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Mathematics 4MB3/6MB3 Mathematical Biology
2019 ASSIGNMENT 1

Group Name: The Plague Doctors

Group Members: Sid Reed, Daniel Segura, Jessa Mallare, Aref Jadda

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1 Analysis of the SI model

The SI model can be written

$$\frac{dI}{dt} = \beta I(N - I), \tag{1}$$

where I denotes prevalence and $N = S + I$ is the total population size.

(a) *Proof.*
Consider the open set $D : \{I \in \mathbb{R} | 0 < I < N + \epsilon\}$ containing the the endemic equilibrium (EE) $I_* = N$ of equation (1), where $\epsilon > 0$ is constant. Suppose a continuous differentiable function $L : D \to \mathbb{R}$ is in the form

$$L(I) = a(N - I)^b \tag{2}$$

where a and b are constants and $a, b > 0$.
Then by Lyapunov's Direct Method,

(a) $L(I_*) = a(N - N)^b = 0$ and $L(I) > 0 \quad \forall I \in D \backslash \{I_*\}$

(b)

$$\begin{aligned}
\dot{L}(I) &= \frac{dL}{dI}\frac{dI}{dt} \\
&= -ab(N - I)^{b-1}\beta I(N - I) \\
&= -ab\beta I(N - I)^b
\end{aligned}$$

Let $a = 1$ and $b = 2$. Assuming $\beta > 0$, then

$$\dot{L}(I) = -2\beta I(N - I)^2 \leq 0 \, \forall I \, in D \backslash \{I_*\}$$

(c) $\dot{L}(I) < 0 \quad \forall I \in D \backslash \{I_*\}$

Since the above (a), (b), and (c) hold true for $L(I) = (N - I)^2$, then by Lyapunov's Stability Theorem, $I_* = N$ is globally asymptotically stable. $\qquad \square$

(b)   (i) *Proof.*

$$\frac{dI}{dt} = \beta I(N - I)$$

$$\int \frac{1}{I(N - I)} = \int \beta dt$$

$$\frac{1}{N} \int (\frac{1}{I} + \frac{1}{N - 1}) dI = \beta t + C$$

$$\frac{1}{N}(ln|I| - ln|N - I|) = \beta t + C$$

$$ln \left| \frac{I}{N - I} \right| = N\beta t + NC$$

$$\frac{N - I}{I} = \pm e^{-NC} e^{-N\beta t}$$

Let $k = \pm e^{-C_2}$

$$\frac{N - I}{I} = ke^{-N\beta t}$$

$$\frac{N}{I} - 1 = ke^{-N\beta t}$$

$$\frac{N}{I} = 1 + ke^{-N\beta t}$$

$$I(t) = \frac{N}{1 + ke^{-N\beta t}}$$

To solve for k, let $I(0) = I_0$.

$$I(0) = I_0 = \frac{N}{1 + k}$$

$$k = \frac{N - I_0}{I_0}$$

So the exact solution of the model is

$$I(t) = \frac{N}{1 + \frac{N - I_0}{I_0}(e^{-\beta N t})} \tag{3}$$

Since

$$\lim_{t \to \infty} e^{-\beta N t} = 0$$

$$\lim_{t \to \infty} I(t) = \frac{N}{1 + \frac{N - I_0}{I_0}(0)} = N$$

then every solution that starts in the interval $(0, N)$ converges to the EE. $\square$

(ii) When $I(t) \in (0, N)$, $\frac{dI}{dt} > 0$ since $\beta > 0, I > 0, and (N - I) > 0$. Hence the function monotonically increases in that interval and is always moving away from 0 and goes towards N (the Endemic Equilibrium). N is also the mathematical (and biological) upper bound of the system of relevant solutions. Therefore, by the Monotone Convergence Theorem all solutions with the intial conditions $I(0) \in (0, N)$ converge to N as $t \to \infty$, and so N is asymptotically stable. Hence, given $\epsilon > 0 \; \exists t < \infty$ such that $I(t) \in [N - \epsilon, N]$ for any $I(0) \in (0, N)$. Considering that "global" here only refers to biologically relevant conditions, only the range of $I(t) \in (0, N)$ needs to be analyzed. Ergo, the Endemic Equilibrium (N) is globally asymptotically stable.

2 Analysis of the basic SIR model

(a) The peak prevalence of the disease will be the maximum number of people infected at a given time t during the infection. This implies that the peak prevalence is the maximum value of the function $I(t)$. In order to get a function of I with the initial conditions (S_0, I_0) as parameters we can consider the phase portrait solution:

$$I_{max} = I_0 + S_0 - \frac{1}{\mathcal{R}_0} + \frac{1}{\mathcal{R}_0} log(\frac{1}{S_0 \mathcal{R}_0}))$$

which can be derived as follows:

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt}$$

$$= -1 + \frac{1}{S\mathcal{R}_0}$$

$$\int_{I_0}^{I} dI = \int_{S_0}^{S} -1 + \frac{1}{S\mathcal{R}_0} dS$$

$$I - I_0 = -(S - S_0) + \frac{1}{\mathcal{R}_0} log(\frac{S}{S_0})$$

$$I = I_0 + S_0 - S + \frac{1}{\mathcal{R}_0} log(\frac{S}{S_0})$$

All maxima and minima of a function occur at points (x, y) such that $f'(x) = 0$, thus

$$\frac{dI/dt}{dS/dt} = \frac{\mathcal{R}_0 SI - I}{-\mathcal{R}_0 SI}$$

$$\frac{dI/dt}{dS/dt} = -1 + \frac{1}{S\mathcal{R}_0}$$

$$0 = -1 + \frac{1}{S\mathcal{R}_0}$$

$$1 = \frac{1}{S\mathcal{R}_0}$$

$$1 = S\mathcal{R}_0$$

This equation is true when $S = \frac{1}{\mathcal{R}_0}$, thus the maximum value for the function $I(S)$ occurs when $S = \frac{1}{\mathcal{R}_0}$. Substituting this into $I(S)$ will give an expression for I_{max} in terms of the initial conditions (S_0, I_0)

$$I_{max} = I_0 + S_0 - \frac{1}{\mathcal{R}_0} + \frac{1}{\mathcal{R}_0} log(\frac{1}{S_0 \mathcal{R}_0}))$$

This quantity may be important to a public health officials for triage. If an epidemic is expected to have a low peak prevalence, fewer health-related resources would need to be allocated to treatment and prevention. If the peak prevalence is estimated to be high then more effort may be directed towards prevention and treatment of the disease. Further if the peak prevalence is estimated the time of peak prevalence can be derived easily using the $I(t)$ function, to estimate how much time exists to prepare for the peak of the infection, when resources (money, health personnel, equipment, etc.) will be most strained.

(b)   (i) First the function $S(R)$ is computed

$$\frac{dS}{dR} = \frac{dS/dt}{dR/dt}$$
$$\frac{dS/dt}{dR/dt} = \frac{-\mathcal{R}_0 SI}{I}$$
$$= -\mathcal{R}_0 S$$
$$\frac{dS}{S} = -\mathcal{R}_0 dR$$
$$\int \frac{dS}{S} = \int -\mathcal{R}_0 dR$$
$$ln(S) = \int -\mathcal{R}_0 R + C$$
$$S = C_1 e^{-\mathcal{R}_0 R}$$
$$S(0) = S_0 = C_1 e^{-\mathcal{R}_0(0)}$$
$$S_0 = C_1(1)$$
$$S(R) = S_0 e^{-\mathcal{R}_0 R}$$

From the function $S(R)$ and the relationship $S + I + R = N$ t(R) can be computed

as an integral using the $\frac{dR}{dt}$ relation.

$$S(R) = S_0 e^{-\mathcal{R}_0 R}$$
$$S + I + R = 1 \implies I = 1 - S - R$$
$$\frac{dR}{dt} = 1 - S_0 e^{-\mathcal{R}_0 R} - R$$
$$\int dR = \int (1 - S_0 e^{-\mathcal{R}_0 R} - R) dt$$
$$R + C_0 = t - t S_0 e^{-\mathcal{R}_0 R} - Rt$$
$$R + C_0 = t(1 - S_0 e^{-\mathcal{R}_0 R} - R)$$
$$t(R) = \frac{R + C_0}{1 - S_0 e^{-\mathcal{R}_0 R} - R}$$
$$t(0) = t_0 = \frac{C_0}{1 - S_0} = 0$$
$$C_0 = t_0(a - S_0)$$
$$C_0 = 0$$
$$t(R) = \frac{R}{1 - S_0 e^{-\mathcal{R}_0 R} - R}$$

(ii) This may be useful as the peak of an infection can calculated be from an estimate of the size of the recovered class.

(iii) I would advise my assistant to do this as it should be a fairly simple comparison and may provide insight in to the validity of the model. Peak prevelance of the time series can be compared to the peak prevelance estimated from the initial conditions (S_0, I_0) of the time series data. If the model's prediction match the time series data very closely that may be indicative of some of the model's assumptions are better suited to modelling deaths as opposed to infections.

Although if deaths are assumed to match infections, then the model closely matching the time series data then that bodes well for the model. That is not a necessairily a reasonable assumption as records about death are necessairily incomplete and not all deaths are attributable to disease, even with a perfect model the data will not match the predictions. This discordance will mean that no matter what there will be uncertainty in assessing the accuracy of the model using the time series data, but it still may be a good qualitative indicator of the model.

(iv) It is possible to find an exact analytical solution fro t as a function of S

$$\frac{dS}{dR} = -R_0 S$$

$$\int \frac{dS}{S} = \int -R_0 dR$$

$$\ln S = -R_0(R) + C_1$$

$$\frac{C_1 - \ln S_0}{R_0} = R(S)$$

$$R(S_0) = \frac{C_1 - \ln S_0}{R_0}$$

$$0 = C_1 - \ln S_0$$

$$C_1 = \ln S - 0$$

$$R(S) = \frac{\ln \frac{S_0}{S}}{R_0}$$

We can then use the fact that $I = 1 - S - R$ and incorporate the previous function $R(S)$ and $\frac{dS}{dt}$ to yield t(S)

$$\frac{dS}{dt} = -R_0 SI$$

$$\frac{dS}{dt} = -R_0 S(1 - S - \frac{\ln \frac{S_0}{S}}{R_0})$$

$$\frac{dS}{dt} = R_0 S^2 - R_0 S + S \ln \frac{S_0}{S}$$

$$\int dS = \int R_0 S^2 - R_0 S + S \ln \frac{S_0}{S} dt$$

$$S + C_2 = (R_0 S^2 - R_0 S + S \ln \frac{S_0}{S})t$$

$$t(S) = \frac{S + C_2}{R_0 S^2 - R_0 S + S \ln \frac{S_0}{S}}$$

$$t(S_0) = \frac{S_0 + C_2}{R_0 S_0^2 - R_0 S_0 + S_0 \ln \frac{S_0}{S_0}}$$

$$0 = \frac{S_0 + C_2}{R_0 S_0^2 - R_0 S_0 + S_0 \ln 1}$$

$$0 = S_0 + C_2$$

$$C_2 = -S_0$$

$$t(S) = \frac{S + S_0}{R_0 S^2 - R_0 S + S \ln \frac{S_0}{S}}$$

(c) $I^* = 0$ is a non-hyperbolic equilibrium, so to prove that all solutions to the SIR model approach $I = 0$ asymptotically stability cannot be assessed using the linearization theorem. However, the existance of a Lyapunov function would be able to prove the stability of these eqiuilibria. A function $L(S, I)$ is a Lyapunov function if:

(i) $L(S^*, I^*) = 0$

(ii) $L(S, I) > 0 \quad \forall (S, I) \in \mathbb{R}^{\not\vDash} \backslash \{S \in \mathbb{R}, I = 0\}$

(iii) $\nabla L(X) \cdot X' < 0 \quad \forall X \in \mathbb{R}^{\not\vDash} \backslash \{X_*\}$

If a Lyapunov function exists for the equilibrium point (S^*, I^*) then that point is asymptotically stable. We consider the function $L(S, I) = I$ and check whether it is a valid Lyapunov function for the eqilibria $(S^*, I^*) = (S, 0) \quad \forall S \in \mathbb{R}$:

$$L(S^*, I^*) = I^* = 0$$
$$L(S, I) = I > 0$$

Note $I > 0$ since we are examining the endemic equilibrium whch includes al values in $(0, 1]$ since $I = 0$ is the equilibrium point and thus not considered. Now we shoe that $\dot{L}(S, I) < 0$

$$\begin{aligned}
\dot{L}(S, I) &= \nabla L(S, I) \cdot X' \\
&= \frac{\partial L}{\partial S} \cdot \frac{dS}{dt} \\
&= 0 + (1) \frac{dI}{dt} \\
&= \mathcal{R}_0 S I - I \\
&= I(\mathcal{R}_0 S - 1)0 \qquad\qquad > I(\mathcal{R}_0 S - 1)
\end{aligned}$$

Since $I > 0$ $L(S, I) = I$ is a Lyapunov function when $\mathcal{R}_0 S - 1 < 0 \implies \mathcal{R}_0 < \frac{1}{S}$ which holds for all cases as $S \to S^*$. This proves that $L(S, I) = I$ is a strict Luapunov function and that the equilibria $(S^*, I^*) = (S, 0) \quad \forall S \in \mathbb{R}$ is asymptotically stable.

As the infection spreads, individuals move from the succeptible class to the infected class while individuals move from the infected class to the recovered class. As there in never an increase in the succeptible class the number of individuals entering the recovered class will be greater than the number of individuals entering the infected class. The number of individuals in the infected class will continuously decrease as people recover and $I \to 0$.

(d) All points $(S, 0) S \in 0 \leq S \leq 1$ are equilibria for the SIR model. The Jacobian of the system is

$$DF_{(S,I)} = \begin{bmatrix} -\mathcal{R}_0 I & -\mathcal{R}_0 S \\ \mathcal{R}_0 I & \mathcal{R}_0 S - 1 \end{bmatrix}$$

Substituting in the equilibrium point gives:

$$DF_{(S,0)} = \begin{bmatrix} 0 & -\mathcal{R}_0 S \\ 0 & \mathcal{R}_0 S - 1 \end{bmatrix}$$

The eigenvalues of the matrix are the roots of the equation $\lambda^2 - T\lambda + D$ where T and D are the trace and determinant respectively.

$$T = \mathcal{R}_0 S - 1$$
$$D = 0$$
$$0 = \lambda^2 - (\mathcal{R}_0 S - 1)\lambda + (0)$$
$$0 = \lambda(\mathcal{R}_0 S - 1 + \lambda)$$

$$\lambda = 0 \qquad or \qquad \lambda = -(\mathcal{R}_0 S - 1)$$

For one of the eigenvalues $R(\lambda) = 0$, therefore the equilibria are non-hyperbolic and the stability must be assessed in some other way. This assessment can be done by examining a Lyapunov function. Lyapunov's theorem states that for an equilibrium point X_* of $X' = F(X)$ and some set $\mathcal{S}$ if $\exists L(X)$ such that

(i) $L(X_*) = 0$

(ii) $L(X) \geq 0 \quad \forall X \in S\backslash\{X_*\}$

(iii) $\dot{L}(S, I) = \nabla L(X) \cdot X' < 0 \quad \forall X \in S\backslash\{X_*\}$

then $L(X)$ is a strict Lyapunov function and X_* is asymptotically stable. The function $L(S, I) = S + I$ satisfies $L(S, I) \geq 0$ and is a candidate for a strict Lyapunov function. Further only values in $[0, 1]$ are considered for S and I as only those values have biological interpretations for the model (i.e $\mathcal{S} = [0, 1]$).

$$\nabla L = (1, 1)$$
$$\nabla L \cdot X' = (1, 1) \cdot (-\mathcal{R}_0 SI, \mathcal{R}_0 SI - I)$$
$$= -\mathcal{R}_0 SI + \mathcal{R}_0 SI - I$$
$$= -I$$

Since $\nabla L \cdot X' = -I < 0 \quad \forall (S, I) \in \mathbb{R}^2\backslash\{(S, 0) \quad \forall S \in \mathbb{R}\}$ all equilibria $(S, 0)$ are globally asymptotically stable by Lyapunov's theorem.

— **END OF ASSIGNMENT** —

Compile time for this document: January 20, 2019 @ 21:44