Mathematics 4MB3/6MB3 Mathematical Biology
2019 ASSIGNMENT 1

Group Name: The Plague Doctore

Group Members: Sid Reed, Daniel Segura, Jessa Mallare, Aref Jadda

1 Analysis of the SI model

The SI model can be written

$$\frac{dI}{dt} = \beta I(N - I)\,, \tag{1}$$

where I denotes prevalence and $N = S + I$ is the total population size.

(a)

(b) (i)

(ii)

2 Analysis of the basic SIR model

(a) The peak prevalence of the disease will be the maximum number of people infected at a given time t during the infection. This implies that the peak prevalence is the maximum value of the function $I(t)$. In order to get a function of I with the initial conditions (S_0, I_0) as parameters we can consider the phase portrait solution:

$$I_{max} = I_0 + S_0 - \frac{1}{\mathcal{R}_0} + \frac{1}{\mathcal{R}_0} log(\frac{1}{S_0\mathcal{R}_0}))$$

which can be derived as follows:

$$\begin{aligned}
\frac{dI}{dS} &= \frac{dI/dt}{dS/dt} \\
&= -1 + \frac{1}{S\mathcal{R}_0} \\
\int_{I_0}^{I} dI &= \int_{S_0}^{S} -1 + \frac{1}{S\mathcal{R}_0} dS \\
I - I_0 &= -(S - S_0) + \frac{1}{\mathcal{R}_0} log(\frac{S}{S_0}) \\
I &= I_0 + S_0 - S + \frac{1}{\mathcal{R}_0} log(\frac{S}{S_0})
\end{aligned}$$

All maxima and minima of a function occur at points (x, y) such that $f'(x) = 0$, thus

$$\begin{aligned}
\frac{dI/dt}{dS/dt} &= \frac{\mathcal{R}_0 SI - I}{-\mathcal{R}_0 SI} \\
\frac{dI/dt}{dS/dt} &= -1 + \frac{1}{S\mathcal{R}_0} \\
0 &= -1 + \frac{1}{S\mathcal{R}_0} \\
1 &= \frac{1}{S\mathcal{R}_0} \\
1 &= S\mathcal{R}_0
\end{aligned}$$

This equation is true when $S = \frac{1}{\mathcal{R}_0}$, thus the maximum value for the function $I(S)$ occurs when $S = \frac{1}{\mathcal{R}_0}$. Substituting this into $I(S)$ will give an expression for I_{max} in terms of the initial conditions (S_0, I_0)

$$I_{max} = I_0 + S_0 - \frac{1}{\mathcal{R}_0} + \frac{1}{\mathcal{R}_0} log(\frac{1}{S_0 \mathcal{R}_0}))$$

This quantity may be important to a public health officials for triage. If an epidemic is expected to have a low peak prevalence, fewer health-related resources would need to be allocated to treatment and prevention. If the peak prevalence is estimated to be high then more effort may be directed towards prevention and treatment of the disease. Further if the peak prevalence is estimated the time of peak prevalence can be derived easily using the $I(t)$ function, to estimate how much time exists to prepare for the peak of the infection, when resources (money, health personnel, equipment, etc.) will be most strained.

(b) (i)

(ii)

(iii)

(iv)

(c)

(d) All points $(S, 0) S \in 0 \leq S \leq 1$ are equilibria for the SIR model. The jacobian of the system is

$$DF_{(S,I)} = \begin{bmatrix} -\mathcal{R}_0 I & -\mathcal{R}_0 S \\ \mathcal{R}_0 I & \mathcal{R}_0 S - 1 \end{bmatrix}$$

Substituting in the equilibrium point gives:

$$DF_{(S,0)} = \begin{bmatrix} 0 & -\mathcal{R}_0 S \\ 0 & \mathcal{R}_0 S - 1 \end{bmatrix}$$

The eigenvalues of the matrix are the roots of the equation $\lambda^2 - T\lambda + D$ where T and D are the trace and determinant respectively.

$$\begin{aligned}
T &= \mathcal{R}_0 S - 1 \\
D &= 0 \\
0 &= \lambda^2 - (\mathcal{R}_0 S - 1)\lambda + (0) \\
0 &= \lambda(\mathcal{R}_0 S - 1 + \lambda)
\end{aligned}$$

$$\lambda = 0 \qquad or \qquad \lambda = -(\mathcal{R}_0 S - 1)$$

For one of the eigenvalues $R(\lambda) = 0$, therefore the equilibria are non-hyperbolic and the stability must be assessed in some other way. This assessment can be done by examining a Lyapunov function. Lyapunov's theorem states that foran equilibrium point X_* of $X' = F(X)$ and some set $\mathcal{S}$ if $\exists L(X)$ such that

$$\begin{aligned}
L(X_*) &= 0 \\
L(X) &\geq 0 \quad \forall X \in S\backslash\{X_*\} \\
\nabla L(X) \cdot X' &< 0 \quad \forall X \in S\backslash\{X_*\}
\end{aligned}$$

then $L(X)$ is a strict Lyapunov function and X_* is asymptotically stable. The function $L(S, I) = S + I$ satisfies $L(S, I) \geq 0$ and is a candidate for a strict Lyapunov function. Further only values in $[0, 1]$ are considered for S and I as only those values have biological interpretations for the model (i.e $\mathcal{S} = [0, 1]$).

$$\begin{aligned}
\nabla L &= (1, 1) \\
\nabla L \cdot X' &= (1, 1) \cdot (-\mathcal{R}_0 SI, \mathcal{R}_0 SI - I) \\
&= -\mathcal{R}_0 SI + \mathcal{R}_0 SI - I \\
&= -I
\end{aligned}$$

Since $\nabla L \cdot X' = -I < 0 \quad \forall (S, I) \in \mathbb{R}^2\backslash\{(S, 0) \quad \forall S \in \mathbb{R}\}$ all equilibria $(S, 0)$ are asymptotically stable.

— END OF ASSIGNMENT —

Compile time for this document: January 19, 2019 @ 20:52