of these, we know that there are two pairs of vertices with the same colors, so n^2 choices for colors. There are 4 of these, so:

$$\mid S_1 \cap S_2 \mid + \mid S_1 \cap S_3 \mid + \mid S_1 \cap S_4 \mid + \mid S_2 \cap S_3 \mid + \mid S_2 \cap S_4 \mid + \mid S_3 \cap S_4 \mid = 6 \cdot n^2$$

Now, we want to find the number of triple intersections between sets. For each of these, we know that there are three pairs of vertices with the same colors, which gives us only n choices for the color of these vertices.

$$\mid S_1 \cap S_2 \cap S_3 \mid + \mid S_1 \cap S_2 \cap S_4 \mid + \mid S_1 \cap S_3 \cap S_4 \mid + \mid S_2 \cap S_3 \cap S_4 \mid = 4 \cdot n$$

Finally, we want to find the number of quadruple intersections between sets. For each of these, we know that there are four pairs of vertices with the same colors, so every vertex has to be the same color–thus, n choices.

$$\mid S_1 \cap S_2 \cap S_3 \cap S_4 \mid = n$$

Now, we can use PIE to find the chromatic polynomial.

$$\begin{aligned}
|(S_1^c \cap S_2^c \cap S_3^c \cap S_4^c)| &= |U| - (|S_1| + |S_2| + |S_3| + |S_4|) \\
&+ (|S_1 \cap S_2| + |S_1 \cap S_3| + |S_1 \cap S_4| + |S_2 \cap S_3| + |S_2 \cap S_4| + |S_3 \cap S_4|) \\
&- (|S_1 \cap S_2 \cap S_3| + |S_1 \cap S_2 \cap S_4| + |S_1 \cap S_3 \cap S_4| + |S_2 \cap S_3 \cap S_4|) \\
&+ (|S_1 \cap S_2 \cap S_3 \cap S_4|) \\
&= n^4 - 4n^3 + 6n^2 - 4n + n \\
&= n^4 - 4n^3 + 6n^2 - 3n
\end{aligned}$$

Thus, the chromatic polynomial of the lollipop graph is:

$$\chi_G(n) = n^4 - 4n^3 + 6n^2 - 3n$$

Question 34: Problem 2 (25 points)

There are $n+2$ people sitting on one side of a long rectangular table. Among these people are two identical twins. The group of people all get up from their seats and shuffle around so that everyone has moved and the resulting configuration looks as though every single person as moved. That is to say, all of the non-twins have moved, and the twins are not sitting in their own seats, nor their sibling's seat. Let T_n denote the number of ways this shuffle could have happened.

(i) Compute T_n for all $n = 0, 1, 2, 3$ by direct counting.

(ii) Compute a formula for T_n for $n \geqslant 4$.

(iii) Use your formula, and a calculator, to provide a precise numerical value for T_{10}.

Solution:

(i) Compute T_n for all $n = 0, 1, 2, 3$ by direct counting.

(Case 1) In the case that $n = 0$, there are exactly 2 people sitting at the table. So, these two people must be the twins; therefore, there is no shuffling of the twins such that neither is sitting in a chair originally occupied by a twin before the shuffle. Therefore, $T_0 = 0$.

(Case 2) In the case that $n = 1$, there are now 3 people sitting at the table–in this case again, however, there is no way that the 2 twins could both be sitting in different spots from their original ones, so $T_1 = 0$ also.

(Case 3) Finally, there is a correct shuffling in this case, as there are enough seats for the two twins to be sitting in 2 new seats after the shuffle. There are 2 open seats for the two twins to move to, say seats a and b. So, there are two cases here in which the first twin moves to a, or the second twin moves to a. And for each of these cases, the two non-twins could sit in any of the 2 remaining seats, adding an additional case within each of the cases listed above. Therefore, in total, there are 4 different possible valid reshufflings. So, $T_2 = 4$

(Case 4) In this case, the two twins now could move to any of 3 remaining seats–this holds 3 different cases, $\binom{3}{2}$. However, since the twins are individuals, we must multiply this number by 2, do account for the fact that for any 2 seats we choose the twins could swap seats. For each of these cases, the 3 non-twin guests must arrange themselves within the remaining 3 chairs–for this, there is 3! different arrangements, for a total of 6 possible ways the remaining 3 people can shuffle themselves.

So, in total for this case there are $6 \cdot 6 = 36$ different possible valid shufflings. So, $T_3 = 36$.

(ii) Compute a formula for T_n for $n \geqslant 4$.

First, we want to determine how many ways the two twins can arrange themselves after the reshuffling. They can choose any 2 of the remaining n seats, and therefore have:

$$\binom{n}{2}$$

Ways to arrange themselves. However, to account for the fact that the twins are unique, and twin 1 sitting in seat 1 + twin 2 sitting in seat 2 is a separate case from twin 1 sitting in seat 2 + twin 2 sitting in seat 1, we can multiply this number by 2.

$$2 \cdot \binom{n}{2}$$

Now, we want to count the number of ways for the remaining n people to choose from the remaining n seats. This is a basic combinatorial problem, and we know that there are $n!$ ways of doing this. This is because the first of the n people can choose any of the n chairs, and the next person can choose any of the remaining $n-1$ chairs, etc. $(n \cdot (n-1) \cdot (n-2) \ldots = n!)$.

Therefore, here is a formula to determine the number of valid reshufflings of the $n+2$ people, as a function of n:

$$T_n = 2 \cdot \binom{n}{2} \cdot n!$$

(iii) Use your formula, and a calculator, to provide a precise numerical value for T_{10}.

$$T_{10} = 2 \cdot \binom{10}{2} \cdot 10! = 2 \cdot 45 \cdot 3,628,800 = 326,592,000$$

Question 35: Problem 3 (25 points)

Prove that there must exist an integer m such that any collection of m integers will either contain a pair whose sum is divisible by 10 , or contain a pair whose difference is divisible by 10 . Once you have accomplished this, compute the smallest such integer m.

Solution:

Proof 34 Integer collection divisible by 10

Consider the integers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. Any integer can be expressed in the form $10k + r$ where $r \in 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ and k is an integer. Notice that there are only 10 possible remainders modulo 10. Therefore, if we have a collection of 11 integers, by the pigeonhole principle, at least two of them will have the same remainder when divided by 10.

Also notice that if two integers have the same remainder modulo 10, then when taking the difference of those numbers, by the subtraction principle of modular arithmetic, the resulting number will be divisible by 10.

So, if we were only looking for the value of m such that any m integers contain a pair in which the difference of the values is divisible by ten, then we would be done.

However, factoring in that we are looking for a pair that could either be added or subtracted, we can further constrict the number of pigeon holes for this problem. Notice that if we have two integers that are in modulo sets that add up to ten, then the addition of those two numbers is divisible by 10.

Therefore, we can combine these cases into single pigeon holes, as having one of each would result in the pair divisibility be true. Thus, we get the following modified pigeon holes, where each value represents the modulus of an integer mod 10:

$$\begin{aligned}
\{0\} \\
\{1, 9\} \\
\{2, 8\} \\
\{3, 7\} \\
\{4, 6\} \\
\{5\}
\end{aligned}$$

Since we know that if any of these holes has a pair values that there is a sum or difference that is divisible by 10, we can conclude that for any 7 integers, there must be a pigeon hole that contains more than 1 value.

So, the smallest possible value for m such that there is always a pair whose sum or difference is divisible by 10, is 7.

😊

Question 36: Problem 4 (25 points)

Suppose that an unlimited number of jelly beans are available in each of five different colors: red, green, yellow, white, and black. Assume that all jelly beans of a given color are identical.

(i) In how many ways can you select 20 jelly beans?

(ii) In how many ways to select 20 jelly beans if you must have at least two of every kind of jelly bean?

Solution:

(i) In how many ways can you select 20 jelly beans?

Solution

To begin, we can think of the 5 different colors as "bins" to place each of the 20 jelly beans in. Therefore if we can find the number of ways to place the 20 indistinguishable jelly beans into the 5 bins, we have counted the number of ways you can select 20 jelly beans.

Note:-

Keep in mind that any of the 5 bins can be empty, as that represents not choosing any jelly beans of that color.

To determine how these jelly beans will be divided into the 5 bins, we can think of the different placement of the boundaries of the bins. For example's sake, lets denote each jelly bean as a 0 and the bin-boundaries as /.

Then, lets take a smaller problem, such as dividing 4 jelly beans into 2 containers. Here are the possible divisions of those jelly beans:

$$\begin{gathered} 0000/ \\ 000/0 \\ 00/00 \\ 0/000 \\ /0000 \end{gathered}$$

If we think of the jelly-beans left of the slash to be red, and the jelly beans to the right of the slash to be blue, then we have just shown all 2-colorations of 4 jelly beans:

$$\begin{aligned} 0000/ &= \text{ 4 red} \\ 000/0 &= \text{ 3 red, 1 blue} \\ 00/00 &= \text{ 2 red, 2 blue} \\ 0/000 &= \text{ 3 red, 1 blue} \\ /0000 &= \text{ 4 blue} \end{aligned}$$

Therefore, this problem is the same as counting all placements of the bin-boundaries. So, for 5 different colors, we will have 4 bins–left of the first bin is the first color, right of the first bin but left of the second is the second color, right of the second but left of the third is the third color, right of the third but left of the fourth is the fourth color, and right of the fourth is the fifth color. So, we want to choose the placement of the 4 boundaries amongst the 20 jelly beans.

We can do this by choosing the 4 positions amongst a collection of 24, where 24 is the number of jellybeans + the number of bin boundaries. Thus, the number of ways to select 20 jelly beans is:

$$\binom{24}{4}$$

(ii) In how many ways to select 20 jelly beans if you must have at least two of every kind of jelly bean?

Solution

This problem is very similar; however, we now know that there are at least 2 of each color. Therefore, for each of the 5 colors, there are 2 jellybeans that cannot be changed, so 10 jellybeans that must be a certain color if no assumptions are made about the remaining 10.

So, we can follow the same method and choose how to divide up the remaining 10 jellybeans into the 5 groups, where any of the groups can be empty, since we are setting aside 2 for each group initially.

Therefore, to count the number of ways to select 20 jelly beans if you must have at least two of every kind of jellybean, we get the following equation:

$$\binom{14}{4}$$

Homework 10

Question 37: Problem 1 (25 points)

In this problem we will compute a Ramsey number.

1. To begin, prove that $R(4,4) \leqslant 18$.
2. Next, label the vertices of K_{17} with the numbers $1, 2, \ldots, 17$ and arrange them in a circle. An edge of our graph will be colored red if and only if the distance between its endpoints traveling along this outer circle is a power of 2 . Prove that this coloring does not produce a monochromatic K_4.

Solution:

1. To begin, prove that $R(4,4) \leqslant 18$.

Proof 35 $R(4,4) \leqslant 18$

As written in class, one of the known Ramses numbers is $R(4,3) = 9$. Therefore, by the definition of Ramses numbers, $R(3,4)$ is also 9. Using the first lemma shown in class about the upper bound of Ramses numbers, we know that:

$$R(m,n) \leqslant R(m-1,n) + R(m,n-1)$$

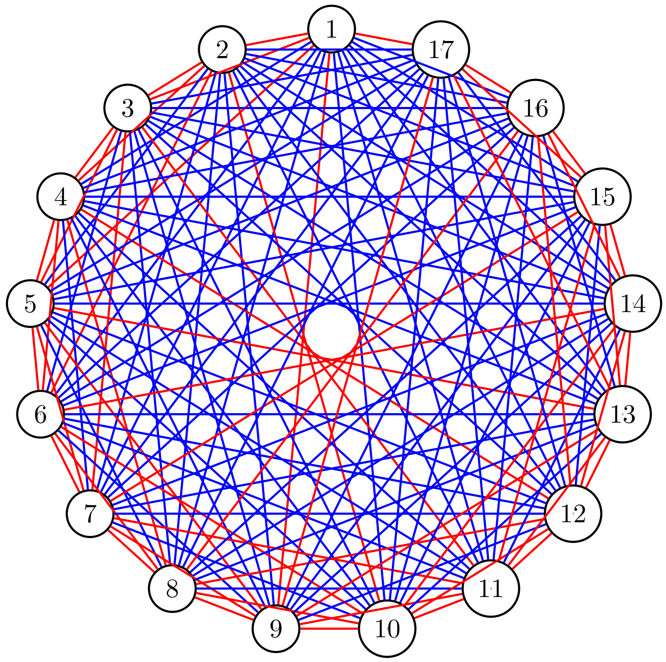
In this scenario, we can use this equation to write the upper bound for $R(4,4)$ as:

$$\begin{aligned} R(4,4) &\leqslant R(3,4) + R(4,3) \\ &= 9 + 9 \\ &= 18 \end{aligned}$$

Therefore, we have shown that $R(4,4) \leqslant 18$.

😄

2. Next, label the vertices of K_{17} with the numbers $1, 2, \ldots, 17$ and arrange them in a circle. An edge of our graph will be colored red if and only if the distance between its endpoints traveling along this outer circle is a power of 2 . Prove that this coloring does not produce a monochromatic K_4.



Graph of K_{17}, where edges are colored red if and only if the distance between its endpoints traveling along this outer circle is a power of 2, and blue otherwise.

Proof 36 No monochromatic K_4

To have a monochromatic K_4, we must be able to find 4 vertices from K_{17} that are all connected by red edges, or all connected by blue edges.

First, we will show that there is no monochromatic K_4 with red edges.

A vertex is then connected to another vertex by a red edge if and only if the distance between them is a power of 2–so, any given vertex is connected to another vertex by a red edge if and only if the distance between them is 1, 2, 4, 8, or 16.

Notice that if we choose any 3 vertices that are a distance of 1 apart, then the remaining vertex cannot be connected to all of them by a red edge.

Similarly, if we choose any 3 vertices that are a distance of 2 apart, then the remaining vertex cannot be connected to all of them by a red edge.

This logic follows for distances of 4, 8, and 16 as well.

Now, if the first 3 vertices are not all the same distance apart, then we can just choose 2 vertices that are the same distance apart, and notice that the remaining 2 vertices cannot be connected to both of them by a red edge.

Now, lets consider the case where all 4 vertices are a distinct distance apart. In this case, we can choose the 3 vertices that are the smallest distance apart, and notice that the remaining vertex cannot be connected to all of them by a red edge.

Alternately, we can approach this problem by listing the set of possible differences when subtracting two vertices:

$$\pm 1, \pm 2, \pm 4, \pm 8$$

If we choose any 3 of these differences, notice that they cannot all be connected to each other by a red edge.

Therefore, we have shown that there is no monochromatic K_4 with red edges.

To show that there is no monochromatic K_4 with blue edges, we can use similar logic to above.

The differences between vertices when subtracting two vertices are, when looking at non power of 2 differences:

$$\pm 3, \pm 5, \pm 6, \pm 7$$

If we choose any 3 of these differences, notice that they cannot all be connected to each other by a blue edge.

Therefore, we have shown that there is no monochromatic K_4 with blue edges.

Since we have shown that there is no monochromatic K_4 with red edges, and no monochromatic K_4 with blue edges, we can conclude that there is no monochromatic K_4 in this graph.

😄

Question 38: Problem 2 (25 points)

Prove that for any $n \geqslant 3, R(n+1, n+1) \leqslant 4^n$.

Solution:

Proof 37 $n \geqslant 3, R(n+1, n+1) \leqslant 4^n$

By the upper bound of Ramsey numbers as discussed in class:

$$\begin{aligned} R(n+1, n+1) &\leqslant R(n, n+1) + R(n+1, n) \\ &\leqslant \binom{2n}{n} \end{aligned}$$

Furthermore, by the binomial theorem and the fact that $n \geqslant 3$ we know that:

$$\begin{aligned} \sum_{n=1}^{2n} \binom{2n}{k} &= 2^{2n} \\ &= 4^n \end{aligned}$$

Therefore:

$$\begin{aligned} R(n+1, n+1) &\leqslant \binom{2n}{n} \\ &\leqslant \binom{2n}{n} + \sum_{n=1}^{2n-1} \binom{2n-1}{k} \end{aligned}$$

And finally:

$$\begin{aligned} R(n+1, n+1) &\leqslant \binom{2n}{n} + \sum_{n=1}^{2n-1} \binom{2n-1}{k} \\ &\leqslant \sum_{n=1}^{2n} \binom{2n}{k} \\ &= 4^n \end{aligned}$$

So, we can conclude that:

$$R(n+1, n+1) \leqslant 4^n$$

😄

Question 39: Problem 3 (50 points)

In this problem we will discuss an "infinite" form of Ramsey Theory.

1. Define $\mathbb{N}^{(2)}$ to be the set of all subsets of $\mathbb{N}$ with size 2. A 2-coloring of $\mathbb{N}^{(2)}$ is an assignment of one of two colors (red and blue) to every element of $\mathbb{N}^{(2)}$. If A is a subset of $\mathbb{N}$ whose every subset of size 2 is the same color, then we call A monochrome. Prove that in any 2 coloring of $\mathbb{N}^{(2)}$ you may always find monochrome sets of arbitrary size.
2. Notice that the above does not tell us that there are infinite monochrome sets. Prove that there must indeed be infinite monochrome sets. (HINT: do what we have done in the past. Start with a point and look at its blue and red neighbors. Then look at the Neighbors' Neighbors, and the Neighbors' Neighbor's Neighbors and so on ...).
3. Define a coloring where $\{x, y\}$ is red if x divides y or y divides x, and blue otherwise. Prove that this coloring produces monochromatic sets of infinite size in both colors.
4. Give an example of a coloring that uses both colors infinitely many times, but only produces infinite monochrome sets in one of the two colors.
5. In truth, given any countably infinite set S we could have considered the subsets of S and defined coloring in this context as well. It is a fact (that you do not need to prove) that if S is any infinite set and $S^{(r)}$ is the set of subsets of size r, then any 2 coloring of these subsets will produce a monochromatic set of infinite size. That is to say, an infinite subset $T \subseteq S$ such that any subset of T is size r is the same color. Using this fact prove the following geometry theorem: If S is any infinite collection of points in $\mathbb{R}^2$, then there must exist an infinite subset T of S such that no three points of T lie on the same line, or otherwise every point of T is on the same line.

Solution:

1. Define $\mathbb{N}^{(2)}$ to be the set of all subsets of $\mathbb{N}$ with size 2. A 2-coloring of $\mathbb{N}^{(2)}$ is an assignment of one of two colors (red and blue) to every element of $\mathbb{N}^{(2)}$. If A is a subset of $\mathbb{N}$ whose every subset of size 2 is the same color, then we call A monochrome. Prove that in any 2 coloring of $\mathbb{N}^{(2)}$ you may always find monochrome sets of arbitrary size.

   Proof 38 Monochrome sets of arbitrary size

   For any two numbers, we can color the edge between them red or blue, according to which color the associated subset of size 2 is. Since there are infinite natural numbers, there are also infinite edges. Now, we will show that there exists a monochrome set of arbitrary size.

   Begin by looking at any natural number, say 1. Then, we know that there are infinite natural numbers that are connected to 1, as 1 is "paired" with every other natural number. Since there are only 2 colors, there must be an infinite number of natural numbers that are connected to 1 that are the same color. Assume that there are an infinite number of natural numbers that are connected to 1 that are red. Let A be the set of natural numbers that are connected to 1 that are red. Then, A is a monochrome set of arbitrary size.

   So, there exists a monochrome set of arbitrary size for any 2 coloring of $\mathbb{N}^{(2)}$.

   😊

2. Notice that the above does not tell us that there are infinite monochrome sets. Prove that there must indeed be infinite monochrome sets. (HINT: do what we have done in the past. Start with a point and look at its blue and red neighbors. Then look at the Neighbors' Neighbors, and the Neighbors' Neighbor's Neighbors and so on ...).

Proof 39 Infinite monochrome sets

Assume we have a 2-coloring of $\mathbb{N}^{(2)}$ using red and blue colors. Start with any natural number, say 1. Look at its neighbors, which are the natural numbers connected to 1 by an edge in the graph.

Among these neighbors, there are infinitely many numbers. Let's say there are k numbers among the neighbors. By the pigeonhole principle, at least one color (either red or blue) must appear at least $\lceil \frac{k}{2} \rceil$ times among these neighbors.

Then, let's assume that the red color appears at least $\lceil \frac{k}{2} \rceil$ times. We can select $\lceil \frac{k}{2} \rceil$ of these red neighbors and form a monochrome set A_1.

Now, consider the neighbors of the numbers in A_1. Among these neighbors, there are again infinitely many numbers. Let's say there are m numbers among the neighbors. By the same argument as before, at least one color (red or blue) appears at least $\lceil \frac{m}{2} \rceil$ times.

Again, assume the red color appears at least $\lceil \frac{m}{2} \rceil$ times. Select $\lceil \frac{m}{2} \rceil$ of these red neighbors and add them to the monochrome set A_1, forming a new monochrome set A_2.

We can continue this process indefinitely. At each step, we consider the neighbors of the numbers in the previous monochrome set, and we select a subset of red neighbors (or blue ones) to form the next monochrome set.

By construction, we have created an infinite sequence of monochrome sets $A_1, A_2, A_3, \ldots$, where each set A_i is a monochrome set of size $\lceil \frac{k}{2^i} \rceil$.

Since the size of each monochrome set A_i is determined by the size of its neighbors in the previous set, and the neighbors are always infinite, each monochrome set A_i is infinite.

Therefore, we have constructed an infinite sequence of monochrome sets, proving that there must be infinite monochrome sets in any 2-coloring of $\mathbb{N}^{(2)}$.

😊

3. Define a coloring where $\{x, y\}$ is red if x divides y or y divides x, and blue otherwise. Prove that this coloring produces monochromatic sets of infinite size in both colors.

Proof 40 Infinite divided monochromatic sets

Say we have a 2-coloring of $\mathbb{N}^{(2)}$ using red and blue colors. We will show that there exists a monochrome set of infinite size in both colors.

To find the infinite red set, lets choose all numbers that are a power of 2. By the definition of the natural numbers, we know that there are infinite natural numbers. We also know that there are infinite powers of 2. Therefore, there are infinite natural numbers that are powers of 2.

To show that this set is monochrome, we will show that any two numbers in this set are connected by a red edge. Let x and y be two numbers in this set. Then, x and y are both powers of 2. Assume that $x \leqslant y$. Then, $y = 2^k x$ for some $k \in \mathbb{N}$. Therefore, x divides y, so $\{x, y\}$ is red.

Therefore, there exists a monochrome set of infinite size in the red color.

Now, we will show that there exists a monochrome set of infinite size in the blue color. To do this, we will choose all prime numbers. We know that there are infinite natural numbers, and we know that there are infinite prime numbers. Therefore, there are infinite natural numbers that are prime.

Furthermore, by the definition of prime numbers, we know that prime numbers are only divisible by 1 and themselves. Therefore, no two prime numbers are connected by a red edge. Therefore, all prime numbers are connected by a blue edge.

😊

4. Give an example of a coloring that uses both colors infinitely many times, but only produces infinite monochrome sets in one of the two colors.

Solution

Consider the 2-coloring of $\mathbb{N}^{(2)}$ where the coloring is as follows:

- If the sum of the two numbers in the subset is even, color it red.
- If the sum of the two numbers in the subset is odd, color it blue.

In this coloring, both red and blue colors are used infinitely many times. However, only the red color produces infinite monochrome sets.

To see this, consider any subset A of natural numbers that are all red. We can observe that for any two numbers x and y in A, the sum $x + y$ will always be even. Therefore, every subset of size 2 within A will also have an even sum, making A a monochrome set.

On the other hand, if we consider any subset B of natural numbers that are all blue, we can observe that for any two numbers x and y in B, the sum $x + y$ will always be odd. Therefore, B cannot be a monochrome set as there will always exist subsets of size 2 within B with an odd sum.

Hence, in this coloring, only the red color produces infinite monochrome sets while the blue color does not.

5. In truth, given any countably infinite set S we could have considered the subsets of S and defined coloring in this context as well. It is a fact (that you do not need to prove) that if S is any infinite set and $S^{(r)}$ is the set of subsets of size r, then any 2 coloring of these subsets will produce a monochromatic set of infinite size. That is to say, an infinite subset $T \subseteq S$ such that any subset of T is size r is the same color. Using this fact prove the following geometry theorem: If S is any infinite collection of points in $\mathbb{R}^2$, then there must exist an infinite subset T of S such that no three points of T lie on the same line, or otherwise every point of T is on the same line.

Proof 41 Geometry theorem

Want to show that if S is any infinite collection of points in $\mathbb{R}^2$, then there must exist an infinite subset T of S such that no three points of T lie on the same line, or otherwise every point of T is on the same line.

Assume we have an infinite collection of points S in $\mathbb{R}^2$. Consider the subsets of S of size 2, denoted as $S^{(2)}$. We will 2-color these subsets.

For each subset $p, q \in S^{(2)}$, where p and q are distinct points in S, we color it red if the line passing through p and q contains at least one more point from S, and we color it blue otherwise.

Now, we have a 2-coloring of $S^{(2)}$. By the given fact, there exists a monochromatic set $T \subseteq S$ of infinite size. Lets assume T is a red monochromatic set.

We claim that no three points of T lie on the same line. Suppose, for the sake of contradiction, that there exist three points $p, q, r \in T$ that lie on the same line.

Since T is a red monochromatic set, it means that the subsets p, q, q, r, and p, r are all colored red. This implies that the lines passing through pairs of these points, namely the lines $\overline{pq}$, $\overline{qr}$, and $\overline{pr}$, each contain at least one more point from S.

However, this contradicts the assumption that p, q, and r are the only points from T on the same line. Therefore, our assumption was false, and no three points of T lie on the same line.

Hence, we have shown that there exists an infinite subset T of S such that no three points of T lie on the same line.

Alternatively, if every point of T is on the same line, then we have the desired property as well.

Therefore, the geometry theorem holds: If S is any infinite collection of points in $\mathbb{R}^2$, there must exist an infinite subset T of S such that no three points of T lie on the same line, or otherwise every point of T is on the same line.

😊