$$\underbrace{\begin{pmatrix}\varepsilon & 1\\ 1 & 1\end{pmatrix}}_{=:A_\varepsilon}\underbrace{\begin{pmatrix}x_1\\ x_2\end{pmatrix}}_{=:x} = \underbrace{\begin{pmatrix}1\\ 2\end{pmatrix}}_{=:b}$$

$$\kappa(A_\varepsilon) := \|A_\varepsilon\|_2 \cdot \|A_\varepsilon^{-1}\| = \left|\frac{\lambda_{max}}{\lambda_{min}}\right|$$

$$A_\varepsilon = A_\varepsilon^T$$

$$\det(A_\varepsilon - \lambda\, Id) = \det\begin{pmatrix}\varepsilon-\lambda & 1\\ 1 & 1-\lambda\end{pmatrix} = (\varepsilon-\lambda)(1-\lambda) - 1 = 0$$

$$\varepsilon - \varepsilon\lambda - \lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - \lambda(1+\varepsilon) - 1 + \varepsilon = 0 \Rightarrow \lambda_{1,2} = \frac{1+\varepsilon \pm \sqrt{1+2\varepsilon+\varepsilon^2-4(\varepsilon-1)}}{2} =$$

$$= \frac{1+\varepsilon \pm \sqrt{\varepsilon^2-2\varepsilon+5}}{2}$$

$$\kappa(A_\varepsilon) = \frac{|1+\varepsilon+\sqrt{\varepsilon^2-2\varepsilon+5}|}{|1+\varepsilon-\sqrt{\varepsilon^2-2\varepsilon+5}|}$$

$$\sqrt{\varepsilon^2-2\varepsilon+5} = \sqrt{5}\sqrt{1+\underbrace{\tfrac{1}{5}(\varepsilon^2-2\varepsilon)}_{=:\delta}} \quad \left[\sqrt{1+\delta} \approx 1+\frac{1}{2}\delta\right] \quad = \sqrt{5}\left(1+\frac{1}{10}(\varepsilon^2-2\varepsilon)\right) =$$

$$= \sqrt{5} + \frac{\varepsilon^2\sqrt{5}}{10} - \frac{\sqrt{5}}{10}2\varepsilon = \sqrt{5} + \frac{\varepsilon^2\sqrt{5}}{10} - \frac{\sqrt{5}}{5}\varepsilon = \sqrt{5} + \frac{\varepsilon^2}{2\sqrt{5}} - \frac{\varepsilon}{\sqrt{5}}$$

$$\kappa(A_\varepsilon) = \frac{1+\varepsilon+\sqrt{5}+\frac{\varepsilon^2}{2\sqrt{5}}-\frac{\varepsilon}{\sqrt{5}}}{1+\varepsilon-\sqrt{5}-\frac{\varepsilon^2}{2\sqrt{5}}+\frac{\varepsilon}{\sqrt{5}}} = \frac{1+\varepsilon+\sqrt{5}+\frac{\varepsilon^2\sqrt{5}}{10}-\frac{2\sqrt{5}}{10}\varepsilon}{1+\varepsilon-\sqrt{5}-\frac{\varepsilon^2\sqrt{5}}{10}+\frac{2\sqrt{5}}{10}\varepsilon} =$$

$$= \frac{1+\sqrt{5}+\varepsilon\left(1-\frac{2\sqrt{5}}{10}\right)+\varepsilon^2\frac{\sqrt{5}}{10}}{1-\sqrt{5}+\varepsilon\left(1+\frac{2\sqrt{5}}{10}\right)-\varepsilon^2\frac{\sqrt{5}}{10}} = \frac{1+\sqrt{5}+\varepsilon\left(\frac{10-2\sqrt{5}}{10}\right)+\varepsilon^2\frac{\sqrt{5}}{10}}{1-\sqrt{5}+\varepsilon\left(\frac{10+2\sqrt{5}}{10}\right)-\varepsilon^2\frac{\sqrt{5}}{10}}$$

$$\Big[\, r_0 + r_1 x + r_2 x^2 = (d_0 + d_1 x + d_2 x^2)(a_0 + a_1 x + a_2 x^2) = d_0 a_0 + d_0 a_1 x + d_0 a_2 x^2 + a_0 d_1 x + d_1 a_1 x^2 +$$

$$+ d_2 a_0 x^2 + O(x^3) = d_0 a_0 + x(d_0 a_1 + a_0 d_1) + (d_0 a_2 + d_1 a_1 + d_2 a_0)x^2 + O(x^3)$$

$$\Rightarrow r_0 = d_0 a_0$$
$$r_1 = d_0 a_1 + a_0 d_1$$
$$r_2 = d_0 a_2 + d_1 a_1 + d_2 a_0 \,\Big]$$

$$d_0 = \frac{r_0}{a_0} = \frac{1+\sqrt{5}}{1-\sqrt{5}}$$

$$\alpha_1 = \frac{r_1 - \alpha_0 a_1}{a_0} = \frac{\frac{10-2\sqrt{5}}{10} - \frac{1+\sqrt{5}}{1-\sqrt{5}}\left(\frac{10+2\sqrt{5}}{10}\right)}{1-\sqrt{5}} = \frac{\frac{10-2\sqrt{5}}{10} + \frac{(1+2\sqrt{5}+5)(10+2\sqrt{5})}{4\cdot 10}}{1-\sqrt{5}} =$$

$$= \frac{40 - 8\sqrt{5} + 60 + 12\sqrt{5} + 20\sqrt{5} + 4\cdot 5}{40-40\sqrt{5}} = \frac{120+24\sqrt{5}}{40-40\sqrt{5}} =$$

$$= \frac{60+12\sqrt{5}}{20-20\sqrt{5}} = \frac{30+6\sqrt{5}}{10-10\sqrt{5}} = \frac{15+3\sqrt{5}}{5-5\sqrt{5}}$$

$$\alpha_2 = \frac{r_2 - \alpha_0 a_2 - \alpha_1 a_1}{a_0} = \frac{\frac{\sqrt{5}}{10} + \frac{1+\sqrt{5}}{1-\sqrt{5}}\,\frac{2\sqrt{5}}{10} - \frac{15+3\sqrt{5}}{5(1-\sqrt{5})}\,\frac{10+2\sqrt{5}}{10}}{1-\sqrt{5}} =$$

$$= \frac{5\cdot(\sqrt{5}-5) + (2\sqrt{5}+10)5 - (15+3\sqrt{5})(10+2\sqrt{5})}{50(1-\sqrt{5})^2} = \frac{5\sqrt{5} - 25 + 10\sqrt{5} + 50 - 150 - 30\sqrt{5} - 30\sqrt{5} - 30}{50(1-2\sqrt{5}+5)} =$$

$$= \frac{-155 - 45\sqrt{5}}{300-100\sqrt{5}} = \frac{31+9\sqrt{5}}{20\sqrt{5}-60}$$

$$\kappa(A_\varepsilon) = \left| \frac{1+\sqrt{5}}{1-\sqrt{5}} + \frac{15+3\sqrt{5}}{5-5\sqrt{5}}\,\varepsilon + \frac{31+9\sqrt{5}}{20\sqrt{5}-60}\,\varepsilon^2 + O(\varepsilon^3) \right|$$

$$\lim_{\varepsilon\to 0} \kappa(A_\varepsilon) = \left| \frac{1+\sqrt{5}}{1-\sqrt{5}} \right|$$

$$\lim_{\varepsilon \to 0} \kappa(A_\varepsilon) = \frac{|1+\sqrt{5}|}{|1-\sqrt{5}|}$$

$\sqrt{x}$ je rastúca $\Rightarrow \sqrt{4} = 2 \Rightarrow \sqrt{5} > 2 \Rightarrow 1-\sqrt{5} < 0 \Rightarrow |1-\sqrt{5}| = \sqrt{5}-1$

$$\Rightarrow \left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right| = \frac{1+\sqrt{5}}{\sqrt{5}-1} = \frac{(1+\sqrt{5})(\sqrt{5}+1)}{5-1} = \frac{2\sqrt{5}+1+5}{4} = \frac{6+2\sqrt{5}}{4} = 1+\frac{2\sqrt{5}}{4} = 1+\frac{\sqrt{5}}{2} > 2$$

$$\left.\begin{aligned} &\sqrt{9} = 3 > \sqrt{5} > 2 \quad \Rightarrow \frac{\sqrt{5}}{2} > 1 \\ &\qquad\qquad\qquad\qquad \Rightarrow 2 > \frac{3}{2} > \frac{\sqrt{5}}{2} > 1 \end{aligned}\right\} \Rightarrow 1+\frac{\sqrt{5}}{2} \in (2,3)$$

$\kappa(A_\varepsilon)$ je rádovo 10^0, preto nie je veľká podmienenosť a teda

<u>matica A_ε pre $\varepsilon \to 0$ je dobre podmienená.</u>

Relatívna chyba je $\kappa(A_\varepsilon) \cdot \varepsilon_{mach}$, teda rádovo 10^{-16}, pre pre st. float.

$$\left(\begin{matrix}\varepsilon & 1 \\ 1 & 1\end{matrix}\middle|\begin{matrix}1 & 0 \\ 0 & 1\end{matrix}\right) \sim \left(\begin{matrix}\varepsilon & 1 \\ 0 & 1-\frac{1}{\varepsilon}\end{matrix}\middle|\begin{matrix}1 & 0 \\ -\frac{1}{\varepsilon} & 1\end{matrix}\right)$$

:)

$A_\varepsilon = L_\varepsilon U_\varepsilon$

$L_\varepsilon^{-1} A_\varepsilon = U_\varepsilon$

$$L_\varepsilon^{-1} = \begin{pmatrix}1 & 0 \\ -\frac{1}{\varepsilon} & 1\end{pmatrix} \quad L_\varepsilon = \begin{pmatrix}1 & 0 \\ \frac{1}{\varepsilon} & 1\end{pmatrix} \quad U_\varepsilon = \begin{pmatrix}\varepsilon & 1 \\ 0 & 1-\frac{1}{\varepsilon}\end{pmatrix} \quad U_\varepsilon^{-1} = \frac{1}{\varepsilon-1}\begin{pmatrix}1-\frac{1}{\varepsilon} & -1 \\ 0 & \varepsilon\end{pmatrix}$$

$x = U_\varepsilon^{-1} L_\varepsilon^{-1} b$

$$x = \frac{1}{\varepsilon-1}\begin{pmatrix}1-\frac{1}{\varepsilon} & -1 \\ 0 & \varepsilon\end{pmatrix}\begin{pmatrix}1 & 0 \\ -\frac{1}{\varepsilon} & 1\end{pmatrix}\begin{pmatrix}1 \\ 2\end{pmatrix} = \begin{pmatrix}-\frac{1}{\varepsilon-1} \\ \frac{2\varepsilon-1}{\varepsilon-1}\end{pmatrix}$$

ekvivalentne

$A_\varepsilon x = b$

$L_\varepsilon U_\varepsilon x = b$

$U_\varepsilon x = L_\varepsilon^{-1} b$

$$\begin{pmatrix}\varepsilon & 1 \\ 0 & 1-\frac{1}{\varepsilon}\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} = \begin{pmatrix}1 & 0 \\ -\frac{1}{\varepsilon} & 1\end{pmatrix}\begin{pmatrix}1 \\ 2\end{pmatrix}$$

$\varepsilon x_1 + x_2 = 1$

$x_2\left(1-\frac{1}{\varepsilon}\right) = 2-\frac{1}{\varepsilon}$

$\Rightarrow x_2 = \frac{2\varepsilon-1}{\varepsilon-1}$

$x_1 = -\frac{1}{\varepsilon-1}$

Nakoľko $\varkappa(A_\varepsilon)$ je malé, úloha je korektná a dobre podmienená. Avšak z výsledku môžeme odpozorovať, že pre $\varepsilon \to 0$ relatívna chyba a reziduum sú veľké a škálujú s rádom ε. Máme dobre podmienenú úlohu, najpresnejšia metóda je riešiť stabilná, ale nie je stabilná v FPA! <u>Algoritmus nie je spätne stabilný (obecne).</u>

$$\prod_{i=1}^{20}(x-i)+\delta x^{19}=\prod_{i=1}^{20}(x-x_i(\delta)) \quad / \frac{d}{d\delta}$$

$$0+x^{19}=\frac{d}{d\delta}\prod_{i=1}^{20}(x-x_i(\delta))=\frac{d}{d\delta}(x-x_1)\left[(x-x_2)\cdots(x-x_{20})\right]+(x-x_1)\frac{d}{d\delta}\left[(x-x_2)\cdots(x-x_{20})\right]=$$

$$=\frac{d}{d\delta}(x-x_1)\left[(x-x_2)\cdots(x-x_{20})\right]+(x-x_1)\left\{\frac{d}{d\delta}(x-x_2)\left[(x-x_3)\cdots(x-x_{20})\right]+(x-x_2)\frac{d}{d\delta}\left[(x-x_3)\cdots\right]\right\}$$

$$=\frac{d}{d\delta}(x-x_1)\left[(x-x_2)\cdots(x-x_{20})\right]+\frac{d}{d\delta}(x-x_2)\left[(x-x_1)(x-x_3)\cdots(x-x_{20})\right]+$$

$$+(x-x_1)(x-x_2)\frac{d}{d\delta}\left[(x-x_3)\cdots(x-x_{20})\right]=\ldots=$$

$$=\frac{d}{d\delta}(x-x_1)\prod_{k\neq 1}(x-x_k)+\frac{d}{d\delta}(x-x_2)\prod_{k\neq 2}(x-x_k)+\ldots+\prod_{k\neq 20}(x-x_k)\left[\frac{d}{d\delta}(x-x_{20})\right]=$$

$$=\sum_{i=1}^{20}\frac{d}{d\delta}(x-x_i)\prod_{k\neq i}(x-x_k)$$

$$\Rightarrow x^{19}=\sum_{i=1}^{20}\left(\frac{d}{d\delta}(x-x_i)\prod_{k\neq i}(x-x_k)\right)=\sum_{i=1}^{20}-\frac{dx_i}{d\delta}\prod_{k\neq i}(x-x_k)$$

$$\frac{d}{d\delta}(x-x_i)=-\frac{dx_i}{d\delta}$$

$$\frac{d}{d\delta}\prod_{i=1}^{20}(x-x_i(\delta))=\frac{d}{d\delta}(x-x_2)\left[(x-x_1)\cdots(x-x_{20})\right]+(x-x_2)\frac{d}{d\delta}\left[(x-x_1)(x-x_3)\cdots(x-x_{20})\right]=$$

$$=\frac{d}{d\delta}(x-x_3)\left[(x-x_1)(x-x_2)(x-x_4)\cdots(x-x_{20})\right]+(x-x_3)\frac{d}{d\delta}\left[(x-x_1)(x-x_2)(x-x_4)\cdots\right]$$

$$\Rightarrow \frac{d}{d\delta}\prod_{i=1}^{20}(x-x_i)=\underbrace{\frac{d}{d\delta}(x-x_i)}\prod_{k\neq i}(x-x_k)+(x-x_i)\frac{d}{d\delta}\prod_{k\neq i}(x-x_k)=x^{19}$$

$$-\frac{dx_i}{d\delta}\prod_{k\neq i}(x-x_k)=x^{19}-(x-x_i)\frac{d}{d\delta}\prod_{k\neq i}(x-x_k)$$

$$-\frac{dx_i}{d\delta}=\frac{x^{19}-(x-x_i)\frac{d}{d\delta}\prod_{k\neq i}(x-x_k)}{\prod_{k\neq i}(x-x_k)}$$

$$x = x_i \Rightarrow -\frac{dx_i}{d\delta} = \frac{x_i^{19} - (x_i - x_i)\frac{d}{d\delta}\prod_{k\neq i}(x_i - x_k)}{\prod_{k\neq i}(x_i - x_k)} = \frac{x_i^{19}}{\prod_{k\neq i}(x_i - x_k)}$$

$$\boxed{\left.\frac{dx_i}{d\delta}\right|_{\delta=0} = \frac{i^{19}}{\prod_{k\neq i}(i - x_k)} = \frac{i^{19}}{\prod_{k\neq i}(i - x_k(0))} = \frac{i^{19}}{\prod_{k\neq i}(i-k)}}$$

$$x_i = x_i(\delta) = x_i(0) = i$$

$$x_k = x_k(\delta) = x_k(0) = k$$

$$x^{19} = \sum_{i=1}^{20} \frac{-dx_i}{d\delta}\prod_{k\neq i}(x - x_k) = \sum_{i=1}^{20}\frac{x_i^{19}}{\prod_{k\neq i}(x_i - x_k)}\prod_{k\neq i}(x - x_k)$$

$$x = x_l;\ l \in \{1, \dots, 20\} \Rightarrow x_l^{19} = \sum_{i=1}^{20}\frac{x_i^{19}}{\prod_{k\neq i}(x_i - x_k)}\prod_{k\neq i}(x_l - x_k)$$

$$x_l^{19} = \frac{x_l^{19}}{\prod_{k\neq l}(x_l - x_k)}\prod_{k\neq l}(x_l - x_k) + \sum_{i\neq l}\frac{x_i^{19}}{\prod_{k\neq i}(x_i - x_k)}\prod_{k\neq i}(x_l - x_k)$$

$$\Rightarrow\quad 0 = \sum_{i\neq l}\frac{x_i^{19}}{\prod_{k\neq i}(x_i - x_k)}\prod_{k\neq i}(x_l - x_k)$$

$$\delta = 0 \Rightarrow 0 = \sum_{i\neq l}\frac{i^{19}}{\prod_{k\neq i}(i-k)}\prod_{k\neq i}(l-k)$$

Interpretácia:

Korene vyšli rádovo nepresné. Tu už nie je problém algoritmus ale priamo podmienenosť úlohy. Táto úloha v FPA nie je dobre podmienená. Preto vyššie korene nie presnejšie ako tie vyšli rádovo, ktoré nie dosť, ale až dosť nepresné.