Principle of
Marine Hydrodynamics I

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Lecture Information

1. Textbook: Lecture notes
2. Time: Friday (19:30 ~)
3. Structure: 15 weeks with Midterm Exam and Term Project
4. Homework: selected problems at each chapter
5. Lecture information: *http//fincl.cnu.ac.kr*

6. Grading: Midterm Exam(30%) + Term Project(50%) + Homework(10%)
   + Attendance(10%) = Total(100%)

7. Reference Books:
   1) Marine Hydrodynamics, J.N. Newman, MIT press, 1977
   2) An Introduction to Fluid Dynamics, G. K. Batchelor, Cambridge University Press, 1974
   3) A First Course in Turbulence, H. Tennekes and J. L. Lumley, MIT Press, 1972

8. Preliminary subject: Fluid Mechanics
9. Inquiry: Monday (3-102) after the lecture
10. Email: bkahn@cnu.ac.kr Tel: 821- 6625

Contents

1. Basic Concepts

1.1 Mechanics

1) 역학 (Mechanics): 물체에 작용하는 힘과 운동의 관계를 연구하는 학문

2) 역학의 구성

① Statics(정역학): (물체)상호간의 상대운동이 없이 정지 또는 등속 운동하는 물체에 작용하는 힘의 평형관계를 다루는 역학

- 외력과 내력과의 관계와 외력에 의한 반작용력을 결정 (압력, 부력 등)

② Kinematics(운동학): 운동하는 물체의 기하학적 거동을 다루는 역학

– 운동을 힘이나 모멘트와 관련시켜 해석하지는 않음 (geometry of motions)

- 유동가시화 (Flow Visualization), 캠(cam), 치차(gear), 연결 봉(connecting rod) 등

③ Kinetics(운동역학): 운동하는 물체에 작용하는 힘과 운동사이의 관계를 다루는 역학

- Newton의 운동법칙을 적용하여 물체의 운동방정식을 구하고 해석함

➲ 동역학(Dynamics) = Kinematics + Kinetics (deals with bodies in motion)

➲ 유체역학(Fluid Mechanics)은 정지상태의 유체를 영의 속도를 갖는 운동으로 간주, 정역학을 포함하여 유체동역학(Fluid Dynamics)로 부르기도 함

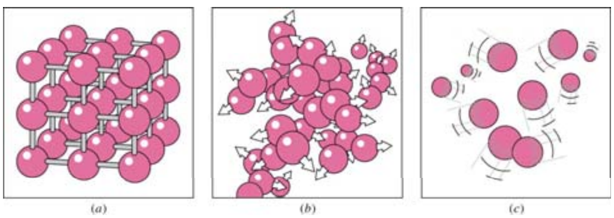
1.2 Fluid

1) 물질의 구분과 정의(분자운동의 관점)

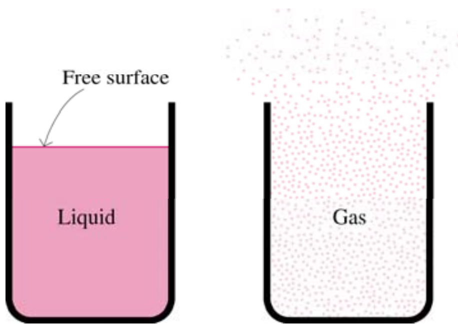
① 고체(Solid): 분자간의 응집력이 강해 자유운동을 할 수 없으며 독립적인 체적을 유지하는 물질

② 기체(Gas): 분자간의 거리가 크고 상대적으로 운동에너지가 크며 독립적인 체적을 유지하지 못하는 물질 (molecules move at random): ex-공기(Air)

③ 액체(Liquid): 기체에 비해 분자가 거리가 짧고 운동에너지가 작으며 정형의 용기 속에서 체적을 유지하는 물질 (makes free surface): ex-물(water)



solid Liquid Gas



유체(Fluid) = 기체(Gas) + 액체(Liquid)

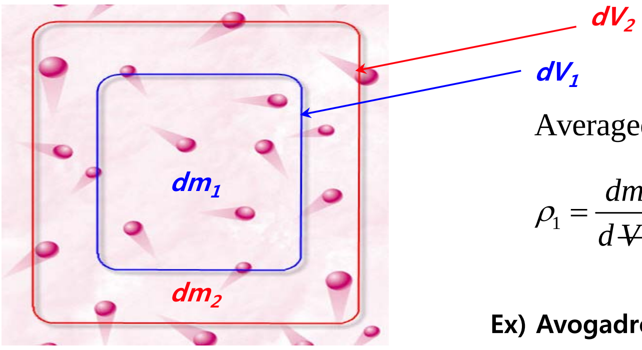
➲ 유체의 운동을 분자운동의 관점에서 정의하고 해석하기 위해서는 통계학적으로 다뤄야 하며 이는 신뢰도의 문제와 연결됨

1.2 Fluid

1) 연속체 : 연속적인 질량분포를 가지는 물질의 집합체

2) 물질적 관측 점†의 평균밀도가 같다($\rho_1 = \rho_2$)면 이 물체는 연속 체로 취급 가능

[† 정의된 공간에 내포되는 분자수가 충분히 많아 분자들의 평균특성이 그 점의 유동특성을 대표할 수 있는 최소공간]



Averaged Density: $\rho = \lim_{\Delta V \to dV} = \frac{dm}{dV}$

$\rho_1 = \frac{dm_1}{dV_1} \qquad \rho_2 = \frac{dm_2}{dV_2}$

Ex) Avogadro's number

Gas 1mol: N=6.024×10^{23} per 22.4 liter ($22.4 \times 10^{-3} m^3$)

- $2.687 \times 10^{19}/1cm^3$ & $2.687 \times 10^7/1\mu m^3$

➲ 유체는 연속체로 취급하기에 충분한 분자 수를 가짐

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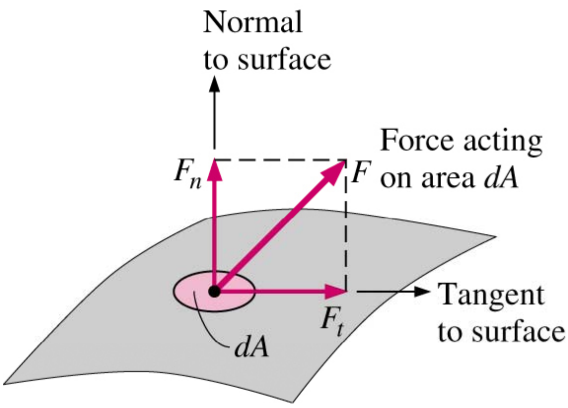
1.2 Fluid

1) Distinction between solid and fluid

① **Fluid:** deforms continuously under applied shear (stress is proportional to strain rate)

② Solid: can resist an applied shear by deforming (stress is proportional to strain)



2) 응력(Sress): Force per unit area

① 수직응력(Normal Stress):

- 작용응력의 수직(Normal)성분

② 수평 or 접선(Tangential) or 전단응력(Shear Stress):

- 작용응력의 접선(Tangent)성분

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1.3 Forces

1) 유체에 작용하는 힘

① 표면력(Surface Force) or 표면응력 – 표면에 직접 접촉하여 작용하는 힘

– 수직응력(Normal Stress, σ)

1) 압축응력(Compressible Stress) – 압력(Pressure)

2) 인장응력(Tensile Stress)

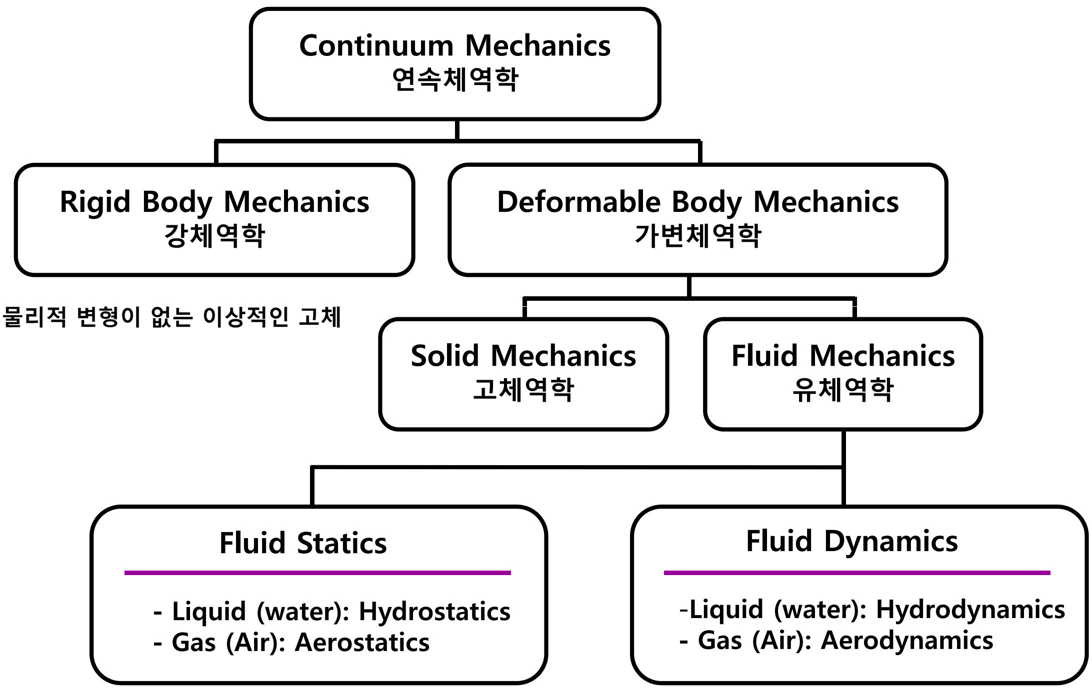
– 전단(접선)응력(Shear Stress, τ)

② 체적력(Body Force) – 표면에 접촉하지 않고 작용하는 힘

- 중력(Gravity), 전/자기력(Electro/Magnetic Force)

유체정역학에서는 유체의 상대 운동이 없는 상태(du/dy=0), 즉 전단응력이 작용하지 않는 상태에서 수직응력(σ - 압력)과 체적력(Gravity)만을 고려

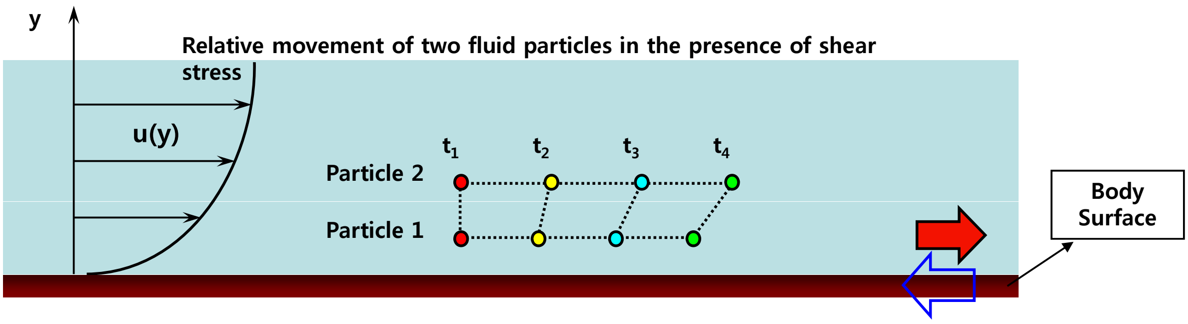
1.3.1 Continuum Mechanics



※ 강체: 물리적 변형이 없는 이상적인 고체

1.4 Viscosity & Shear Stress

1) **점성(Viscosity):** 유체 층 사이에 상대 운동이 생길 때 이 상대운동에 대응하는 저항력의 원인이 되는 물질의 상태량
2) **점성**에 의한 저항력은 접선(유동) 방향으로 작용하여 **전단응력(shear stress)**을 발생시키며 이 힘을 표면 마찰력(surface friction force) 또는 **점성항력(viscous drag force)**이라 부름
3) 모든 유체는 고유의 상태량인 점성 (viscosity)을 가짐



1.4 Shear stress

1) 점성에 대한 관계식을 얻기 위해 평판 사이의 유체 층을 고려해 보면
2) **전단응력(shear stress): $\tau = F/A$**
3) 유체 요소 AD의 각변형 속도 또는 변형률 (strain rate)

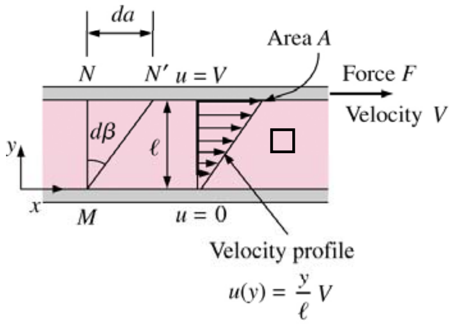
$$\dot{\theta} = \frac{d\theta}{dt} \quad \rightarrow \; d\theta = \dot{\theta} dt$$

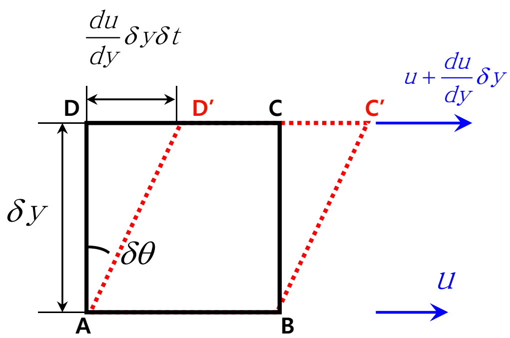
$$\tan(d\theta) = \tan(\dot{\theta} dt) \simeq \dot{\theta} dt = \frac{du/dy \; \delta y \delta t}{\delta y} = \frac{du}{dy} dt$$

$$\therefore \; \dot{\theta} = \frac{d\theta}{dt} = \frac{du}{dy}$$

4) **유체 요소의 변형률 (strain rate)은 속도구배 (velocity gradient)와 같다**
5) 대부분의 유체의 **전단응력은 유체 요소의 변형률 즉 속도구배에 비례 함** (Hooke's Law)

$$\tau \propto \frac{d\theta}{dt} \quad \rightarrow \quad \tau \propto \frac{du}{dy}$$





1.4 Dynamic Viscosity

1) Newtonian fluids: 변형률이 전단응력에 비례하는 유체

   - 물, 가솔린, 기름 등 대부분의 유체

2) Non-newtonian fluids: 변형률이 전단응력에 비례하지 않는 유체 - 혈액, 액체 플라스틱 등

3) 뉴턴유체의 전단응력 (선형적인 관계식)

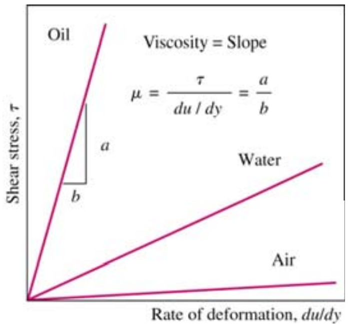
$$\tau = \mu \frac{du}{dy} \quad \left[\mathrm{N/m^2} \right]$$

   - 점성계수는 속도구배에 독립적

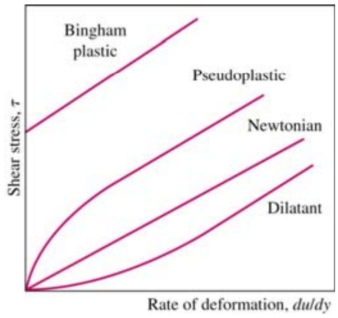
4) 점성계수(coefficient of viscosity) 또는 동역학적 점성계수 (dynamic viscosity)

$$\mu = \tau \frac{dy}{du}$$

$$\text{unit:} \left[\mathrm{N/m^2} \frac{\mathrm{m}}{\mathrm{m/s}} = \frac{\mathrm{N \cdot s}}{\mathrm{m^2}} = \frac{\mathrm{kg}}{\mathrm{m \cdot s}} = \mathrm{Pa \cdot s} \right] \text{ (poise)}$$



(Newtonian fluids)



(Non-newtonian fluids)

1.4 Kinematic Viscosity

1) 동점성계수 (kinematic viscosity): 밀도에 대한 동역학적 점성계수의 비율

$$\upsilon = \frac{\mu}{\rho} \qquad \text{unit:} \left[\frac{N \cdot s / m^2}{N \cdot s^2 / m^4} \right] = \left[\frac{m^2}{s} \right]$$

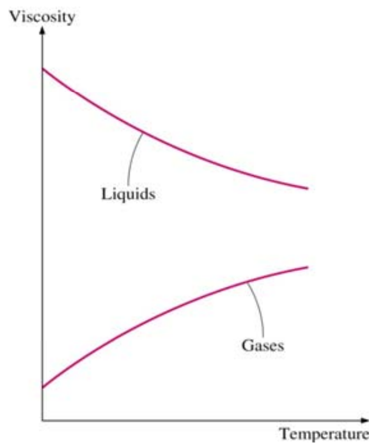
➲ 운동학(kinematics)적 차원을 가짐: Stoke [cm^2/s]

Fluid (15ºC, 1atm)	Density(ρ)	Viscosity(μ)	Kinematic Viscosity(ν)
Water	999.1 [kg/m^3]	1.144×10^{-3} [kg/ms]	1.144×10^{-6} [m^2/s]
Air	1.224 [kg/m^3]	1.785×10^{-5} [kg/ms]	1.458×10^{-5} [m^2/s]

➲ Reynolds number:

$$R_e = \frac{\rho}{\mu} VL = \frac{VL}{\upsilon}$$

$$\upsilon_a \cong \Theta(1.5\text{x}10^{-5}) \;\; \& \;\; \upsilon_w \cong \Theta(1.0\text{x}10^{-6})$$



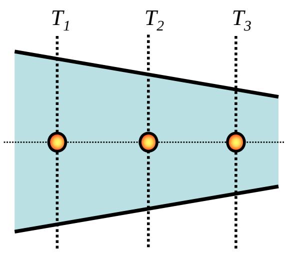
The viscosity of liquids decreases and the viscosity of gases increase with temperature

2. Description of Fluid Flow

2.1 Lagrangian and Eulerian Description

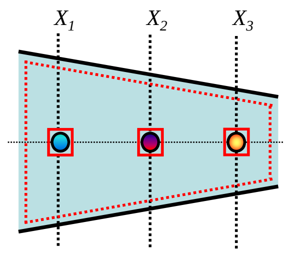
1) **Lagrangian description: 유체입자를 따라가면서 시간경과에 따른 유동장 변수(위치, 속도, 압력 등)의 변화를 기술하는 방법 (물질좌표계, Lagrange 좌표계): *N = f(t)***
   - ① 수많은 개별 유체입자의 거동을 파악하기는 어려움
2) **Eulerian description: 정해진 시간/공간 (Control Volume)을 지나는 유체(연속체로서의)의 유동장 변수의 변화를 기술하는 방법 (공간좌표계, Euler 좌표계): *N = f(x,t)***
   - ① 유동변수를 유체가 유/출입하는 검사체적(Control Volume) 내에서 위치와 시간의 함수로 정의
   - ② 유체역학에서는 일반적으로 Euler 기술방법이 적합하며 편리함

Time (T)	Velocity ($\underline{V}$)
T_1	$\underline{V}_1$
T_2	$\underline{V}_2$
T_3	$\underline{V}_3$



Lagrangian description: 측정장치를 유체와 함께 흐르게 하면서 유동장 변수를 측정

Time	Velocity ($\underline{V}$)		
(T)	X_1	X_2	X_3
T_1	$\boldsymbol{\underline{V}_{11}}$	$\underline{V}_{12}$	$\underline{V}_{13}$
T_2	$\underline{V}_{21}$	$\boldsymbol{\underline{V}_{22}}$	$\underline{V}_{23}$
T_3	$\underline{V}_{31}$	$\underline{V}_{32}$	$\boldsymbol{\underline{V}_{33}}$



Eulerian description: 측정장치를 유동의 한 위치에서 고정시키고 유동장 변수를 측정

2.2 Acceleration Field

1) **Newton의 제2법칙:**

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

2) **유체입자의 가속도는 입자 속도의 시간 도함수(time derivative):**

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$

3) **유체입자는 유동장을 따라 운동하기 때문에 주어진 시간(*t*)에서의 유체입자 속도는 유체입자의 위치(*x(t), y(t), z(t)*)에서의 속도장과 같다**

$$\vec{V}_{particle} = \vec{V}\left(x_{particle}(t), y_{particle}(t), z_{particle}(t),\ t\right)$$

4) **시간 도함수를 연쇄법칙(chain rule)을 사용하여 표현하면:**

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt} = \frac{\partial\vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial\vec{V}}{\partial x}\frac{dx_{particle}}{dt} + \frac{\partial\vec{V}}{\partial y}\frac{dy_{particle}}{dt} + \frac{\partial\vec{V}}{\partial z}\frac{dz_{particle}}{dt}$$

2.2 Acceleration Field

5) **입자의 시간에 대한 변화율은:**

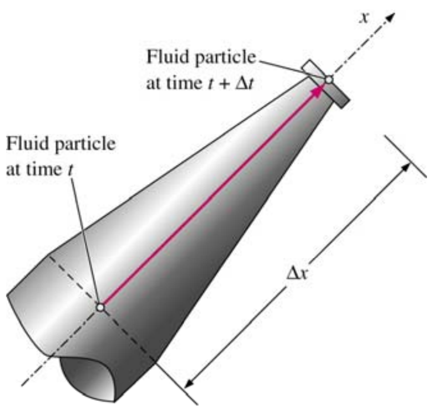
$$\frac{dx_{particle}}{dt} = u, \frac{dy_{particle}}{dt} = v, \frac{dz_{particle}}{dt} = w$$

6) **유동장의 가속도는 그 주어진 시간(*t*)에서 위치(*x,y,z*)를 점유하는 유체입자의 가속도와 같다.**

$$\vec{a}_{particle} = \vec{a}(x, y, z, t) = \frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}$$

7) **유동장 변수로 표현한 유체입자의 가속도:**

$$\vec{a}(x, y, z, t) = \frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + \left(\vec{V}\bullet\vec{\nabla}\right)\vec{V}$$



Euler 좌표계의 관찰자의 관점에서는 정상이지만, Lagrange좌표계에서는 정상이 아님

2.3 Material Derivative

1) A field variable as functions of space and time, within the control volume:

$$f = f(x,y,z,t)$$

2) Time derivative of the field variable (*just consider x component*):

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(x+\Delta x, t+\Delta t) - f(x,t)}{\Delta t}$$

3) Taylor series expansion for the space:

$$f(x+\Delta x, t+\Delta t) = f(x, t+\Delta t) + \frac{\partial f(x,t+\Delta t)}{\partial x}\Delta x + \ldots$$

4) Taylor series expansion for the time:

$$f(x, t+\Delta t) = f(x,t) + \frac{\partial f(x,t)}{\partial t}\Delta t + \ldots$$

$$\frac{\partial f(x,t+\Delta t)}{\partial x}\Delta x = \frac{\partial f(x,t)}{\partial x}\Delta x + \frac{\partial^2 f(x,t)}{\partial t \partial x}\Delta x \Delta t + \ldots$$

2.3 Material Derivative

4) 3) becomes

$$f(x+\Delta x, t+\Delta t) = f(x,t) + \frac{\partial f(x,t)}{\partial t}\Delta t + \frac{\partial f(x,t)}{\partial x}\Delta x + HOT$$

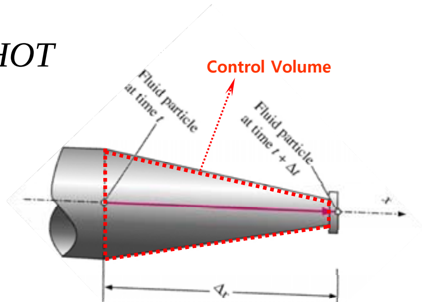
$$= f(x,t) + \frac{\partial f(x,t)}{\partial t}\Delta t + \frac{\partial f(x,t)}{\partial x}u\Delta t + HOT$$

5) Therefore

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(x+\Delta x, t+\Delta t) - f(x,t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{\partial f(x,t)}{\partial t}\Delta t + u\frac{\partial f(x,t)}{\partial x}\Delta t}{\Delta t}$$

$$= \frac{\partial f(x,t)}{\partial t} + u\frac{\partial f(x,t)}{\partial x} = \frac{\partial f(x,t)}{\partial t} + (u \cdot \nabla) f$$



The flow is steady in the Eulerian reference frame, but it is unsteady in the Lagrangian reference frame. Note that the accelerations are same in both frames

2.3 Material Derivative

1) **Material (total, particle & substantial) Derivative:**

$$\underbrace{\boxed{\frac{D\{\ \}}{Dt}}}_{\text{Material Derivative}} = \frac{d\{\ \}}{dt} = \underbrace{\boxed{\frac{\partial\{\ \}}{\partial t}}}_{\text{Local Term}} + \underbrace{\boxed{\left(\vec{V}\bullet\vec{\nabla}\right)\{\ \}}}_{\text{Convective Term}}$$

2) **물질가속도(Material acceleration):**

① **국소가속도**

② **대류가속도**

$$\vec{a}(x,y,z,t) = \frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + \left(\vec{V}\bullet\vec{\nabla}\right)\vec{V}$$

3) **압력의 물질 도함수(Material derivative of pressure):**

① **국소압력**

② **대류압력**

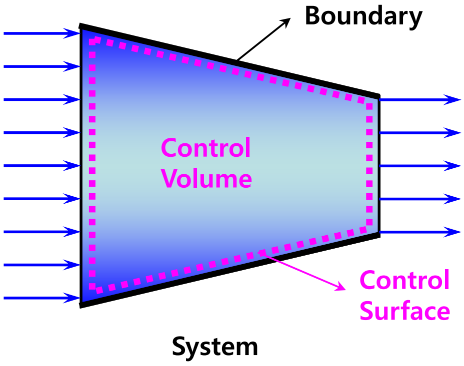
$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + \left(\vec{V}\bullet\vec{\nabla}\right)P$$

2.4 Control Volume & Surface

1) **열역학 및 고체역학의 경우 일정한 질량을 가지는 물질의 집합체로 정의 되는 시스템(System or Closed System)을 이용하여 해석**

2) **유체역학의 경우 해석을 위해 선택한 공간상의 영역으로 정의되는 검사체적(Control Volume) 또는 개방 시스템(Open System)으로 해석**

3) **시스템의 크기와 모양은 변할 수 있지만, 질량은 시스템의 경계를 통과할 수 없다. 반면, 검사체적에서 질량은 검사체적의 경계 면으로 정의되는 검사면(Control Surface)을 출입할 수 있다.**



- System(계): 해석하고자 하는 대상
- Surrounding(외계): 계를 둘러싼 주위의 모든 것
- Boundary(경계): 계와 외계의 경계(면)
- Control Volume(검사,제한체적): 주어진 공간좌표계에 고정된 형상불변의 체적
- Control Surface(검사면): 검사체적을 둘러싸고 있는 표면

2.5 Reynolds Transport Eqn.

1) $\vec{F} = \dfrac{d(m\vec{V})}{dt}$ $\quad\leftarrow\quad m = \displaystyle\int_{\forall} \rho\, d\forall$

$$= \frac{D}{Dt}\int_{\forall} \rho\vec{V} d\forall = \lim_{\Delta t \to 0} \frac{\displaystyle\int_{\forall_0+\Delta\forall} \left(\rho\vec{V}\right)_{t_0+\Delta t} d\forall - \int_{\forall_0} \left(\rho\vec{V}\right)_{t_0} d\forall}{\Delta t}$$

2) by Taylor series expension

$$\left(\rho\vec{V}\right)_{t_0+\Delta t} = \left(\rho\vec{V}\right)_{t_0} + \frac{\partial}{\partial t}\left(\rho\vec{V}\right)_{t_0} \Delta t + \text{HOT}$$

3) $$\int_{\forall_0+\Delta\forall} \left(\rho\vec{V}\right)_{t_0+\Delta t} d\forall = \int_{\forall_0} \left(\rho\vec{V}\right)_{t_0+\Delta t} d\forall + \int_{\Delta\forall} \left(\rho\vec{V}\right)_{t_0+\Delta t} d\forall$$

negligible when $\Delta\forall$ is small

0

$$= \int_{\forall_0} \left[\left(\rho\vec{V}\right)_{t_0} + \frac{\partial}{\partial t}\left(\rho\vec{V}\right)_{t_0} \Delta t\right] d\forall + \int_{\Delta\forall} \left[\left(\rho\vec{V}\right)_{t_0} + \cancel{\frac{\partial}{\partial t}\left(\rho\vec{V}\right)_{t_0} \Delta t}\right] d\forall$$

4) $$\vec{F} = \int_{\forall_0} \frac{\partial}{\partial t}\left(\rho\vec{V}\right)_{t_0} d\forall + \lim_{\Delta t \to 0} \int_{\Delta\forall} \left(\rho\vec{V}\right)_{t_0} \frac{d\forall}{dt}$$

5) $$\mathrm{d}\forall = dA(\vec{V}\bullet\vec{n})dt = dA\vec{V}_n dt \qquad \therefore \frac{\mathrm{d}\forall}{dt} = dA\vec{V}_n$$

2.5 Reynolds Transport Eqn.

6) $$\vec{F} = \frac{D}{Dt}\int_{\forall} \rho\vec{V} d\forall = \int_{\forall_0} \frac{\partial}{\partial t}\left(\rho\vec{V}\right)_{t_0} d\forall + \int_{S_0} \left(\rho\vec{V}\right)_{t_0} \vec{V}_n dA = \frac{\partial}{\partial t}\int_{CV} \rho\vec{V} d\forall + \int_{CS} \rho\vec{V}\vec{V}_n dA$$

7) Reynolds Transport Eqn.

$$\frac{D}{Dt}\int_{MV} f d\forall = \frac{\partial}{\partial t}\int_{CV} f d\forall + \int_{CS} f\vec{V}_n dA$$

8) Gauss's Divergence Theorem

$$\int_{\forall} \nabla\bullet\left(f\vec{V}\right) d\forall = \int_{S} f\vec{V}_n dA$$

9) Vector Identity

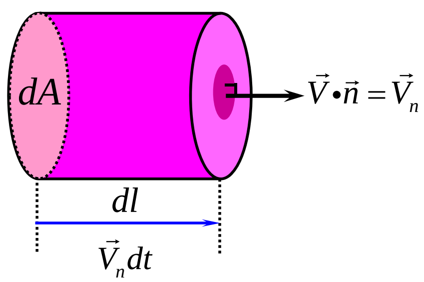
$$\nabla\bullet\left(f\vec{V}\right) \equiv \vec{V}\bullet\nabla f + f\nabla\bullet\vec{V}$$

$\nabla\cdot\vec{V} = 0$

10) Reynolds Transport Eqn.

for a fixed CV in an incompressible flow

$$\int_{MV} \frac{Df}{Dt} d\forall = \int_{CV} \left[\frac{\partial f}{\partial t} + \vec{V}\bullet\nabla f\right] d\forall$$



$$d\forall = dAdl = \vec{V}_n dAdt$$

(법선 속도벡터와 체적 변화량)

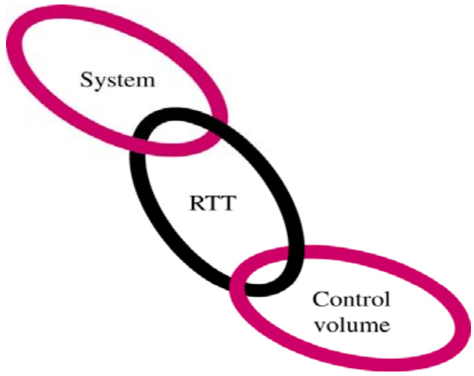
2.5 Reynolds—Transport Theorem (RTT)

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b)\, d\forall + \int_{CS} \rho b \vec{V}_n dA$$

Time rate of change of the property B of the system is equal to (Term 1) + (Term 2)

1st Term: the time rate of change of B of the control volume

2nd Term: the net flux of B out of the control volume by mass crossing the control surface



2.5 Material Derivative & RTT

The Material Derivative is used to transform from Lagrangian to Eulerian descriptions for Differential Analysis

Lagrangian Description ⇨

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\vec{V} \bullet \nabla) f$$

Differential Analysis

⇨ **Eulerian Description**

Integral Analysis

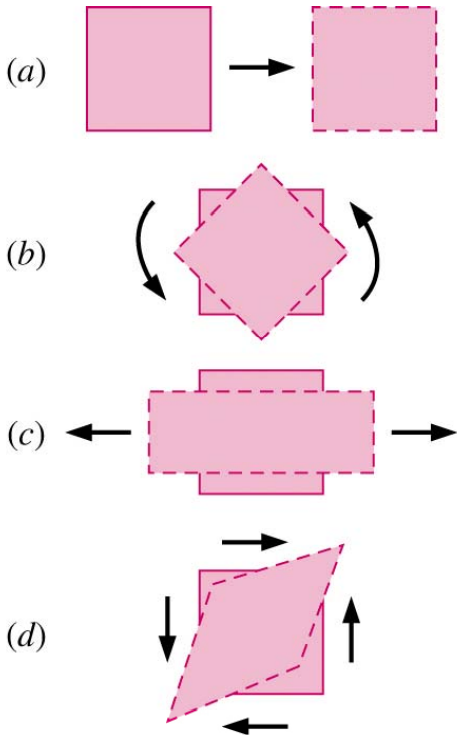
System Analysis ⇨

$$\int_{MV} \frac{Df}{Dt} d\forall = \int_{CV} \left[\frac{\partial f}{\partial t} + (\vec{V} \bullet \nabla) f \right] d\forall$$

⇨ **Control Volume Analysis**

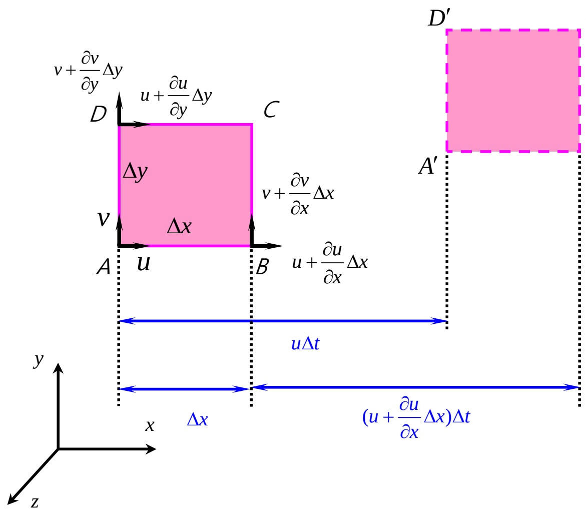
The Reynolds transport theorem (RTT) is used to transform from system to control volume for Integral Anlaysis

2.6 Motion & Deformation



1) 유체요소의 운동과 변형의 기본 형태:
   a) 이동(Translation)
   b) 회전(Rotation)
   c) 선형변형(Linear strain)
   d) 전단변형(Shear strain)
2) 유체역학에서는 운동과 변형이 동시에 발생하여 복잡하므로 이들을 율(rate)의 형태로 표현하는 것이 필요
   a) 속도(velocity): rate of translation
   b) 각속도(angular velocity): rate of rotation
   c) 선형변형률(linear strain rate): rate of linear strain
   d) 전단변형률(shear strain rate): rate of shear strain
3) 변형률(deformation rates)을 속도와 속도의 도함수로 표현

2.6.1 이동 (Translation)



1) 이동(Translation): 변형이나 회전없이 위치의 변화만을 수반하는 운동.
2) 선분 *AB*가 시간 Δt 동안 변화된 량:

$$\overline{A'B'}-\overline{AB}=\left[\left\{\Delta x+\left(u+\frac{\partial u}{\partial x}\Delta x\right)\Delta t\right\}-u\Delta t\right]-\Delta x$$
$$=\frac{\partial u}{\partial x}\Delta x\Delta t$$

3) 수직변형률(Normal strain rate):

$$\varepsilon_{xx}=\lim_{\Delta x,\Delta t\to 0}\frac{\overline{A'B'}-\overline{AB}}{\Delta x\Delta t}=\frac{\partial u}{\partial x}$$

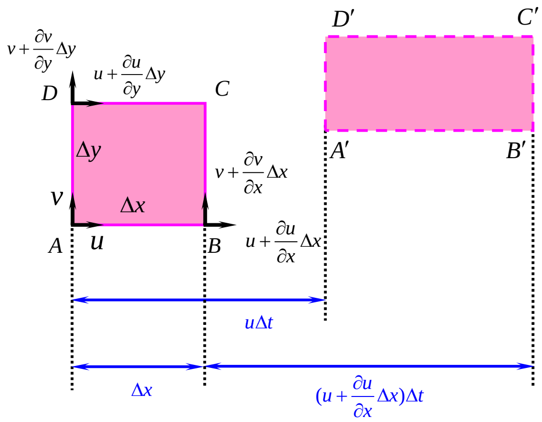
4) y, z 방향의 수직변형률:

$$\varepsilon_{yy}=\frac{\partial v}{\partial y},\quad \varepsilon_{zz}=\frac{\partial w}{\partial z}$$

5) 이동의 경우 정의(크기의 변화없음)에 따라

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}=0,\quad \varepsilon_{yy}=\frac{\partial v}{\partial y}=0,\quad \varepsilon_{zz}=\frac{\partial w}{\partial z}=0$$

2.6.2 선형변형 (Linear Strain)



1) 선형변형(Linear strain): 각변형 없이 수직변형을 수반하는 운동.

2) 선형변형률(Linear strain rate)

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}=a\neq 0,\ \ \varepsilon_{yy}=\frac{\partial v}{\partial y}=b,\ \ \varepsilon_{zz}=\frac{\partial w}{\partial z}=c$$

3) 선형변형에 의한 체적의 변화:

$$x:\ \overline{A'B'}=\Delta x+\frac{\partial u}{\partial x}\Delta x\Delta t$$

$$y:\ \overline{A'D'}=\Delta y+\frac{\partial v}{\partial y}\Delta y\Delta t$$

$$z:\ \overline{A'E'}=\Delta z+\frac{\partial w}{\partial z}\Delta z\Delta t$$

4) $\triangle$t 동안 선형변화에 의한 체적변화:

$$\Delta\cancel{V}=\left(\overline{A'B'}\,\overline{A'D'}\,\overline{A'E'}\right)-\left(\overline{AB}\,\overline{AD}\,\overline{AE}\right)$$

$$=(\Delta x+\frac{\partial u}{\partial x}\Delta x\Delta t)(\Delta y+\frac{\partial v}{\partial y}\Delta y\Delta t)(\Delta z+\frac{\partial w}{\partial z}\Delta z\Delta t)-\Delta x\Delta y\Delta z$$

$$\approx\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\Delta x\Delta y\Delta z\Delta t$$

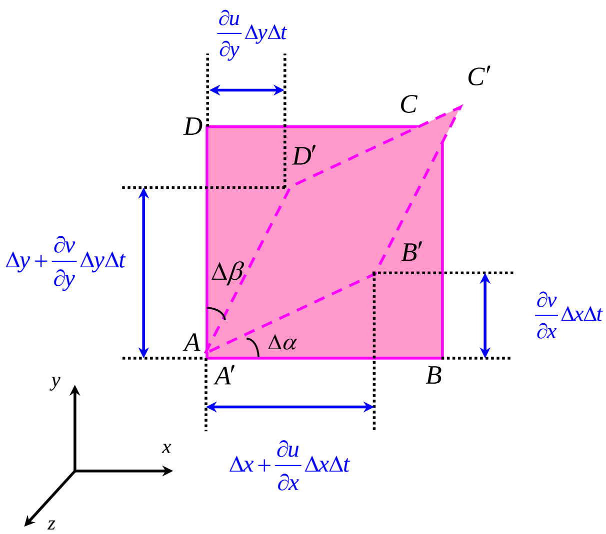
5) 체적변형률(Volumetric strain rate)

$$\frac{\Delta\cancel{V}}{\Delta x\Delta y\Delta z\Delta t}=\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=div(\vec{V})=\nabla\bullet\vec{V}$$

6) 비압축성유체의 체적변형률: $\cancel{V}=\rho m$

$$\Delta\cancel{V}=0\ \ \rightarrow\ \ \nabla\bullet\vec{V}=0$$

2.6.3 전단변형 (Shear Strain)



1) 전단변형(Shear strain): 수직변형과 각변형을 수반하는 운동 - 각변형.

2) 전단(각)변형률(Shear strain rate): 처음에 수직으로 교차하는 두 직선간의 각도의 감소율의 반(1/2)

$$\varepsilon_{xy}=\frac{1}{2}\frac{\tan(\angle DAB-\angle D'A'B')}{\Delta t}=\frac{\tan(\pi/2-\angle D'A'B')}{\Delta t}$$

$$=\frac{1}{2}\frac{\tan(\Delta\alpha+\Delta\beta)}{\Delta t}=\frac{1}{\Delta t}\frac{1}{2}\left(\frac{\tan\Delta\alpha+\tan\Delta\beta}{1-\tan\Delta\alpha\tan\Delta\beta)}\right)$$

$$\approx\frac{1}{2}\frac{\tan\Delta\alpha+\tan\Delta\beta}{\Delta t}$$

where,

$$\tan\Delta\alpha=\frac{\partial v/\partial x\Delta x\Delta t}{\Delta x(1+\partial u/\partial x\Delta t)}=\frac{\partial v}{\partial x}\Delta t$$

$$\tan\Delta\beta=\frac{\partial u/\partial y\Delta y\Delta t}{\Delta y(1+\partial v/\partial y\Delta t)}=\frac{\partial u}{\partial y}\Delta t$$

$$\therefore\ \ \varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=\varepsilon_{yx},\ \ \varepsilon_{yz}=\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)=\varepsilon_{zy},\ \ \varepsilon_{zx}=\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=\varepsilon_{xz}$$

2.6.3 Shear Strain Rate Tensor

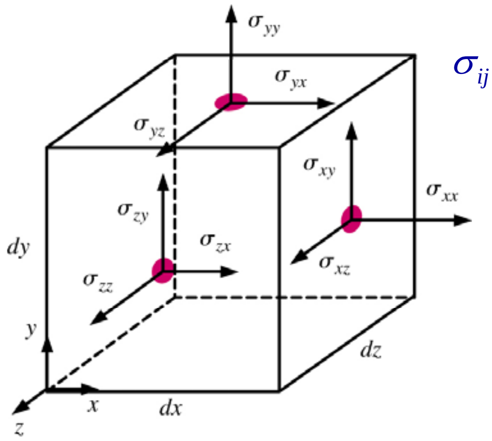
1) 텐서량(Tensor Quantities):

① 0계텐서(0th order tensor)–스칼라(Scalar)량(n^0): 크기만으로 정의되는 물리량 - Speed

② 1계텐서(1st order tensor)–벡터(Vector)량(n^1): 크기와 방향으로 정의되는 물리량 - Velocity

③ 2계텐서(2^{nd} order tensor)–텐서(Tensor)량(n^2): 크기/방향/작용면으로 정의되는 물리량 - Stress

2) 응력텐서(Stress Tensor): 선형/전단 변형률 텐서(Shear strain rate tensor)와의 관계:

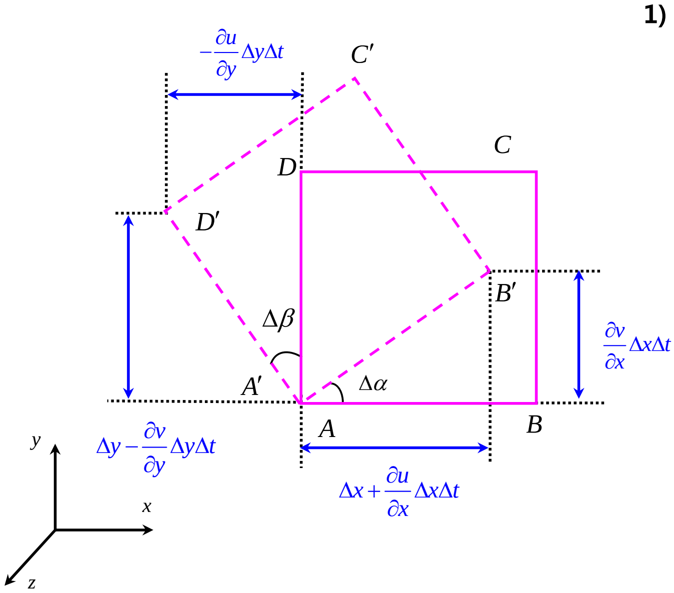


$$\sigma_{ij} \propto \varepsilon_{ij}$$

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right) & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

2.6.4 회전 (Rotation)



1) 각속도: 오른손 좌표계에서 두 직교선의 반시계 방향의 평균속도를 양으로 표시

$$\omega_z = \frac{1}{2}\left(\frac{\Delta\alpha}{\Delta t}+\frac{\Delta\beta}{\Delta t}\right)$$

$$\tan\Delta\alpha = \frac{\partial v/\partial x\Delta x\Delta t}{(\Delta x+\partial u/\partial x\Delta x\Delta t)} \simeq \frac{\partial v}{\partial x}\Delta t$$

$$\tan\Delta\beta = \frac{-\partial u/\partial y\Delta y\Delta t}{(\Delta y-\partial v/\partial y\Delta y\Delta t)} \simeq -\frac{\partial u}{\partial y}\Delta t$$

$$\frac{\partial u}{\partial x}\Delta x\Delta t \ll 1,\ \frac{\partial v}{\partial y}\Delta y\Delta t \ll 1$$

$$\tan\Delta\alpha \cong \Delta\alpha, \tan\Delta\beta \cong \Delta\beta$$

$$\frac{\Delta\alpha}{\Delta t}=\frac{\partial v}{\partial x},\ \frac{\Delta\beta}{\Delta t}=\frac{\partial u}{\partial y}$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\quad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

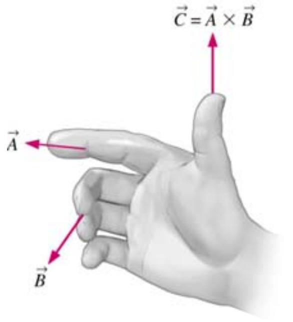
$$\therefore \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

2.7 Vorticity

1) 각속도 $\vec{\omega} = \omega_x i + \omega_y j + \omega_z k$

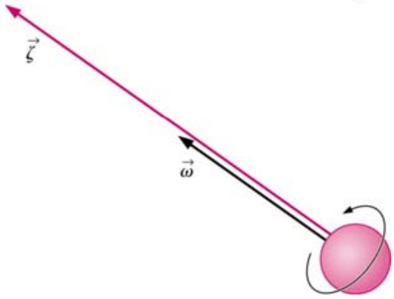
$$= \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k\right]$$

$$= \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\mathrm{curl}(\vec{V})$$



2) 와도(Vorticity): 유체 입자의 각속도의 2배

$$\vec{\zeta} = \vec{\nabla} \times \vec{V} = 2\vec{\omega}$$



3) 비회전 유동(Irrotational Flow): $\vec{\zeta} = 0$

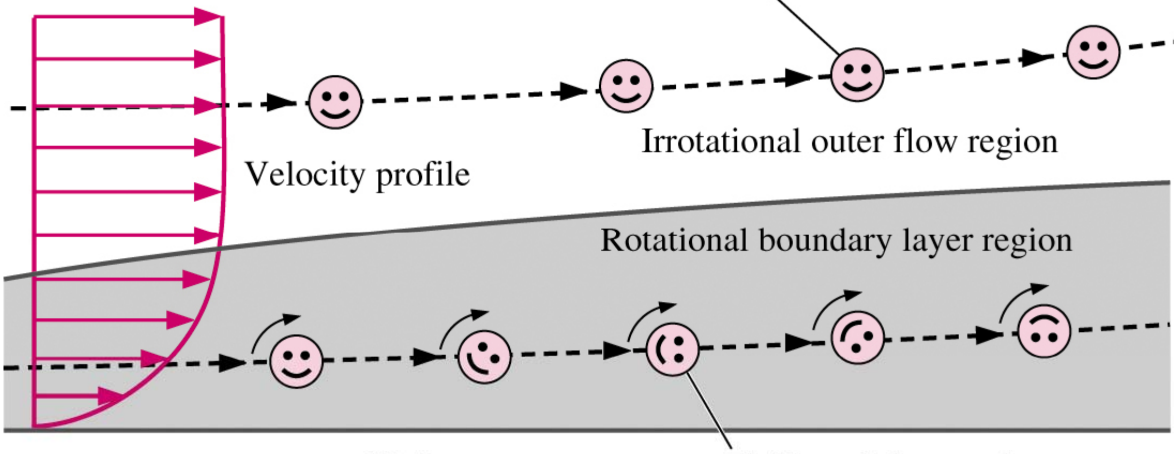
4) 회전 유동(Rotational Flow):

$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

$$\vec{\zeta} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z$$

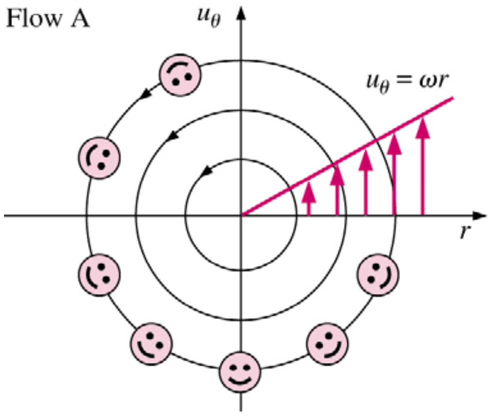
2.7.1 회전과 와도 (Vorticity)

1) 고체 벽면 근처 점성 경계층 내의 유체입자는 회전하지만 경계층 외부의 유체요소는 회전하지 않는다(와도가 영이다)



2.8 Circular Flows

A: 강체회전(Solid-body rotation)

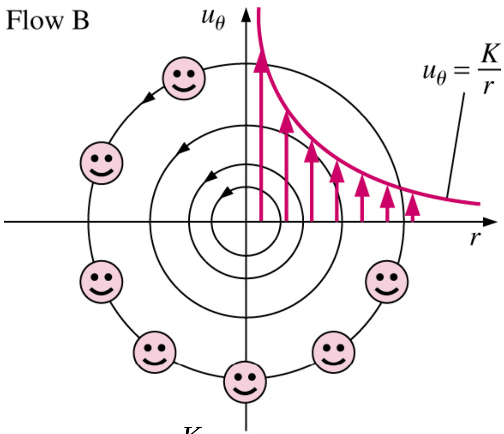


$$u_r = 0, \quad u_\theta = \omega r$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(\omega r^2)}{\partial r} - 0\right)\vec{e}_z = 2\omega\vec{e}_z$$

Rotational Flow

B: 선 와류(Line vortex)



$$u_r = 0, \quad u_\theta = \frac{K}{r}$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(K)}{\partial r} - 0\right)\vec{e}_z = 0\vec{e}_z$$

Irrotational Flow

3. Conservation Laws

3.1 Conservation Laws

1) **Conservation of Mass:** mass of a system remains constant

$$\frac{Dm}{Dt} = 0$$

2) **Conservation of Momentum:** momentum of a system remains constant when the net force acting on it is zero

① Conservation of Linear Momentum

$$\sum F = \frac{D(m\vec{V})}{Dt}$$

② Conservation of Angular Momentum

$$\sum M = \frac{D(\vec{r} \times m\vec{V})}{Dt}$$

3) **Conservation of Energy:** net energy transfer to or from a system during a process be equal to the change in the energy content of the system

$$\frac{DE}{Dt} = \frac{\partial Q}{\partial t} + \frac{\partial W}{\partial t}$$

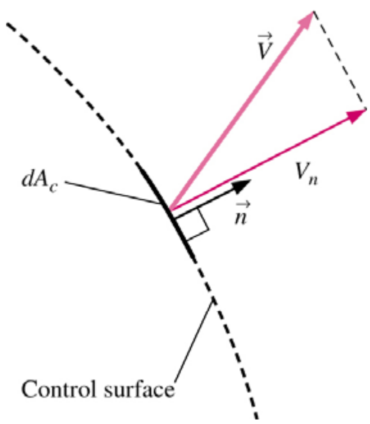
3.2 Conservation of Mass

1) Conservation of mass: Mass, like energy, is a conserved property, and it cannot be created or destroyed during a process.

2) **Mass flow rate:** amount of mass flowing through a cross section per unit time

$$\delta \dot{m} = \rho \vec{V}_n dA_c$$

$$\dot{m} = \int_{A_c} \delta \dot{m} = \int_{A_c} \rho \vec{V}_n dA_c = \rho \vec{V}_{avg} A_c$$



3) **Volume flow rate:** volume of the fluid flowing through a cross section per unit time

$$\dot{\mathcal{V}} = \int_{A_c} \vec{V}_n dA_c = \vec{V}_{avg} A_c \qquad \dot{m} = \rho \dot{\mathcal{V}} = \frac{\dot{\mathcal{V}}}{\mathcal{V}_s} \qquad (\mathcal{V}_s = \frac{1}{\rho})$$

4) The net mass transfer to or from a control volume during a time interval is equal to the net change(increase or decrease) in the total mass within the control volume during the time interval

$$\Delta m_{CV} = m_{in} - m_{out} \quad \rightarrow \quad \frac{dm_{CV}}{dt} = \dot{m}_{in} - \dot{m}_{out}$$

3.2 Conservation of Mass

5) **Total mass within the CV:** $m_{CV} = \int_{CV} \rho \, d\cancel{V}$

6) **Rate of change of mass within the CV:**

$$\frac{dm_{CV}}{dt} = \frac{d}{dt}\int_{CV} \rho d\cancel{V}$$

7) **Net mass flow rate:**

$$\dot{m}_{net} = \int_{CS} \delta\dot{m} = \int_{CS} \rho\vec{V}_n dA$$

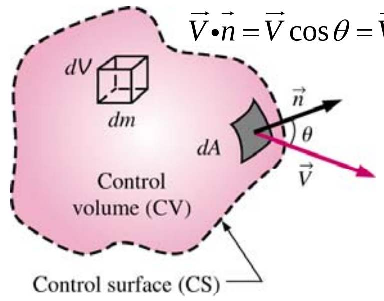
8) **General conservation of mass:**

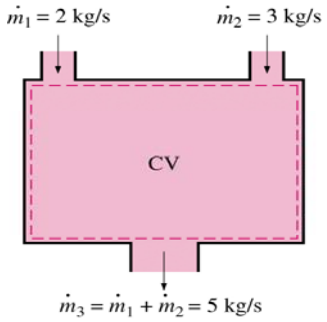
$$\frac{Dm}{Dt} = \frac{d}{dt}\int_{CV} \rho d\cancel{V} + \int_{CS} \rho\vec{V}_n dA = 0$$

9) **In case of the steady flow:**

$$\dot{m}_1 - \dot{m}_2 \quad \rightarrow \quad \rho_1\vec{V}_1 A_1 = \rho_2\vec{V}_2 A_2$$

$$\vec{V}\bullet\vec{n} = \vec{V}\cos\theta = \vec{V}_n$$





$$\sum_{in} V_n A_n = \sum_{out} V_n A_n$$

Ex) Conservation of Energy

1) **Obtain relations for the conservation of energy of the ball for the cases of frictionless and actual motions**

2) **The general energy balance for any system:**

$$\Delta E_{system} = E_{in} - E_{out}$$

3) **The energy balance for the ball for a process from 1 to 2:**

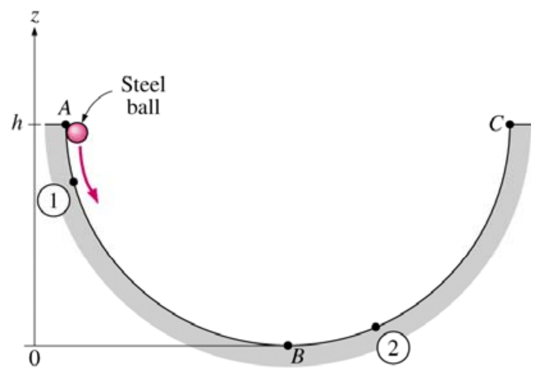
$$(K_{e2} + P_{e2}) - (K_{e1} + P_{e1}) = -W_{friction}$$

$$\frac{V_1^2}{2} + gz_1 = \frac{V_2^2}{2} + gz_2 + W_{friction}$$

4) **For the frictionless motion:**

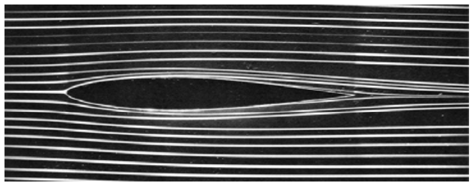
$$\frac{V_1^2}{2} + gz_1 = \frac{V_2^2}{2} + gz_2 \quad \rightarrow \quad \frac{V^2}{2} + gz = C = \text{constant}$$

If the frictional effects are negligible, the sum of the Kinetic & Potential energies of the ball remains constant.



3.3 Bernoulli Equation

1) Bernoulli Equation: relation between pressure and velocity and elevation in regions of steady, incompressible and inviscid flow.



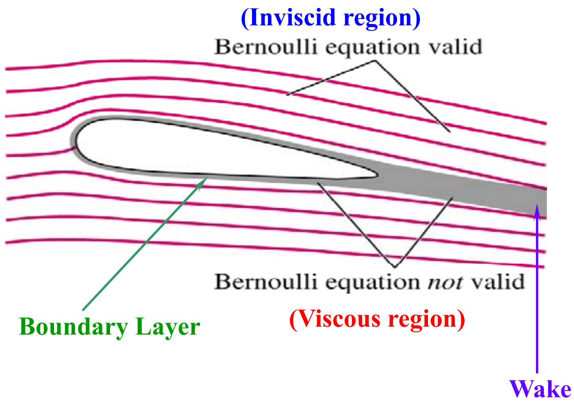
2) Viscous effects are negligibly small compared to inertial, gravitational and pressure effects.

3) Conservation of linear momentum principle.

4) When the flow is steady, all particles that pass through the same point follow the same path, and the velocity vectors remain tangent to the path at every point.

5) Bernoulli Equation [J]: Sum of the kinetic, potential, and flow energies of a fluid particle is constant along a streamline of steady, incompressible and inviscid flow.

$$\frac{P_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{P_2}{\rho}+\frac{V_2^2}{2}+gz_2$$



3.3.1 Derivation of the Bernoulli Equation

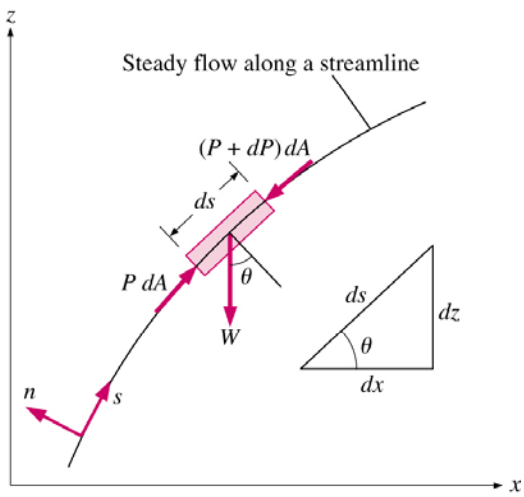
1) Conservation of Linear Momentum:

$$\sum F_s = ma_s$$

2) Acceleration along a streamline (s-direction):

$$d\vec{V}=\frac{\partial \vec{V}}{\partial s}ds+\frac{\partial \vec{V}}{\partial t}dt \;\rightarrow\; \frac{d\vec{V}}{dt}=\frac{\partial \vec{V}}{\partial s}\frac{ds}{dt}+\underbrace{\cancel{\frac{\partial \vec{V}}{\partial t}}}_{\text{steady}}$$

$$\therefore a_s=\frac{d\vec{V}}{dt}=\frac{\partial \vec{V}}{\partial s}\frac{ds}{dt}=\frac{\partial \vec{V}}{\partial s}\vec{V}=\vec{V}\frac{d\vec{V}}{ds}$$



3) Forces acting along a streamline:

$$\sum F_s = PdA-(P+dP)dA-W\sin\theta$$

4) $$\therefore -dPdA-\rho gdAds\frac{dz}{ds}=\rho dAds\left(V\frac{dV}{ds}\right)$$

$$-dP-\rho gdz=\rho VdV$$

$$\frac{dP}{\rho}+\frac{1}{2}d(V^2)+gdz=0 \;\rightarrow\; \frac{1}{\rho}\int dP+\frac{1}{2}\int dV^2+g\int dz=C$$

$$\boxed{\frac{P}{\rho}+\frac{V^2}{2}+gz=C}$$

3.3.2 Euler & Bernoulli Equation

1) The Bernoulli equation was first stated in words by the Swiss mathematician Daniel Bernoulli (1700~1782) in a text written in 1738. It was later derived in general equation from by Leonhard Euler (1707~1783) in 1755.

2) The Bernoulli equation is derived assuming <u>incompressible & inviscid flow</u>, and thus is should not be used fro flows with significant compressibility effects.

3) Unsteady, Compressible Flow:

$$\int \frac{dP}{\rho} + \int \frac{\partial V}{\partial t} ds + \frac{V^2}{2} + gz = C$$

4) Steady, Compressible Flow:

$$\int \frac{dP}{\rho} + \frac{V^2}{2} + gz = C$$

5) Steady, Incompressible Flow:

$$\boxed{\frac{P}{\rho} + \frac{V^2}{2} + gz = C \quad [\mathrm{J}]}$$

The sum of the kinetic, potential, and flow energies of a fluid particle is constant ***<u>along a streamline</u>*** during steady flow

3.3.3 Static, Dynamic & Stagnation Pressures

1) Ke and Pe can be converted to flow energy (vice versa) during flow:

$$\boxed{P + \rho\frac{V^2}{2} + \rho gz = C \quad [\mathrm{Pa}]}$$

① P : Static pressure(정압)

② $\rho\frac{V^2}{2}$: Dynamic pressure(동압)

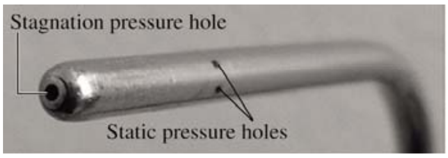
③ ρgz : Hydrostatic pressure(정수압)

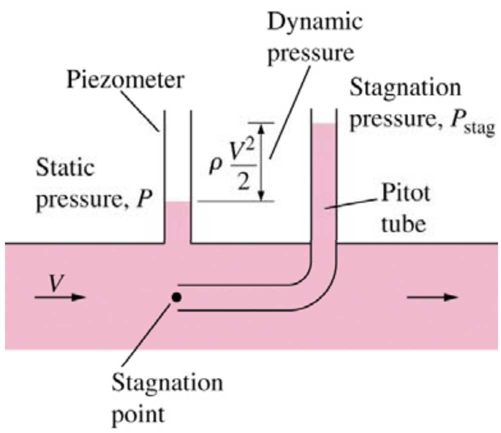
2) Bernoulli equation [Pa]: total pressure (static + dynamic + hydrostatic pressures) along a streamline is constant along a streamline of steady, incompressible and inviscid flow.

3) Stagnation pressure(정체압)= static pressure(정압) + dynamic pressures(동압)

$$P_{stag} = P + \rho\frac{V^2}{2}$$

4) Measuring apparatus: Pitot tube, Piezometer, U type manometer, Pitot-static probe etc.





$$V = \sqrt{\frac{2(P_{stag} - P)}{\rho}}$$

3.3.4 Hydraulic Grad Line (HGL)

1) Bernoulli equation: expresed in terms of heads (level of mechanical energy using heights)
   ① $\frac{P}{\rho g}$: Pressure head (압력수두)
   ② $\frac{V^2}{2g}$: Velocity head (속도수두)
   ③ z : Elevation head (위치수두)
   ④ H : Total head (전체 수두)

$$\boxed{\frac{P}{\rho g}+\frac{V^2}{2g}+z=H \quad [\mathrm{m}]}$$

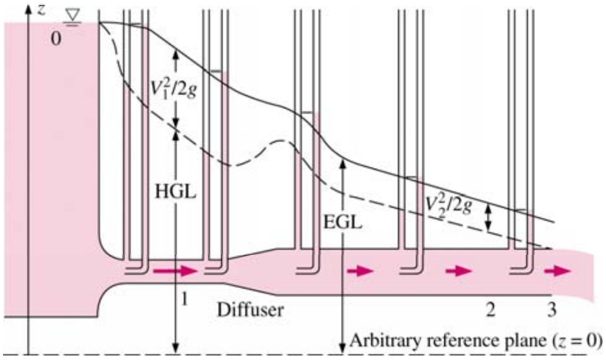
2) Bernoulli equation [H]: Sum of the pressure, velocity and elevation heads along a stream is constant along a streamline of steady, incompressible and inviscid flow.

3) Hydraulic Grade Line (수력 구배선): line that represents the sum of the static pressure and the elevation heads

$$\frac{P}{\rho g}+z \;\rightarrow\; \text{HGL}$$

4) Energy Grade Line (에너지 구배선): line that represents the total heads of the fluid

$$\frac{P}{\rho g}+\frac{V^2}{2g}+z \;\rightarrow\; \text{EGL}$$



3.3.5 Application of the Bernoulli eqn.

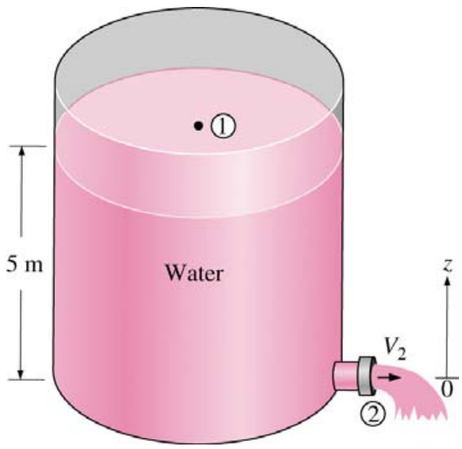
1) Determine the water velocity at the outlet
2) Assumptions:
   ① The flow is incompressible & irrotational
   ② The water drains slowly enough that the flow can be approximated as steady (actually quasi-steady)
3) The Bernoulli equation:

$$\frac{P_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{P_2}{\rho g}+\frac{V_2^2}{2g}+z_2$$

$$z_1=\frac{V_2^2}{2g}$$

4) Solving for outlet velocity: (Toricellie equation)

$$V_2=\sqrt{2gz_1}=\sqrt{2(9.81\times 5)}=9.9\ m/s$$

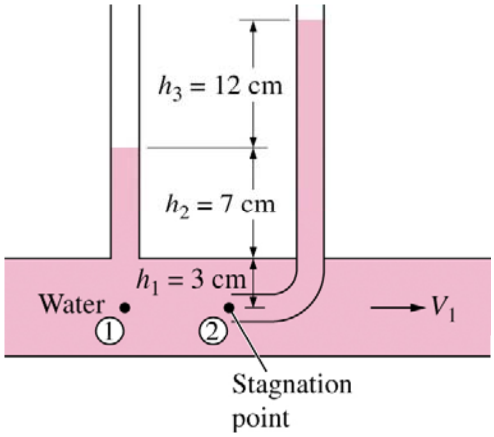


3.3.5 Application of the Bernoulli eqn.

1) Determine the water velocity at the center of the pipe.
2) Assumptions:
   ① The flow is steady & incompressible.
   ② Points 1 & 2 are close enough together that the irreversible energy loss is negligible.
3) Gage pressures at points 1 & 2:

$$P_1 = \rho g(h_1 + h_2)$$
$$P_2 = \rho g(h_1 + h_2 + h_3)$$

4) The Bernoulli equation:



$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + \cancel{z_1} = \frac{P_2}{\rho g} + \cancel{\frac{V_2^2}{2g}} + \cancel{z_2}$$

$$\frac{V_1^2}{2g} = \frac{P_2 - P_1}{\rho g} = \frac{\rho g(h_1 + h_2 + h_3) - \rho g(h_1 + h_2)}{\rho g} = h_3 \qquad \therefore V_1 = \sqrt{2gh_3} = 1.53 m/s$$

4. Dimensional Analysis

4.1 Dimensions and Units

1) **Dimension:** Measure of a physical quantity,
e.g., length, mass, time
2) **Units:** Assignment of a number to a dimension
e.g., [m], [kg], [second] MKS
3) **3 & 7 Primary Dimensions:**
4) All non-primary dimensions can be formed by a combination of the 7 primary dimensions

①	Mass	m	[kg]	3 Primary Dimensions (MKS)
②	Length	L	[m]	
③	Time	t	[sec]	
④	Temperature	T	[K]	
⑤	Current	I	[A]	
⑥	Amount of Light	C	[cd]	
⑦	Amount of matter	N	[mol]	

4.2 Dimensional Homogeneity

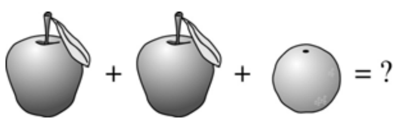
1) Law of **Dimensional Homogeneity**: **every additive term in an equation must have the same dimensions**
2) Example: Bernoulli equation

$$p + \frac{1}{2}\rho V^2 + \rho g z = C$$

① $[p]$ = [force/area]=[mass x length/time^2 x 1/length^2] = $[m/(t^2L)]$

② $[1/2\rho V^2]$ = [mass/length^3 x (length/time)^2] = $[m/(t^2L)]$

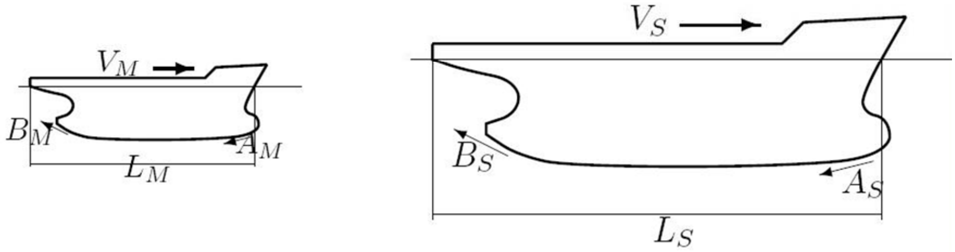
③ $[\rho g z]$ = [mass/length^3 x length/time^2 x length] = $[m/(t^2L)]$



You can not add apples and oranges !

4.3 Principle of Similarity

1) **Geometric Similarity:** the model must be the same shape as the prototype. Each dimension must be scaled by the same factor.
2) **Kinematic Similarity:** velocity as any point in the model must be proportional
3) **Dynamic Similarity:** all forces in the model flow scale by a constant factor to corresponding forces in the prototype flow.
4) Complete Similarity is achieved only if all three above conditions are met.



Geometric similarity

Kinematic similarity

Dynamic similarity

4.4 Dimension of the force

1) Consider all forces acting on a body in fluid flows

$$\sum F = ma \quad \rightarrow \quad F_G + F_P + F_V + F_T + F_E = F_I$$

$$\text{Inertial force } (F_I) = ma = \rho L^3 V^2 / L = \rho V^2 L^2$$

$$\text{Gravitational force } (F_G) = mg = \rho L^3 g$$

$$\text{Pressure force } (F_P) = PA = \rho V^2 A = \rho V^2 L^2$$

$$\text{Viscous force } (F_V) = \sigma A = \mu V / L \, A = \mu V L$$

$$\text{Surface tension force } (F_T) = \sigma L = \rho V^2 L$$

$$\text{Elastic force } (F_E) = \kappa A = \kappa L^2$$

4.5 Nondimensional parameters

1) Common established nondimensional parameters:

Froude number $\left[\dfrac{\text{Intertial force}}{\text{Gravitational force}}\right]$ $F_r = \left[\dfrac{F_I}{F_G}\right] = \left[\dfrac{\rho V^2 L^2}{\rho g L^3}\right] = \left[\dfrac{V^2}{gL}\right] = \left[\dfrac{V}{\sqrt{gL}}\right]$

Reynolds number $\left[\dfrac{\text{Intertial force}}{\text{Viscous force}}\right]$ $R_e = \left[\dfrac{F_I}{F_V}\right] = \left[\dfrac{\rho V^2 L^2}{\mu V L}\right] = \left[\dfrac{\rho V L}{\mu}\right] = \left[\dfrac{VL}{\upsilon}\right]$

Euler number $\left[\dfrac{\text{Pressure force}}{\text{Inertial force}}\right]$ $E_u = \left[\dfrac{F_P}{F_I}\right] = \left[\dfrac{PL^2}{\rho V^2 L^2}\right] = \left[\dfrac{P}{\rho V^2}\right]$

Weber number $\left[\dfrac{\text{Intertial force}}{\text{Surface tension force}}\right]$ $W_e = \left[\dfrac{F_I}{F_T}\right] = \left[\dfrac{\rho V^2 L^2}{\sigma L}\right] = \left[\dfrac{\rho V^2 L}{\sigma}\right]$

Mach number $\left[\dfrac{\text{Intertial force}}{\text{Elastic force}}\right]$ $M_a = \left[\dfrac{F_I}{F_E}\right] = \left[\dfrac{\rho V^2 L^2}{\kappa L^2}\right] = \left[\dfrac{V^2}{C^2}\right] = \left[\dfrac{V}{C}\right]$

where κ is a bulk modulus of elasticity $\left[\kappa = c^2 \rho\right]$

4.6 Nondimensional parameters

2) Common established nondimensional parameters:

Cavitation number $\left[\dfrac{\text{Pressure-Vapor P}}{\text{Dynamic pressure}}\right]$ $\sigma_c = \left[\dfrac{P - P_v}{1/2\,\rho V^2}\right]$

Strouhal number $\left[\dfrac{\text{Characteristic flow time}}{\text{Period of oscillation}}\right]$ $S_t = \left[\dfrac{fL}{V}\right]$

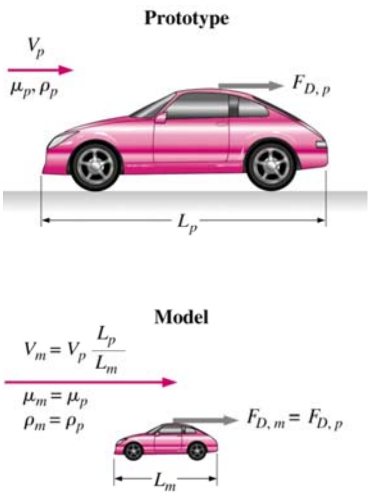
Lift coefficient $\left[\dfrac{\text{Lift force}}{\text{Dynamic force}}\right]$ $C_L = \left[\dfrac{F_L}{1/2\,\rho V^2 A}\right]$

Drag coefficient $\left[\dfrac{\text{Drag force}}{\text{Dynamic force}}\right]$ $C_D = \left[\dfrac{F_D}{1/2\,\rho V^2 A}\right]$

Pressure coefficient $\left[\dfrac{\text{Static pressure diff.}}{\text{Dynamic pressure}}\right]$ $C_p = \left[\dfrac{P - P_\infty}{1/2\,\rho V^2}\right]$

Friction coefficient $\left[\dfrac{\text{Wall friction force}}{\text{Dynamic pressure}}\right]$ $C_f = \left[\dfrac{F_f}{1/2\,\rho V^2 A}\right]$

4.7 Principle of Similarity



Aerodynamic drag force on the prototype should be equal to that on the scale model !

$$[R_e]_P = \left[\frac{F_I}{F_V}\right]_P = \left[\frac{VL}{\upsilon}\right]_P$$

$$\Downarrow$$

$$\left[\frac{V_P L_P}{\upsilon_P}\right] = \left[\frac{V_M L_M}{\upsilon_M}\right] \Rightarrow V_M = V_P \frac{L_P}{L_M}$$

$$\Uparrow$$

$$[R_e]_M = \left[\frac{F_I}{F_V}\right]_M = \left[\frac{VL}{\upsilon}\right]_M$$

4.7 Principle of Similarity

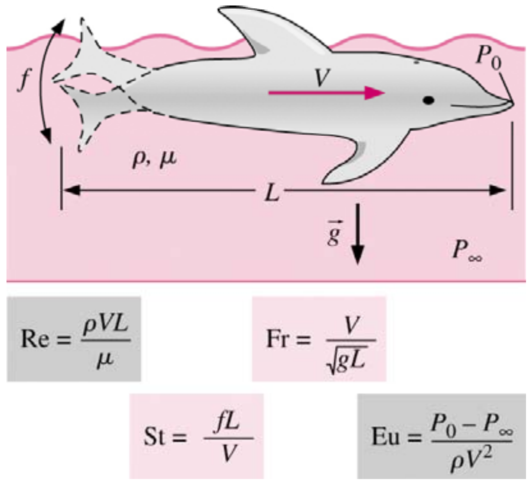
Is it possible for us to do experiments in conditions satisfying all these parameters at once?

$$[F_r]_m = [F_r]_p$$

$$[R_e]_m = [R_e]_p$$

$$[S_t]_m = [S_t]_p$$

$$[E_u]_m = [E_u]_p$$



Dimensionless parameters which appear in an unsteady fluid flow problem with a free surface

4.8 Buckingham Pi(Π) theorem

1) Buckingham Π theorem: when number of physical parameters is { *n* } and number of primary dimensions is { *j* }, then number of independent nondimensional parameters Π is { *k* } = { *n* } - { *j* }.

$$n_1 = f(n_2, n_3, \cdots, n_n) \qquad g(\Pi_1, \Pi_2, \Pi_3, \cdots, \Pi_{n-j}) = 0$$

$$g(n_1, n_2, n_3, \cdots, n_n) = 0 \quad \Rightarrow \quad \Pi_1 = f(\Pi_2, \Pi_3, \cdots, \Pi_{n-j})$$

2) Consider car experiment

① Drag force:

$$F_D = f(V, \rho, \mu, L) \;\rightarrow\; g(F_D, V, \rho, \mu, L) = 0$$

② Nondimensional parameters

$$k = n - j \quad \rightarrow 2 = 5 - 3$$

$$g(\Pi_1, \Pi_2) = 0 \;\rightarrow\; \Pi_1 = f(\Pi_2) \qquad \Rightarrow \qquad C_D = \{R_e\}$$

4.9 Method of Repeating Variables

1) Nondimensional parameters Π can be generated by several methods.

2) Method of Repeating Variables:

Step 1: List the parameters in the problem and count their total number n.

Step 2: List the primary dimensions of each of the n parameters.

Step 3: Set the reduction j as the number of primary dimensions. Calculate k, the expected number of Π's, *k* = *n* - *j*.

Step 4: Choose j repeating(scaling) parameters.

Step 5: Construct the *k* Π's, and manipulate as necessary.

Step 6: Write the final functional relationship and check algebra.

Example: Drag of a car

Step 1: List relevant parameters:

$$F_D = f(V, \rho, \mu, L) \quad \rightarrow \quad n = 5$$

Step 2: Primary dimensions of each parameter:

$$F_D\{m^1L^1t^{-2}\} \quad V\{L^1t^{-1}\} \quad L\{L^1\}$$

$$\rho\{m^1L^3\} \quad \mu\{m^1L^{-1}t^{-1}\}$$

Step 3: Primary parameters: Number of Π:

$$(m, L, t) \quad \rightarrow \quad k = n - j = 5 - 3 = 2$$

$$g(\Pi_1, \Pi_2) = 0$$

Step 4: Repeating parameters:

$$V, \; L, \; \rho$$

Why these parameter?
A wise choice for most fluid flow problems!

Example: Drag of a car

Step 5: Finding 2 nondimensional parameters: $\Pi_1 = f(\Pi_2)$

① **Dependent Π_1 :**

$$\Pi_1 = F_D V^a L_C^b \rho^c \quad \rightarrow \quad \{\Pi_1\} = \{(m^1L^1t^{-2})(L^1t^{-1})^a(L^1)^b(m^1L^{-3})^c\}$$

$$\Pi_1 = F_D V^a L_C^b \rho^c = \frac{F_L}{\rho V^2 L_C^2}$$

$$\Pi_{1-\text{modified}} = \frac{F_D}{\frac{1}{2}\rho V^2 A} = C_D$$

$$\begin{aligned} m&: & 1 + c &= 0 & \rightarrow \quad c &= -1 \\ L&: & 1 + a + b - 3c &= 0 & \rightarrow \quad b &= -2 \\ t&: & -2 - a &= 0 & \rightarrow \quad a &= -2 \end{aligned}$$

② **Independent Π_2:**

$$\Pi_2 = \mu V^a L^b \rho^c \quad \rightarrow \quad \{\Pi_2\} = \{(m^1L^{-1}t^{-1})(L^1t^{-1})^a(L^1)^b(m^1L^{-3})^c\}$$

$$\Pi_2 = \mu V^a L^b \rho^c = \frac{\mu}{\rho V L}$$

$$\Pi_{2-\text{modified}} = \frac{\rho V L}{\mu} = R_e$$

$$\begin{aligned} m&: & 1 + c &= 0 & \rightarrow \quad c &= -1 \\ L&: & -1 + a + b - 3c &= 0 & \rightarrow \quad b &= -1 \\ t&: & -1 - a &= 0 & \rightarrow \quad a &= -1 \end{aligned}$$

Example: Drag of a car

Step 6: Final functional relationship:

$$\Pi_1 = f(\Pi_2)$$

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = f(R_e)$$

Ex: How much drag force is acting on the car racing with 25.0m/s?

$L_p = 5m, \ \ A_p = 4m^2$ (frontal area)

$V_p = 25.0 \, m/s$

Scale ratio $= 1/5 \ \rightarrow \ L_m = 1m$

$$\left[\frac{V_P L_P}{\upsilon_P}\right] = \left[\frac{V_M L_M}{\upsilon_M}\right] \qquad V_M = V_P \frac{L_P}{L_M} = 25 \times 5 = 125[\text{m/s}]$$

Example: Drag of a car

Model test (wind tunnel):

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = 0.25$$

Prediction of Drag force of the prototype

$$[C_D]_m = [C_D]_p$$

$$[C_D]_p = 0.25 = \frac{F_D}{\frac{1}{2}\rho V^2 A}$$

$$F_D = C_D \times (\tfrac{1}{2}\rho V^2 A)$$

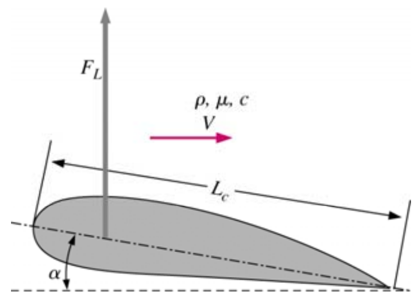
$$= 0.25 \times (0.5 \times 1.0[\text{kg/m}^3] \times 25^2[\text{m}^2/\text{s}^2] \times 4[\text{m}^2])$$

$$= 312.5[\text{kg m/s}^2] = 312.5[\text{N}]$$

Example: Lift on a Wing

Step 1: List relevant parameters:

$$F_L = f(V, L_C, \rho, \mu, c, \alpha) \quad \rightarrow \quad n = 7$$



Step 2: Primary dimensions of each parameter:

$$F_L\{m^1 L^1 t^{-2}\} \quad V\{L^1 t^{-1}\} \quad L_C\{L^1\}$$

$$\rho\{m^1 L^3\} \quad \mu\{m^1 L^{-1} t^{-1}\} \quad c\{L^1 t^{-1}\} \quad \alpha\{1\}$$

Step 3: Primary parameters: Number of Π:

$$g(\Pi_1, \Pi_2, \Pi_3, \Pi_4) = 0$$

$$(m, L, t) \quad \rightarrow \quad k = n - j = 7 - 3 = 4$$

Step 4: Repeating parameters:

$$V, \ L_C, \ \rho$$

Example: Lift on a Wing

Step 5: Finding 4 nondimensional parameters:

$$\Pi_1 = f(\Pi_2)$$

① **Dependent Π_1 :**

$$\Pi_1 = F_L V^a L^b \rho^c \quad \rightarrow \quad \{\Pi_1\} = \{(m^1 L^1 t^{-2})(L^1 t^{-1})^a (L^1)^b (m^1 L^{-3})^c\}$$

$$\Pi_1 = F_L V^a L^b \rho^c = \frac{F_L}{\rho V^2 L^2}$$

$$\Pi_{1-\text{modified}} = \frac{F_L}{\frac{1}{2}\rho V^2 A} = C_L$$

$$m: \quad 1 + c = 0 \quad \rightarrow \quad c = -1$$

$$L: \quad 1 + a + b - 3c = 0 \quad \rightarrow \quad b = -2$$

$$t: \quad -2 - a = 0 \quad \rightarrow \quad a = -2$$

② **First independent Π_2:**

$$\Pi_2 = \mu V^a L_C^b \rho^c \quad \rightarrow \quad \{\Pi_2\} = \{(m^1 L^{-1} t^{-1})(L^1 t^{-1})^a (L^1)^b (m^1 L^{-3})^c\}$$

$$\Pi_2 = \mu V^a L_C^b \rho^c = \frac{\mu}{\rho V L_C}$$

$$\Pi_{2-\text{modified}} = \frac{\rho V L_C}{\mu} = R_e$$

$$m: \quad 1 + c = 0 \quad \rightarrow \quad c = -1$$

$$L: \quad 1 + a + b - 3c = 0 \quad \rightarrow \quad b = -1$$

$$t: \quad -1 - a = 0 \quad \rightarrow \quad a = -1$$

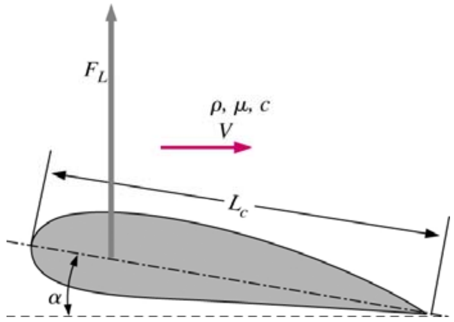
③ **Other independent Π_3 , Π_4:**

$$\Pi_3 = \frac{V}{c} = M_a \qquad \Pi_4 = \alpha = \text{Angle of attack}$$

Example: Lift on a Wing

Step 6: Final functional relationship:

$$C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A} = f(R_e, M_a, \alpha)$$



Ex7-8:

$\Delta P_{\text{tunnel}} = 5atm$

$L_{C\,\text{m}} = 1.12m, \quad A = 10.7m^2$

$V_p = 52.0\, {}^{m}\!/_{s}, \quad T = 25^\circ C$

$\text{Scale ratio} = 10.0 \quad \rightarrow \quad L_p = 1.12 \times 10m$

① Match Reynolds number:

$$[R_e]_m = [R_e]_p \quad \rightarrow \quad \left[\rho VL/\mu\right]_m = \left[\rho VL/\mu\right]_p$$

$$V_m = V_p \frac{L_p}{L_m} = 52.0 \times 10\, {}^{m}\!/_{s}$$

$$[M_a]_p = V_p / c = 52.0/346 = 0.15 \quad \rightarrow \text{subsonic}$$

$$[M_a]_m = V_m / c = 520.0/346 = 1.50 \rightarrow \text{supersonic}$$

② Match Mach number:

$$[M_a]_m = [M_a]_p \quad \rightarrow \quad \left[V/c\right]_m = \left[V/c\right]_p$$

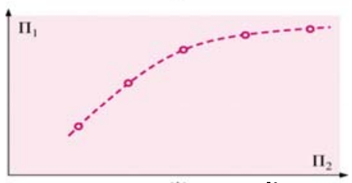
$$V_m = V_p, \quad \frac{L_p}{L_m} = 10$$

$$\left.\begin{aligned} [R_e]_p &= 52.0 \times 11.2 \times \rho_p / \mu_p \\ [R_e]_m &= 52.0 \times 1.12 \times \rho_m / \mu_m \end{aligned}\right\} [R_e]_m \ll \tfrac{1}{10}[R_e]_p$$

4.10 Exp. Testing and Incomplete Similarity

1) One of the most useful applications of dimensional analysis is in designing physical experiments, and in reporting the results.
2) Setup of an experiment and correlation of data.

(a) Linear



(b) Non-linear

① Consider a problem with 5 parameters: one dependent and 4 independent.

② Full test matrix with 5 data points for each independent parameter would require 5^4=625 experiments

③ If we can reduce to 2 Π's, the number of independent parameters is reduced from 4 to 1, which results in 5^1=5 experiments

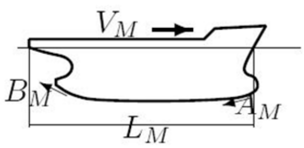
3) **Incomplete Similarity:** not always possible to match all the Πs of a model to that of the prototype. One is able to extrapolate model tests to obtain reasonable prototype predictions.

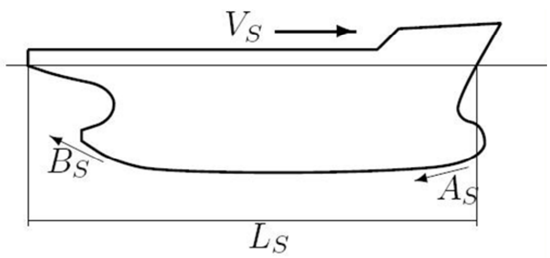
4.10 Exp. Testing and Incomplete Similarity

Similarity can be achieved even when the model fluid is different than the prototype fluid

Submarine model can be tested in a wind tunnel

4.11 Flows with Free Surfaces





4.11 Flows with Free Surfaces

1) List relavant parameters:

$R = f(V, L, \rho, g, \mu) \quad \rightarrow \quad n = 6$

2) Determine primary parameters:

(m, L, t)

3) Determine the number of nondimensional parameters:

$k = n - j = 6 - 3 = 3$

4) Choose repeating parameters:

V, L, ρ

5) Find two parameters:

$g(\Pi_1, \Pi_2, \Pi_3) = 0 \quad \rightarrow \quad \Pi_1 = f(\Pi_2, \Pi_3)$

$\Pi_1 = RV^a L^b \rho^c$

$\{\Pi_1\} = \{(m^1 L^1 t^{-2})(L^1 t^{-1})^a (L^1)^b (m^1 L^{-3})^c\}$

$$\begin{cases} m: & 1 + c = 0 & \rightarrow & c = -1 \\ L: & 1 + a + b - 3c = 0 & \rightarrow & b = -2 \\ t: & -2 - a = 0 & \rightarrow & a = -2 \end{cases}$$

$$\Pi_1 = RV^a L^b \rho^c = \frac{R}{\rho V^2 L^2}$$

$$\Pi_{1-\text{modified}} = \frac{R}{\frac{1}{2}\rho V^2 A} = C_R$$

$\Pi_2 = gV^a L^b \rho^c$

$\{\Pi_2\} = \{(L^1 t^{-2})(L^1 t^{-1})^a (L^1)^b (m^1 L^{-3})^c\}$

$$\begin{cases} m: & c = 0 & \rightarrow & c = 0 \\ L: & 1 + a + b - 3c = 0 & \rightarrow & b = 1 \\ t: & -2 - a = 0 & \rightarrow & a = -2 \end{cases}$$

$$\Pi_2 = gV^a L^b \rho^c = \frac{gL}{V^2}$$

$$\Pi_{2-\text{modified}} = \frac{V}{\sqrt{gL}} = F_n$$

$\Pi_3 = \mu V^a L^b \rho^c$

$\{\Pi_3\} = \{(m^1 L^{-1} t^{-1})(L^1 t^{-1})^a (L^1)^b (m^1 L^{-3})^c\}$

$$\begin{cases} m: & 1 + c = 0 & \rightarrow & c = -1 \\ L: & 1 + a + b - 3c = 0 & \rightarrow & b = -1 \\ t: & -1 - a = 0 & \rightarrow & a = -1 \end{cases}$$

$$\Pi_3 = \mu V^a L^b \rho^c = \frac{\mu}{\rho VL}$$

$$\Pi_{3-\text{modified}} = \frac{\rho VL}{\mu} = R_e$$

$$\therefore C_R = f(F_n, R_e)$$

4.11 Flows with Free Surfaces

1) For ship hydrodynamics:

$$F_D = f(V, L, \rho, g, \mu)$$

$$C_D = f(R_e, F_r)$$

2) Complete similarity?:

$$[R_e]_m = [R_e]_p \rightarrow \left[\frac{VL}{\upsilon}\right]_m = \left[\frac{VL}{\upsilon}\right]_p \rightarrow V_m = V_p \frac{L_p \upsilon_m}{L_m \upsilon_p}$$

$$[F_r]_m = [F_r]_p \rightarrow \left[\frac{V}{\sqrt{gL}}\right]_m = \left[\frac{V}{\sqrt{gL}}\right]_p \rightarrow V_m = V_p \sqrt{\frac{L_m}{L_p}} = V_p \sqrt{\lambda}$$

3) To match both *Re* and *Fr*, viscosity in the model test is a function of scale ratio! This is not feasible to change the kinematic viscosity each model test.

$$\frac{\upsilon_m}{\upsilon_p} = \left(\frac{L_m}{L_p}\right)^{3/2} \quad \rightarrow \quad \upsilon_m = \upsilon_p \left(\frac{L_m}{L_p}\right)^{3/2}$$

4.11 Flows with Free Surfaces

Resistance Test

$V_M \quad R_{TM}$

2D method	**3D method**
$C_{RM} = C_{RS}$	$C_{WM} = C_{WS}$

$$C_{TM} = \frac{R_{TM}}{\frac{1}{2}\rho_M V_M^2 S_M} \qquad C_{FM} = \frac{0.075}{\left(\log R_{nM} - 2\right)^2}$$

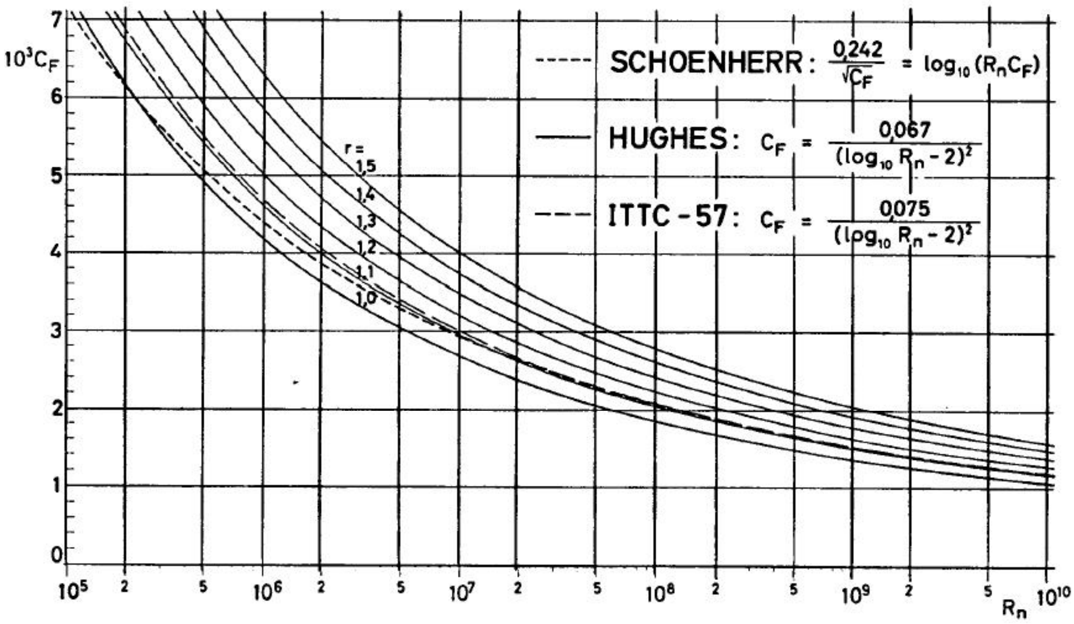
2D method	3D method
$C_{RM} = C_{TM} - C_{FM}$	$C_{WM} = C_{TM} - (1+k)C_{FM}$

$$C_{FS} = \frac{0.075}{\left(\log R_{nS} - 2\right)^2}$$

2D method	3D method
$C_{TS} = C_{RS} + C_{FS} + C_{AS}$	$C_{TS} = C_{WS} + (1+k)C_{FS} + C_{AS} + C_{AAS}$

$$R_{TS} = C_{TS}\left(\frac{1}{2}\rho_S V_S^2 S_S\right)$$

4.11 Flows with Free Surfaces

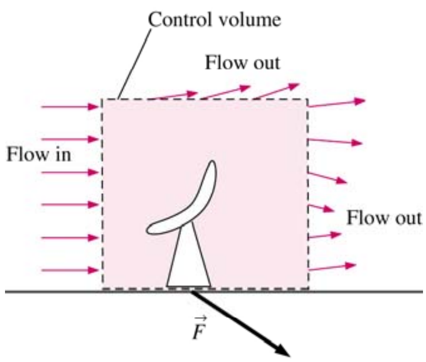


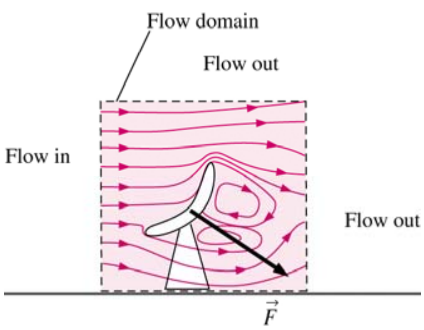
Model-Ship correlation curve

5. Differential Analysis

5.1 Introduction

1) Control Volume Analysis:
   ① Control volume (CV) analysis based on the laws of conservation of mass and energy
   ② Control volume (CV) analysis based on the law of conservation of momentum
2) Integral forms of equations are useful for determining overall effects
3) However, we cannot obtain detailed information on the flow field inside the CV
4) Motivation for Differential Analysis





5.1 Introduction

1) Navier-Stokes equations for incompressible fluid flow

① Conservation of Mass: Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V}\right) = 0 \quad \rightarrow \quad \nabla \cdot \vec{V} = 0$$

② Conservation of Momentum: Momentum Equation

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho\left(\vec{V} \cdot \nabla\right)\vec{V} = -\nabla p + \mu \nabla^2 \vec{V} + \rho \vec{g}$$

2) We will learn:

① Physical meaning of each term

② How to derive

③ How to solve

5.2 Conservation of Mass - Continuity Equation

1) Conservation of Mass

$$\frac{Dm}{Dt} = \int_{CV} \frac{\partial \rho}{\partial t} d\cancel{V} + \int_{CS} \rho \vec{V}_n dA = 0$$

$$\int_{CV} \frac{\partial \rho}{\partial t} d\cancel{V} = \sum_{in} \dot{m} - \sum_{out} \dot{m}$$

2) Two methods to derive differential form of conservation of mass

① Divergence (Gauss's) Theorem: Volume Integral ↔ Surface Integral

$$\int_V \left(\nabla \cdot \vec{f}\right) d\cancel{V} = \int_A \left(\vec{f} \cdot \vec{n}\right) dA$$

② Differential CV & Taylor series expansions

$$\vec{f}(b) = \vec{f}(a) + \sum_{n=1}^{\infty} \frac{(b-a)^n}{n!} \vec{f}^{\,n}(a)$$

5.2.1 Continuity Equation: *Divergence Theorem*

1) Conservation of Mass using Divergence Theorem:

$$\frac{Dm}{Dt} = 0 = \int_{CV} \frac{\partial \rho}{\partial t} d\cancel{V} + \int_{CS} \rho \vec{V}_n dA = \int_{CV} \left[\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \vec{V}) \right] d\cancel{V}$$

2) Integral holds for <u>any CV</u>: General form of the Continuity Equation

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \vec{V}) = 0$$

3) This differential equation is available for all type of flows (Steady, Unsteady, Viscous, Inviscid, Compressible, Incompressible)

① for Steady / Compressible flow:

$$\nabla \cdot \rho \vec{V} = 0$$

② for (Steady) / Incompressible flow: density is not *f(time, space)*

$$\nabla \cdot \vec{V} = 0$$

5.2.2 Continuity Equation: *Taylor series*

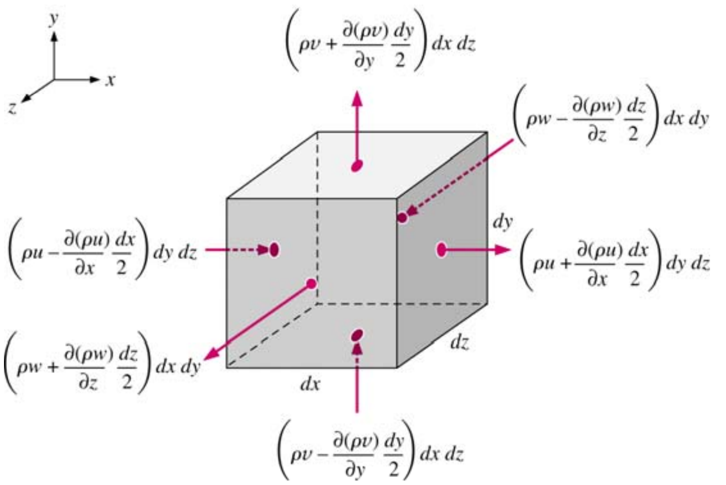
1) Consider an infinitesimal control volume *dxdydz*

2) Consider an infinitesimal control volume *dx* x *dy* x *dz*
3) Approximate the mass flow rate into or out of each of the 6 faces using Taylor series expansions around the center point:
4) Ignore HOTs than $O(dx)$
5) Conservation of Mass:

$$0 = \int_{CV} \frac{\partial \rho}{\partial t} d\cancel{V} + \int_{CS} \rho \vec{V}_n dA$$

$$\int_{CV} \frac{\partial \rho}{\partial t} d\cancel{V} = \sum_{in} \dot{m} - \sum_{out} \dot{m}$$

$$(\rho u)\big|_{right} = (\rho u) + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} + \left(\frac{dx}{2}\right)^2 \frac{1}{2!}\frac{\partial^2(\rho u)}{\partial x^2} + \cdots$$



5.2.2 Continuity Equation: *Taylor series*

1) Rate of change of mass within CV:

$$\int_{CV} \frac{\partial \rho}{\partial t} d\cancel{V} = \frac{\partial \rho}{\partial t} dxdydz$$

2) Net mass flow rate into & out of CV:

$$\sum_{in} \dot{m} = \left(\rho u - \frac{\partial \rho u}{\partial x}\frac{dx}{2} \right) dydz + \left(\rho v - \frac{\partial \rho v}{\partial y}\frac{dy}{2} \right) dxdz + \left(\rho w - \frac{\partial \rho w}{\partial z}\frac{dz}{2} \right) dxdy$$

$$\sum_{out} \dot{m} = \left(\rho u + \frac{\partial \rho u}{\partial x}\frac{dx}{2} \right) dydz + \left(\rho v + \frac{\partial \rho v}{\partial y}\frac{dy}{2} \right) dxdz + \left(\rho w + \frac{\partial \rho w}{\partial z}\frac{dz}{2} \right) dxdy$$

$$\frac{\partial \rho}{\partial t} dxdydz = \left(-\frac{\partial \rho u}{\partial x} - \frac{\partial \rho v}{\partial y} - \frac{\partial \rho w}{\partial z} \right) dxdydz$$

3) Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \quad \rightarrow \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

5.2.4 Continuity Equation: *Alternative Form*

1) Use product rule on divergence term:

Material Derivative

$$\frac{\partial \rho}{\partial t} + \underline{\nabla \cdot (\rho \vec{V})} = \frac{\partial \rho}{\partial t} + \underline{\vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V}} = 0$$

2) Alternative Form of the Continuity Equation:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0$$

① **for Steady / Compressible flow:** $\nabla \cdot \rho \vec{V} = 0$

② **for (Steady) / Incompressible flow:** $\nabla \cdot \vec{V} = 0$

3) Continuity Equation in cylindrical coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r \rho u_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho u_\theta)}{\partial \theta} + \frac{\partial (\rho u_z)}{\partial z} = 0$$

5.3 Conservation of Momentum – Momentum Eqn.

1) Conservation of Linear Momentum: derivation using the divergence theorem

$$\sum \vec{F} = \underbrace{\int_{CV} \rho \vec{g}\, d\forall}_{\text{Body Force}} + \underbrace{\int_{CS} \sigma_{ij} \cdot \vec{n}\, dA}_{\text{Surface Force}} = \int_{CV} \frac{\partial}{\partial t}(\rho \vec{V})\, d\forall + \int_{CS} (\rho \vec{V}) \vec{V} \cdot \vec{n}\, dA$$

Divergence Theorem

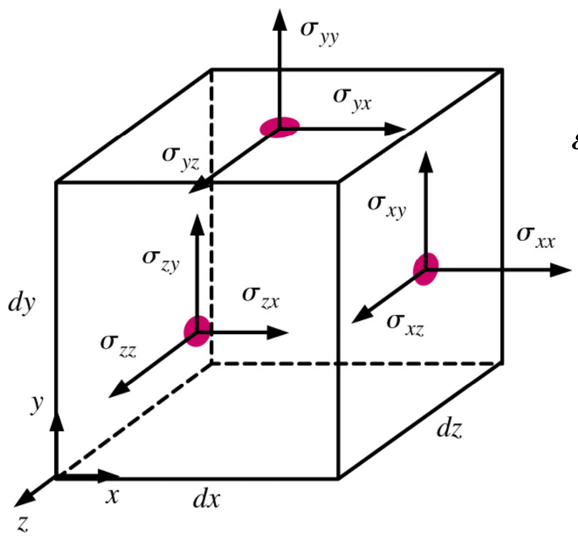
$$\int_{CV} \nabla \cdot \sigma_{ij}\, d\forall \qquad\qquad \int_{CV} \nabla \cdot (\rho \vec{V}\vec{V})\, d\forall$$

$$\int_{CV} \left[\frac{\partial}{\partial t}(\rho \vec{V}) + \nabla \cdot (\rho \vec{V}\vec{V}) - \rho \vec{g} - \nabla \cdot \sigma_{ij} \right] d\forall = 0$$

This equation holds for any control volume regardless of its size or shape if only if the integrand is identically zero.

5.3.0 Review of the Stress Tensor

1) Stress tensor



$$\boxed{\sigma_{ij} \propto \varepsilon_{ij}}$$

2) Shear strain rate tensor

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

5.3.1 Momentum Equation: *Divergence Theorem*

1) Cauchy's Equation:

$$\frac{\partial}{\partial t}(\rho\vec{V}) + \nabla\cdot(\rho\vec{V}\vec{V}) = \nabla\cdot\sigma_{ij} + \rho\vec{g}$$

2) Alternate form of the Cauchy Equation:

a) $\displaystyle \frac{\partial}{\partial t}(\rho\vec{V}) = \rho\frac{\partial\vec{V}}{\partial t} + \vec{V}\frac{\partial\rho}{\partial t}$ (chain rule)

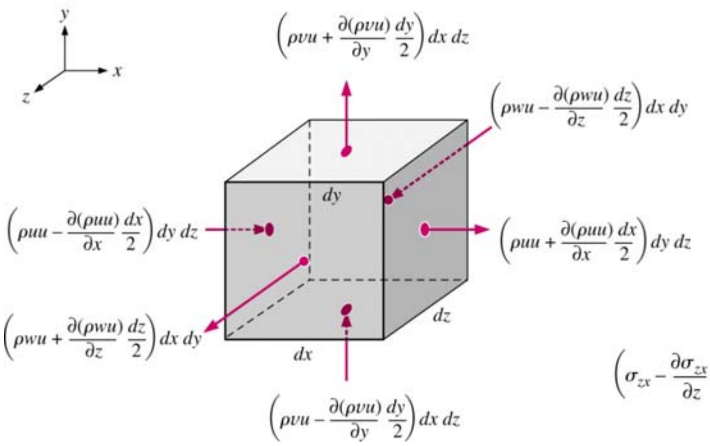
b) $\nabla\cdot(\rho\vec{V}\vec{V}) = \vec{V}\nabla\cdot(\rho\vec{V}) + \rho(\vec{V}\cdot\nabla)\vec{V}$

0 Conservation of Mass

$$\rho\frac{\partial\vec{V}}{\partial t} + \vec{V}\left[\boxed{\cancel{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V})}}\right] + \rho(\vec{V}\cdot\nabla)\vec{V} = \nabla\cdot\sigma_{ij} + \rho\vec{g}$$

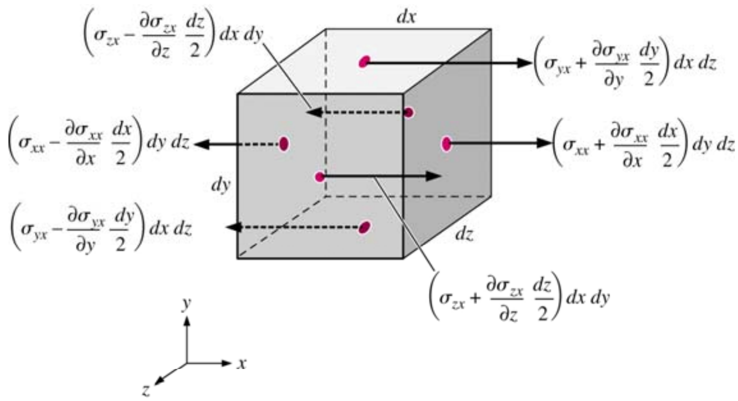
$$\rho\left[\frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right] = \nabla\cdot\sigma_{ij} + \rho\vec{g} \quad \Rightarrow \quad \rho\frac{D\vec{V}}{Dt} = \nabla\cdot\sigma_{ij} + \rho\vec{g}$$

5.3.2 Momentum Equation: *Taylor series*



Inflow and outflow of the x-component of linear momentum through each face of an infinitesimal control volume (differential CV)

Surface forces acting in the x-direction due to the appropriate stress tensor component on each face of the infinitesimal control volume (differential CV)



5.3.2 Momentum Equation: *Taylor series*

1) Consider x-momentum:

$$\sum F = \underbrace{\sum F_{x-surface}}_{(c)} + \underbrace{\sum F_{x-body}}_{(d)} = \underbrace{\int_{CV} \frac{\partial}{\partial}(\rho u)\,dV\!\!\!\!-}_{(a)} + \underbrace{\sum_{out} \beta \dot{m} u - \sum_{in} \beta \dot{m} u}_{(b)}$$

(a) $\int_{CV} \frac{\partial}{\partial t}(\rho u)\,dV\!\!\!\!- = \frac{\partial}{\partial t}(\rho u)\,dxdydz$

(b) $\sum_{out} \beta \dot{m} u - \sum_{in} \beta \dot{m} u = \left[\frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho vu) + \frac{\partial}{\partial z}(\rho wu)\right] dxdydz$

(c) $\sum F_{x\text{-}surface} = \left[\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{yx} + \frac{\partial}{\partial z}\sigma_{zx}\right] dxdydz$

(d) $\sum F_{x\text{-}body} = \rho g_x\, dxdydz$

$$\frac{\partial(\rho u)}{\partial t} + \left[\frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} + \frac{\partial(\rho wu)}{\partial z}\right] = \left[\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{yx} + \frac{\partial}{\partial z}\sigma_{zx}\right] + \rho g_x$$

5.3.2 Momentum Equation: *Taylor series*

1) Consider x-y-z momentum:

$$x: \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} + \frac{\partial(\rho wu)}{\partial z} = \frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{yx} + \frac{\partial}{\partial z}\sigma_{zx} + \rho g_x$$

$$y: \frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} + \frac{\partial(\rho wv)}{\partial z} = \frac{\partial}{\partial x}\sigma_{xy} + \frac{\partial}{\partial y}\sigma_{yy} + \frac{\partial}{\partial z}\sigma_{zy} + \rho g_y$$

$$z: \frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho ww)}{\partial z} = \frac{\partial}{\partial x}\sigma_{xz} + \frac{\partial}{\partial y}\sigma_{yz} + \frac{\partial}{\partial z}\sigma_{zz} + \rho g_z$$

2) Cauchy's Equation:

$$\frac{\partial}{\partial t}(\rho \vec{V}) + \nabla \cdot (\rho \vec{V}\vec{V}) = \nabla \cdot \sigma_{ij} + \rho \vec{g}$$

5.3.2 Momentum Equation: Newton's 2nd Law

1) Consider *x-y-z* momentum:

$$x:\ \rho\frac{Du}{Dt}=\frac{\partial}{\partial x}\sigma_{xx}+\frac{\partial}{\partial y}\sigma_{yx}+\frac{\partial}{\partial z}\sigma_{zx}+\rho g_x$$

$$y:\ \rho\frac{Dv}{Dt}=\frac{\partial}{\partial x}\sigma_{xy}+\frac{\partial}{\partial y}\sigma_{yy}+\frac{\partial}{\partial z}\sigma_{zy}+\rho g_y$$

$$z:\ \rho\frac{Dw}{Dt}=\frac{\partial}{\partial x}\sigma_{xz}+\frac{\partial}{\partial y}\sigma_{yz}+\frac{\partial}{\partial z}\sigma_{zz}+\rho g_z$$

2) Cauchy's Equation

$$\rho\frac{D\vec{V}}{Dt}=\nabla\cdot\sigma_{ij}+\rho\vec{g}$$

3) Closure Problem: unknowns are more than equations

① 10 unknowns

- Stress tensor σ_{ij} : 6 independent components
- Density: ρ
- Velocity: (u,v,w) 3 independent components

② 4 equations: (1 Continuity Equation + 3 Momentum Equations)

③ 6 more equations required to close problem!

5.4 Navier-Stokes Equation

1) Separate σ_{ij} into pressure and viscous stresses

$$\sigma_{ij}=\begin{pmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{pmatrix}=\begin{pmatrix}-p & 0 & 0\\ 0 & -p & 0\\ 0 & 0 & -p\end{pmatrix}+\begin{pmatrix}\tau_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \tau_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \tau_{zz}\end{pmatrix}$$

Mechanical pressure

$$P_m=-\frac{1}{3}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})$$

Viscous (Deviatoric) Stress Tensor

2) Mechanical Pressure:

① Incompressible flows: Mean Pressure

② Compressible flows: Thermodynamic Pressure

3) Equations are not improved!

6 unknowns in σ_{ij} $\Rightarrow$ *6 unknowns in τ_{ij} + P = 7 unknowns*

5.4 Navier-Stokes Equation

1) Rheology: the study of the deformation of flowing fluids.

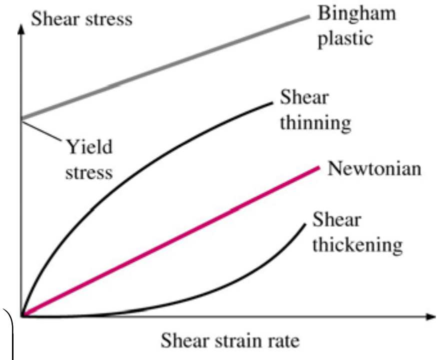
2) Newtonian fluids: fluids for which the shear stress is linearly proportional to the shear strain rate (air, other gases, water, etc).

3) Reduction in the number of variables is achieved by relating shear stress to strain-rate tensor.

4) Viscous stress tensor for incompressible Newtonian fluid with constant properties

$$\tau_{ij} = 2\mu\varepsilon_{ij}$$

Rheological behavior of fluids – shear stress as a function of shear strain rate.



$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$

5) Newtonian closure is analogous to Hooke's Law for elastic solids

5.4 Navier-Stokes Equation

1) Stress tensor with Newtonian closure:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\varepsilon_{ij} \qquad \delta_{ij}\begin{cases} 1: & i = j \\ 0: & i \neq j \end{cases}$$

Kronecker Delta

2) Using the definition of ε_{ij}

$$\sigma_{ij} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} 2\mu\frac{\partial u}{\partial x} & \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 2\mu\frac{\partial v}{\partial y} & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & 2\mu\frac{\partial w}{\partial z} \end{pmatrix}$$

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial \vec{V}_j}{\partial x_i} + \frac{\partial \vec{V}_i}{\partial x_j}\right)$$

5.4 Navier-Stokes Equation

1) **Navier-Stokes equations for incompressible & Isothermal flow:**

① **Continuity Equation:**

$$\nabla \cdot \vec{V} = \mathrm{div}(\vec{V}) = 0$$

② **Momentum Equation:**

$$\rho \frac{D\vec{V}}{Dt} = \nabla \cdot \sigma_{ij} + \rho \vec{g} = \nabla \cdot \left[-p\delta_{ij} + \mu \left(\frac{\partial \vec{V}_j}{\partial x_i} + \frac{\partial \vec{V}_i}{\partial x_j} \right) \right] + \rho \vec{g}$$

$$\rho \left(\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} \right) = -\nabla p + \mu \nabla^2 \vec{V} + \rho \vec{g}$$

2) **Closed system of equations:**

① **4 equations** (continuity and *x-y-z* momentum equations)

② **4 unknowns** (*u, v, w, p*)

5.4 Navier-Stokes Equation

1) **Continuity Equaton:**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Calculate velocity (*U, V, W*) and pressure (*P*) for known geometry with Boundary Conditions (BC), and Initial Conditions (IC)

2) **Momentum Equations:**

$$x: \quad \rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x$$

$$y: \quad \rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y$$

$$z: \quad \rho \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho g_z$$

5.4 Navier-Stokes Equation

1) Continuity Equaton:

① **Vector notation:**

$$\nabla \cdot \vec{V} = 0$$

② **Tensor notation:**

$$\frac{\partial V_i}{\partial x_i} = 0$$

2) Momentum Equations:

① **Vector notation:**

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V}\right) = -\nabla p + \mu \nabla^2 \vec{V} + \rho \vec{g}$$

$$\rho\left(\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu\left(\frac{\partial^2 V_i}{\partial x_j \partial x_j}\right) + \rho g_{x_i}$$

6. Viscous Flow

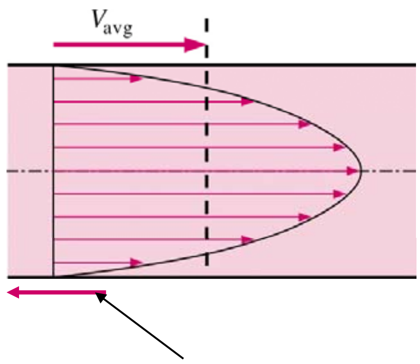
6.1 Introduction

1) Average velocity in a pipe

   ① Recall - because of the no-slip condition, the velocity at the walls of a pipe or duct flow is zero

   ② We are often interested only in V_{avg}, which we usually call just V (drop the subscript for convenience)

   ③ Keep in mind that the no-slip condition causes shear stress and friction along the pipe walls



Friction force of wall on fluid

6.1 Introduction

1) For pipes of constant diameter and incompressible flow

2) V_{avg} stays the same down the pipe, even if the velocity profile changes

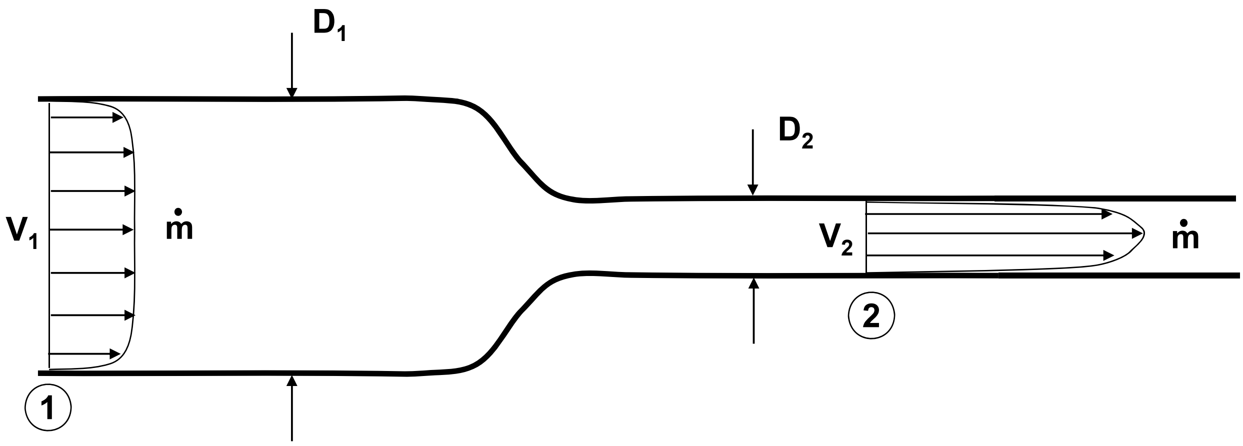
3) Why? Conservation of Mass

$$\dot{m} = \rho V_{avg} A = constant$$

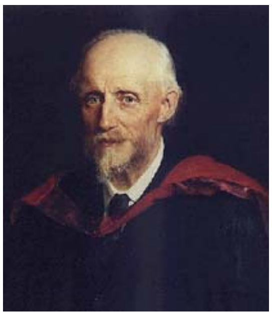
same　same　same

6.1 Introduction

For pipes with variable diameter, mass flow rate is still the same due to conservation of mass, but $V_1 \neq V_2$



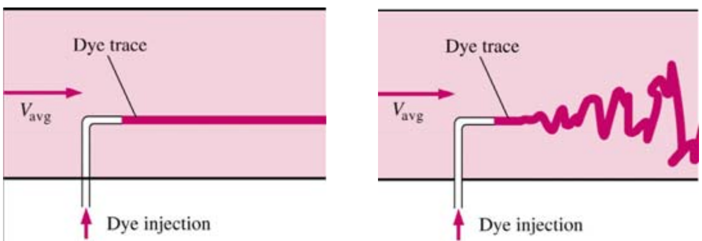
6.1 Introduction



- **Osborne Reynolds (1842 ~ 1912)**
- **1842: born at Belfast, Ireland**
- **1867: graduate at Queens' College, Univ. of Cambridge, in mathematics**
- **1868: 1st professor of engineering at Owens College, Manchester**
- **1878: `Improvements in Machinery for Propelling Ships or Vessels'**

Reynolds number $\left[\dfrac{\text{Intertial force}}{\text{Viscous force}}\right]$

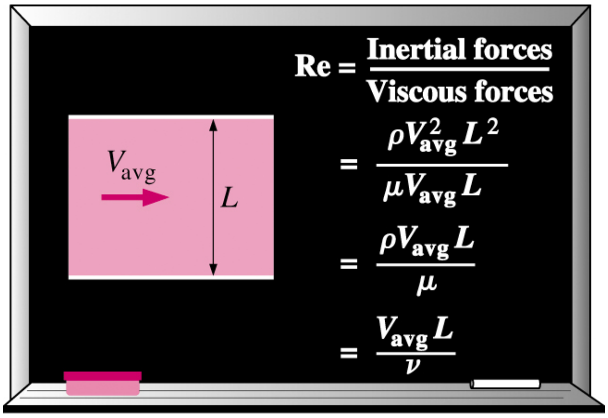
$$R_e = \left[\frac{F_I}{F_V}\right] = \left[\frac{\rho V^2 L^2}{\mu V L}\right] = \left[\frac{\rho V L}{\mu}\right] = \left[\frac{VL}{\upsilon}\right]$$



(a) Laminar flow (b) Turbulent flow

6.2 Laminar and Turbulent Flows

Definition of Reynolds number



1) Critical Reynolds number (Re_{cr}) for flow in a round pipe

   $Re < 2300 \Rightarrow$ laminar

   $2300 \le Re \le 4000 \Rightarrow$ transitional

   $Re > 4000 \Rightarrow$ turbulent

2) Note that these values are approximate.

3) For a given application, Re depends upon

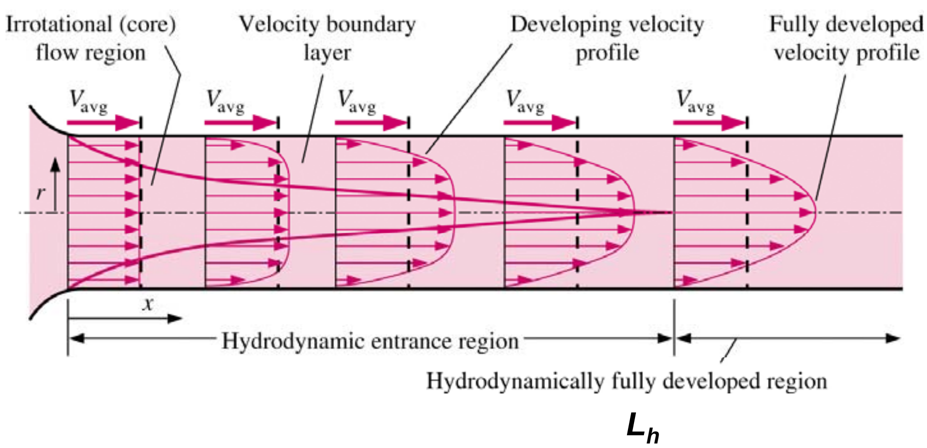
   ① Pipe roughness

   ② Vibrations

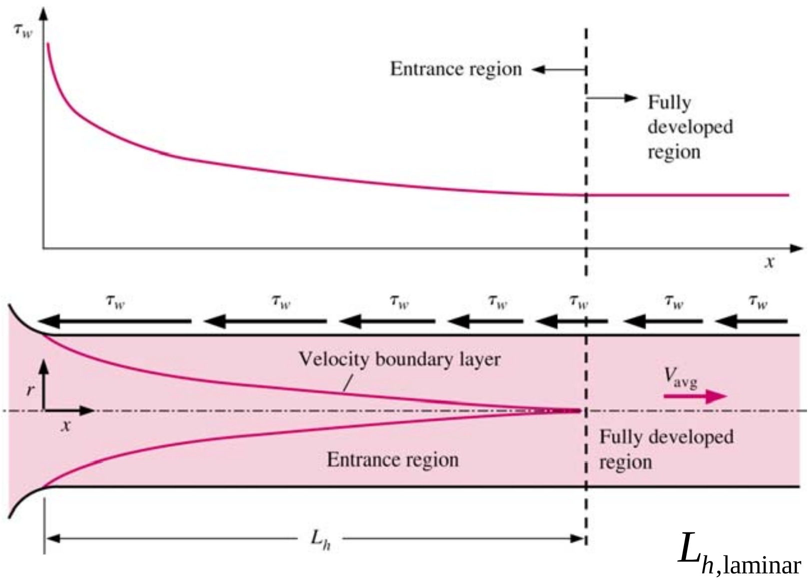
   ③ Upstream fluctuations, disturbances (valves, elbows, etc. that may disturb the flow)

6.3 The Entrance Region

Consider a round pipe of diameter D. The flow can be laminar or turbulent. In either case, the profile develops downstream over several diameters called the *entry length* L_h. L_h/D is a function of Re.



6.3 The Entrance Region



$$L_{h,\text{laminar}} \cong 0.05 D\,\text{Re}$$

$$L_{h,\text{turbuent}} \cong 1.359 D\,\text{Re}^{1/4}$$

$$L_{h,\text{turbuent}} \approx 10D$$

6.4 Laminar Flow in Pipes

Average velocity:

$$(2\pi r dr P)_x - (2\pi r dr P)_{x+dx} + (2\pi r dx\tau)_r - (2\pi r dx \tau)_{r+dr}$$

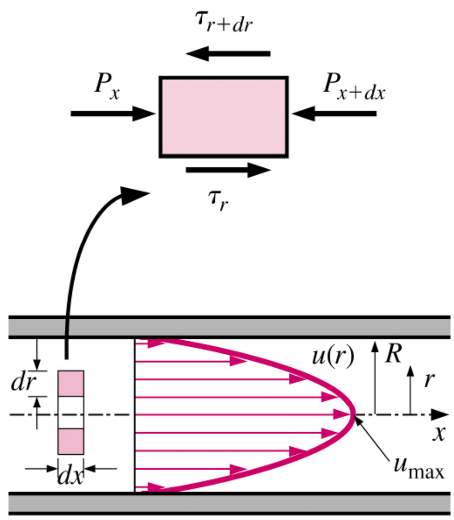
$$r\frac{P_{x+dx}-P_x}{dx} + \frac{(r\tau)_{r+dr}-(r\tau)_r}{dr} = 0$$

$$dx, dr \to 0$$

$$r\frac{dP}{dx} + \frac{d(r\tau)}{dr} = 0$$

$$\tau \equiv -\mu\frac{du}{dr}$$

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{dP}{dx}$$



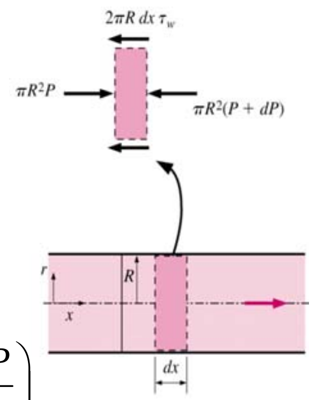
6.4 Laminar Flow in Pipes

$$\frac{dP}{dx} = -\frac{2\tau_w}{R}$$

$$u(r) = \frac{r^2}{4\mu}\left(\frac{dP}{dx}\right) + C_1 \ln r + C_2$$

$$u(r) = -\frac{R^2}{4\mu}\left(\frac{dP}{dx}\right)\left[1 - \frac{r^2}{R^2}\right]$$

$$V_{avg} = \frac{2}{R^2}\int_0^R u(r) r dr = -\frac{2}{R^2}\int_0^R -\frac{R^2}{4\mu}\left(\frac{dP}{dx}\right)\left[1 - \frac{r^2}{R^2}\right] r dr = -\frac{R^2}{8\mu}\left(\frac{dP}{dx}\right)$$



$$u(r) = 2V_{avg}\left[1 - \frac{r^2}{R^2}\right]$$

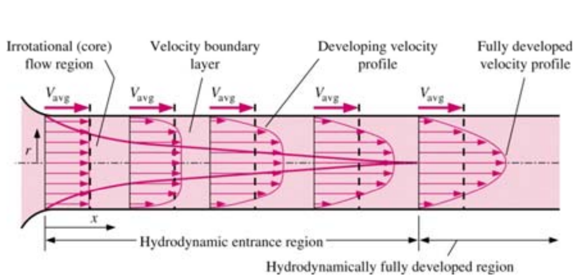
$$V_{avg} = \frac{\int_{A_c} \rho u(r) dA_c}{\rho A_c} = \frac{\int_0^R \rho u(r) 2\pi r dr}{\rho \pi R^2} = \frac{2}{R^2}\int_0^R u(r) r dr$$

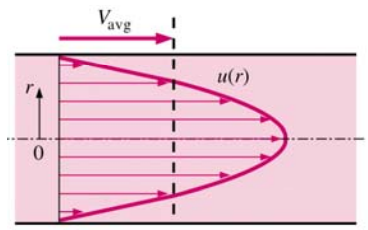
at $r = 0$

$$u_{\max} = 2V_{avg}$$

The average velocity in fully developed laminar pipe flow is one half of the maximum velocity

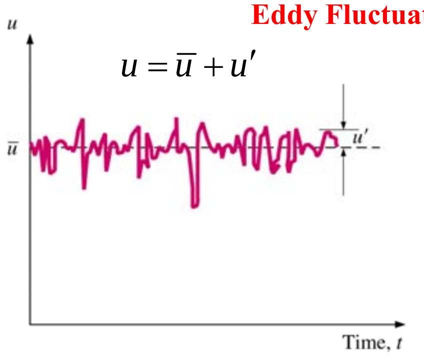
6.5 Turbulent Flow in Pipes

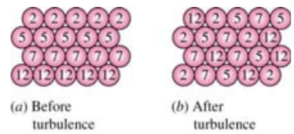


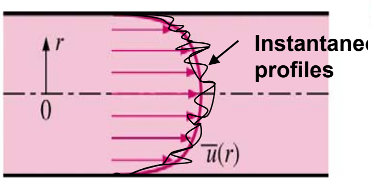


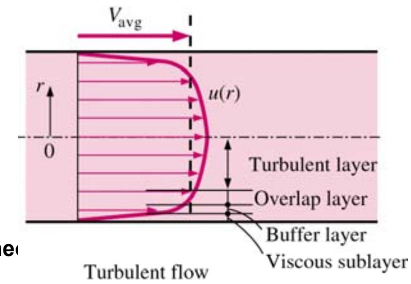
Laminar flow

Eddy Fluctuation





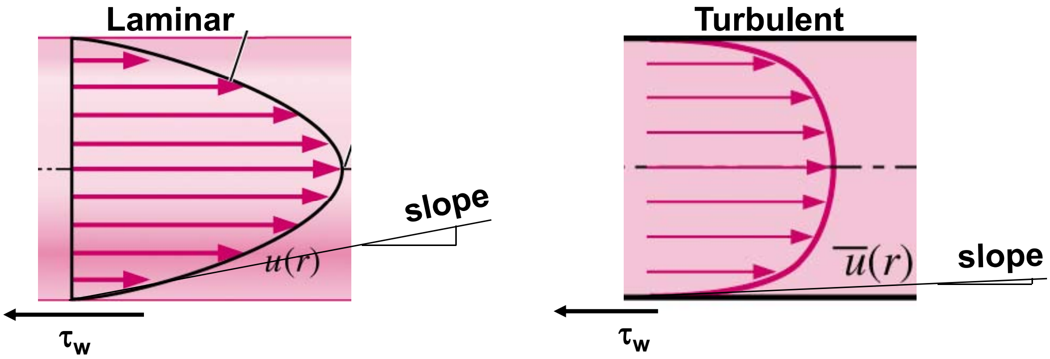




Turbulent flow

6.5 Turbulent Flow in Pipes

τ_w = shear stress at the wall, acting on the fluid

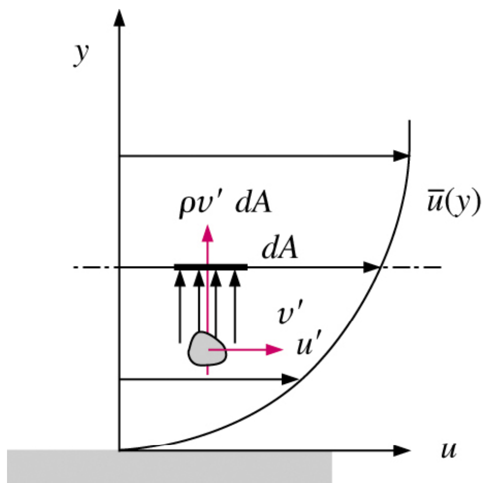


$$\tau = \mu \frac{du}{dr}$$

$$\tau_{lam} < \tau_{tur}$$

6.5 Turbulent Flow in Pipes

Turbulent shear stress:



Tangential(Shear) Force:

$$\delta F = \left(\rho v' dA\right)\left(-u'\right) = -\rho u'v'dA$$

$$\frac{\delta F}{dA} = -\rho u'v'$$

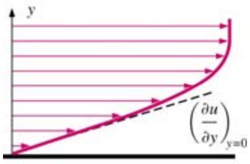
$$\tau_{turb} \equiv -\rho\overline{u'v'}$$

Reynolds stress or Turbulent stress

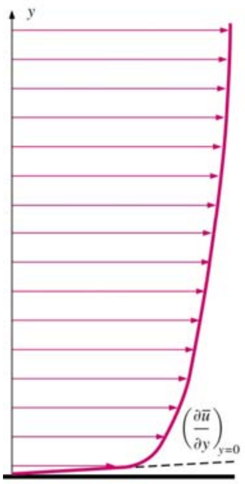
$$\overline{u'} = 0, \quad \overline{v'} = 0, \quad \overline{u'}\,\overline{v'} = 0, \quad \overline{u'v'} \neq 0$$

$$\tau_{total} = \tau_{lam} + \tau_{tur} = -\left(\mu \frac{d\overline{u}}{dy} + \rho\overline{u'v'}\right)$$

6.5 Turbulent Flow in Pipes



Laminar flow



Turbulent flow

Joseph Boussinesq:

$$\tau_{turb} = -\rho\overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y}$$

μ_t : eddy viscosity or turbulent viscosity

L. Prandtl:

$$\tau_{turb} = \mu_t \frac{\partial \bar{u}}{\partial y} = \rho l_m^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2$$

l : mixing length

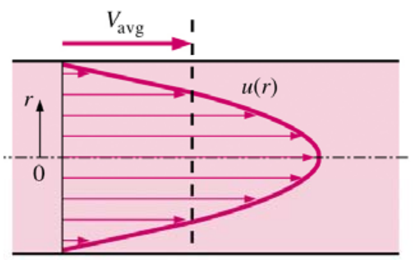
$$\tau_{toal} = (\mu + \mu_t)\frac{\partial \bar{u}}{\partial y} = \rho(\nu + \nu_t)\frac{\partial \bar{u}}{\partial y}$$

$\nu_t \equiv \mu_t / \rho$: kinematic eddy viscosity

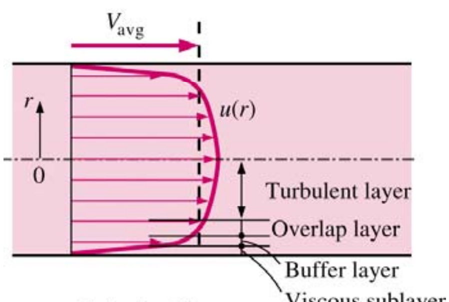
kinematic turbulent viscosity

eddy diffusivity of momentum

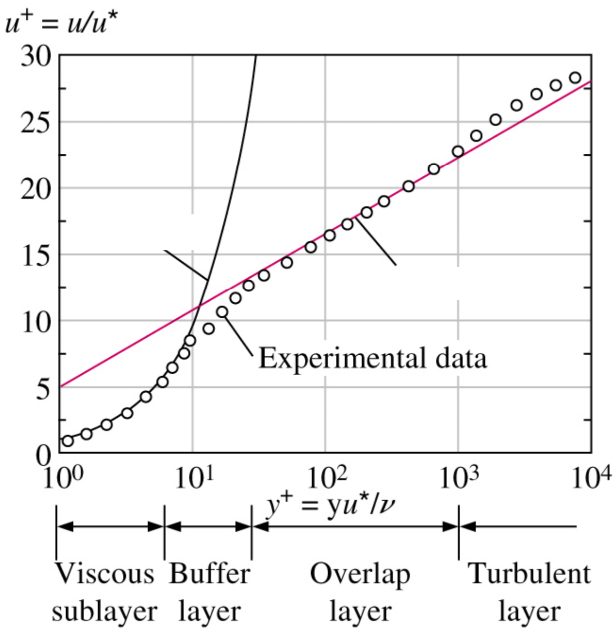
6.5 Turbulent Flow in Pipes



Laminar flow



Turbulent flow



6.5 Turbulent Flow in Pipes

Viscous sub-layer:

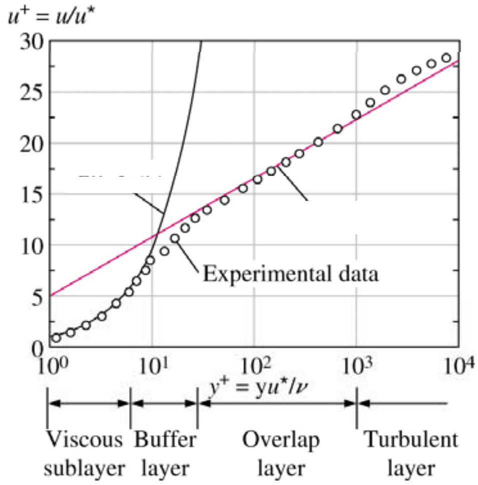
$$\tau_w = \mu \frac{du}{dy} = \rho \nu \frac{u}{y} \quad \rightarrow \quad \frac{\tau_w}{\rho} = \frac{\nu u}{y}$$

$$u_* = \sqrt{\tau_w / \rho}$$

u_* : friction velocity

$$\frac{u}{u_*} = \frac{y u_*}{\nu} \quad \text{law of the wall}$$

$$\text{y}=\delta_{sublayer} = \frac{5\nu}{u_*} = \frac{25\nu}{u_\delta} \qquad 0 \le \frac{y u_*}{\nu} \le 5$$



The thickness of the viscous sub-layer is proportional to the kinematic viscosity and inversely proportional to the friction velocity

6.5 Turbulent Flow in Pipes

Nondimensional variables:

$$\frac{\nu}{u_*} : \text{ viscous lenght}$$

$$\left. \begin{array}{l} \text{y}^+ = \dfrac{y u_*}{\nu} \\ u^+ = \dfrac{u}{u_*} \end{array} \right\} \quad u^+ = y^+$$

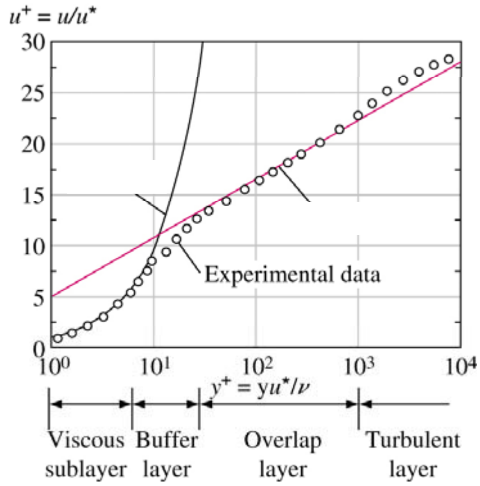
Nomalized law of the wall:

6.5 Turbulent Flow in Pipes

Overlap layer:

the logarithmic law

$$\frac{u}{u_*} = \frac{1}{\kappa}\ln\frac{yu_*}{\nu} + B$$



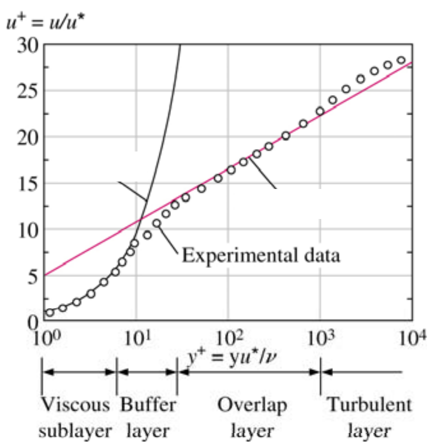
$$(\kappa = 0.4,\ B = 5.0)$$

$$\frac{u}{u_*} = 2.5\ln\frac{yu_*}{\nu} + 5.0$$

$$u^+ = 2.5\ln y^+ + 5.0$$

6.5 Turbulent Flow in Pipes

Outer turbulent layer:



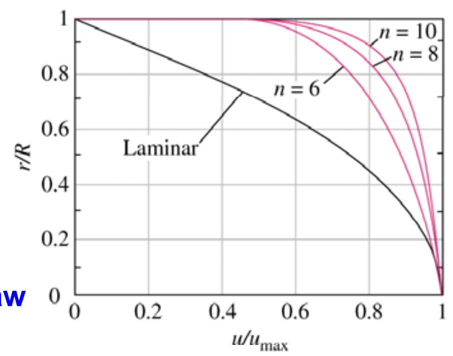
velocity defect law:

$$\frac{u_{max} - u}{u_*} = 2.5\ln\frac{R}{R-r}$$

power-law velocity profile:

$$\frac{u}{u_{max}} = \left(\frac{y}{R}\right)^{1/n}$$

$$\frac{u}{u_{max}} = \left(1 - \frac{r}{R}\right)^{1/n}$$

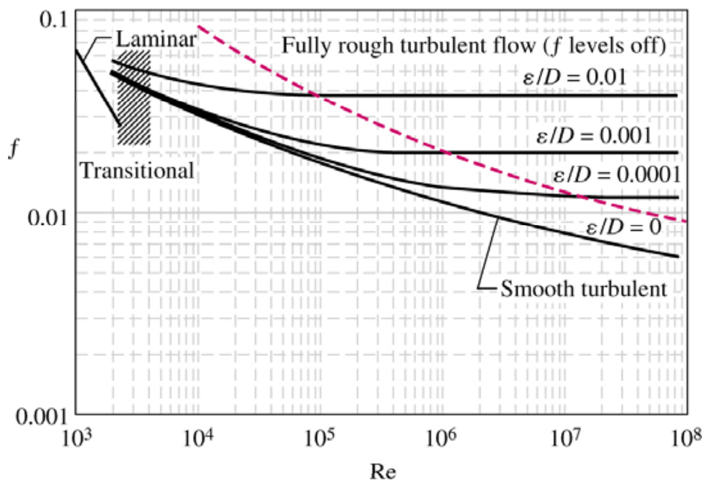


one-seventh power-law

6.6 Moody chart

Colebrook:

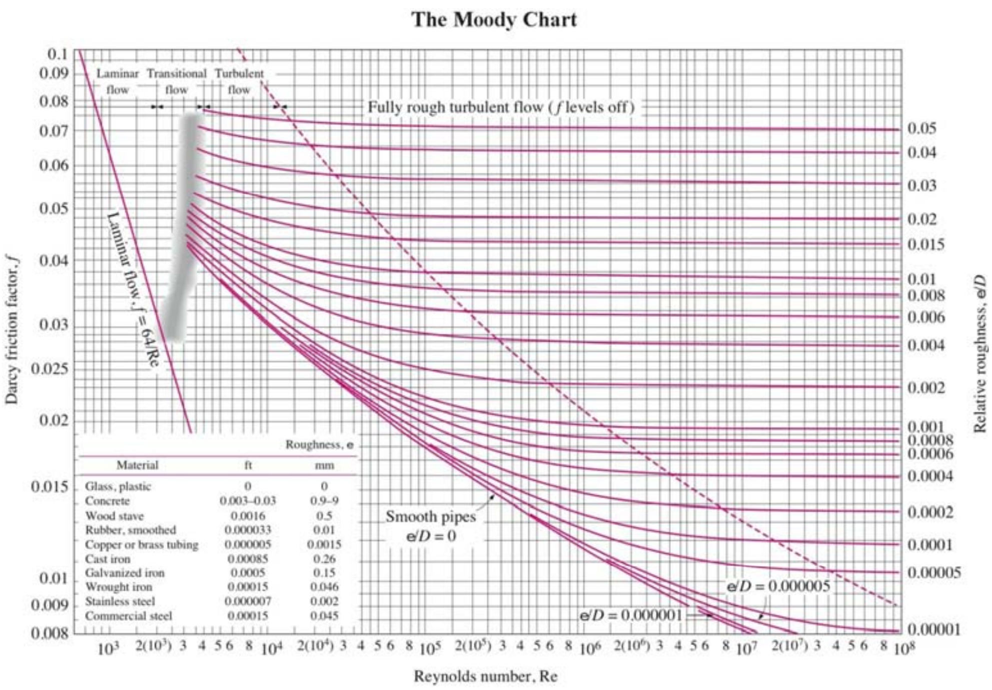
$$\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$$



Haaland:

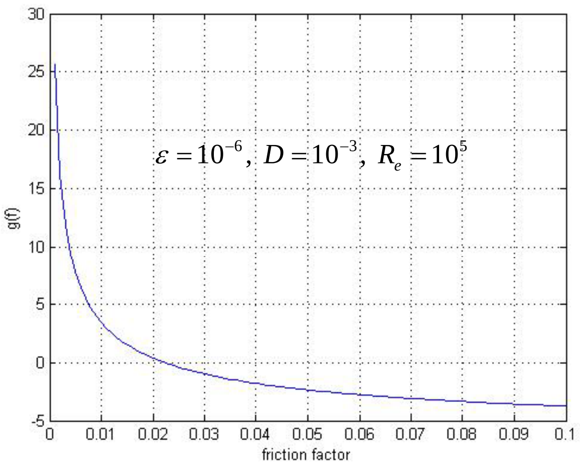
$$\frac{1}{\sqrt{f}} \cong -1.8\log\left(\frac{6.9}{\mathrm{Re}} + \left(\frac{\varepsilon/D}{3.7}\right)^{1.11}\right)$$

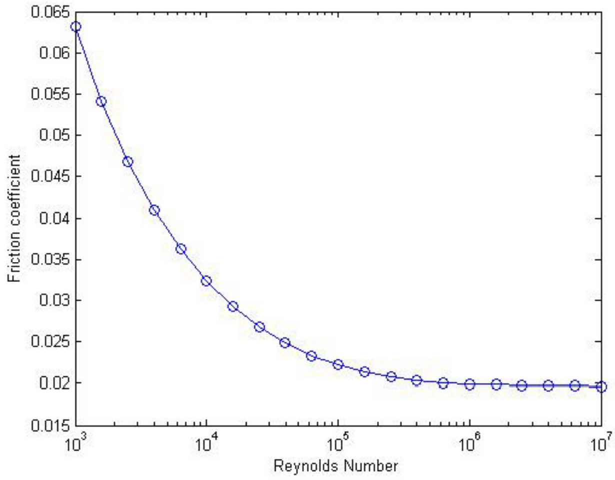
6.6 Moody chart



6.6 Moody chart

$$\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right) \;\rightarrow\; g(f) = \frac{1}{\sqrt{f}} + 2.0\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right) = 0$$

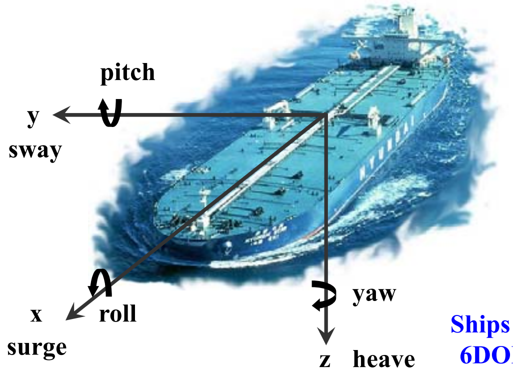




7. Flow over bodies: Lift & Drag

7.1 Introduction

1) Flow over bodies (External flows): aircrafts, automobiles, buildings, ships, submarines, turbo-machines, etc.
2) Fuel economy, speed, acceleration, maneuverability, stability, and control are directly related to the aerodynamic/hydrodynamic forces and moments.
3) General 6DOF motion of bodies is described by 6 equations for the linear (surge, heave, sway) and angular (roll, pitch, yaw) momentum.
4) Objects of this chapter: to understand the fundamentals of flow over bodies, and calculate the drag and lift forces acting on bodies.



Ships in waves: 6DOF motion

7.1 Introduction

1) Classifications of the flow over bodies:

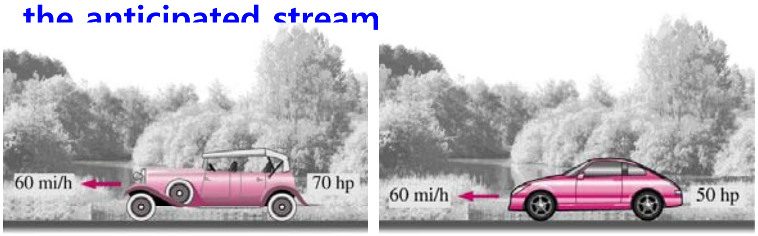
① 2D Flow: body is very long and of constant cross section and the flow is normal to the body.

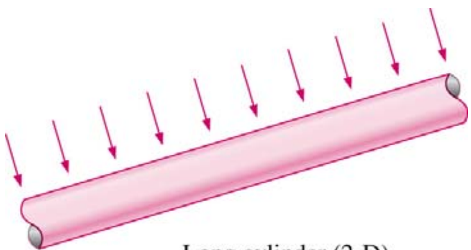
② Axisymmetric Flow: body possesses rotational symmetry about an axis in the flow direction.

③ 3D Flow: body that cannot be modeled as above two cases.

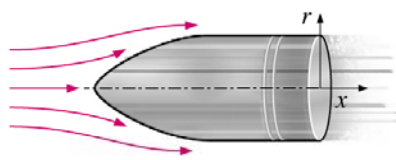
④ Incompressible and Compressible Flows:

2) Streamlined body: align the body shape with the anticipated stream





Long cylinder (2-D)



Bullet (axisymmetric)

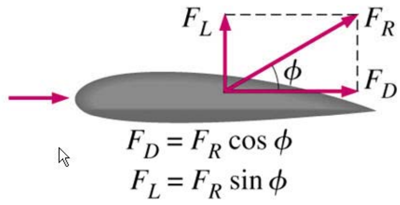
Car (3-D)

7.2 Drag and Lift

1) **Fluid dynamic forces are due to pressure and viscous forces acting on the body surface.**

   ① **Drag: component parallel to flow direction.**

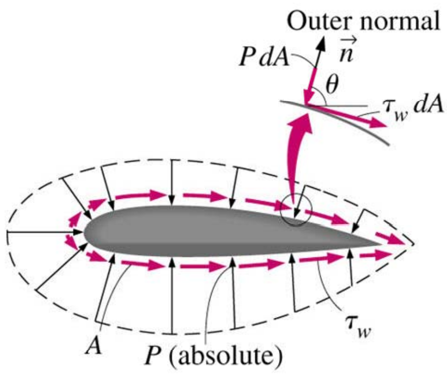
   ② **Lift: component normal to flow direction.**

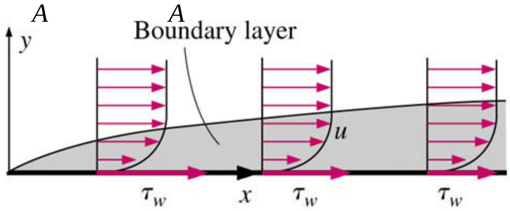


2) **Lift and drag forces can be found by integrating pressure and wall-shear stress.**

$$F_D = \int_A dF_D = \int_A (-P\cos\theta + \tau_w \sin\theta)dA$$

$$F_L = \int_A dF_L = \int_A (-P\sin\theta - \tau_w \cos\theta)dA$$

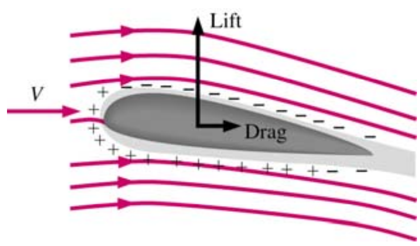




7.2 Drag and Lift

1) **Drag and Lift forces are a function of velocity and density:**

$$F_L \ \& \ F_D = f(\vec{V}, \rho)$$



2) **Dimensional analysis gives 2 dimensionless parameters:**

   ① **Drag coefficient:** $C_D = \dfrac{F_D}{1/2\rho V^2 A}$ $\xrightarrow{\textbf{Average}}$ $C_D = \dfrac{1}{L_S}\int_0^{L_S} C_{Dx}dx$

   ② **Lift coefficient:** $C_L = \dfrac{F_L}{1/2\rho V^2 A}$ $\qquad C_L = \dfrac{1}{L_S}\int_0^{L_S} C_{Lx}dx$

3) **Area (*A*) can be frontal area (drag applications), planform area (wing aerodynamics), or wetted-surface area (ship hydrodynamics).**

4) **The Drag and Lift coefficients are primarily functions of the shape of the body, but in some cases they also depend on the Reynolds number and the surface roughness.**

7.3 Drag: Friction and Pressure Drag

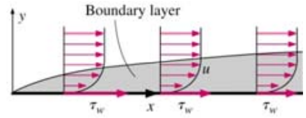
1) Fluid dynamic drag forces are comprised of pressure and friction effects.

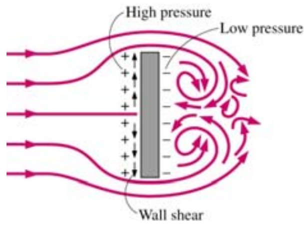
   ① Skin Friction Drag (Viscous Drag)

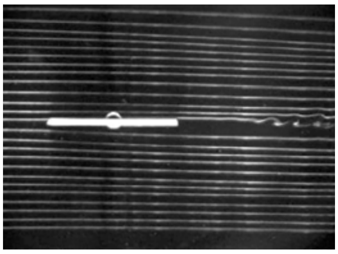
   ② Pressure Drag (Form Drag)

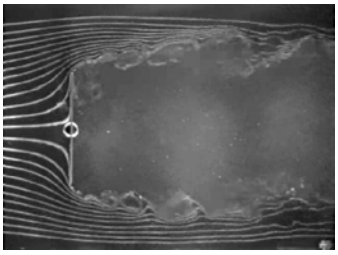
$$F_D = F_{D(friction)} + F_{D(pressure)}$$

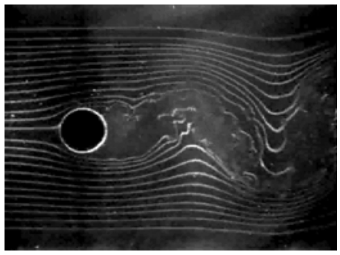
$$C_D = C_{D(friction)} + C_{D(pressure)}$$






Friction drag


Pressure drag

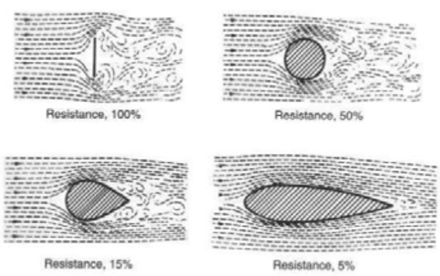

Friction & pressure drag

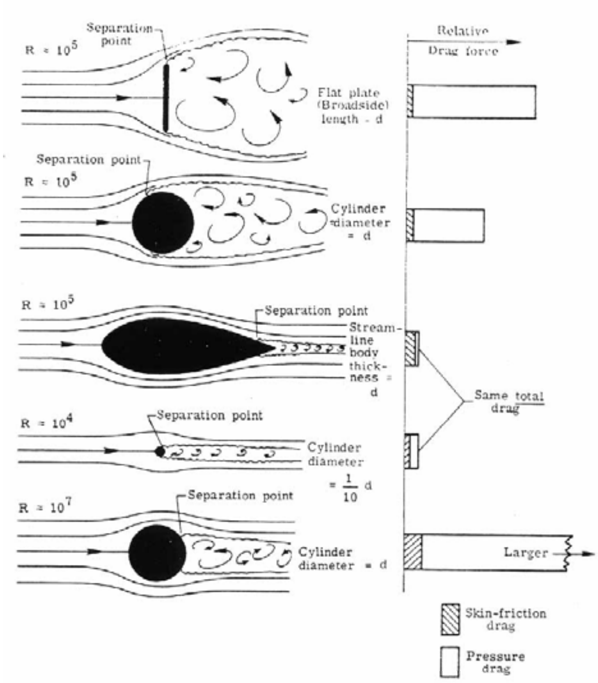
7.3.1 Reducing Drag by Streamlining

1) Streamlining reduces drag by reducing $F_{D(pressure)}$ at the cost of increasing wetted surface area and $F_{D(friction)}$

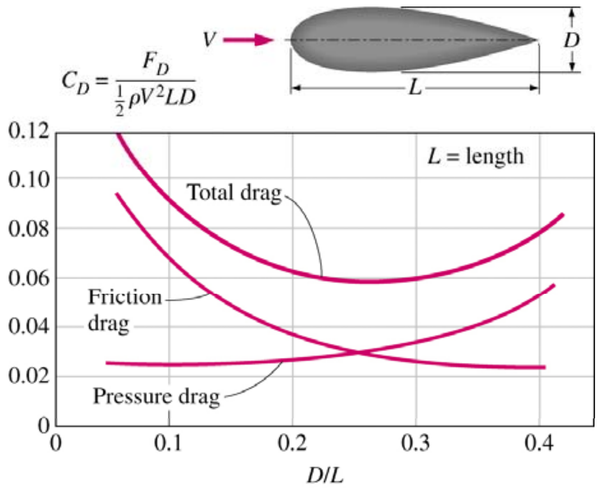
2) Goal is to eliminate flow separation and minimize total drag F_D

3) Also improves structural acoustics since separation and vortex shedding can excite structural modes.

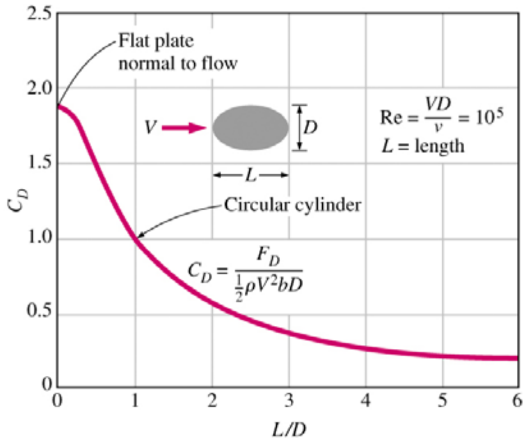




7.3.1 Reducing Drag by Streamlining

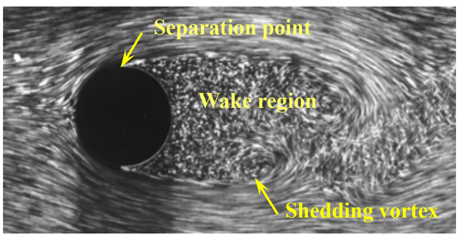


The variation of friction, pressure and total drag coefficients of a streamlined airfoil with thickness-to-chord length ratio for Re=4x10^4

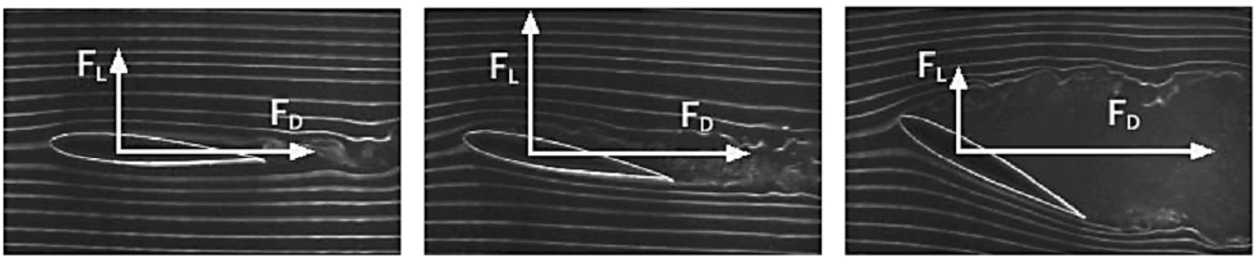


The variation of the drag coefficient of a long elliptical cylinder with aspect ratio.

7.3.2 Flow Separation



The Location of the separation point depends on several factors such as Reynolds number, the surface roughness, and the level of fluctuations in the free stream.

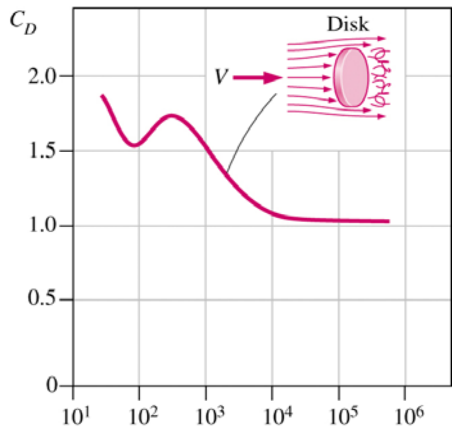


(*a*) 5° (*b*) 15° (*c*) 30°

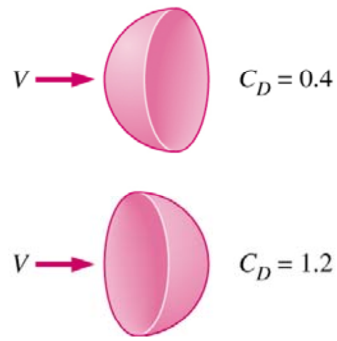
At large angles of attack (usually larger than 15°), flow may separate completely from the top surface of an airfoil reducing lift drastically and causing the airfoil to stall.

7.4 C_D of Common Geometries

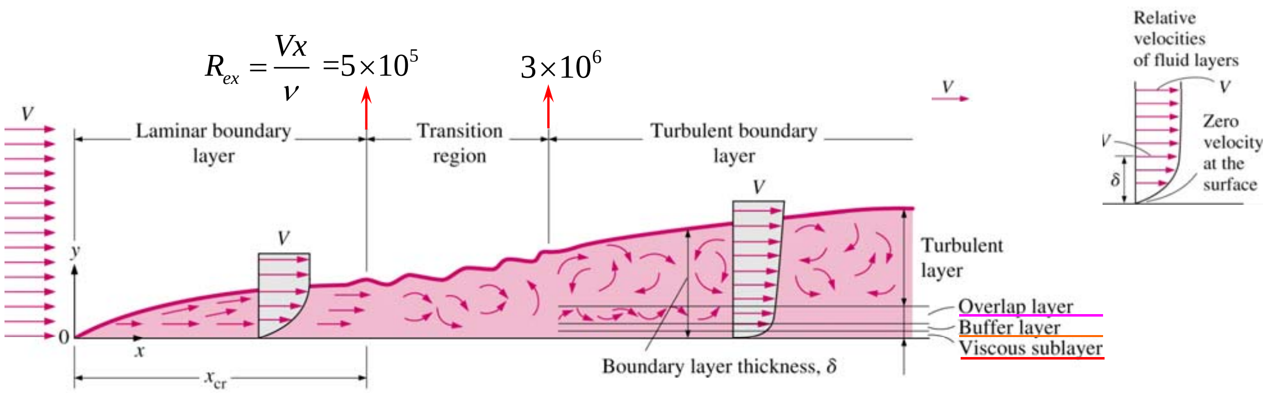
1) For many geometries, total drag C_D is constant for $Re > 10^4$
2) C_D can be very dependent upon orientation of body.
3) As a crude approximation, superposition can be used to add C_D from various components of a system to obtain overall drag. However, there is no mathematical reason (e.g., linear PDE's) for the success of doing this.



A hemisphere at two different orientations for Re > 10^4



7.5 Flow over Flat Plate



1) Boundary layer thickness (δ): $u = 0.99V$
2) Viscous sub-layer: very thin layer next to the wall where viscous effects are dominant.
3) Drag on flat plate is solely due to friction created by laminar, transitional, and turbulent boundary layers.

$$C_D = C_{D(friction)} = C_f \quad \rightarrow \quad F_D = F_f = 1/2 C_f A \rho V^2$$

7.5.1 Friction Coefficient

1) Local friction coefficient

① **Laminar:**

$$\delta = \frac{4.91x}{R_{ex}^{1/2}} \qquad C_{f,x} = \frac{0.666}{R_{ex}^{1/2}}$$

② **Turbulent:**

$$\delta = \frac{0.38x}{R_{ex}^{1/5}} \qquad C_{f,x} = \frac{0.059}{R_{ex}^{1/5}}$$

2) Average friction coefficient: $C_{f,x} = \frac{1}{L}\int_0^L C_{f,x}\,dx$

① **Laminar:**

$$C_f = \frac{1.33}{R_{eL}^{1/2}} \qquad R_{eL} < 5\times10^5$$

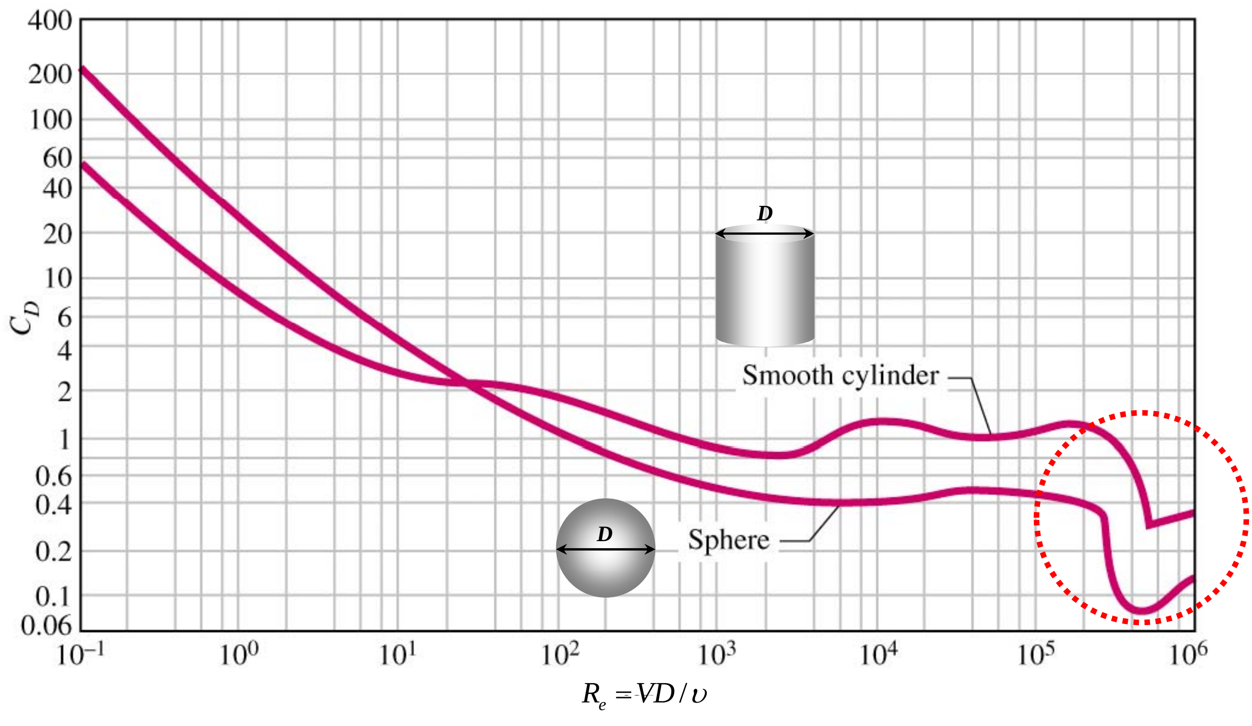
② **Turbulent:**

$$C_f = \frac{0.074}{R_{eL}^{1/5}} \qquad 5\times10^5 \le R_{eL} \le 10^7$$

3) For some cases, plate is long enough for turbulent flow, but not long enough to neglect laminar portion

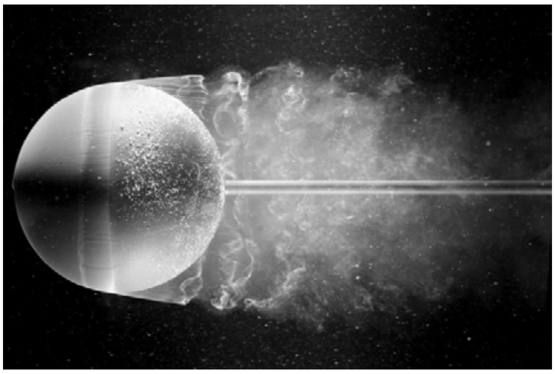
$$C_f = \frac{1}{L}\left(\int_0^{x_{cr}} C_{f,x,\mathrm{lam}}\,dx + \int_{x_{cr}}^{L} C_{f,x,\mathrm{tur}}\,dx\right) \;\rightarrow\; C_f = \frac{0.074}{R_{eL}^{1/5}} - \frac{1742}{R_{eL}}$$

7.6 Flow over Cylinders & Spheres

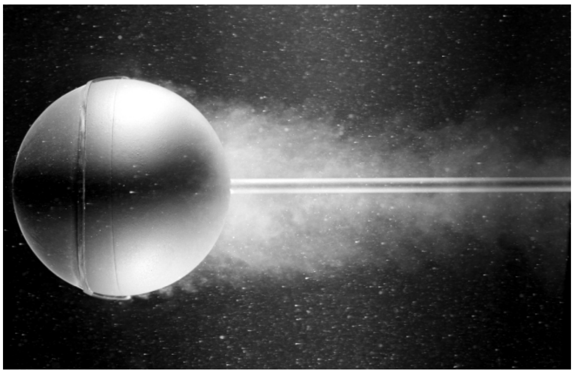


7.6.1 Flow over Cylinders & Spheres

1) Flow is strong function of Re.
2) The delay of separation in turbulent flow enables the TBL (turbulent boundary layer) to travel father along the surface, resulting in a narrow wake and a smaller pressure drag (adverse pressure gradient).
3) This is in contrast to streamlined bodies, which experience an increase in drag coefficient due to friction drag.

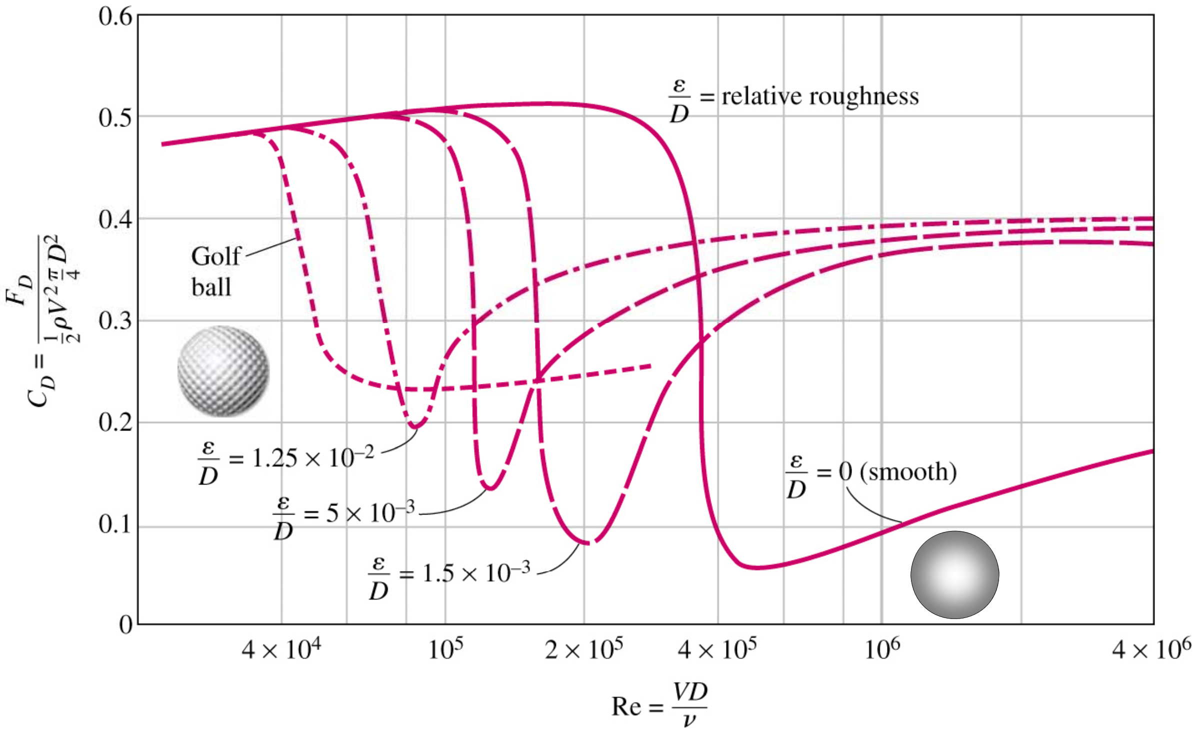


Re=15,000 (Laminar) $\theta_{sep} \approx 80^o$



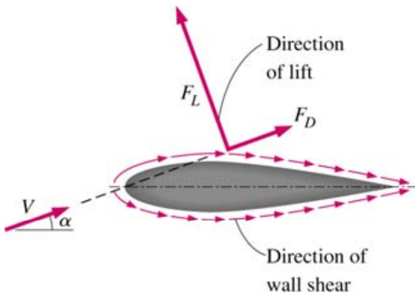
Re=30,000 (Turbulent) $\theta_{sep} \approx 140^o$

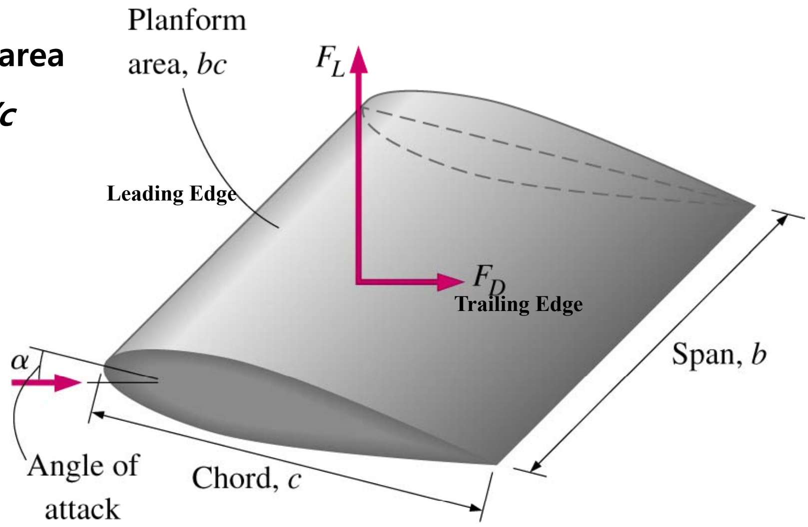
7.6.2 Effect of Surface Roughness



7.7 Lift

1) Lift is the net force (due to pressure and viscous forces) perpendicular to flow direction.
2) Lift coefficient: $C_L = \dfrac{F_L}{\frac{1}{2}\rho V^2 A}$
3) *A=bc* is the plan-form area
4) Aspect Ratio (*AR*) = *b/c*

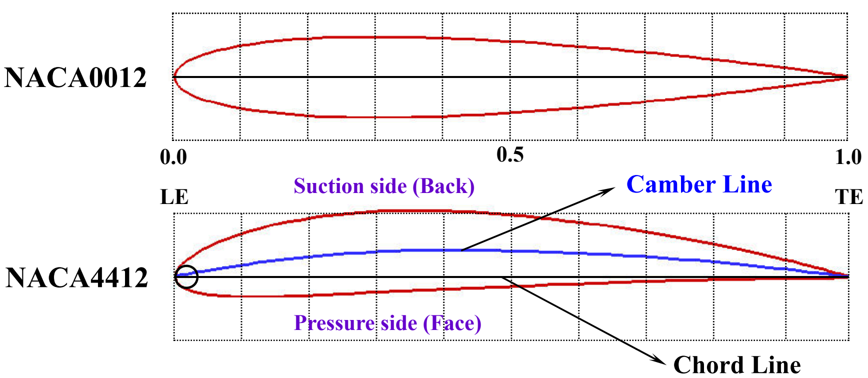




7.7.1 Wing Sections

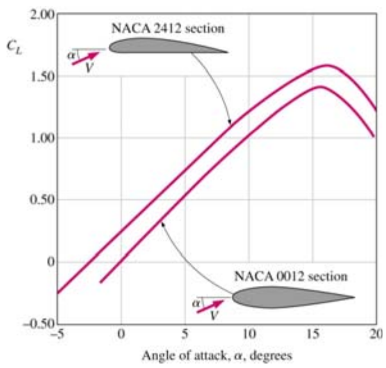
1) NACA (National Advisory Committee for Aeronautics)
2) NACA 4-Digit Wing Section:

NACA4412
- Maximum Thickness (12% of the Chord)
- Location of the maximum Camber (40% of the Chord)
- Maximum Camber (4% of the Chord)

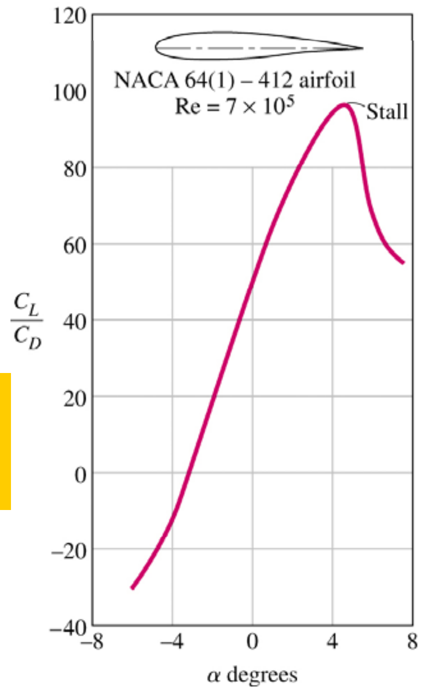


7.7.2 Effect of Angle of Attack & Foil Shape

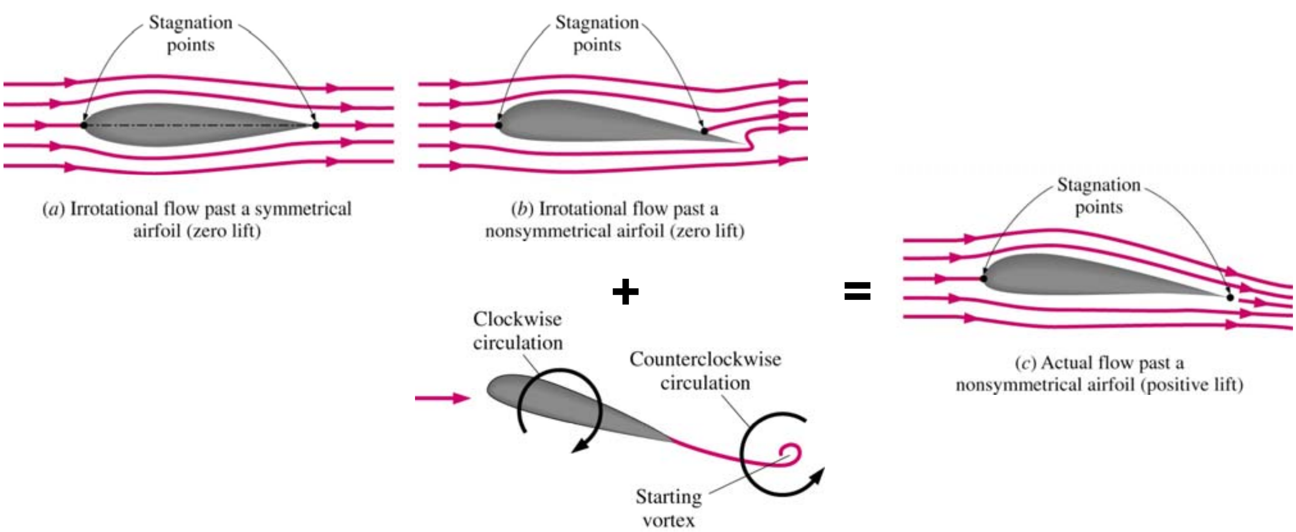
1) Thin-foil theory shows that $C_L \approx 2\pi\alpha$ for $\alpha < \alpha_{stall}$
2) Therefore, lift increases linearly with α
3) Objective for most applications is to achieve maximum Lift-to-Drag ratio (C_L/C_D) ratio.
4) (C_L/C_D) increases (up to order 100) until stall.



Thickness and camber influences pressure distribution and location of flow separation.



7.7.3 Irrotational & actual flow past airfoils

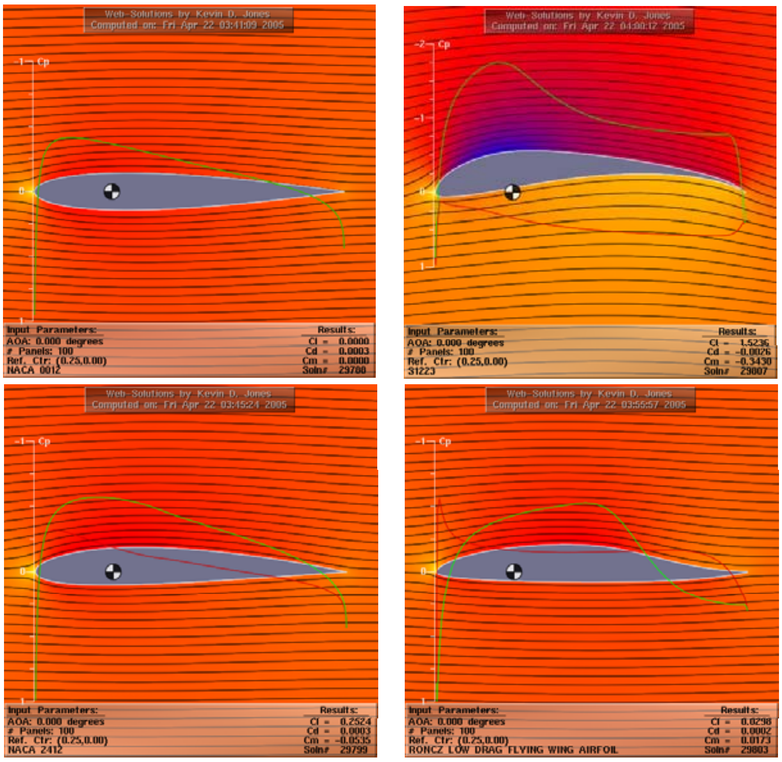


1) Potential-Flow approximation gives accurate CL for angles of attack below stall: boundary layer can be neglected.
2) Kutta condition required at trailing edge: fixes stagnation pt at TE.

7.7.4 C_L resulted from Potential Flow Theory

NACA0012
$C_L = 0.0000$
$C_D = 0.0003$

S1223
$C_L = 1.5236$
$C_D = -0.0026$

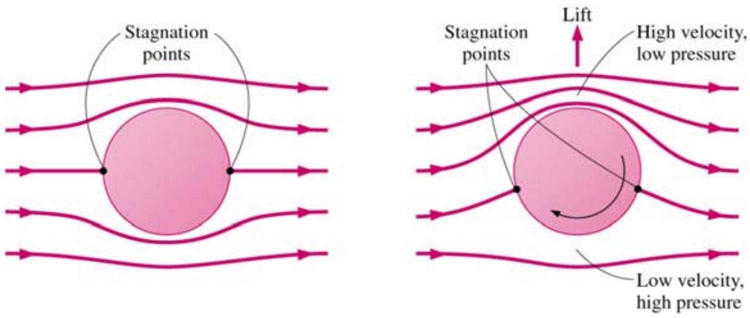


NACA2412
$C_L = 0.2524$
$C_D = 0.0003$

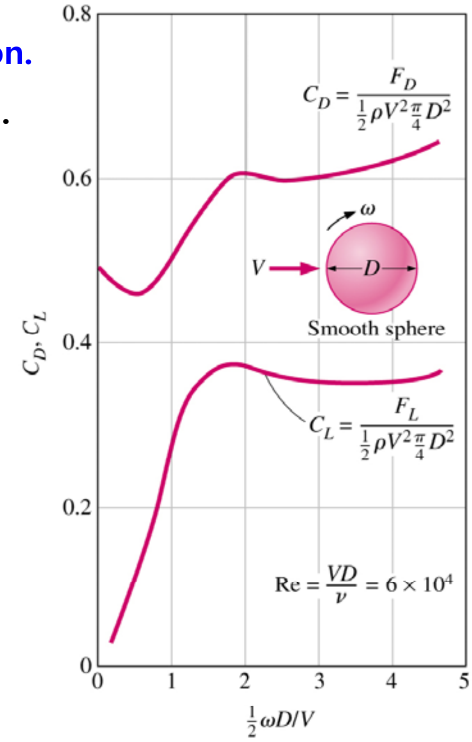
RONCZ
$C_L = 0.0298$
$C_D = 0.0002$

7.7.5 Lift Generated by Spinning

1) **Lift (C_L) strongly depends on rate of rotation.**
2) **The effect of rate of rotation on C_D is small.**
3) **Baseball, golf, soccer, tennis players utilize spin.**
4) **Magnus Effect: Lift generated by rotation.**



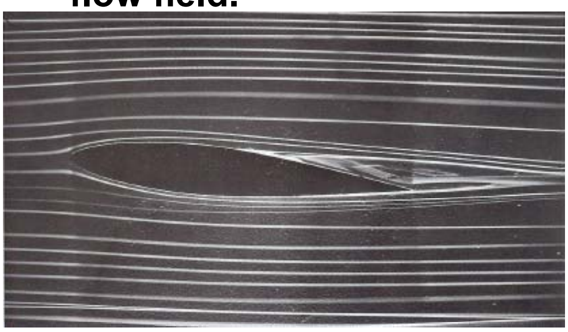
(*a*) Potential flow over a stationary cylinder   (*b*) Potential flow over a rotating cylinder

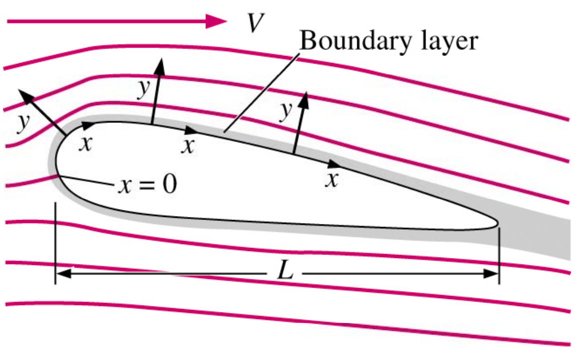


8. Potential Flow

8.1 Introduction

1) We derived the NSE and developed several exact solutions.
2) In this Chapter, we will study several methods for simplifying the NSE, which permit use of mathematical analysis and solution: these approximations often hold for certain regions of the flow field.





A particular approximation of the Navier-Stokes equation is appropriate only in certain regions of the flow field: other approximations may be appropriate in other regions of the flow field.

8.2 Nondimensional N-S Equation

1) **Purpose: Order-of-magnitude analysis of the terms in the N-S equation, which is necessary for simplification and approximate solutions.**

2) **Incompressible NSE:**

$$\rho\frac{D\vec{V}}{Dt}=\rho\left(\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\nabla)\vec{V}\right)=-\nabla P+\mu\nabla^2\vec{V}+\rho\vec{g}$$

3) **Each term is *dimensional, and* each variable or property (ρ, *V, t,* μ, etc.) is also dimensional.**

4) **What are the primary dimensions of each term in the N-S equation?**

5) **To nondimensionalize, choose scaling parameters as follows**

$$\boxed{\left\{\frac{m}{L^2t^2}\right\}}$$

8.2 Nondimensional N-S Equation

1) **Nondimensional variables, using the scaling parameters in Table 10-1**

$$t^\star = ft \quad \vec{x}^\star=\frac{\vec{x}}{L} \quad \vec{V}^\star=\frac{\vec{V}}{V} \quad P^\star=\frac{P-P_\infty}{P_0-P_\infty} \quad \vec{g}^\star=\frac{\vec{g}}{g} \quad \nabla^\star=L\nabla$$

2) **To plug the nondimensional variables into the NSE, we need to first rearrange the equations in terms of the dimensional variables:**

$$t=\frac{1}{f}t^\star \quad \vec{x}=L\vec{x}^\star \quad \vec{V}=V\vec{V}^\star \quad \nabla=\frac{1}{L}\nabla^\star \quad P=P_\infty+(P_0-P_\infty)P^\star \quad \vec{g}=g\vec{g}^\star$$

3) **Substitute into the NSE to obtain:**

$$\rho V f\frac{\partial\vec{V}^\star}{\partial t^\star}+\frac{\rho V^2}{L}\left(\vec{V}^\star\cdot\nabla^\star\right)\vec{V}^\star=-\frac{P_0-P_\infty}{L}\nabla^\star P^\star+\rho g\vec{g}^\star+\frac{\mu V}{L^2}\nabla^{\star 2}\vec{V}^\star$$

4) **Every additive term has primary dimensions $\{m^1L^{-2}t^{-2}\}$. To nondimensionalize, multiply every term by $L/(\rho V^2)$, which has primary dimensions $\{m^{-1}L^2t^2\}$, so that the dimensions cancel. After rearrangement:**

$$\left[\frac{fL}{V}\right]\frac{\partial\vec{V}^\star}{\partial t^\star}+\left(\vec{V}^\star\cdot\nabla^\star\right)\vec{V}^\star=-\left[\frac{P_0-P_\infty}{\rho V^2}\right]\nabla^\star P^\star+\left[\frac{gL}{V^2}\right]\vec{g}^\star+\left[\frac{\mu}{\rho VL}\right]\nabla^{\star 2}\vec{V}^\star$$

8.2 Nondimensional N-S Equation

Terms in [ ] are nondimensional parameters:

$$\boxed{\left[\frac{fL}{V}\right]}\frac{\partial \vec{V}^\star}{\partial t^\star} + \left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -\boxed{\left[\frac{P_0 - P_\infty}{\rho V^2}\right]}\nabla^\star P^\star + \boxed{\left[\frac{gL}{V^2}\right]}\vec{g}^\star + \boxed{\left[\frac{\mu}{\rho V L}\right]}\nabla^{\star 2}\vec{V}^\star$$

Strouhal number

Euler number

Inverse of Froude number squared

Inverse of Reynolds number

$$\boxed{[St]\frac{\partial \vec{V}^\star}{\partial t^\star} + \left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -[Eu]\,\nabla^\star P^\star + \left[\frac{1}{Fr^2}\right]\vec{g}^\star + \left[\frac{1}{Re}\right]\nabla^{\star 2}\vec{V}^\star}$$

8.3 Creeping Flow

1) Also known as "Stokes Flow" or "Low Reynolds number flow"
2) Occurs when Re << 1
   ① ρ, V, or L are very small, e.g., micro-organisms, MEMS, nano-tech, particles, bubbles
   ② μ is very large, e.g., honey, lava
3) To simplify NSE

$$[Eu]\,\nabla^\star P^\star = \left[\frac{1}{Re}\right]\nabla^{\star 2}\vec{V}^\star$$

Pressure forces **Viscous forces**

$$P^\star \sim 1, \quad \nabla^\star \sim 1$$

4) Since

$$Eu = \frac{P_0 - P_\infty}{\rho V^2} \sim \frac{1}{Re} = \frac{\mu}{\rho V L} \Rightarrow \boxed{P_0 - P_\infty \sim \frac{\mu V}{L}}$$

8.3 Creeping Flow

1) This is important $\boxed{P_0 - P_\infty \sim \dfrac{\mu V}{L}}$

   ① Very different from inertia dominated flows where

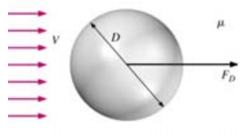
   $$P_0 - P_\infty \sim \rho V^2$$

   ② Density has completely dropped out of NSE. To demonstrate this, convert back to dimensional form.

   $$\nabla P = \mu \nabla^2 \vec{V}$$

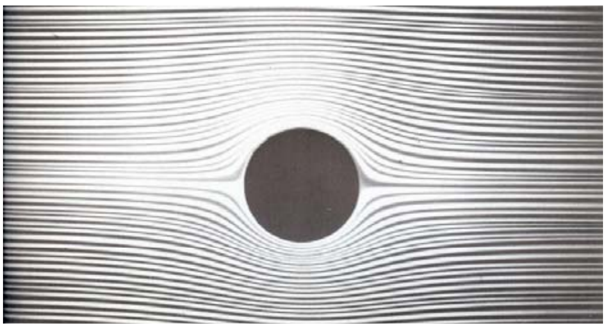
   ③ This is now a **LINEAR EQUATION** which can be solved for simple geometries.

2) Analytical solution for flow over a sphere gives a drag coefficient which is a linear function of velocity V and viscosity m.



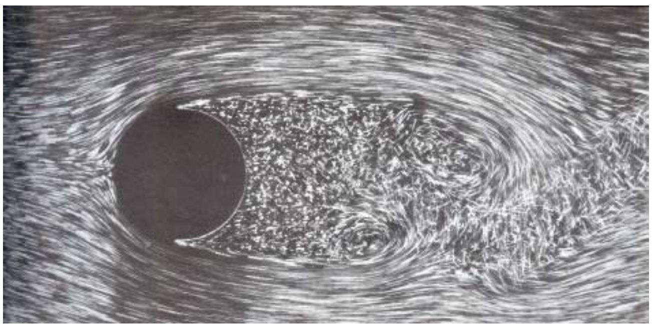
$$F_D = 3\pi\mu V D$$

8.3 Creeping Flow



Flowing 1mm/s between glass plates spaced 1mm apart.

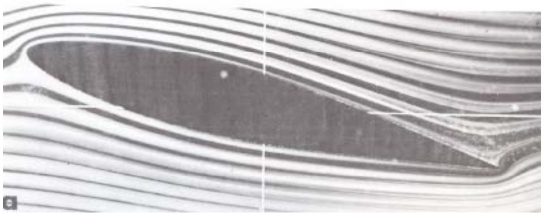
Creeping Flow (Hele-Shaw Flow)

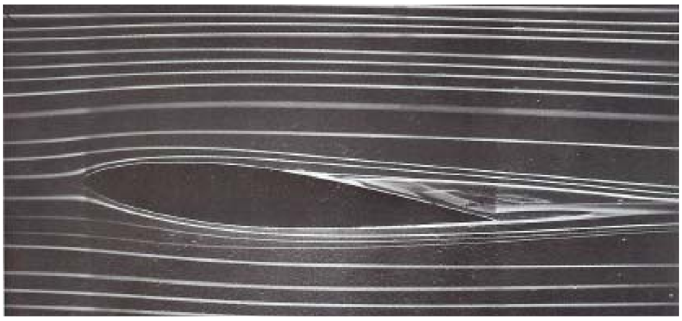


Re = 2,000

Laminar & Turbulent Flow

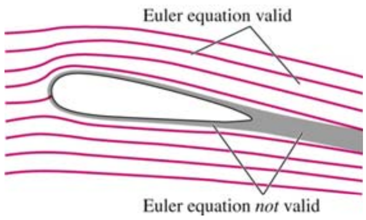
8.3 Creeping Flow





8.4 Inviscid Regions of Flow

1) **Inviscid Regions: where net viscous forces are negligible compared to pressure and/or inertia forces**



2) **Euler Equation:**

~0 if *Re* large

$$[St]\,\frac{\partial \vec{V}^{\star}}{\partial t^{\star}} + \left(\vec{V}^{\star}\cdot\nabla^{\star}\right)\vec{V}^{\star} = -\,[Eu]\,\nabla^{\star}P^{\star} + \left[\frac{1}{Fr^{2}}\right]\vec{g}^{\star} + \cancel{\left[\frac{1}{Re}\right]}\,\nabla^{\star 2}\vec{V}^{\star}$$

3) **Euler equation often used in aerodynamics and hydrodynamics**

4) **Elimination of viscous term changes PDE from mixed elliptic-hyperbolic to hyperbolic. This affects the type of analytical and computational tools used to solve the equations.**

5) **Must "relax" wall boundary condition from no-slip to slip**

<u>No-slip BC</u>
$u = v = w = 0$

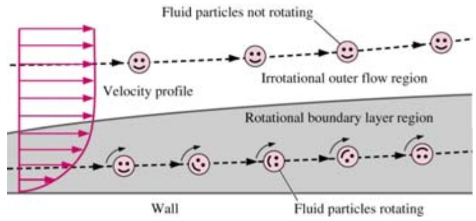
<u>Slip BC</u>
$\tau_w = 0,\ V_n = 0$

V_n = normal velocity

8.4.1 Potential Flow

1) Irrotational approximation: vorticity is negligibly small

$$\vec{\zeta} = \vec{\nabla} \times \vec{V} = 2\vec{\omega} \cong 0$$



2) In general, inviscid regions are also irrotational.

3) <u>Continuity equation</u>

① Use the vector identity: $\nabla \times \nabla \phi = 0$

② Since the flow is irrotational: $\nabla \times \vec{V} = 0$

③ ϕ is a scalar (velocity) potential function $\vec{V} = \nabla \phi$

4) Regions of irrotational flow are also called regions of potential flow.

5) Ideal Flow = Incompressible + Inviscid Flow

6) Potential Flow = Ideal Flow + Irrotational Flow

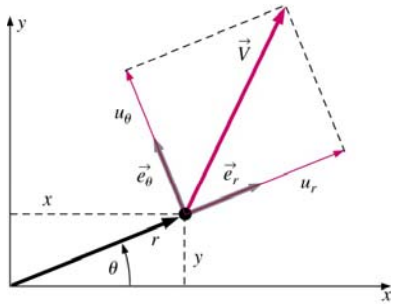
8.4.2 Potential Flow: velocity potential

1) Coordinate Systems

① Cartesian coordinates:

$$\vec{V}(u,v,w) = \nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

② Polar coordinates



$$\vec{V}(u_r,u_\theta,u_z) = \frac{\partial \phi}{\partial r} e_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta} e_\theta + \frac{\partial \phi}{\partial z} e_z$$

2) Substituting into the continuity equation gives: Laplace Equation

$$\nabla \cdot \vec{V} = 0 \quad \rightarrow \quad \nabla \cdot \nabla \phi = \nabla^2 \phi = 0$$

3) This means we only need to solve 1 linear scalar equation to determine all 3 components of velocity!

4) Laplace equation appears in numerous fields of science, engineering, and mathematics. This means there are well developed tools for solving this equation.

8.4.2 Potential Flow: stream function

1) In general, continuity equation cannot be used by itself to solve for flow field, however it can be used to find the missing velocity component if the flow field is incompressible.

2) Consider the continuity equation for an incompressible 2D flow:

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

3) Definition of the **Stream Function: one dependent variable (ψ) instead of two dependent variables (u, v)**

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

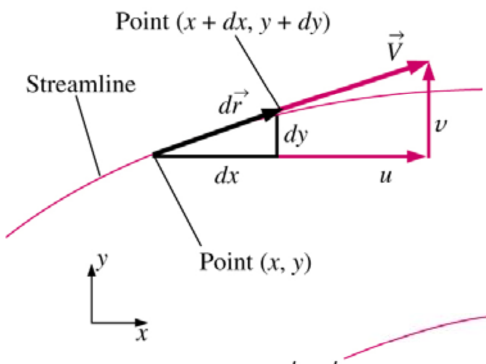
4) Substitution 3) into 2) yields: which is identically satisfied for any smooth function ψ(x, y).

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

8.4.2 Potential Flow: stream function

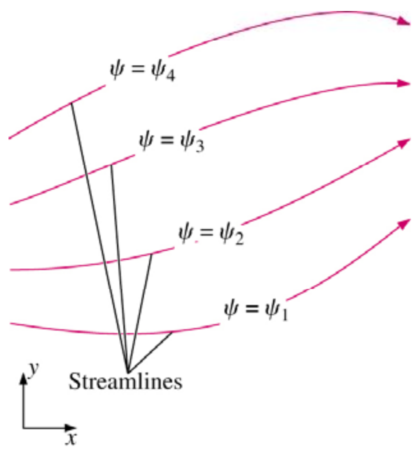
1) Consider streamlines:

$$\frac{dy}{dx} = \frac{v}{u} \quad \rightarrow \quad udy - vdx = 0$$



2) Total change of ψ:

$$\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$$

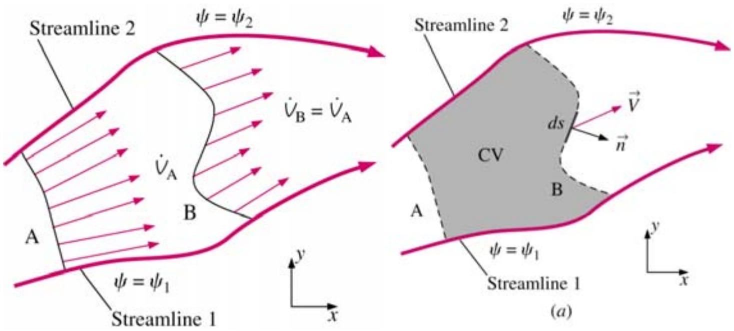


- ψ is constant along streamlines
- Change in ψ along streamlines is zero
- Curves of constant ψ are streamlines of the flow

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$$

8.4.2 Potential Flow: stream function

1) **Difference in ψ from one streamline to another is equal to the volume flow rate per unit width between the two streamlines**

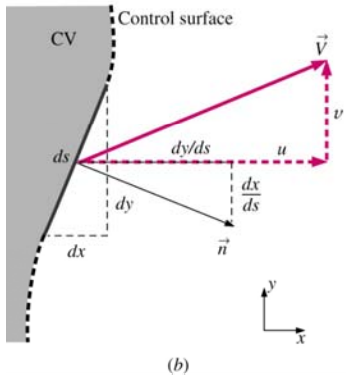


$$\vec n = \frac{dy}{ds}\vec i - \frac{dx}{ds}\vec j$$

$$d\dot{V} = \vec V \cdot \vec n dA = (u\vec i + v\vec j)\cdot\left(\frac{dy}{ds}\vec i - \frac{dx}{ds}\vec j\right)ds$$

$$d\dot{V} = udy - vdx = \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial x}dx = d\psi$$

$$\dot{V}_B = \int_B \vec V \cdot \vec n dA = \int_B d\dot{V} = \int_{\psi=\psi_1}^{\psi=\psi_2} d\psi = \psi_2 - \psi_1$$



8.4.2 Potential Flow: Momentum Equation

1) Momentum equation

① **If we can compute ϕ from the Laplace equation (which came from continuity) and velocity from the definit $\vec V = \nabla\phi$, why do we need the NSE? Answer: To compute Pressure.**

② **Apply irrotational approximation to viscous term of the NSE:**

$$\mu\nabla^2\vec V = \mu\nabla^2(\nabla\phi) = \mu\nabla(\nabla^2\phi) = 0$$

2) Therefore, the NSE reduces to the Euler equation for irrotational flow

$$\rho\left(\frac{\partial \vec V}{\partial t} + \boxed{(\vec V\cdot\nabla)\vec V}\right) = -\nabla P + \cancel{\mu\nabla^2\vec V} + \rho\vec g$$

3) Instead of integrating to find P, use vector identity to derive Bernoulli equation

$$(\vec V\cdot\nabla)\vec V = \nabla\left(\frac{\vec V\cdot\vec V}{2}\right) - \vec V\times(\nabla\times\vec V) = \boxed{\nabla\left(\frac{V^2}{2}\right) - \vec V\times\vec\varsigma}$$

8.4.2 Potential Flow: Momentum Equation

1) <u>Steady</u> Euler equation to be written as

$$\nabla\left(\frac{V^2}{2}\right)-\vec{V}\times\vec{\varsigma}=-\frac{\nabla P}{\rho}+\vec{g}$$

2) This form of Bernoulli equation is valid for inviscid and irrotational flow since we've shown that NSE reduces to the Euler equation.

$$\nabla\left(\frac{P}{\rho}+\frac{V^2}{2}+gz\right)=\vec{V}\times\vec{\varsigma}$$

3) Inviscid Flow: along a streamline

4) Irrotational Flow: everywhere

$$\frac{P}{\rho}+\frac{V^2}{2}+gz=C$$

Irrotational Flow Analysis

Calculate Velocity Potential ϕ form the Lapace Eqn.

Calculate Velocity from its definition

Calculate Pressure from the Bernoulli Eqn.

8.4.2 Potential Flow: Stream Function & Velocity Potential

1) Velocity components and Lapalce Eqn. in Cartesian coordinates:

$$u=\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}\qquad v=\frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$$

$$\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

2) Velocity components and Lapalce Eqn. in Cylinderical coordinates:

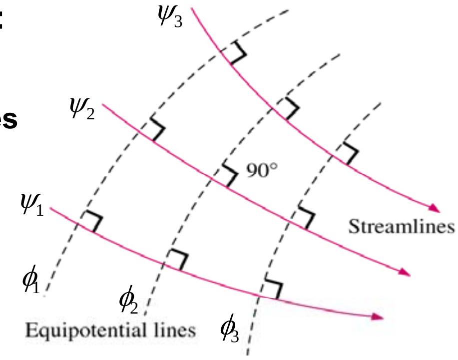
$$u_r=\frac{\partial\phi}{\partial r}=\frac{1}{r}\frac{\partial\psi}{\partial\theta}\qquad u_\theta=\frac{1}{r}\frac{\partial\phi}{\partial\theta}=-\frac{\partial\psi}{\partial r}$$

$$\nabla^2\phi=\frac{1}{r}\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0$$

8.4.2 Potential Flow: Stream Function & Velocity Potential

1) Recall the definition of streamfunction:

- ① Constant values of ψ: streamlines
- ② Constant values of ϕ: equipotential lines
- ③ ψ and ϕ are mutually orthogonal
- ④ ψ and ϕ are harmonic functions
- ⑤ ψ is defined by continuity; $\nabla^2\psi$ results from irrotationality
- ⑥ ϕ is defined by irrotationality; $\nabla^2\phi$ results from continuity



$$\phi \perp \psi$$

2) Since vorticity is zero: $\varsigma_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0$

$$u = \frac{\partial \psi}{\partial y},\; v = -\frac{\partial \psi}{\partial x} \;\rightarrow\; \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Flow solution can be achieved by solving either $\nabla^2\phi$ or $\nabla^2\psi$, however, BC are easier to formulate for ψ.

3) Laplace equation holds for the streamfunction and the velocity potential:

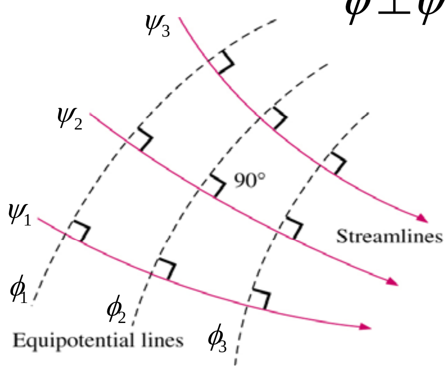
8.4.2 Potential Flow: Stream Function & Velocity Potential

1) Streamlines: $d\psi=0$ (ψ is constant along streamlines)

$$\phi \perp \psi$$

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$$

$$\left.\frac{dy}{dx}\right|_{d\psi=0} = \frac{-\partial\psi/\partial x}{\partial\psi/\partial y} = \frac{v}{u}$$



2) Equipotential lines: $d\phi=0$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0$$

$$\left.\frac{dy}{dx}\right|_{d\phi=0} = \frac{-\partial\phi/\partial x}{\partial\phi/\partial y} = \frac{-u}{v}$$

$$\left(\left.\frac{dy}{dx}\right|_{d\psi=0}\right) \times \left(\left.\frac{dy}{dx}\right|_{d\phi=0}\right) = \left(\frac{v}{u}\right) \times \left(\frac{-u}{v}\right) = -1$$

Thus ψ **and** ϕ **are mutually orthogonal**

8.5 2D P0tential Flows

1) Method of Superposition

① Since $\nabla^2\phi=0$ is linear, a linear combination of two or more solutions is also a solution, e.g., if ϕ_1 and ϕ_2 are solutions, then ($A\phi_1$), ($A+\phi_1$), ($\phi_1+\phi_2$), ($A\phi_1+B\phi_2$) are also solutions

② Also true for y in 2D flows ($\nabla^2\psi=0$)

③ Velocity components are also additive

$$u=\frac{\partial\phi}{\partial x}=\frac{\partial(\phi_1+\phi_2)}{\partial x}=\frac{\partial\phi_1}{\partial x}+\frac{\partial\phi_2}{\partial x}$$

2) Given the principal of superposition, there are several elementary planar irrotational flows which can be combined to create more complex flows.

① Uniform stream

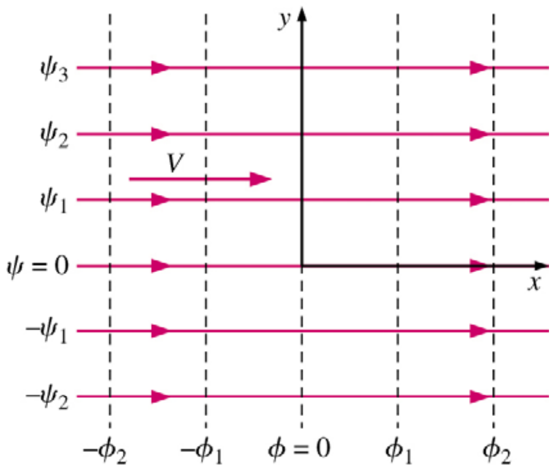
② Line source/sink

③ Line vortex

④ Doublet

8.5.1 Uniform Stream

$$V(u,v)=ui+0j$$



1) In Cartesian coordinates

$$u=\frac{\partial\phi}{\partial x}=V,\quad v=\frac{\partial\phi}{\partial y}=0$$

$$\phi=\int Vdx+c=Vx$$

$$u=\frac{\partial\psi}{\partial y}=V,\ v=-\frac{\partial\psi}{\partial x}=0$$

$$\left[\frac{m^2}{s}\right]$$

$$\psi=\int Vdy+c=Vy$$

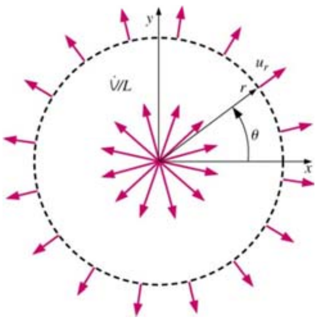
2) In Cylindrical coordinates

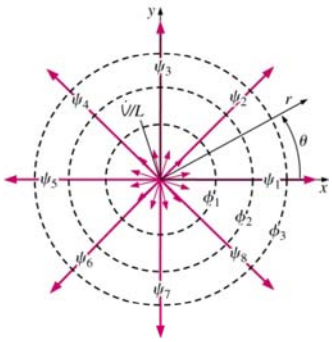
$$x=r\cos x\ \rightarrow\ \phi=Vr\cos\theta$$

$$y=r\sin x\ \rightarrow\ \psi=Vr\sin\theta$$

8.5.2 Line Source

$Q = \dot{V} / L$





1) **Potential and streamfunction are derived by observing that volume flow rate across any circle:**

2) **Velocity components:**

$$\phi = \frac{Q}{2\pi}\ln r \qquad \Leftrightarrow \qquad u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{Q}{2\pi}\frac{1}{r}$$

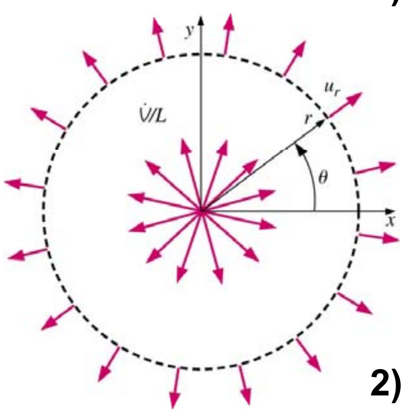
$$\psi = \frac{Q}{2\pi}\theta \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = 0$$

3) **If the source is placed at *(a, b)***

$$\phi = \frac{Q}{2\pi}\ln r_1 = \frac{Q}{2\pi}\ln\sqrt{(x-a)^2 + (y-b)^2}$$

$$\psi = \frac{Q}{2\pi}\theta = \frac{Q}{2\pi}\tan^{-1}\left(\frac{(y-b)}{(x-a)}\right)$$

8.5.2 Line Source & Sink



1) **Stream function:**

$$\frac{\partial \psi}{\partial r} = -u_\theta = 0 \qquad \therefore \psi = f(\theta)$$

$$\frac{\partial \psi}{\partial \theta} = f'(\theta) = r u_r = \frac{Q}{2\pi}$$

$$\therefore \psi = \frac{Q}{2\pi}\theta + c \qquad (c = 0)$$

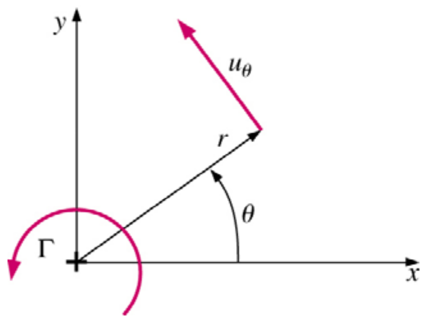
2) **Source strength: volume flow rate:** $\dot{V} / L \quad [m^2 / s]$

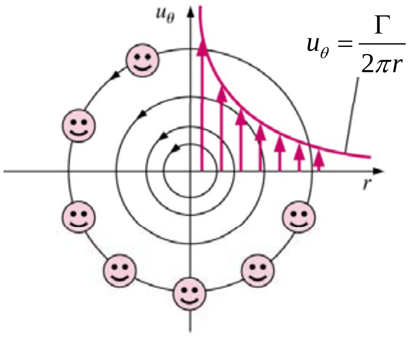
$$\int u_r dl = \int_0^{2\pi} u_r r d\theta = \int_0^{2\pi} \frac{Q}{2\pi} d\theta = Q$$

3) **Sink:** $Q < 0$

$$\phi = -\frac{Q}{2\pi}\ln r, \quad \psi = -\frac{Q}{2\pi}\theta$$

8.5.3 Line Vortex





1) Vortex at the origin. Velocity components

$$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0 \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

2) These can be integrated to give ϕ and ψ

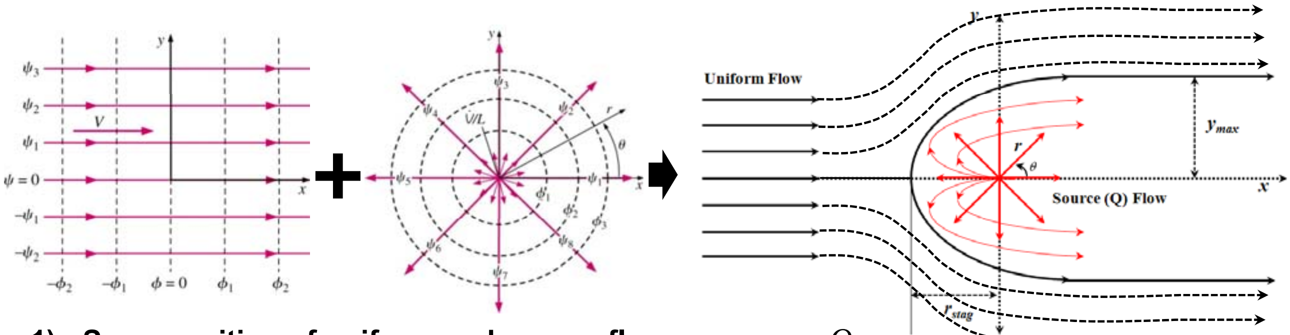
$$\phi = \frac{\Gamma}{2\pi}\theta, \qquad \psi = -\frac{\Gamma}{2\pi}\ln r\theta$$

3) If vortex is moved to *(x, y) = (a, b)*

$$\phi = \frac{\Gamma}{2\pi}\theta_1 = \frac{\Gamma}{2\pi}\tan^{-1}\left(\frac{(y-b)}{(x-a)}\right)$$

$$\psi = -\frac{\Gamma}{2\pi}r_1 = -\frac{\Gamma}{2\pi}\ln\sqrt{(x-a)^2 + (y-b)^2}$$

8.5.4. Superposition (Uniform + Source)



1) **Superposition of uniform and source flows:** $\phi = Vr\cos\theta + \frac{Q}{2\pi}\ln r$

2) **Velocity** $u_r = \frac{\partial \phi}{\partial r} = V\cos\theta + \frac{Q}{2\pi}\frac{1}{r}, \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -V\sin\theta$

3) **Stagnation point(r_{stag})**

$$u_r = u_\theta = 0 \quad (\text{at } \theta = \pi)$$

$$u_r(\pi)_{stag} = -V + \frac{Q}{2\pi}\frac{1}{r_{stag}} = 0, \quad u_\theta(\pi) = 0$$

$$\therefore r_{stag} = \frac{Q}{2\pi V}$$

4) Stream function

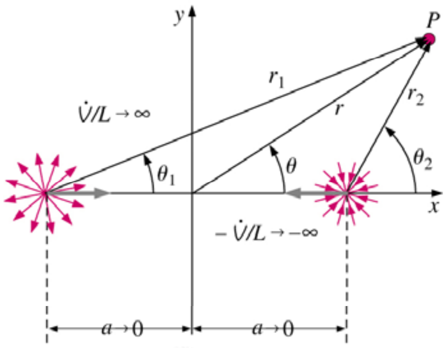
$$\psi = Vy + \frac{Q}{2\pi}\theta = Vr\sin\theta + \frac{Q}{2\pi}\theta$$

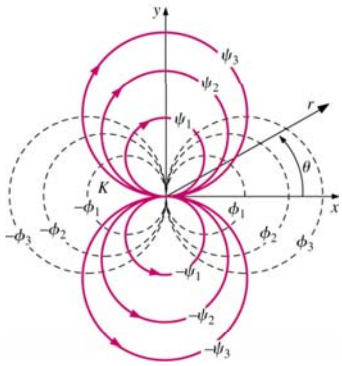
$$\psi_{stag} = 0 + \frac{Q}{2\pi}\pi = \frac{Q}{2}$$

$$\therefore y_{max} = \frac{Q}{2V}$$

$$Vy + \frac{Q}{2\pi}\theta = \frac{Q}{2} \;\rightarrow\; y = \frac{Q}{2\pi V}(\pi - \theta) = \frac{Q}{2V}\left(1 - \frac{\theta}{\pi}\right)$$

8.5.4. Superposition (Source + Sink = Doublet)





1) **A doublet is a combination of a line source and sink of equal magnitude**

2) **Source:**

$$\phi_1 = \frac{Q}{2\pi}\ln r_1, \quad \psi_1 = \frac{Q}{2\pi}\theta_1, \quad \theta_1 = \tan^{-1}\left(\frac{y}{x+a}\right)$$

3) **Sink:**

$$\phi_2 = \frac{Q}{2\pi}\ln r_2, \quad \psi_2 = \frac{Q}{2\pi}\theta_2, \quad \theta_2 = \tan^{-1}\left(\frac{y}{x-a}\right)$$

4) **Adding ψ_1 and ψ_2 together, taking $a \to 0$**

$$\phi = \lim_{2a\to 0}\frac{Q}{2\pi}\left(\ln r_1 + \ln r_2\right)$$

$$= \lim_{2a\to 0}\frac{Q}{2\pi}\left(\ln\sqrt{(x+a)^2+y^2} - \ln\sqrt{(x-a)^2+y^2}\right)$$

$$= \lim_{2a\to 0}\frac{2aQ}{2\pi}\left(\frac{\ln\sqrt{(x+a)^2+y^2} - \ln\sqrt{(x-a)^2+y^2}}{2a}\right)$$

8.5.4. Superposition (Source + Sink = Doublet)

1) **Velocity potential**

let $\quad 2aQ = \Lambda, \quad 2a \to 0$

$$\phi = \frac{\Lambda}{2\pi}\lim_{2a\to 0}\left(\frac{\ln\sqrt{(x+a)^2+y^2} - \ln\sqrt{(x-a)^2+y^2}}{2a}\right)$$

$$= \frac{\Lambda}{2\pi}\left.\frac{\partial}{\partial x}\right|_{a\to 0}\left(\ln\sqrt{(x+a)^2+y^2}\right)$$

$$= \frac{\Lambda}{2\pi}\frac{x}{\left(x^2+y^2\right)} = \frac{\Lambda}{2\pi}\frac{r\cos\theta}{r^2}$$

$$\therefore \phi = \frac{\Lambda}{2\pi}\frac{\cos\theta}{r}$$

2) Stream function

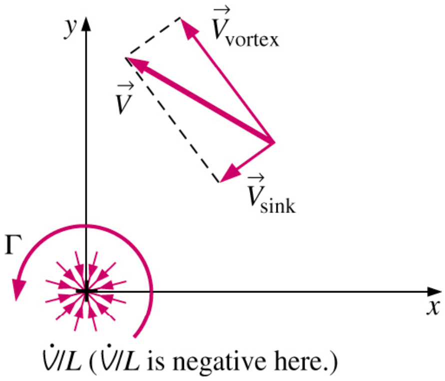
$$u_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

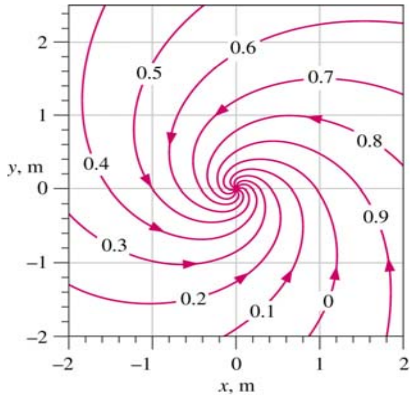
$$u_r = -\frac{\Lambda}{2\pi}\frac{\cos\theta}{r^2} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$\frac{\partial\psi}{\partial\theta} = -\frac{\Lambda}{2\pi}\frac{\cos\theta}{r}$$

$$\therefore \psi = -\frac{\Lambda}{2\pi}\frac{\sin\theta}{r}$$

8.5.5 Superposition (Sink + Vortex)





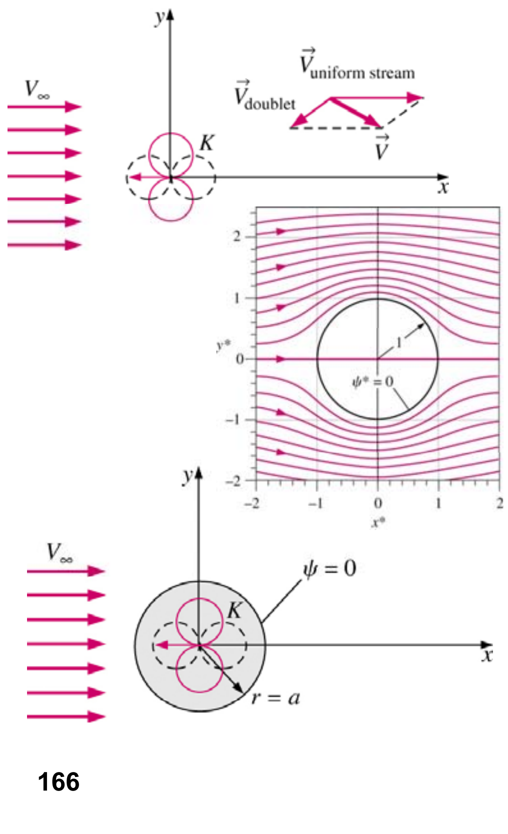
1) **Superposition of sink and vortex : Spiral vortex**

$$\psi = \underbrace{\frac{Q}{2\pi}\theta}_{\textbf{Sink}} - \underbrace{\frac{\Gamma}{2\pi}\ln r}_{\textbf{Vortex}}$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{Q}{2\pi}\frac{1}{r}$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

8.5.6 Superposition (Uniform + Doublet)



1) **Flow over a circular cylinder: Free stream + doublet**

$$\phi = Vr\cos\theta + \frac{\Lambda}{2\pi}\frac{\cos\theta}{r}, \quad \psi = Vr\sin\theta - \frac{\Lambda}{2\pi}\frac{\sin\theta}{r}$$

2) **Assume body is $\psi = 0$ (at $r = a$)**

$$\psi_{body} = \sin\theta\left(V - \frac{\Lambda}{2\pi}\frac{1}{r^2}\right) = 0 \quad \rightarrow \quad \therefore \Lambda = 2\pi a^2 V$$

$$\phi = Vr\cos\theta + \frac{a^2 V\cos\theta}{r} = V\cos\theta\left(r + \frac{a^2}{r}\right)$$

$$\psi = Vr\sin\theta - \frac{a^2 V\sin\theta}{r} = V\sin\theta\left(r - \frac{a^2}{r}\right)$$

3) **Velocity fields:**

$$u_r = \frac{\partial \phi}{\partial r} = V\cos\theta\left(1 - \frac{a^2}{r^2}\right)$$

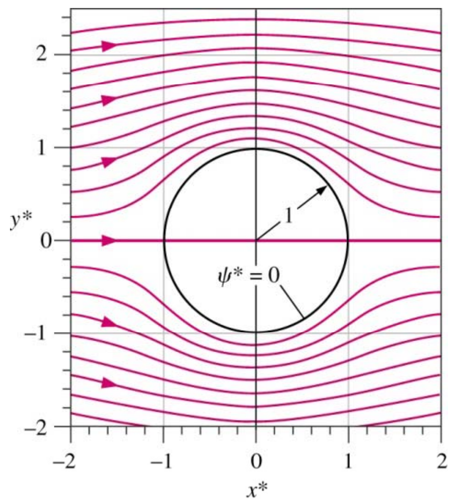
$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -V\sin\theta\left(1 + \frac{a^2}{r^2}\right)$$

8.5.6 Superposition (Uniform + Doublet)

1) **On the cylinder surface (r=a)**

$$u_r = \frac{\partial \phi}{\partial r} = V\cos\theta\left(1 - \frac{a^2}{a^2}\right) = 0$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -V\sin\theta\left(1 + \frac{a^2}{a^2}\right) = -2V\sin\theta$$



2) **Compute pressure using Bernoulli equation and velocity on cylinder surface**

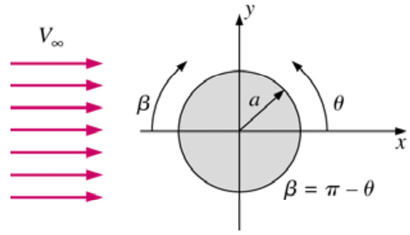
$$\frac{P}{\rho} + \frac{V^2}{2} + gy = \frac{P_\infty}{\rho} + \frac{V_\infty^{\ 2}}{2} + gy_\infty$$

$$C_p = \frac{P - P_\infty}{\rho V^2} = \left(1 - \frac{V^2}{V_\infty^{\ 2}}\right)$$

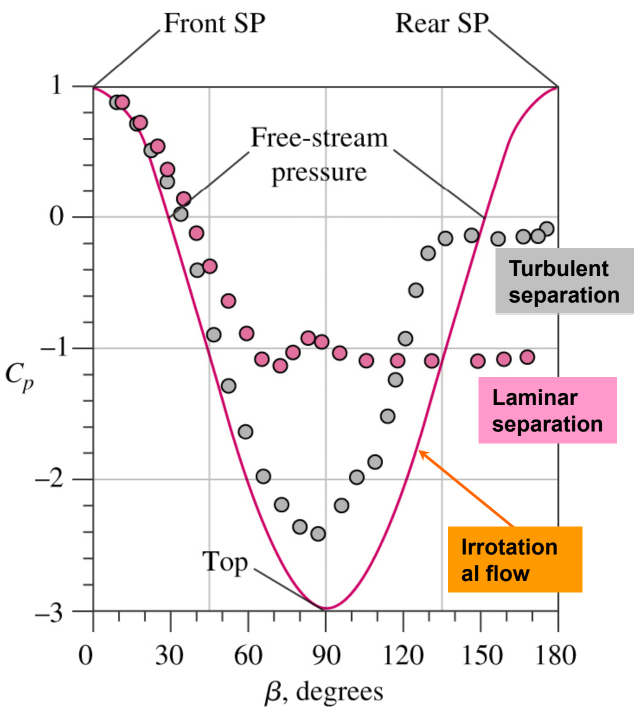
$$u_r = 0, \quad u_\theta = -2V_\infty \sin\theta$$

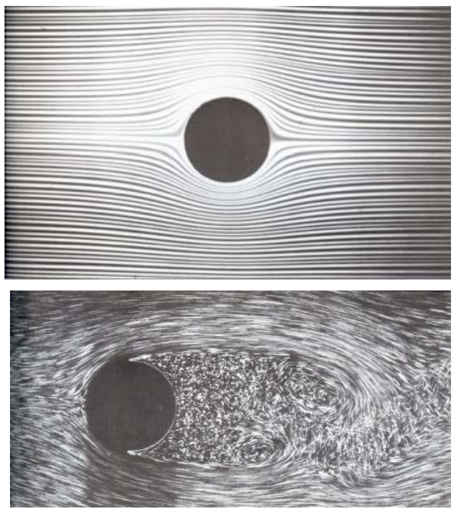
$$V^2 = u_r^2 + u_\theta^2 = 4V_\infty^2 \sin^2\theta$$

$$\therefore C_p = 1 - 4\sin^2\theta = 1 - 4\sin^2\beta$$



8.5.7 Velocity on cylinder surface





$$u_r = 0, \quad u_\theta = -2V_\infty \sin\theta$$

$$V^2 = u_r^2 + u_\theta^2 = 4V_\infty^2 \sin^2\theta$$

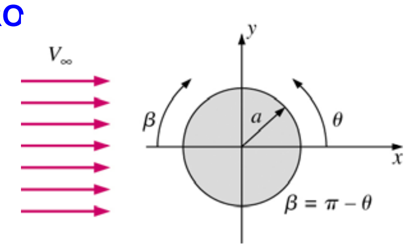
$$\therefore C_p = 1 - 4\sin^2\theta = 1 - 4\sin^2\beta$$

8.5.8 D'Alembert's Paradox

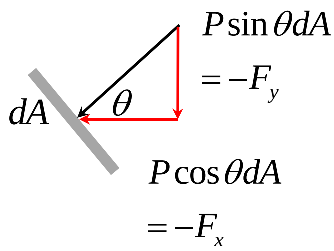
1) **D'Alembert's Paradox:** Integration of surface pressure (which is symmetric in x), reveals that the **DRAG is ZERO**

$$P = P_\infty + \frac{\rho V_\infty^{\,2}}{2} - \frac{\rho V^2}{2} = P_\infty + \frac{\rho V_\infty^{\,2}}{2} - 2\rho V_\infty^2 \sin^2\theta$$

$$\begin{cases} F_x = -\int P\cos\theta dA = -a\int_0^{2\pi} P\cos\theta d\theta = 0 \\ F_x = -\int P\sin\theta dA = -a\int_0^{2\pi} P\sin\theta d\theta = 0 \end{cases}$$



2) For the irrotational flow approximation, the drag force on ***any*** non-lifting body of ***any*** shape immersed in a uniform stream is <u>ZERO</u>



3) Why?

① Viscosity and the Viscous effects have been neglected.

② no-slip condition are responsible for

a. Flow separation (pressure drag)

b. Wall-shear stress (friction drag)