QP/LP Solvers

Barrier Interior-point Method

$$\min_z \{f(z) \mid g(z) \le 0\}$$

Input: strictly feasible z, $\kappa := \kappa^{(0)}$, $0 < \mu < 1$, tolerance $\epsilon > 0$
repeat
1. **Centering step: Compute $z^*(\kappa)$ by minimizing $f(z) + \kappa\phi(z)$ starting from z**
2. *Update* $z := z^*(\kappa)$
3. *Stopping criterion:* Stop if $m\kappa < \epsilon$
4. *Decrease barrier parameter:* $\kappa := \mu\kappa$

- Gradient descent: $\Delta z := -\nabla f(z)$

$$\nabla f(z)^T \Delta z = -\nabla f(z)^T \nabla f(z) = -\|\nabla f(z)\| < 0$$

- Best: Newton method $\quad \Delta z = -\nabla^2 f(z)^{-1} \nabla f(z)$

Newton step for Quadratic Programming

$$\min_z \{\frac{1}{2} z^T H z \mid Gz \le d\}$$

- Barrier method:

$$\min_z f(z) + \kappa\phi(z) = \min_z \frac{1}{2} z^T H z - \kappa \sum_{i=1}^m \log(d_i - g_i z)$$

where $g_1, \ldots, g_m$ are the rows of G.

- The gradient and Hessian of the barrier function are:

$$\nabla\phi(z) = \sum_{i=1}^m \frac{1}{d_i - g_i z} g_i^T, \nabla^2\phi(z) = \sum_{i=1}^m \frac{1}{(d_i - g_i z)^2} g_i^T g_i$$

- Newton step: $(\nabla^2 f(z) + \kappa \nabla^2 \phi(z)) \Delta z_{nt} = -\nabla f(z) - \kappa \nabla \phi(z)$

$$(H + \kappa \sum_{i=1}^m \frac{1}{(d_i - g_i z)^2} g_i^T g_i) \Delta z_{nt} = -Hz - \kappa \sum_{i=1}^m \frac{1}{d_i - g_i z} g_i^T$$

Logarithmic Barrier Function

$$\phi(z) = -\sum_{i=1}^m \log(-g_i(z)), \quad \text{domain } \phi = \{z \mid g_1(z) \le 0, \ldots, g_m(z) \le 0\}$$

$$\nabla\phi(z) = \sum_{i=1}^m \frac{1}{-g_i(z)} \nabla g_i(z)$$

$$\nabla^2\phi(z) = \sum_{i=1}^m \frac{1}{g_i(z)^2} \nabla g_i(z) \nabla g_i(z)^T + \frac{1}{-g_i(z)} \nabla^2 g_i(z)$$

descent direction : $\nabla f(z)^T \Delta z_{nt} = -\nabla f(z)^T \nabla^2 f(z)^{-1} \nabla(z) < 0$

Lyapunov equations

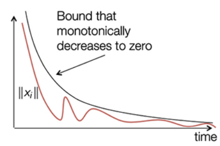
We assume $A \in \mathbf{R}^{n\times n}$, $P = P^T \in \mathbf{R}^{n\times n}$. It follows that $Q = Q^T \in \mathbf{R}^{n\times n}$.

Discrete-time linear system: for $x(t+1) = Ax(t)$, $V(z) = z^T P z$, we have $\Delta V(z) = -z^T Q z$, where P, Q satisfy (discrete-time) Lyapunov equation $A^T P A - P + Q = 0$.

Lyapunov theorems

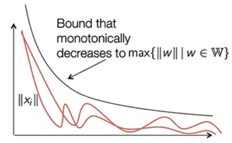
- If $P > 0$, $Q > 0$, then system is (globally asymptotically) stable.
- If $P > 0$, $Q \ge 0$, and (Q, A) observable, then system is (globally asymptotically) stable.
- If $P > 0$, $Q \ge 0$, then all trajectories of the system are bounded
- If $Q \ge 0$, then the sublevel sets $\{z \mid z^T P z \le a\}$ are invariant. (These are ellipsoids if $P > 0$.)
- If $P \not\ge 0$ and $Q \ge 0$, then A is not stable.

Asymptotic stability



System converges to zero

ISS stability



Converges to set around zero, who's size is determined by size of the noise

Invariant sets :

Positive Invariant set

A set $\mathcal{O}$ is said to be a positive invariant set for the autonomous system $x_{i+1} = f(x_i)$ if

$$x_i \in \mathcal{O} \;\Rightarrow\; x_i \in \mathcal{O}\ ,\ \forall i \in \{0, 1, \ldots\}$$

Pre Set

Given a set S and the dynamic system $x^+ = f(x)$, the **pre-set** of S is the set of states that evolve into the target set S in one time step:

$$\text{pre}(S) := \{x \mid f(x) \in S\}$$

Theorem: Geometric condition for invariance

A set $\mathcal{O}$ is a positive invariant set if and only if

$$\mathcal{O} \subseteq \text{pre}(\mathcal{O})$$

Polyhedron

A **polyhedron** is the intersection of a finite number of halfspaces.

$$P := \{x \mid a_i^T x \le b_i,\ i = 1, \ldots, n\}$$

A **polytope** is a bounded polyhedron.

Conceptual Algorithm to Compute Invariant Set

Input: f, $\mathbb{X}$
Output: $\mathcal{O}_\infty$

$\Omega_0 \leftarrow \mathbb{X}$
loop
  $\Omega_{i+1} \leftarrow \text{pre}(\Omega_i) \cap \Omega_i$
  if $\Omega_{i+1} = \Omega_i$ **then**
    return $\mathcal{O}_\infty = \Omega_i$
  end if
end loop

The algorithm generates the set sequence $\{\Omega_i\}$ satisfying $\Omega_{i+1} \subseteq \Omega_i$ for all $i \in \mathbb{N}$ and it terminates when $\Omega_{i+1} = \Omega_i$ so that Ω_i is the maximal positive invariant set $\mathcal{O}_\infty$ for $x^+ = f(x)$.

Convex hull

For any subset S of $\mathbb{R}^d$, the convex hull conv (S) of S is the intersection of all convex sets containing S. Since the intersection of two convex sets is convex, it is the smallest convex set containing S.

Proposition: Convex hull

The convex hull of a set $S \subseteq \mathbb{R}^d$ is
Given a set of points $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^d$, their **convex hull** is

$$\text{conv}(\{v_1, \ldots, v_k\}) := \left\{x \;\middle|\; x = \sum_i \lambda_i v_i,\ \lambda_i \ge 0,\ \sum_i \lambda_i = 1\ \forall i = 1, \ldots, k\right\}$$

Intersection

The intersection $I \subseteq \mathbb{R}^n$ of sets $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^n$ is

$$I = S \cap T := \{x \mid x \in S \text{ and } x \in T\}$$

Intersection of polytopes in **inequality form** is easy:

$$S := \{x \mid Cx \le c\} \qquad S \cap T = \left\{x \;\middle|\; \begin{bmatrix} C \\ D \end{bmatrix} x \le \begin{bmatrix} c \\ d \end{bmatrix}\right\}$$
$$T := \{x \mid Dx \le d\}$$

Pre-Set Computation: Controlled System

Consider the system $x^+ = Ax + Bu$ under the constraints $u \in \mathbb{U} := \{u \mid Gu \le g\}$ and the set $S := \{x \mid Fx \le f\}$.

$\text{pre}(S) := \{x \mid \exists u \in \mathbb{U} \text{ s.t. } f(x, u) \in S\}$

$$\begin{aligned} \text{pre}(S) &= \{x \mid \exists u \in \mathbb{U},\ Ax + Bu \in S\} \\ &= \{x \mid \exists u \in \mathbb{U},\ FAx + FBu \le f\} \\ &= \left\{x \;\middle|\; \exists u, \begin{bmatrix} FA & FB \\ 0 & G \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \le \begin{bmatrix} f \\ g \end{bmatrix}\right\} \end{aligned}$$

Polytopic Projection

Given a polytope $P = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^d \mid Cx + Dy \le b\}$, find a matrix E and vector e, such that the polytope

$$P_\pi = \{x \mid Ex \le e\} = \{x \mid \exists y,\ (x, y) \in P\}$$

Robust MPC:

Robust Pre Set

Given a set Ω and the dynamic system $x^+ = f(x, w)$, the **pre-set** of Ω is the set of states that evolve into the target set Ω in one time step *for all values of the disturbance* $w \in \mathbb{W}$:

$$\text{pre}^{\mathbb{W}}(\Omega) := \{x \mid f(x, w) \in \Omega \text{ for all } w \in \mathbb{W}\}$$

Computing Robust Pre-Sets for Linear Systems

Goal: Given the system $f(x, w) = Ax + w$, and the set $\Omega := \{x \mid Fx \le f\}$, compute $\text{pre}^{\mathbb{W}}(\Omega)$.

$$\text{pre}^{\mathbb{W}}(\Omega) = \{x \mid Ax + w \in \Omega,\ \forall w \in \mathbb{W}\} = \{x \mid FAx + Fw \le f,\ \forall w \in \mathbb{W}\}$$

$$\text{pre}^{\mathbb{W}}(\Omega) = \left\{x \;\middle|\; FAx \le f - \max_{w \in \mathbb{W}} Fw\right\} = \{x \mid FAx \le f - h_{\mathbb{W}}(F)\} = A(\Omega \ominus \mathbb{W})$$

Computing Robust Invariant Sets

Tube-MPC Assumptions

Much the same as for nominal MPC:

1. The stage cost is a positive definite function, i.e. it is strictly positive and only zero at the origin
2. The terminal set is invariant **for the nominal system** under the local control law $\kappa_f(z)$:
$$z^+ = Az + B\kappa_f(z) \in \mathcal{X}_f \quad \text{for all } z \in \mathcal{X}_f$$
All **tightened state and input constraints** are satisfied in $\mathcal{X}_f$:
$$\mathcal{X}_f \subseteq \mathbb{X} \ominus \mathcal{E},\ \kappa_f(z) \in \mathbb{U} \ominus \mathcal{E} \text{ for all } z \in \mathcal{X}_f$$
3. Terminal cost is a continuous Lyapunov function in the terminal set $\mathcal{X}_f$:
$$V_f(Az + B\kappa_f(z)) - V_f(z) \le -l(z, \kappa_f(z)) \text{ for all } z \in \mathcal{X}_f$$

Uncertain constrained linear system

$$x^+ = Ax + Bu + w \qquad (x, u) \in \mathbb{X}, \mathbb{U} \qquad w \in \mathbb{W}$$

Robust Positive Invariant set

A set $\mathcal{O}^{\mathbb{W}}$ is said to be a robust positive invariant set for the autonomous system $x_{i+1} = f(x_i, w)$ if

$$x \in \mathcal{O}^{\mathbb{W}} \;\Rightarrow\; f(x, w) \in \mathcal{O}^{\mathbb{W}}\ ,\ \text{for all } w \in \mathbb{W}$$

Robust Open-Loop MPC

$$\begin{aligned} \min_{\mathbf{u}} \ & \sum_{i=0}^{N-1} l(x_i, u_i) + V_f(x_N) \\ \text{s.t. } & x_{i+1} = Ax_i + Bu_i \\ & x_i \in \mathbb{X} \ominus \mathcal{A}_i \mathbb{W}^i \\ & u_i \in \mathbb{U} \\ & x_N \in \tilde{\mathcal{X}}_f \end{aligned}$$

where $\mathcal{A}_i := \begin{bmatrix} A^0 & A^1 & \ldots & A^i \end{bmatrix}$ and $\tilde{\mathcal{X}}_f$ is a robust invariant set for the system $x^+ = (A + BK)x$ for some stabilizing K.

Putting it all together: Tube MPC

— Offline —

1. Choose a stabilizing controller K so that $\|A + BK\| < 1$
2. Compute the minimal robust invariant set $\mathcal{E} = F_\infty$ for the system $x^+ = (A + BK)x + w$, $w \in \mathbb{W}^1$
3. Compute the tightened constraints $\tilde{\mathbb{X}} := \mathbb{X} \ominus \mathcal{E}$, $\tilde{\mathbb{U}} := \mathbb{U} \ominus \mathcal{E}$
4. Choose terminal weight function V_f and constraint $\mathcal{X}_f$ satisfying assumptions

— Online —

1. Measure / estimate state x
2. Solve the problem $(\mathbf{v}^*(x), \mathbf{z}^*(x)) = \text{argmin}_{\mathbf{v},\mathbf{z}} \{V(\mathbf{z}, \mathbf{v}) \mid (\mathbf{z}, \mathbf{v}) \in \mathcal{Z}(x)\}$
3. Set the input to $u = K(x - z_0^*(x)) + v_0^*(x)$

Minimal Invariant Set

Input: A
Output: F_∞
$\Omega_0 \leftarrow \{0\}$
loop
  $\Omega_{i+1} \leftarrow \Omega_i \oplus A^i \mathbb{W}$
  if $\Omega_{i+1} = \Omega_i$ **then**
    return $F_\infty = \Omega_i$
  end if
end loop

Pontryagin Difference

Let A and B be subsets of $\mathbb{R}^n$. The Pontryagin Difference is

$$A \ominus B := \{x \mid x + e \in A\ \forall e \in B\}$$

Lemma

$$x \in A \ominus B \Rightarrow x + e \in A\ \forall e \in B$$

Tube-MPC Problem Formulation

Tube-MPC

Feasible set: $\mathcal{Z}(x_0) := \left\{\mathbf{z}, \mathbf{v} \;\middle|\; \begin{array}{ll} z_{i+1} = Az_i + Bv_i & i \in [0, N-1] \\ z_i \in \mathbb{X} \ominus \mathcal{E} & i \in [0, N-1] \\ v_i \in \mathbb{U} \ominus K\mathcal{E} & i \in [0, N-1] \\ z_N \in \mathcal{X}_f & \\ x_0 \in z_0 \oplus \mathcal{E} & \end{array}\right\}$

Cost function: $V(\mathbf{z}, \mathbf{v}) := \sum_{i=0}^{N-1} l(z_i, v_i) + V_f(z_N)$

Optimization problem: $(\mathbf{v}^*(x_0), \mathbf{z}^*(x_0)) = \text{argmin}_{\mathbf{v},\mathbf{z}} \{V(\mathbf{z}, \mathbf{v}) \mid (\mathbf{z}, \mathbf{v}) \in \mathcal{Z}(x_0)\}$

Control law: $\mu_{\text{tube}}(x) := K(x - z_0^*(x)) + v_0^*(x)$

NMPC

Runge Kutta 2

Consider the ODE $\quad \dot{x} = f(x)$
Given $x = x(t)$, we want to compute $x^+ = x(t + h)$

Compute a second-order Taylor series expansion

$$x^+ = x + h\dot{x} + \frac{h^2}{2}\ddot{x} + \mathcal{O}(h^3)$$

Take Jacobian of f to compute $\ddot{x}$

$$\ddot{x} = J_f(x)\dot{x} = J_f(x) f(x)$$

The Taylor series expansion is now

$$\begin{aligned} x^+ &= x + hf(x) + \frac{h^2}{2} J_f(x) f(x) + \mathcal{O}(h^3) \\ &= x + \frac{h}{2} f(x) + \frac{h}{2}(f(x) + hJ_f(x) f(x)) + \mathcal{O}(h^3) \end{aligned}$$

Runge-Kutta 4 - The most common version

Consider the time dependent ODE: $\quad \dot{x} = f(t, x)$

$$x_{k+1} = x_k + h\left(\frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}\right)$$

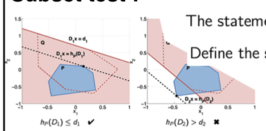
$$k_1 = f(t_k, x_k)$$
$$k_2 = f(th_k + \frac{h}{2}, x_k + x_k + \frac{h}{2} k_1)$$
$$k_2 = f(t_k + \frac{h}{2}, x_k + x_k + \frac{h}{2} k_2)$$
$$k_2 = f(t_k + h, x_k + x_k + hk_3)$$

Minkovski sum :

Given $P := \{x \mid Tx \le t\}$ and $Q := \{x \mid Rx \le r\}$, the Minkowski sum is:

$$\begin{aligned} P \oplus Q &:= \{x + y \mid x \in P,\ y \in Q\} \\ &= \{z \mid \exists x, y\ z = x + y,\ Tx \le t,\ Ry \le r\} \\ &= \{z \mid \exists y\ Tz - Ty \le t,\ Ry \le r\} \\ &= \left\{z \;\middle|\; \exists y \begin{bmatrix} T & -T \\ 0 & R \end{bmatrix} \begin{pmatrix} z \\ y \end{pmatrix} \le \begin{pmatrix} t \\ r \end{pmatrix}\right\} \end{aligned}$$

Subset test :



Problem: Is $P := \{x \mid Cx \le c\}$ contained in $Q := \{x \mid Dx \le d\}$?

The statement is true if $P \subset \{x \mid D_i x \le d_i\}$ for each row D_i of D.

Define the **support function** of the set P:

$$h_P(D_i) := \max_x D_i x \quad \text{s.t. } Cx \le c$$

LQR Regulation

Linear Quadratic Regulator

$$V^\star(x_0) := \min_{\mathbf{u}} \sum_{k=0}^{\infty} x_k{}^T Q x_k + u_k{}^T R u_k \quad \text{s.t. } x_{k+1} = A x_k + B u_k$$

Can solve the infinite-horizon predictive control problem in closed-form:

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$

The optimal input is the constant state feedback

$$u = Kx \qquad K = -(R + B^T P B)^{-1} B^T P A$$

The optimal cost function $V^\star(x) = x^T P x$ is a Lyapunov function for the closed-loop system $x^+ = (A + BK)x$.

Lemma: Lyapunov function for LQR

The optimal value function $V^\star(x) = x^T P x$ is a Lyapunov function for the system $x^+ = (A + BK)x$ where $K = -(R + B^T P B)^{-1} B^T P A$ and P solves

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$

for some $Q \succeq 0$, $R \succ 0$.

Solving Infinite-Horizon LQR

Fact: $V^\star(x)$ is quadratic, $V^\star(x) = x^T P x$ for $P \succ 0^2$

Bellman equation:

$$V(x) = \min_u x^T Q x + u^T R u + V(Ax + Bu)$$
$$x^T P x = \min_u x^T Q x + u^T R u + (Ax + Bu)^T P (Ax + Bu)$$

minimizing gives $u^\star = -(R + B^T P B)^{-1} B^T P A x$, giving

$$x^T P x = x^T Q x + u^{\star T} R u^\star + (Ax + Bu^\star)^T P (Ax + Bu^\star)$$
$$x^T P x = x^T (Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A) x$$

Consider the DP iteration:

$$V_i^\star(x_i) := \min_{u_i} \, l(x_i, u_i) + V_{i+1}^\star(Ax_i + Bu_i)$$

If $V_i^\star(\cdot) = V_{i+1}^\star(\cdot)$, then $V_j^\star(\cdot) = V_{i+1}^\star(\cdot)$

for all $j \le i$.

Lyapunov Function for LQR-Controlled System

Lemma: Lyapunov function for LQR

The optimal value function $V^\star(x) = x^T P x$ is a Lyapunov function for the system $x^+ = (A + BK)x$ where $K = -(R + B^T P B)^{-1} B^T P A$ and P solves

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$

for some $Q \succeq 0$, $R \succ 0$.

Dynamic Programming

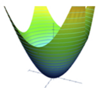
1. Start at step N and compute $V_N^\star(x_N) := \min_{u_N} x_N{}^T Q x_N + u_N{}^T R u_N = x_N{}^T Q x_N \quad H_N := Q$
2. Iterate *backwards* for $i = N-1 \dots 0$ (*DP iteration*)
$$V_i^\star(x_i) := \min_{u_i} x_i{}^T Q x_i + u_i{}^T R u_i + V_{i+1}^\star(Ax_i + Bu_i)$$
$$u_i^\star(x_i) = K_i x_i \qquad K_i = -(R + B^T H_{i+1} B)^{-1} B^T H_{i+1} A$$
$$V_i^\star(x_i) = x_i^T H_i x_i \qquad H_i := Q + K_i^T R K_i + (A + BK_i)^T H_{i+1} (A + BK_i)$$
3. $V^\star(x_0) := V_0^\star(x_0)$ and the optimal controller is the optimizer $u_0^\star(x_0)$

MPC Properties

Lyapunov function

Let $\mathcal{X}$ be a positively invariant set for system $x(k+1) = f_\kappa(x(k))$ containing a neighborhood of the origin in its interior. A function $V : \mathcal{X} \to \mathbb{R}_+{}^1$ is called a **Lyapunov function** in $\mathcal{X}$ if for all $x \in \mathcal{X}$:

$$V(x) > 0 \; \forall x \neq 0, \; V(0) = 0,$$
$$V(x(k+1)) - V(x(k)) \le 0$$



Theorem: *(e.g., [Vidyasager, 1993])*

If a system admits a Lyapunov function in $\mathcal{X}$, then the equilibrium point at the origin is **(Lyapunov) stable** in $\mathcal{X}$.

Definition: Feasible set

The **feasible set** $\mathcal{X}_N$ is defined as the set of initial states x for which the MPC problem with horizon N is feasible, i.e.

$\mathcal{X}_N := \{x \mid \exists [u_0, \dots, u_{N-1}]$ such that $Cu_i + Dx_i \le b, i = 1, \dots, N\}$

Linear MPC with Quadratic Cost

Standard formulation:

- Quadratic performance measure
- Linear system dynamics
- $\mathbb{X}$, $\mathcal{X}_f$ and $\mathbb{U}$ are polyhedra

$$\begin{aligned}
\min_{\mathbf{u}} \; & \sum_{i=0}^{N-1} x_i{}^T Q x_i + u_i{}^T R u_i + x_N{}^T Q_f x_N \\
\text{s.t.} \; & x_i \in \mathbb{X} \qquad i \in \{1, \dots, N-1\} \\
& u_i \in \mathbb{U} \qquad i \in \{0, \dots, N-1\} \\
& x_N \in \mathcal{X}_f \\
& x_{i+1} = Ax_i + Bu_i
\end{aligned}$$

Assumptions: $Q = Q^T \succeq 0$, $Q_f = Q_f^T \succ 0$, $R = R^T \succ 0$

Delta-Formulation for tracking

- Define deviation variables that (in the linear case) satisfy the same model equations:
$$\Delta x = x - x_s \qquad \Delta u = u - u_s \quad \Rightarrow \quad \begin{aligned} \Delta x_{k+1} &= x_{k+1} - x_s \\ &= Ax_k + Bu_k - (Ax_s + Bu_s) \\ &= A\Delta x_k + B\Delta u_k \end{aligned}$$
- Constraints for deviation variables:
$$H_x x \le k_x \Rightarrow \quad H_x \Delta x \le k_x - H_x x_s$$
$$H_u \le k_u \Rightarrow \quad H_u \Delta u \le k_u - H_u u_s$$
- Apply regulation problem to new system in Delta-Formulation:
$$\begin{aligned}
\min \; & \sum_{i=0}^{N-1} \Delta x_i{}^T Q \Delta x_i + \Delta u_i{}^T R \Delta u_i + V_f(\Delta x_N) \\
\text{s.t. } & \Delta x_0 = \Delta x \\
& \Delta x_{i+1} = A\Delta x_i + B\Delta u_i \\
& H_x \Delta x_i \le k_x - H_x x_s \\
& H_u \Delta u_i \le k_u - H_u u_s \\
& \Delta x_N \in \mathcal{X}_f
\end{aligned}$$

Formalize Goals: Feasibility and Stability

Goal 1: Feasibility for all time

Definition: Recursive feasibility

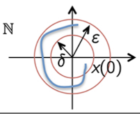
The MPC problem is called **recursively feasible**, if for all feasible initial states feasibility is guaranteed at every state along the closed-loop trajectory.

Goal 2: Stability

Definition: Lyapunov stability

The equilibrium point at the origin of system $x_{k+1} = Ax_k + B\kappa(x_k) = f_\kappa(x_k)$ is said to be **(Lyapunov) stable** in $\mathcal{X}$ if for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that, for every $x(0) \in \mathcal{X}$:

$$\|x(0)\| \le \delta(\epsilon) \Rightarrow \|x(k)\| < \epsilon \; \forall k \in \mathbb{N}$$



Stability of MPC - Main Result

If we can choose/find an $\mathcal{X}_f$, κ_f, V_f and l such that:

1. The stage cost is a positive definite function, i.e. it is strictly positive and only zero at the origin
2. The terminal set is **invariant** under the local control law $\kappa_f(x)$:
$$x^+ = Ax + B\kappa_f(x) \in \mathcal{X}_f \quad \text{for all } x \in \mathcal{X}_f$$
All state and input **constraints are satisfied** in $\mathcal{X}_f$:
$$\mathcal{X}_f \subseteq \mathbb{X}, \; \kappa_f(x) \in \mathbb{U} \text{ for all } x \in \mathcal{X}_f$$
3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$:
$$V_f(x^+) - V_f(x) \le -l(x, \kappa_f(x)) \text{ for all } x \in \mathcal{X}_f$$

MPC problem for tracking

Convergence

Assume target is feasible with $x_s \in \mathbb{X}, u_s \in \mathbb{U}$ and choose terminal weight $V_f(x)$ and constraint $\mathcal{X}_f$ as in the regulation case satisfying:

- $\mathcal{X}_f \subseteq \mathbb{X}$, $Kx \in \mathbb{U}$ for all $x \in \mathcal{X}_f$
- $V_f(x^+) - V_f(x) \le -l(x, Kx)$ for all $x \in \mathcal{X}_f$

If in addition the target reference x_s, u_s is such that

- $x_s \oplus \mathcal{X}_f \subseteq \mathbb{X}$, $K\Delta x + u_s \in \mathbb{U}$ for all $\Delta x \in \mathcal{X}_f$

then the closed-loop system converges to the target reference, i.e. $x_k \to x_s$ and therefore $y_k = Cx_k \to r$ for $k \to \infty$

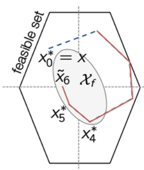
Soft constrained MPC problem setup

$$\begin{aligned}
\min_{\mathbf{u}} \; & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + \rho(\epsilon_i) + x_N^T P x_N + \rho(\epsilon_N) \\
\text{s.t. } & x_{i+1} = Ax_i + Bu_i, \\
& H_x x_i \le k_x + \epsilon_i, \\
& H_u u_i \le k_u, \\
& \epsilon_i \ge 0
\end{aligned}$$

- Relax state constraints by introducing so called **slack variables** $\epsilon_i \in \mathbb{R}^p$
  - Quadratic penalty: $\rho(\epsilon_i) = \epsilon_i^T S \epsilon_i$
  - Quadratic and linear norm penalty: $\rho(\epsilon_i) = \epsilon_i^T S \epsilon_i + s\|\epsilon_i\|_{1/\infty}$

Recursive Feasability / Stability

- Assume feasibility of x and let $[u_0^\star, \; u_1^\star, \; \dots, \; u_{N-1}^\star]$ be the optimal control sequence computed at x
- At x^+, $[u_1^\star, \; u_2^\star, \; \dots, \; \kappa_f(x_N^\star)]$ is feasible:
$$x_N \text{ is in } \mathcal{X}_f \to \kappa_f(x_N^\star) \text{ is feasible}$$
$$\text{and } x_{N+1} = Ax_N^\star + B\kappa_f(x_N^\star) \text{ in } \mathcal{X}_f$$



$\Rightarrow$ **Terminal constraint provides recursive feasibility**

$$J^\star(x_0) = \sum_{i=0}^{N-1} l(x_i^*, u_i^*) + V_f(x_N^*)$$

Feasible, sub-optimal sequence for x_1 : $[u_1^\star, \; u_2^\star, \; \dots, \; \kappa_f(x_N^\star)]$

$$\begin{aligned}
J^\star(x_1) &\le \sum_{i=1}^{N} l(x_i^*, u_i^*) + V_f(\tilde{x}_{N+1}) \\
&= \sum_{i=0}^{N-1} l(x_i^*, u_i^*) + V_f(x_N^*) - l(x_0^*, u_0^*) + V_f(\tilde{x}_{N+1}) - V_f(x_N^*) + l(x_N^*, \kappa_f(x_N^*)) \\
&= J^*(x_0) - l(x, u_0^*) + \underbrace{V_f(\tilde{x}_{N+1}) - V_f(x_N^*) + l(x_N^*, \kappa_f(x_N^*))}_{V_f(x) \text{ is a Lyapunov function: } \le 0}
\end{aligned}$$

$J^\star(x)$ is a Lyapunov function $\to$ (Lyapunov) Stability

Augmented model

Incorporate disturbance model assuming integral disturbance dynamics

$$\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + B_d d_k \\
d_{k+1} &= d_k \\
y_k &= Cx_k + C_d d_k
\end{aligned}$$

Only restriction on choice of B_d, C_d: observability of the augmented model

The augmented system is observable if and only if (A, C) is observable and

$$\begin{bmatrix} A - I & B_d \\ C & C_d \end{bmatrix} \text{ has full column rank, i.e. } \operatorname{rank} = n_x + n_d$$

$\Rightarrow$ Maximal dimension of the disturbance: $n_d \le n_y$

Linear state estimation

Design state and disturbance estimator based on the augmented model:

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} L_x \\ L_d \end{bmatrix} (C\hat{x}_k + C_d \hat{d}_k - y_k)$$

where $\hat{x}$, $\hat{d}$ are estimates of the state and disturbance.

Error dynamics:

$$\begin{aligned}
\begin{bmatrix} x_{k+1} - \hat{x}_{k+1} \\ d_{k+1} - \hat{d}_{k+1} \end{bmatrix} &= \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k \\
&\quad - \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} u_k - \begin{bmatrix} L_x \\ L_d \end{bmatrix} (C\hat{x}_k + C_d \hat{d}_k - Cx_k - C_d d_k) \\
&= \left(\begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} + \begin{bmatrix} L_x \\ L_d \end{bmatrix} \begin{bmatrix} C & C_d \end{bmatrix} \right) \begin{bmatrix} x_k - \hat{x}_k \\ d_k - \hat{d}_k \end{bmatrix}
\end{aligned}$$

$\Rightarrow$ Choose $L = \begin{bmatrix} L_x \\ L_d \end{bmatrix}$ such that the error dynamics are stable and converge to zero, i.e. the estimator is stable.

pLCPs

Parametric Linear Complementarity Problem

Given matrices M, q and Q, find functions $w(x)$, $z(x)$ such that

$$w - Mz = q + Qx$$
$$w^T z = 0$$
$$w, z \ge 0$$

Definition 9.1. *KKT Necessary and Sufficient Optimality Conditions*

Consider the following problem : $\min f(x)$, $Ax \le b$,

1. ***Stationarity :*** $\nabla f(x^*) + A^T \lambda = 0$
   Gradient is in the normal cone.
2. ***Primal Feasibility :*** $Ax^* \le b$
   Ensures that x^* is in the feasible polytope.
3. ***Dual Feasibility :*** $\lambda_i \ge 0 \; \forall i$
   Ensures that Stationarity is enforced.
4. ***Complementarity*** $\lambda^T (Ax - b) = 0$
   Normal cone contains only active constraints. Same as saying that either $\lambda_i = 0$ or $a_i x = b_i$, $\forall i$

Parametric quadratic optimization problem:

$$\begin{aligned}
J^\star(x) := \min_u \; & \frac{1}{2} u^T Q u + (Fx + f)^T u \\
\text{s.t. } & Gu \ge Ex + e \\
& u \ge 0
\end{aligned}$$

KKT Conditions:

$Qu + Fx + f - G^T\lambda - \nu = 0$	Stationarity
$-s + Gu = Ex + e \,, \; u \ge 0$	Primary feasibility
$\lambda, \; \nu \ge 0$	Dual feasibility
$\nu^T u = 0 \,, \; \lambda^T s = 0$	Complementarity

Stationarity
Primal feasibility
$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} \nu \\ s \end{pmatrix} - \begin{bmatrix} Q & -G^T \\ G & 0 \end{bmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{bmatrix} F \\ -E \end{bmatrix} x + \begin{bmatrix} f \\ -e \end{bmatrix}$$

Primal and dual feasibility $\quad \nu, s, u, \lambda \ge 0$

Complementarity $\quad \nu^T u = s^T \lambda = 0$

Standard pLCP:

$$Iw - Mz = Qx + q$$
$$w, z \ge 0 \,, \; w^T z = 0$$

Note: Optimizer is linear transform of pLCP solution $u^\star(x) = \begin{bmatrix} I & 0 \end{bmatrix} z(x)$

Geometric Series

$$\sum_{n=0}^{N-1} q^n = \frac{1-q^N}{1-q} \qquad \sum q^n = \frac{1}{1-q}$$

Algebraic Identities

$$\begin{aligned}
(\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\
(\mathbf{ABC}\dots)^{-1} &= \dots\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \\
(\mathbf{A}^T)^{-1} &= (\mathbf{A}^{-1})^T \\
(\mathbf{A}+\mathbf{B})^T &= \mathbf{A}^T + \mathbf{B}^T \\
(\mathbf{AB})^T &= \mathbf{B}^T\mathbf{A}^T \\
(\mathbf{ABC}\dots)^T &= \dots\mathbf{C}^T\mathbf{B}^T\mathbf{A}^T
\end{aligned}$$

$$\frac{\partial \mathbf{x}^T \mathbf{a}}{\partial \mathbf{x}} = \frac{\partial \mathbf{a}^T \mathbf{x}}{\partial \mathbf{x}} =$$

Hessian Matrix

$$H_{ij}(f) = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Or compute each collumn by derivating the gradient for each variable