Mathematics 4MB3/6MB3 Mathematical Biology

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2019 ASSIGNMENT 3

Group Name: The Plague Doctors

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This assignment is **due in class** on **Monday 25 February 2019 at 9:30am**.

Analysis of the standard SIR model with vital dynamics

Consider the standard SIR model with vital dynamics,

$$\frac{dS}{dt} = \mu N - \frac{\beta}{N} SI - \mu S \tag{1a}$$

$$\frac{dI}{dt} = \frac{\beta}{N} SI - \gamma I - \mu I \tag{1b}$$

$$\frac{dR}{dt} = \gamma I - \mu R \tag{1c}$$

where S, I and R denote the numbers of susceptible, infectious and removed individuals, respectively, and $N = S + I + R$ is the total population size. The *per capita* rates of birth and death are the same (both are equal to μ). As usual, β is the transmission rate and γ is the recovery rate.

(a) Since equations (1) represent all changes in the size of each population compartment, the net change in the total population should be the sum of the change in each compartment, i.e. the sum of all equations (1). If the sum of all equations (1) is zero, $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$, the the change in total population size must be zero and the total population size N must be constant.

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = \mu N - \frac{\beta}{N} SI - \mu S + \frac{\beta}{N} SI - \gamma I - \mu I + \gamma I - \mu R \tag{2}$$

$$= \mu N - \mu S - \mu I - \mu R(-\frac{\beta}{N} SI + \frac{\beta}{N} SI)(-\gamma I + \gamma I) \tag{3}$$

$$= \mu(N - (S + I + R)) \tag{4}$$

Since $N = S + I + R$, i.e. the sum of all population compartments is equal tot the total population size, 4 evaluates to 0. Thus the sum of population changes in all population compartments is 0 and the total population size remains constant.

Definition 1. *Forward Invariant Set*
Given a dynamical system $\dot{x} = f(x)$*, a solution* $x(t, x_0)$ *with initial condition* x_0*, a set* $\Delta = \{x \in \mathbb{R} \, |\phi(x) = 0\}$ *for some positive definite function* $\phi(x)$ *is forward invariant if* $x_0 \in \Delta \implies x(t, x_0) \in \Delta \; \forall \; t \geq 0$*.*

Since the population size has been shown to be constant and equal to N, the function $\phi(S, I, R) = N - (S + I + R)$ is always equal to zero, given any initial condition.

Definition 2. *Biologically Meaningful States*
Define the set $\Delta = \{(S, I, R) \mid 0 \leq S, I, R \text{ and } \phi(S, I, R) = 0\}$ *where* $\phi(S, I, R) = N - (S + I + R)$*, to be the set of biologically meaningful states for this model.*

Once the total population is equal to N, it will remain equal to N in all subsequent time steps due to the population size constancy. If all initial conditions are defined such that they satisfy $\phi(S, I, R) = 0$, then they can only evolve towards other states that satisfy $\phi(S, I, R) = 0$ due to the constant population size. Thus if the set Δ is defined to include all initial conditions x_0 that have a total population equal to N (i.e. all 3-tuples of positive integers $x_0 = (S, I, R)$ such that $\phi(S, I, R) = 0$), then they must necessarily include all possible time steps for solutions to the dynamical system with initial condition x_0, since the total population must remain constant over all time.

(b) Set the following variables:

$$S_p = \frac{S}{N} \tag{5a}$$

$$I_p = \frac{I}{N} \tag{5b}$$

$$R_p = \frac{R}{N} \tag{5c}$$

$$N_p = \frac{N}{N} = 1 \tag{5d}$$

Then substituting equations 5 into equations (1)

$$\begin{aligned}
\frac{dS_p}{dt} &= \mu N_p - \frac{\beta}{N_p} S_p I_p - \mu S_p \\
&= \mu 1 - \frac{\beta}{1} S_p I_p - \mu S_p \\
&= \frac{1}{N}(\mu - \beta S I - \mu S) \\
&= \frac{1}{N}\frac{dS}{dt} && (6) \\
\frac{dI_p}{dt} &= \frac{\beta}{N_p} S_p I_p - \gamma I_p - \mu I_p \\
&= \frac{\beta}{1} S_p I_p - \gamma I_p - \mu I_p \\
&= \frac{1}{N}(\beta S I - \gamma I - \mu I) \\
&= \frac{1}{N}\frac{dI}{dt} && (7) \\
\frac{dR_p}{dt} &= \gamma I_p - \mu R_p \\
&= \frac{1}{N}(\gamma I - \mu R) \\
&= \frac{1}{N}\frac{dR}{dt} && (8)
\end{aligned}$$

From equations 6,7 and 8 it is clear that the proportional equations are equivalent to the original equations (1) scaled by a constant factor of $\frac{1}{N}$, and thus will retain the same dynamical behaviour.

(c) First we express dt in terms of τ

$$\begin{aligned}
\tau &= t(\gamma + \mu) \\
d\tau &= dt(\gamma + \mu) \\
\frac{d\tau}{dt} &= (\gamma + \mu) \\
\frac{d\tau}{dt}\frac{d}{d\tau} &= \frac{d}{dt} \\
(\gamma + \mu)\frac{d}{d\tau} &= \frac{d}{dt} \\
\frac{d}{d\tau} &= \frac{1}{(\gamma + \mu)}\frac{d}{dt} && (9)
\end{aligned}$$

Next we isolate γ, β, μ,

$$\begin{aligned} \varepsilon &= \frac{\mu}{\gamma+\mu} \\ \varepsilon(\gamma+\mu) &= \mu \qquad\qquad\qquad \gamma+\mu = \frac{\mu}{\varepsilon} \\ \varepsilon\gamma + \varepsilon\mu &= \mu \\ \varepsilon\gamma &= \mu(1-\varepsilon) \\ \gamma &= \frac{\mu}{\varepsilon}(1-\varepsilon) \\ \gamma &= (\gamma+\mu)(1-\varepsilon) \qquad (10) \end{aligned}$$

From (2b) it follows that $\beta = (\gamma+\mu)\mathcal{R}_0$ (11)

From (2c) it follows that $\mu = (\gamma+\mu)\varepsilon$ (12)

Next, expressing $\frac{dS}{dt}$ in terms of τ using 9 and substituting equations 10,11 and 12 gives

$$\begin{aligned} \frac{dS}{d\tau} &= \frac{1}{\gamma+\mu}\frac{dS}{dt} \\ &= \frac{1}{\gamma+\mu}[\mu - \beta SI - \mu S] \\ &= \frac{1}{\gamma+\mu}[\mu(1-S) - \beta SI] \qquad (13) \\ &= \frac{1}{\gamma+\mu}[\varepsilon(\gamma+\mu)(1-S) - (\gamma+\mu)\mathcal{R}_0 SI] \\ &= \frac{1}{\gamma+\mu}(\gamma+\mu)[\varepsilon(1-S) - \mathcal{R}_0 SI] \\ &= \varepsilon(1-S) - \mathcal{R}_0 SI \qquad (14) \end{aligned}$$

Solving for $\frac{dI}{d\tau}$ using 9, 10, 11 and 12 gives

$$\begin{aligned} \frac{dI}{d\tau} &= \frac{1}{\gamma+\mu}\frac{dI}{dt} \\ &= \frac{1}{\gamma+\mu}[\beta SI - \gamma I - \mu I] \\ &= \frac{1}{\gamma+\mu}[(\gamma+\mu)\mathcal{R}_0 SI - (\gamma+\mu)(1-\varepsilon)I - (\gamma+\mu)\varepsilon I] \\ &= \frac{1}{\gamma+\mu}(\gamma+\mu)[\mathcal{R}_0 SI - (1-\varepsilon)I - \varepsilon I] \\ &= \mathcal{R}_0 SI - (1-\varepsilon)I - \varepsilon I \\ &= \mathcal{R}_0 SI - (1-2\varepsilon)I \qquad (15) \end{aligned}$$

Finally, solving for $\frac{dR}{d\tau}$ using 9, 10, 11 and 12 gives

$$\begin{aligned}
\frac{dR}{d\tau} &= \frac{1}{\gamma+\mu}\frac{dR}{dt} \\
&= \frac{1}{\gamma+\mu}[\gamma I - \mu R] \\
&= \frac{1}{\gamma+\mu}[(\gamma+\mu)(1-\varepsilon)I - (\gamma+\mu)\varepsilon R] \\
&= \frac{1}{\gamma+\mu}(\gamma+\mu)[(1-\varepsilon)I - \varepsilon R] \\
&= (1-\varepsilon)I - \varepsilon R
\end{aligned}$$

The biological meanings of $\tau, \mathcal{R}_0$ and ε are

- τ is the average proportion of the population infected
- $\mathcal{R}_0$ is the number of secondary infections per infection
- ε is the average period being infected until death or recovery

For ebola the mean infectious period (ε) has been estimated to be 5.33 ± 4.03 days from wet symptom onset until death.[1]. For influenza the mean infectious period (ε) has been estimated to be 1.0 days until death.[2].

(d) tmp

(e) tmp

(f) tmp

(g) tmp

(h)

(i)

(j)

(k) tmp

References

1. Velasquez, G. E. *et al.* Time From Infection to Disease and Infectiousness for Ebola Virus Disease, a Systematic Review. *Clin. Infect. Dis.* **61,** 1135–1140 (Oct. 2015).
2. Cori, A. *et al.* Estimating influenza latency and infectious period durations using viral excretion data. *Epidemics* **4,** 132–138 (Aug. 2012).

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