**Mathematics 4MB3/6MB3 Mathematical Biology**
**2019 ASSIGNMENT 1**

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# 1 Analysis of the SI model

The SI model can be written

$$\frac{dI}{dt} = \beta I(N - I)\,, \tag{1}$$

where $I$ denotes prevalence and $N = S + I$ is the total population size.

(a) *Proof.*
Consider the open set $D : \{I \in \mathbb{R} | 0 < I < N + \epsilon\}$ containing the the endemic equilibrium (EE) $I_* = N$ of equation (**??**), where $\epsilon > 0$ is constant. Suppose a continuous differentiable function $L : D \to \mathbb{R}$ is in the form

$$L(I) = a(N - I)^b \tag{2}$$

where $a$ and $b$ are constants and $a, b > 0$.
Then by Lyapunov's Direct Method,

(a) $L(I_*) = a(N - N)^b = 0$ and $L(I) > 0 \quad \forall I \in D\backslash\{I_*\}$

(b)

$$\begin{aligned}\dot{L}(I) &= \frac{dL}{dI}\frac{dI}{dt} \\ &= -ab(N - I)^{b-1}\beta I(N - I) \\ &= -ab\beta I(N - I)^b\end{aligned}$$

Let $a = 1$ and $b = 2$. Assuming $\beta > 0$, then

$$\dot{L}(I) = -2\beta I(N - I)^2 \leq 0 \forall I \; in D\backslash\{I_*\}$$

(c) $\dot{L}(I) < 0 \quad \forall I \in D\backslash\{I_*\}$

Since the above (a), (b), and (c) hold true for $L(I) = (N - I)^2$, then by Lyapunov's Stability Theorem, $I_* = N$ is globally asymptotically stable.

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(b) (i) *Proof.*

$$\begin{aligned}\frac{dI}{dt} &= \beta I(N-I) \\ \int \frac{1}{I(N-I)} &= \int \beta dt \\ \frac{1}{N}\int (\frac{1}{I} + \frac{1}{N-1})dI &= \beta t + C \\ \frac{1}{N}(ln|I| - ln|N-I|) &= \beta t + C \\ ln\left|\frac{I}{N-I}\right| &= N\beta t + NC \\ \frac{N-I}{I} &= \pm e^{-NC}e^{-N\beta t}\end{aligned}$$

Let $k = \pm e^{-C_2}$

$$\begin{aligned}\frac{N-I}{I} &= ke^{-N\beta t} \\ \frac{N}{I} - 1 &= ke^{-N\beta t} \\ \frac{N}{I} &= 1 + ke^{-N\beta t} \\ I(t) &= \frac{N}{1+ke^{-N\beta t}}\end{aligned}$$

To solve for $k$, let $I(0) = I_0$.

$$I(0) = I_0 = \frac{N}{1+k}$$
$$k = \frac{N - I_0}{I_0}$$

So the exact solution of the model is

$$I(t) = \frac{N}{1 + \frac{N-I_0}{I_0}(e^{-\beta N t})} \tag{3}$$

Since

$$\lim_{t\to\infty} e^{-\beta N t} = 0$$
$$\lim_{t\to\infty} I(t) = \frac{N}{1 + \frac{N-I_0}{I_0}(0)} = N$$

then every solution that starts in the interval $(0, N)$ converges to the EE.

$$\frac{dI}{dt} = \beta I(N - I)$$
$$\int \frac{1}{I(N-I)} dI = \int \beta dt$$
$$\frac{1}{N}\int (\frac{1}{I} + \frac{1}{N-1}) dI = \beta t + C$$
$$\frac{1}{N}(ln|I| - ln|N - I|) = \beta t + C$$
$$ln\left|\frac{I}{N-I}\right| = N\beta t + NC$$
$$\frac{N-I}{I} = \pm e^{-NC} e^{-N\beta t}$$

Let $k = \pm e^{-C_2}$

$$\frac{N-I}{I} = k e^{-N\beta t}$$
$$\frac{N}{I} - 1 = k e^{-N\beta t}$$
$$\frac{N}{I} = 1 + k e^{-N\beta t}$$
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So the exact solution of the model is

$$I(t) = \frac{N}{1 + \frac{N-I_0}{I_0}(e^{-\beta N t})} \tag{4}$$

Since

$$\lim_{t\to\infty} e^{-\beta N t} = 0$$
$$\lim_{t\to\infty} I(t) = \frac{N}{1 + \frac{N-I_0}{I_0}(0)} = N$$

then every solution that starts in the interval $(0, N)$ converges to the EE. □

(ii) tmp

# 2 Analysis of the basic SIR model

(a) The peak prevalence of the disease will be the maximum number of people infected at a given time $t$ during the infection. This implies that the peak prevalence is the maximum value of the function $I(t)$. In order to get a function of $I$ with the initial conditions $(S_0, I_0)$ as parameters we can consider the phase portrait solution:

$$I_{max} = I_0 + S_0 - \frac{1}{\mathcal{R}_0} + \frac{1}{\mathcal{R}_0} log(\frac{1}{S_0 \mathcal{R}_0}))$$

which can be derived as follows:

$$\begin{aligned}
\frac{dI}{dS} &= \frac{dI/dt}{dS/dt} \\
&= -1 + \frac{1}{S\mathcal{R}_0} \\
\int_{I_0}^{I} dI &= \int_{S_0}^{S} -1 + \frac{1}{S\mathcal{R}_0} dS \\
I - I_0 &= -(S - S_0) + \frac{1}{\mathcal{R}_0} log(\frac{S}{S_0}) \\
I &= I_0 + S_0 - S + \frac{1}{\mathcal{R}_0} log(\frac{S}{S_0})
\end{aligned}$$

All maxima and minima of a function occur at points $(x, y)$ such that $f'(x) = 0$, thus

$$\begin{aligned}
\frac{dI/dt}{dS/dt} &= \frac{\mathcal{R}_0 SI - I}{-\mathcal{R}_0 SI} \\
\frac{dI/dt}{dS/dt} &= -1 + \frac{1}{S\mathcal{R}_0} \\
0 &= -1 + \frac{1}{S\mathcal{R}_0} \\
1 &= \frac{1}{S\mathcal{R}_0} \\
1 &= S\mathcal{R}_0
\end{aligned}$$

This equation is true when $S = \frac{1}{\mathcal{R}_0}$, thus the maximum value for the function $I(S)$ occurs when $S = \frac{1}{\mathcal{R}_0}$. Substituting this into $I(S)$ will give an expression for $I_{max}$ in terms of the initial conditions $(S_0, I_0)$

$$I_{max} = I_0 + S_0 - \frac{1}{\mathcal{R}_0} + \frac{1}{\mathcal{R}_0} log(\frac{1}{S_0 \mathcal{R}_0}))$$

This quantity may be important to a public health officials for triage. If an epidemic is expected to have a low peak prevalence, fewer health-related resources would need to be allocated to treatment and prevention. If the peak prevalence is estimated to be high then more effort may be directed towards prevention and treatment of the disease. Further if the peak prevalence is estimated the time of peak prevalence can be derived easily using the $I(t)$ function, to estimate how much time exists to prepare for the peak of the infection, when resources (money, health personnel, equipment, etc.) will be most strained.

(b) (i)

$$\begin{aligned}
&S(R) = S_0 e^{-\mathcal{R}_0 R} \\
&S + I + R = 1 \implies I = 1 - S - R
\end{aligned}$$

(ii) This may be useful as the peak of an infection can calculated be from an estimate of the size of the recovered class.

(iii) I would advise my assistant to do this as it should be a fairly simple comparison and may provide insight in to the validity of the model. Peak prevelance of the time series can be compared to the peak prevelance estimated from the initial conditions $(S_0, I_0)$ of the time series data. If the model's prediction match the time series data very closely that may be indicative of some of the model's assumptions are better suited to modelling deaths as opposed to infections.

Although if deaths are assumed to match infections, then the model closely matching the time series data then that bodes well for the model. That is not a necessairily a reasonable assumption as records about death are necessairily incomplete

and not all deaths are attributable to disease, even with a perfect model the data will not match the predictions. This discordance will mean that no matter what there will be uncertainty in assessing the accuracy of the model using the time series data, but it still may be a good qualitative indicator of the model.

(iv) tmp

(c) tmp

(d) All points $(S, 0)S \in 0 \leq S \leq 1$ are equilibria for the SIR model. The jacobian of the system is

$$DF_{(S,I)} = \begin{bmatrix} -\mathcal{R}_0 I & -\mathcal{R}_0 S \\ \mathcal{R}_0 I & \mathcal{R}_0 S - 1 \end{bmatrix}$$

Substituting in the equilibrium point gives:

$$DF_{(S,0)} = \begin{bmatrix} 0 & -\mathcal{R}_0 S \\ 0 & \mathcal{R}_0 S - 1 \end{bmatrix}$$

The eigenvalues of the matrix are the roots of the equation $\lambda^2 - T\lambda + D$ where $T$ and $D$ are the trace and determinant respectively.

$$\begin{aligned} T &= \mathcal{R}_0 S - 1 \\ D &= 0 \\ 0 &= \lambda^2 - (\mathcal{R}_0 S - 1)\lambda + (0) \\ 0 &= \lambda(\mathcal{R}_0 S - 1 + \lambda) \end{aligned}$$

$$\lambda = 0 \qquad or \qquad \lambda = -(\mathcal{R}_0 S - 1)$$

For one of the eigenvalues $R(\lambda) = 0$, therefore the equilibria are non-hyperbolic and the stability must be assessed in some other way. This assessment can be done by examining a Lyapunov function. Lyapunov's theorem states that foran equilibrium point $X_*$ of $X' = F(X)$ and some set $\mathcal{S}$ if $\exists L(X)$ such that

$$\begin{aligned} L(X_*) &= 0 \\ L(X) &\geq 0 \quad \forall X \in S \backslash \{X_*\} \\ \nabla L(X) \cdot X' &< 0 \quad \forall X \in S \backslash \{X_*\} \end{aligned}$$

then $L(X)$ is a strict Lyapunov function and $X_*$ is asymptotically stable. The function $L(S, I) = S + I$ satisfies $L(S, I) \geq 0$ and is a candidate for a strict Lyapunov function. Further only values in $[0, 1]$ are considered for $S$ and $I$ as only those values have biological interpretations for the model (i.e $\mathcal{S} = [0, 1]$).

$$\begin{aligned} \nabla L &= (1, 1) \\ \nabla L \cdot X' &= (1, 1) \cdot (-\mathcal{R}_0 SI, \mathcal{R}_0 SI - I) \\ &= -\mathcal{R}_0 SI + \mathcal{R}_0 SI - I \\ &= -I \end{aligned}$$

Since $\nabla L \cdot X' = -I < 0 \quad \forall (S, I) \in \mathbb{R}^2 \backslash \{(S, 0) \quad \forall S \in \mathbb{R}\}$ all equilibria $(S, 0)$ are asymptotically stable by Lyapunov's theorem.

**— END OF ASSIGNMENT —**

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