MPC

Modelling MPC

Nonlinear : $x_+ = f(x,u)\ ,\ y = h(x,u)$
Linear : $x_+ = Ax + Bu\ ,\ y = Cx + Du$

Description

Equilibrium Point EP : x_s is an EP if $x_s = f(x_s)$
Asymptotic stability AS : an EP is AS if it is :
- (Lyapunov) stable
- Attractive : $\lim_{x\to\infty} ||x_k - x_s|| = 0\ for\ all\ x(0) \in \mathbb{R}^n$

AS for linear systems : A necessary and suf cond for AS of EP at the origin of an LTI is $|\lambda_i| < 1\ \ \forall i\ (\lambda_i : eigenvalues\ of\ A)$

MPC Formulation

$$u^*(x) := argmin\ x_N^T Q_f x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i$$

$s.t.$	$x_0 = x$	measurement
	$x_{i+1} = Ax_i + Bu_i$	system model
	$Cx_i + Du_i \le b$	constraints
	$R > 0, Q > 0$	perf weights

To be done at each sample time $\Rightarrow$ find opti u seq for entire planning window N $\Rightarrow$ Implement only first u

LQR

Goal: Move from state x to the origin. (i.e., keep x 'small')
Consider N inputs into the future

$$\mathbf{u} := \{u_0, \ldots, u_{N-1}\}$$

Express the 'cost' of being in state x and applying input u with the function

$$l(x,u) := x^T Q x + u^T R u$$

Cost of following a trajectory:

$$V(x_0, \mathbf{u}) = \sum_{i=0}^{N} x_i^T Q x_i + u_i^T R u_i$$

Lemma : Lyapunov function for LQR
$V^*(x) = x^T P x$ is a LF for the system $x_+ = (A + BK)x$
$where\ K = -(R + B^T P B)^{-1} B^T P A$
$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$
For some $Q \ge 0, R > 0$

Constrained Minimization Problem

Consider the following problem with inequality constraints

$$\min f(z) \quad \text{s.t. } g_i(z) \le 0,\ i = 1, \ldots, m$$

- f, g_i convex, twice continuously differentiable
- We assume p^* is finite and attained
- We assume problem is strictly feasible: there exists a $\tilde{z}$ with

$$\tilde{z} \in \text{domain of } f, \quad g_i(\tilde{z}) < 0, i = 1, \ldots, m$$

Idea: There exist many methods for unconstrained minimization
$\Rightarrow$ Reformulate problem as an unconstrained problem

Barrier Method : $\min f(z) + \kappa\phi(z)$
Indicator function : $\phi(z) = \sum_{i=1}^m I\_(g_i(z))\ and\ \kappa = 1$
$I\_(u) = 0\ if\ u < 0\ and\ I\_(u) = \infty\ otherwise$
Log function : $\phi(z) = -\sum_i^m \log(-g_i(z))$
- $argmin_z(\phi(z))$ is called analytic center of $g_i < 0$

Central Path :
- Define $z^*(\kappa)$ as the solution of
$$\min_z f(z) + \kappa\phi(z)$$
(assume minimizer exists and is unique for each $\kappa > 0$)
- Barrier parameter κ determines relative weight of objective and barrier
- Barrier 'traps' $z(\kappa)$ in strictly feasible set
- **Central path** is defined as $\{z^*(\kappa) \mid \kappa > 0\}$
- For given κ can compute $z^*(\kappa)$ by solving smooth unconstrained minimization problem
- Intuitively $z^*(\kappa)$ converges to optimal solution as $\kappa \to 0$

Path-following Method :
Idea: Follow central path to the optimal solution
Solve sequence of smooth unconstrained problems:
$$z^*(\kappa) = \text{argmin}_z\ f(z) + \kappa\phi(z)$$
- Assume current solution is on the central path $z^{(k)} = z^*(\kappa^{(k)})$
- Update $\kappa^{(k+1)}$ by decreasing $\kappa^{(k)}$ by some amount
- Solve for $z^*(\kappa^{(k+1)})$ starting from $z^*(\kappa^{(k)})$
- If method converges, it converges to the optimal solution, $z^{(k)} \to z^*$ for $\kappa \to 0$

Barrier Interior-point Method :
$$\min_z \{f(z) \mid g(z) \le 0\}$$
Input: strictly feasible z, $\kappa := \kappa^{(0)}$, $0 < \mu < 1$, tolerance $\epsilon > 0$
repeat
1. **Centering step: Compute $z^*(\kappa)$ by minimizing $f(z) + \kappa\phi(z)$ starting from z**
2. *Update* $z := z^*(\kappa)$
3. *Stopping criterion:* Stop if $m\kappa < \epsilon$
4. *Decrease barrier parameter:* $\kappa := \mu\kappa$

Controller constraints

Infinite horizon optimal control :
$$V^*(x_0) = \min \sum_{i=0}^{\infty} l(x_i, u_i)\ , \quad s.t.\ x_{i+1} = f(x_i, u_i)$$
Finite-time : $V_N^*(x_0) = \min \sum_{i=0}^{N-1} l(x_i, u_i) + V_f(x_N)$,
s.t. $\quad x_{i+1} = f(x_i, u_i)$ and $x_N \in \mathcal{X}_f$
V_f approximates the tail of the cost.
$\mathcal{X}_f$ approximates the tail of the constraints

Lyapunov Stability

AN Ep is LS if $\forall\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that, for every $x_0 : ||x_0 - x_s|| \le \delta(\epsilon) \Rightarrow ||x_k - x_s|| < \epsilon\ \ \forall k \in \mathbb{N}$
Lyapunov Function (LF) : Continuous function $V: \mathbb{R}^n \to \mathbb{R}_+$ is called (asymptotic) Lyapunov function if :
- $||x|| \to \infty \Rightarrow V(x) \to \infty$
- $V(0) = 0\ and\ V(x) > 0\ \forall x \in \mathbb{R}^n \backslash \{0\}$
- $V(f(x)) - V(x) < 0\ \forall x \in \mathbb{R}^n \backslash \{0\}$

Theorem : If a system admits a $LF \Rightarrow x(0)\ is\ AS$
Lemma : Lin sys $x_+ = Ax$ is stable $\Leftrightarrow$ there is a quadratic LF $V = x^T P x, P > 0$ which satisfies the Lyap equation $A^T P A - P = -Q$ for some $Q > 0$

Dynamic Prog

Algo :
Procedure:
1. Start at step N and compute
$$V_N^*(x_N) := \min_{u_N} l(x_N, u_N)$$
2. Iterate backwards for $i = N-1 \ldots 0$ (*DP iteration*)
$$V_i^*(x_i) := \min_{u_i} l(x_i, u_i) + V_{i+1}^*(Ax_i + Bu_i)$$
3. $V^*(x_0) := V_0^*(x_0)$ and the optimal controller is the optimizer $u_0^*(x_0)$

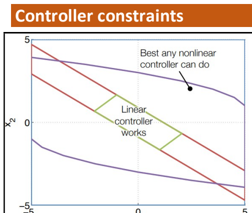
Requirements:
- Closed-form representation of the function $V_i^*(x)$
- Ability to compute a DP iteration

Normally impossible. Some special cases (e.g., LQR).

Bellman Recursion :
1. Start at step N and compute
$$V_N^*(x_N) := \min_{u_N} x_N{}^T Q x_N + u_N{}^T R u_N = x_N{}^T Q x_N$$
$$H_N := Q$$
2. Iterate backwards for $i = N-1 \ldots 0$ (*DP iteration*)
$$V_i^*(x_i) := \min_{u_i} x_i{}^T Q x_i + u_i{}^T R u_i + V_{i+1}^*(Ax_i + Bu_i)$$
$$u_i^*(x_i) = K_i x_i \qquad K_i = -(R + B^T H_{i+1} B)^{-1} B^T H_{i+1} A$$
$$V_i^*(x_i) = x_i{}^T H_i x_i \qquad H_i := Q + K_i{}^T R K_i + (A + BK_i)^T H_{i+1}(A + BK_i)$$
3. $V^*(x_0) := V_0^*(x_0)$ and the optimal controller is the optimizer $u_0^*(x_0)$

Optimal Control law : $\quad u_0^*(x) = K_0 x\ ,\ V_0^*(x) = x^T H_0 x$
Need : $V_i^*(x)$ to have nice form (quadratic)
Ability to solve the DP iteration in closed form

Controller constraints



System:
$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u$$
Constraints:
$$\mathbb{X} := \{x \mid \|x\|_\infty \le 5\}$$
$$\mathbb{U} := \{u \mid \|u\|_\infty \le 1\}$$
Consider an LQR controller, with $Q = I$, $R = 1$.

Positive Invariance Set :
Set $\mathcal{O}$ is said to be a positive invariant set for autonomous sys $x_{i+1} = f(x_i)$ if : $\quad x_i \in \mathcal{O} \Rightarrow x_j \in \mathcal{O}, \quad \forall j \in \{0,1,\ldots,i\}$
Maximal Positive Invariant Set :
Set $\mathcal{O}_\infty \subset \mathbb{X}$ is the max invariant set with respect to $\mathbb{X}$ if $0 \in \mathcal{O}_\infty$, $\mathcal{O}_\infty$ is invariant and contains all invariant sets that contain the origin
Pre-Sets :
Given a set S and dynamic system $x_+ = f(x)$, the pre-set of S is the set of states that evolve into S in one time step :
$$pre(S) := \{x | f(x) \in S\}$$
Or for $x_+ = f(x,u)$:
$$pre(S) := \{x | \exists u \in \mathbb{U}\ s.t.\ f(x,u) \in S\}$$
Theorem : Set $\mathcal{O}$ is a positive invariant set $\Leftrightarrow\ \mathcal{O} \subseteq pre(\mathcal{O})$
Controlled Invariance Set :
Set $\mathcal{C} \subseteq \mathbb{X}$ is said to be a control invariant set if
$$x_i \in \mathcal{C}\ \Leftrightarrow \exists u_i \in \mathbb{U}\ such\ that\ f(x_i, u_i) \in \mathcal{C} \quad \forall i \in \mathbb{N}^+$$
Maximal Control Invariant Set $\mathcal{C}_\infty$:
Set $\mathcal{C}_\infty$ is said to be the maximal control invariant set for the system $x_+ = f(x,u)$ subject to the constraints $(x,u) \in \mathbb{X} \times \mathbb{U}$ if it is control invariant and contains all control invariant sets contained in $\mathbb{X}$
Theorem : Set $\mathcal{C}$ is a control invariant set $\Leftrightarrow\ \mathcal{C} \subseteq pre(\mathcal{C})$
Note : $\mathcal{C}_\infty > \mathcal{O}_\infty$, but more difficult to compute
$\mathcal{C}_\infty$ is the best any controller can do
Control Law :
$$\kappa(x) := \text{argmin}\ \{g(x,u) \mid f(x,u) \in C\}$$
where g is any function (including $g(x,u) = 0$).
This doesn't ensure that the system will converge, but it will satisfy constraints.

Controller constraints

Feasible set $\mathcal{X}_N$: $\mathcal{X}_N$ is defined as the set of initial states x for which the MPC problem with horizon N is feasible :
$$\mathcal{X}_N := \{x \mid \exists[u_0, \ldots, u_{N-1}]\ such\ that\ Cu_i + Dx_i < b\}$$
The values of P and $\mathcal{X}_f$ are chosen to simulate an infinite horizon
Recursive Feasibility : MPC prob is RF, if for all feasible initial states feasibility is guaranteed at all state along the closed-loop traj
Lyapunov stability : EP at origin of sys $x_{k+1} = Ax_k + B\kappa(x_k)$ is said to be LS in $\mathcal{X}$ if $\forall\epsilon > 0, \exists\ \delta(\epsilon) > 0$ such that :
$$||x(0)|| \le \delta(\epsilon) \Rightarrow ||x(k)|| < \epsilon, \quad \forall k \in \mathbb{N}$$

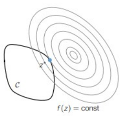
Optimization (intro)

Mathematical Optimization

Mathematical optimization problem is generally formulated as

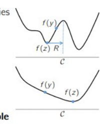
$$\begin{aligned} &\text{minimize } f(z) \\ &\text{s.t. } g_i(z) \le 0,\ i = 1, \ldots, m \\ &\quad\ \ h_i(z) = 0,\ i = 1, \ldots, p \end{aligned}$$

- $z = [z_1, \ldots, z_n]$: optimization variables
- $f : \mathbb{R}^n \to \mathbb{R}$: objective or cost function
- $g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$: inequality constraint functions
- $h_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$: equality constraint functions
- z is **feasible** or admissible if it satisfies the constraints
- $\mathcal{C} := \{z \mid g_i(z) \le 0, i = 1, \ldots, m, h_i(z) = 0, i = 1, \ldots, p\}$: set of feasible or admissible decisions, or **feasible set**



Optimality

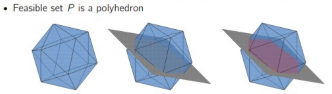
Optimal value: smallest possible cost
$p^* \triangleq \inf \{f(z) \mid g_i(z) \le 0\ i = 1, \ldots, m, h_i(z) = 0, i = 1, \ldots, p\}$
Optimizer: feasible z that achieves smallest cost p^*, i.e., $z^* \in \mathcal{C}$ with $p^* = f(z^*)$; set of all optimizers is denoted by Z_{opt} (optimizer is not always unique).
- $z \in \mathcal{C}$ is **locally optimal** if, for some $R > 0$, it satisfies
$$y \in \mathcal{C}, \|y - z\| \le R \Rightarrow f(y) \ge f(z)$$
- $z \in \mathcal{C}$ is **globally optimal** if it satisfies
$$y \in \mathcal{C} \Rightarrow f(y) \ge f(z)$$
- If $p^* = -\infty$ the problem is **unbounded below**
- If $\mathcal{C}$ is empty, then the problem is said to be **infeasible** (convention: $p^* = \infty$)
- If $m = p = 0$ the problem is said to be **unconstrained**



Local optimization methods :
- Fast, can handle large problem
- Require intial guess and no info on dist to glob opti

Global optimization methods :
- Worst case complex grows expo with problem size

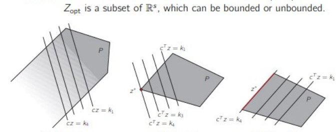
Linear Program :
$$\begin{aligned} \min\ & c^T z \\ \text{s.t. } & Gz \le d \\ & Cz = b \end{aligned}$$
where $z \in \mathbb{R}^n$.
- Convex optimization problem with affine objective and constraint functions
- Feasible set P is a polyhedron



(a) $Gz \le d$ (b) $C_i^T z = b_i$ (c) $Gz \le d \cap C_i^T z = b_i$

- If P is empty, then the problem is infeasible

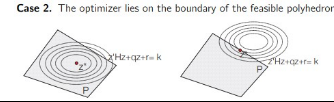
Denote by p^* the optimal value and by Z_{opt} the set of optimizers
Case 1. The LP solution is unbounded, i.e., $p^* = -\infty$.
Case 2. The LP solution is bounded, i.e., $p^* > -\infty$ and the optimizer is unique. Z_{opt} is a singleton.
Case 3. The LP solution is bounded and there are multiple optima. Z_{opt} is a subset of $\mathbb{R}^s$, which can be bounded or unbounded.



(a) Case 1 (b) Case 2 (c) Case 3

Quadratic Program :
$$\begin{aligned} \min\ & \frac{1}{2} z^T H z + q^T z + r \\ \text{s.t. } & Gz \le d \\ & Cz = b \end{aligned}$$
where $z \in \mathbb{R}^n$, $H \in \mathbb{R}^{n \times n}$.
- Convex if $H \succeq 0$ (hard problem if $H \not\succeq 0$)
- Let P be the feasible set.

Two cases can occur if P is not empty:
Case 1. The optimizer lies strictly inside the feasible polyhedron
Case 2. The optimizer lies on the boundary of the feasible polyhedron

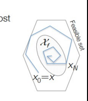


Infinite vs Finite Horizon

Terminal cost :
Terminal constraint provides a sufficient condition for constraint satisfaction :

$$J^*(x) = \min_{x,u} \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + \boxed{x_N^T P x_N}$$
Infinite horizon cost starting from x_N

s.t.	x_{i+1}	$= Ax_i + Bu_i$
	$Cx_i + Du_i$	$\le b$
	$\boxed{x_N}$	$\boxed{\in \mathcal{X}_f}$
	x_0	$= x$



- All input and state constraints are satisfied for the closed-loop system using the LQR control law for $x \in \mathcal{X}_f$
- Terminal set is often defined by linear or quadratic constraints

$\to$ The bound holds in the **terminal set** and is used as a **terminal cost**
$\to$ The terminal set defines the **terminal constraint**

Theorem Vidysager : if a system admits a LF in $\mathcal{X}$, the EP at the origin is (Lyapunov) stable in $\mathcal{X}$.
Stability of MPC :
Standing assumptions hold:
1. The stage cost is a positive definite function, i.e. it is strictly positive and only zero at the origin
2. The terminal set is **invariant** under the local control law $\kappa_f(x)$:
$$x^+ = Ax + B\kappa_f(x) \in \mathcal{X}_f \quad \text{for all } x \in \mathcal{X}_f$$
All state and input **constraints are satisfied** in $\mathcal{X}_f$:
$$\mathcal{X}_f \subseteq \mathbb{X},\ \kappa_f(x) \in \mathbb{U} \text{ for all } x \in \mathcal{X}_f$$
3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$:
$$V_f(x^+) - V_f(x) \le -l(x, \kappa_f(x)) \text{ for all } x \in \mathcal{X}_f$$

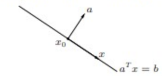
Thm: The closed-loop system under the MPC control law $u_0^*(x)$ is stable and the system $x^+ = Ax + Bu_0^*(x)$ is invariant in the feasible set $\mathbb{X}_N$.

Note :
Finite-horizon MPC may not be stable and not satisfy constraints for all time

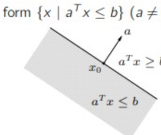
Geometric definition

Convex Set : A set S is convex if
$$\lambda z_1 + (1-\lambda) z_2 \in S, for\ all\ z_1, z_2\ and\ \lambda \in [0,1]$$

- **Hyperplane:** Set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$

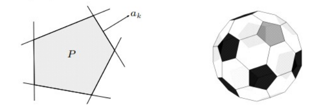


- **Halfspace:** Set of the form $\{x \mid a^T x \le b\}$ $(a \neq 0)$

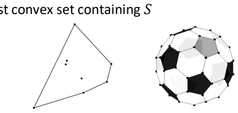


Polyhedron
A **polyhedron** is the intersection of a finite number of halfspaces.
$$P := \{x \mid a_i^T x \le b_i,\ i = 1, \ldots, n\}$$
A **polytope** is a bounded polyhedron.

Often written as $P := \{x \mid Ax \le b\}$, for matrix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, where the inequality is understood row-wise.



Convex function : $f: S \to \mathbb{R}$ is convex if S is convex and
$$f(\lambda z_1 + (1-\lambda) z_2 \le \lambda f(z_1) + (1-\lambda) f(z_2)$$
Strictly convex : $\le$ instead of $<$
Concave : S is convex and $-$f is convex
Lemma : local opti of a convex prob is glob opti
Convex hull : For any $S \subset R^d$, the convex hull $conv(S)$ is the intersection of all convex set containing S and is the smallest convex set containing S



2D convex hull 3D convex hull

Theorem Minkowski-Weyl :

Theorem: Minkowski-Weyl Theorem
For $P \subseteq \mathbb{R}^d$, the following statements are equivalent:
- P is a polytope, i.e., P is bounded and there exist $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$ such that $P = \{x \mid Ax \le b\}$
- P is finitely generated, i.e., there exist a finite set of vectors $\{v_i\}$ such that $P = \text{conv}(\{v_1, \ldots, v_s\})$

Input saturation
$$u_{lb} \le u \le u^{ub}$$
$$\Downarrow$$
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} u \le \begin{bmatrix} u^{ub} \\ -u_{lb} \end{bmatrix}$$

Magnitude constraints
$$\|Cx\|_\infty \le \alpha$$
$$\Downarrow$$
$$\begin{bmatrix} C \\ -C \end{bmatrix} x \le \mathbf{1}\alpha$$

Rate constraints
$$\|x_i - x_{i+1}\|_\infty \le \alpha$$
$$\Downarrow$$
$$\begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \begin{pmatrix} x_i \\ x_{i+1} \end{pmatrix} \le \mathbf{1}\alpha$$

Integral constraints
$$\|x\|_1 \le \alpha$$
$$\Downarrow$$
$$x \in \text{conv}(e_i\alpha)$$

Polytopic Projection : Given a polytope $P = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^d | Cx + Dy \le b\}$, find a matrix E and vector e, such that the polytope
$$P_\pi = \{x | Ex \le e\} = \{x | \exists y, (x,y) \in P\}$$

Linear MPC with quadra cost

$$\begin{aligned} \min_z\ & \frac{1}{2} z^T H z \\ \text{s.t. } & Gz \le g \\ & Tz = t \end{aligned}$$

$$\begin{aligned} \min_u\ & \sum_{i=0}^{N-1} x_i{}^T Q x_i + u_i{}^T R u_i + x_N{}^T Q_f x_N \\ \text{s.t. } & x_i \in \mathbb{X} && i \in \{1, \ldots, N-1\} \\ & u_i \in \mathbb{U} && i \in \{0, \ldots, N-1\} \\ & x_N \in \mathcal{X}_f \\ & x_{i+1} = Ax_i + Bu_i \end{aligned}$$

Define variables:
$$\mathbf{z} := [x_1^T \ \ldots\ x_N^T\ u_0^T\ \ldots\ u_{N-1}^T]^T$$

Equalities (T, t) from system dynamics $x_{i+1} = Ax_i + Bu_i$:

$$T := \left[\begin{array}{cccc|cccc} I & & & & -B & & & \\ -A & I & & & & -B & & \\ & -A & I & & & & -B & \\ & & \ddots & \ddots & & & & \ddots \\ & & -A & I & & & & -B \end{array}\right]$$

$$t := \begin{bmatrix} A \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_0 \qquad t \text{ is a linear function of the current state } x_0!$$

Assume $\mathbb{X}$ and $\mathbb{U}$ given by:
$$\mathbb{X} := \{x \mid Fx \le f\} \quad \mathbb{U} := \{u \mid Mu \le m\} \quad \mathcal{X}_f := \{x \mid F_f x \le f_f\}$$
Form matrices G and g

$$G := \left[\begin{array}{ccccc|cccc} F & & & & & 0 & & & \\ & F & & & & & 0 & & \\ & & \ddots & & & & & \ddots & \\ & & & F & & & & & 0 \\ & & & & F_f & & & & 0 \\ \hline 0 & & & & & M & & & \\ & 0 & & & & & M & & \\ & & \ddots & & & & & \ddots & \\ & & & 0 & & & & & M \\ & & & & 0 & & & & M \end{array}\right] \qquad g := \begin{bmatrix} f \\ f \\ \vdots \\ f \\ f_f \\ m \\ m \\ \vdots \\ m \\ m \end{bmatrix}$$

$$H := \left[\begin{array}{cccc|ccc} Q & & & & & & \\ & \ddots & & & & & \\ & & Q & & & & \\ & & & Q_f & & & \\ \hline & & & & R & & \\ & & & & & \ddots & \\ & & & & & & R \end{array}\right]$$

Terminal Sets and Functions

For $f(x,u) = Ax + Bu$ and $I(x,u) = x^TQx + u^TRu$

Define terminal controller as the opti unconstrained LQR control law, same for the terminal weight :

$$\kappa_f(x) = Kx, \qquad K = -(R + B^TPB)^{-1}B^TPA$$
$$P = Q + A^TPA - A^TPB(R + B^TPB)^{-1}B^TPA$$

Terminal weight :

$$V_f(x) := x^TPx = \sum_{i=0}^{\infty} x_i^TQx_i + x_i^TK^TRKx_i$$

Choose for terminal Set $\mathcal{X}_f$ the max invariant set for the closed-loop system $x_+ = (A + BK)x$

MPC tracking

Compute steady-state target : (x_s, u_s)

$$\min u_s^T R_s u_s$$
$$\text{s.t.} \quad \begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix}\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$
$$H_x x_s \le k_x$$
$$H_u u_s \le k_u$$

Replace in MPC problem $\boldsymbol{x}$ and $\boldsymbol{u}$ by $\boldsymbol{\Delta x = x - xs}$ and $\boldsymbol{\Delta u}$

MPC problem :

$$\min \sum_{i=0}^{N-1} \Delta x_i^TQ\Delta x_i + \Delta u_i^TR\Delta u_i + V_f(\Delta x_N)$$
$$\text{s.t.} \quad \Delta x_0 = x$$
$$\Delta x_{i+1} = A\Delta x_i + B\Delta u_i$$
$$H_x\Delta x_i \le k_x - H_x x_s$$
$$H_u\Delta u_i \le k_u - H_u u_s$$
$$\Delta x_N \in \mathcal{X}_f$$

- Find optimal seq of $\Delta\boldsymbol{u}^*$
- Input applied to the system is $u_0^* = \Delta u_o^* + u_s$

Offest free control (constant dist rejection) :

Idea : include disturbance in model :

$$x_{k+1} = Ax_k + Bu_k + B_d d_k$$
$$d_{k+1} = d_k$$
$$y_k = Cx_k + C_d d_k$$

Only restriction on choice of B_d, C_d: observability of the augmented model

The augmented system is observable if and only if (A, C) is observable and

$$\begin{bmatrix} A-I & B_d \\ C & C_d \end{bmatrix} \text{ has full column rank, i.e. } \text{rank} = n_x + n_d$$

$\Rightarrow$ Maximal dimension of the disturbance: $n_d \le n_y$

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix}\begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} L_x \\ L_d \end{bmatrix}(C\hat{x}_k + C_d\hat{d}_k - y_k)$$

- For L use of pole placement to obtain it

Offset + tracking :

Compute steady-state target : (x_s, u_s)

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix}\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} B_d\hat{d} \\ r - C_d\hat{d} \end{bmatrix}$$

Robust MPC (1/2):

MPC vs real world : MPC system evolves in a predictable ways but in real world random noise w ($\propto t, unknown$), model structure is unknown and unknown parameters θ ($const\ and\ unknown$) impact the dynamics

$$x_+ = f(x, u; w; \theta), \qquad (x,u) \in \mathbb{X}, \mathbb{U} \ \ w \in \mathbb{W} \ and \ \theta \in \Theta$$

Common uncertainty models :

Measurement/input bias $\quad g(x,u,w;\theta) = f(x,u) + \theta$

- Handled generally by estimating offset

Linear parameter varying system

$$g(x,u,w;\theta) = \sum_{k=0}^{t} \theta_k A_k x + \sum_{k=0}^{t} \theta_k B_k u,$$
$$\boldsymbol{1}^T\theta = 1 \ and \ \theta \ge 0$$

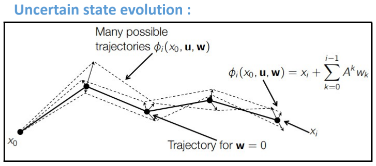
Additive Stochastic Noise

$$g(x,u,w;\theta) = Ax + Bu + w, \qquad distrib\ of\ w\ known$$

Additive Bounded Noise $\quad g(x,u,w;\theta) = Ax + Bu + w$

- Noise is persistent

Uncertain state evolution :



Cost :

$$J(x_0, \boldsymbol{u}, \boldsymbol{w}) := \sum_{i=0}^{N-1} I(\phi_i(x_0, \boldsymbol{u}, \boldsymbol{w}), u_i) + V_f(\phi_N(x_0, \boldsymbol{u}, \boldsymbol{w}))$$

$$with\ \phi_i = A^i x_0 + \sum_{k=0}^{i-1} A^k Bu_{i-k} + \sum_{k=0}^{i-1} A^k w_{i-k}$$
$$= x_i + \sum_{k=0}^{i-1} A^k w_{i-k}$$

Need to eliminate the dependence on $\boldsymbol{w}$:

- Minimize the expected value (requires some assumption on the distribution)

$$V_N(x_0, \mathbf{u}) := \mathbf{E}[J(x_0, \mathbf{u}, \mathbf{w})]$$

- Minimize the variance (requires some assumption on the distribution)

$$V_N(x_0, \mathbf{u}) := \text{Var}(J(x_0, \mathbf{u}, \mathbf{w}))$$

- Take the worst-case

$$V_N(x_0, \mathbf{u}) := \max_{\mathbf{w} \in \mathbb{W}^{N-1}} J(x_0, \mathbf{u}, \mathbf{w})$$

- Take the nominal case

$$V_N(x_0, \mathbf{u}) := J(x_0, \mathbf{u}, 0)$$

Removing of Terminal Set/Terminal constraints

Possible to remove terminal constraint while maintaining stab if

- Initial state lies in sufficiently small subset of feasible set
- N is sufficiently large

Note : Feasible set without terminal constraint is not invariant

Warning : Region of attraction without term constraint may be larger than for MPC with terminal constraint

Soft constrained MPC

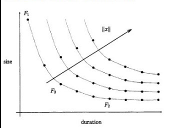
Idea : allow "small" violation of some constrained :

- Input constraints often represent actuator limit : can't be soften
- State constraints often represent perf or comfort constraints : can be soften

Objective : minimize duration and size of violation (conflicting goal)

Pareto optimal curve :

For given system and horizon can plot pareto optimal size/duration curve for different initial conditions:



Best operation points lie on pareto optimal curve:

- points below cannot be attained
- points above are inferior

$\rightarrow$ Operation at pareto optimality is in general difficult and only approximately achieved

Soft constrained MPC problem setup :

$$\min_{\mathbf{u}} \sum_{i=0}^{N-1} x_i^TQx_i + u_i^TRu_i + \rho(\epsilon_i) + x_N^TPx_N + \rho(\epsilon_N)$$

s.t. $x_{i+1} = Ax_i + Bu_i$	System model
$H_x x_i \le k_x + \epsilon_i$,	State constraint (soften)
$H_u u_i \le k_u$,	Input constraint
$\epsilon_i \ge 0$	Slack variable (use to soften)

- Penalize amount of of constraint violation in the cost function by means of penalty : $\rho(\epsilon_N)$

Quadratic penalty : $\quad \rho(\epsilon_i) = \epsilon_i^T S \epsilon_i$

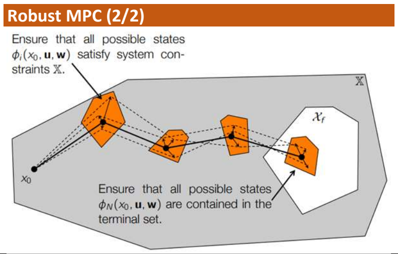
Quadratic and linear norm penalty : $\rho(\epsilon_i) = \epsilon_i^T S \epsilon_i + s||\epsilon_i||_{1/\infty}$

Effect of S : Increase in S lead to 'hardening' of the soft constraints

Effect of s : (same a S)

- allow for exact penalties : if s large enough constraints are satisfied if possible
- large linear penalties make tuning difficult and cause numerical problems

Robust MPC (2/2)



$$\left.\begin{aligned} \phi_{i+1} &= A\phi_i + Bu_i + w_i \\ u_i &\in \mathbb{U} \\ \phi_i &\in \mathbb{X} \ \forall \mathbf{w} \in \mathbb{W}^N \end{aligned}\right\}$$

- $i = 0, \ldots, N-1$
- Optimize over control actions $\{u_0, \ldots, u_{N-1}\}$
- Enforce constraints explicitly by imposing $\phi_i \in \mathbb{X}$ and $u_i \in \mathbb{U}$ for all sequences $\mathbf{w}$

$$\left.\begin{aligned} \phi_{i+1} &= (A + BK)\phi_i + w_i \\ \phi_N &\in \mathcal{X}_f \end{aligned}\right\}$$

- $i = N, \ldots$
- Assume control law to be linear $u_i = K\phi_i$
- Enforce constraints implicitly by constraining ϕ_N to be in an *robust invariant set* $\mathcal{X}_f \subseteq \mathbb{X}$ and $K\mathcal{X}_f \subseteq \mathbb{U}$ for the system $\phi^+ = (A + BK)\phi + w$

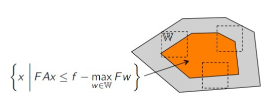
Robust Invariant Set : set $\mathcal{O}^W$ is said to be a robust positive invariant set for the autonomous system $x_+ = f(x,w)$ if :

$$x \in \mathcal{O}^W \Rightarrow f(x,w) \in \mathcal{O}^W, \qquad \forall w \in \mathbb{W}$$

Robust Pre-Sets : Given a set Ω and the dyn system $x_+ = f(x,w)$, pre-set of Ω is the set of states that evolve into it in one step for all $w \in \mathbb{W}$: $\qquad pre^{\mathbb{W}}(\Omega) := \{x|\ f(x,w) \in \Omega, \forall w \in \mathbb{W}\}$

Goal: Given the system $f(x,w) = Ax + w$, and the set $\Omega := \{x \mid Fx \le f\}$, compute $\text{pre}^{\mathbb{W}}(\Omega)$.

$$\text{pre}^{\mathbb{W}}(\Omega) = \{x \mid Ax + w \in \Omega, \ \forall w \in \mathbb{W}\} = \{x \mid FAx + Fw \le f, \ \forall w \in \mathbb{W}\}$$



$$\text{pre}^{\mathbb{W}}(\Omega) = \left\{x \,\middle|\, FAx \le f - \max_{w \in \mathbb{W}} Fw\right\} = \{x \mid FAx \le f - h_{\mathbb{W}}(F)\} = A(\Omega \ominus \mathbb{W})$$

Theorem Robust Invariant Set : set $\mathcal{O}$ is a robust positive invariant set $\Leftrightarrow \quad \mathcal{O} \subseteq pre^{\mathbb{W}}(\mathcal{O})$

Robust MPC :

Robust Open-Loop MPC

$$\min_{\mathbf{u}} \sum_{i=0}^{N-1} l(x_i, u_i) + V_f(x_N)$$
$$\text{s.t.} \quad x_{i+1} = Ax_i + Bu_i$$
$$x_i \in \mathbb{X} \ominus \mathcal{A}_i\mathbb{W}^i$$
$$u_i \in \mathbb{U}$$
$$x_N \in \mathcal{X}_f \ominus \mathcal{A}_N\mathbb{W}^N$$

where $\mathcal{A}_i := \begin{bmatrix} A^0 & A^1 & \ldots & A^i \end{bmatrix}$ and $\tilde{\mathcal{X}}_f$ is a robust invariant set for the system $x^+ = (A + BK)x$ for some stabilizing K.

We do **nominal MPC**, but with tighter constraints on the states and inputs.

We can be sure that if the nominal system satisfies the tighter constraints, then the uncertain system will satisfy the real constraints.

Robust Control Invariance : if $\boldsymbol{u}^*(x)$ is the opti of the robust open-loop MPC prob, then the sys $Ax + Bu_0^*(x) + w \in \mathbb{X}, \forall w \in \mathbb{W}$

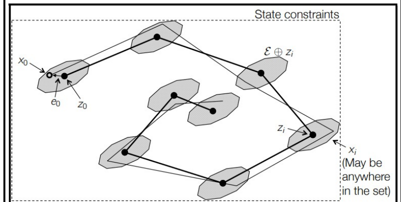
Robust MPC II

Problem : $\mathcal{A}^i\mathbb{W}^i$ can be very large $\Rightarrow$ reduce heavily feasible set

Solution idea : react to given disturbance :

1. Controller choose u_0
2. Disturbance choose w_0
3. Controller choose u_1 as a function of w_0
4. …

Tube-MPC:



- Compute the set $\mathcal{E}$ that the error will remain inside
- Modify constraints on nominal trajectory $\{z_i\}$ so that $z_i \oplus \mathcal{E} \subset \mathbb{X}$ and $v_i \in \mathbb{U} \ominus K\mathcal{E}$
- Formulate as convex optimization problem

Minimum robust invariant set :

As sum goes to infinity, we arrive at the **minimum robust invariant set** mRPI

$$F_\infty = \bigoplus_{k=0}^{\infty} A^k\mathbb{W}, \quad F_0 := \{0\}$$

If there exists an n such that $F_n = F_{n+1}$, then $F_n = F_\infty$

Minkowski sum :

Given $P := \{x \mid Tx \le t\}$ and $Q := \{x \mid Rx \le r\}$, the Minkowski sum is:

$$P \oplus Q := \{x + y \mid x \in P, \ y \in Q\}$$
$$= \{z \mid \exists x, y \ z = x + y, \ Tx \le t, \ Ry \le r\}$$
$$= \{z \mid \exists y \ Tz - Ty \le t, \ Ry \le r\}$$
$$= \left\{z \,\middle|\, \exists y \begin{bmatrix} T & -T \\ 0 & R \end{bmatrix}\begin{pmatrix} z \\ y \end{pmatrix} \le \begin{pmatrix} t \\ r \end{pmatrix}\right\}$$

Constraint Tightening : $we\ want\ x_i \in z_i \oplus \mathcal{E}$

$$z_i \in \mathbb{X} \ominus \mathcal{E} \Rightarrow z_i \oplus \mathcal{E} \in \mathbb{X}$$
$$v_i \in \mathbb{U} \ominus K\mathcal{E} \Rightarrow u_i \in K\mathcal{E} \oplus v_i \subset \mathbb{U}$$

Problem Formulation :

Tube-MPC

Feasible set: $\mathcal{Z}(x_0) := \left\{ \mathbf{z}, \mathbf{v} \,\middle|\, \begin{array}{ll} z_{i+1} = Az_i + Bv_i & i \in [0, \ N-1] \\ z_i \in \mathbb{X} \ominus \mathcal{E} & i \in [0, \ N-1] \\ v_i \in \mathbb{U} \ominus K\mathcal{E} & i \in [0, \ N-1] \\ z_N \in \mathcal{X}_f & \\ x_0 \in z_0 \oplus \mathcal{E} & \end{array} \right\}$

Cost function: $V(\mathbf{z}, \mathbf{v}) := \sum_{i=0}^{N-1} l(z_i, v_i) + V_f(z_N)$

Optimization problem: $(\mathbf{v}^\star(x_0), \mathbf{z}^\star(x_0)) = \text{argmin}_{\mathbf{v},\mathbf{z}} \{V(\mathbf{z}, \mathbf{v}) \mid (\mathbf{z}, \mathbf{v}) \in \mathcal{Z}(x_0)\}$

Control law: $\mu_{\text{tube}}(x) := K(x - z_0^\star(x)) + v_0^\star(x)$

Assumptions :

1. Stage cost is a positive def function
2. Terminal set is invariant for the nominal system under local control law $\kappa_f(z)$
   All tightened state and input const are satisfied in $\mathcal{X}_f$:
   $$\mathcal{X}_f \subseteq \mathbb{X} \ominus \mathcal{E}, \ \ \kappa_f(z) \in \mathbb{U} \ominus \mathcal{E} \ \forall z \in \mathcal{X}_f$$
3. Terminal cost is a continuous LF in the therminal set $\mathcal{X}_f$:
   $$V_f\left(Az + B\kappa_f(z)\right) - V_f(z) \le -i\left(z, \kappa_f(z)\right) \ \forall z \in \mathcal{X}_f$$

Theoreme (Robust inv of tube-MPC) : set $\mathcal{Z} := \{x|\mathcal{Z}(x) \neq \emptyset\}$ is a robust invariant set of the sys $x_+ = Ax + B\mu_{tube(x)} + w$ subject to the constraint $(x, u) \in \mathbb{X} \times \mathbb{U}$

Let $(\{v_0^\star, \ldots, v_{N-1}^\star\}, \{z_0^\star, \ldots, z_N^\star\})$ be the optimal solution for time x_0.

At the next point in time, the state is:

$$x_1 = Ax_0 + BK(x_0 - z_0^\star) + Bv_0^\star + w \quad \text{for some } w \in \mathbb{W}$$

i.e., the state x_1 may have many possible values. We need to show that there exists a feasible solution for **all of them**.

By construction, the state x_1 is in the set $z_1 \oplus \mathcal{E}$ for all $\mathbb{W}$. Therefore (as in standard MPC), the sequence

$$(\{v_1^\star, \ldots, v_{N-1}^\star, \kappa_f(z_N^\star)\}, \{z_1^\star, \ldots, z_N^\star, Az_N^\star + B\kappa_f(z_N^\star)\})$$

is feasible for all x_1.

Theoreme (Robust Stab of tube-MPC) : state x of the sys $x_+ = Ax + B\mu_{tube}(x) + w$ converges in the limit to the set $\mathcal{E}$

$$J^\star(x_0) = \sum_{i=0}^{N-1} l(z_i^\star, v_i^\star) + V_f(z_N^\star)$$
$$J^\star(x_1) \le \sum_{i=1}^{N} l(z_i^\star, v_i^\star) + V_f(z_{N+1}^\star)$$
$$= J^\star(x_0) - \underbrace{l(z_0^\star, v_0^\star)}_{\ge 0} + \underbrace{V_f(z_{N-1}^\star) - V_f(z_N^\star) + l(z_N^\star, \kappa_f(z_N^\star))}_{\le 0 \ (V_f \text{ is a Lyapunov function in } \mathcal{X}_f)}$$

Tube MPC – Resume :

— Offline —

1. Choose a stabilizing controller K so that $\|A + BK\| < 1$
2. Compute the minimal robust invariant set $\mathcal{E} = F_\infty$ for the system $x^+ = (A + BK)x + w, \ w \in \mathbb{W}^1$
3. Compute the tightened constraints $\tilde{\mathbb{X}} := \mathbb{X} \ominus \mathcal{E}$, $\tilde{\mathbb{U}} := \mathbb{U} \ominus \mathcal{E}$
4. Choose terminal weight function V_f and constraint $\mathcal{X}_f$ satisfying assumptions on slide 35

— Online —

1. Measure / estimate state x
2. Solve the problem $(\mathbf{v}^\star(x), \mathbf{z}^\star(x)) = \text{argmin}_{\mathbf{v},\mathbf{z}} \{V(\mathbf{z}, \mathbf{v}) \mid (\mathbf{z}, \mathbf{v}) \in \mathcal{Z}(x)\}$ (Slide 33)
3. Set the input to $u = K(x - z_0^\star(x)) + v_0^\star(x)$