Drexel University

Office of the Dean of the College of Engineering

ENGR 232 – Dynamic Engineering Systems

Week 1 Laboratory Assignment

In this lab we will investigate the use of direction fields to evaluate solutions to first-order differential equations and introduce many important Matlab commands for working exactly with integrals and differential equations. We will explore these qualitative and exact analysis tools via the example of population growth using the **Gompertz** equation as a model. This equation is used to model the growth of tumor cells, and gives an expression of the dependence of tumor size on time.

The differential equation governing the growth is:

$$\frac{dV}{dt} = \beta V \ln\left(\frac{k}{V}\right)$$

Here β is the growth rate and k is a reflection of the carrying capacity of the tumor. Its growth is limited by the supply of nutrients.

Let $\beta = 2$; $k = 100$; and V range from 0 to 140

Part A. Open Matlab, and review basic techniques like symbolic integration and solution of differential equations using `dsolve`.

It is always a good idea to initialize your workspace when you start a new problem. This especially helps clear out any variables that may have been used in previous problems. Just to review how the `clear` command works, enter the following two values for the growth rate and carrying capacity of the tumor.

```
B = 2;   K = 100;
```

If you type in either B or K, Matlab returns the value you just entered. OK, let's clear these values. Enter `clear`, then see if Matlab still knows B and K. Next type in clc. What does it do? Explore the help facility by entering: `help clear` and `help clc`.

Solving differential equations usually involves finding integrals. Matlab's int command can help! First, get help on the `int` command.

```
help sym/int
```

Note: You must always declare any symbolic variables before attempting a symbolic integration.

Here is how to find for example, the integral of x^2.

```
syms x
int(x^2, x)
```

Alert: If you don't declare x to be a symbolic value you will get an error.

Question 1: Find this integral and display the answer in the box below. Find: $\int \frac{2x^5}{1+x^2}\,dx$

Note that by default, Matlab does not display the constant of integration c. You should include that in your answer.

Also include the code you used in the answer box below. Both x and c must first be declared as symbolic.

Question 1: The integral is: _ _ _

Paste code here:

Matlab can also solve many differential equations exactly using the `dsolve` command. First `clear` your variables and command window, then get help on the `dsolve` command. Spend a few minutes reviewing that help info. There is lots of good stuff!

Try this simple example to solve the second-order differential equation: $\frac{d^2x}{dt^2} = -4x$

You may recognize this as simple harmonic motion and should expect two constants of integration.

```
syms x(t)
DE = diff(x,t,t) == -4*x
dsolve(DE)
```

Note the use of == so you can give Matlab actual equations!

Now find the solution with the initial conditions $x(0) = 1$, $dx/dt(0) = 0$.

```
Dx = diff(x)
dsolve(DE, x(0)==1, Dx(0)==0)
```

Question 2: Find the general solution to this new differential equation. Place your code and answers in the numbered box below. No initial condition is given so your answer will involve a constant c. You can save space by placing multiple commands on the same line separated by a ;

$$\frac{dy}{dt} = y - t$$

Question 2. The general solution is: _ _ _

Paste your code here:

Question 3: Find the specific solution to this differential equation $\frac{dy}{dt} = y - t$ with the initial condition that: $y(0) = 1$

Question 3. The solution satisfying $y(0) = 1$ is: $y(t) =$ _ _ _

Ungraded Practice: Now find the specific solution satisfying $y'(0) = 2$ instead. Note the derivative!
You do not need to record the answer. You should find: $y(t) = e^t + (1 + t)$

Question 4: The Gompertz differential equation at the top of the first page is a difficult non-linear, autonomous differential equation. But let's see if Matlab can solve it exactly. No harm in at least trying. Clear all your variables. Now enter the values for the growth rate and carrying capacity of the tumor once again.

```
B = 2; K = 100;
```

The volume V of the tumor is a function of time – it's growing!

```
syms V(t);
```

Complete the next line to enter the Gompertz DE into Matlab. You can use B and K in your code. Matlab uses `log` for the natural log. Don't forget to use `==` to denote equality in the equation.

$$\frac{dV}{dt} = \beta V \ln\left(\frac{k}{V}\right)$$

```
DE = diff(V, t) ==  _ _ _ _ _        ← add the RHS to complete this line.
```

Now solve the Gompertz exactly using `dsolve` and record your command and the specific solution starting with a volume of 1 at time 0. Record your code and the answer in the box below. Be sure to apply `simplify` to obtain your answer in nicer form.

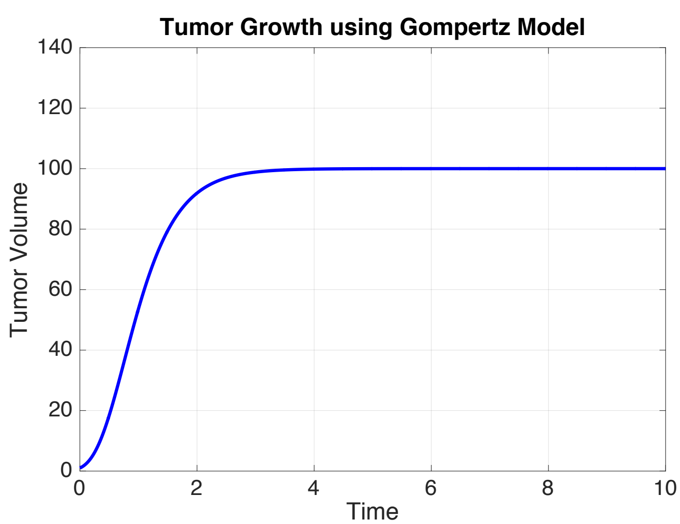
Question 4. The solution satisfying the initial condition $V(0) = 1$ is: $V(t) =$ _ _ _

Paste your code here:

Ungraded Practice: If you stored the answer in a variable, say using: `sol = dsolve( _ _ _ _ )` you can convert that answer to a MATLAB function using:

```
V = matlabFunction(sol);
```

Now create the following plot using a **blue** line with **'LineWidth'** of 3. Add a **title** as shown and label both axes using **xlabel** and **ylabel**. Turn on the **grid**, and adjust the **'fontsize'** to 20. Use the **axis** command to set limits as shown. This plot is not graded, but it's still an important review exercise. Don't skip!!

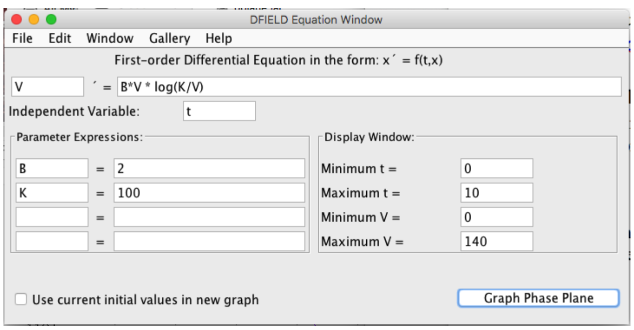


Part B: Qualitative Analysis

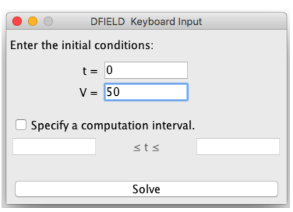
The **Gompertz** differential equation is notoriously difficult to solve, but as we just saw, Matlab quickly dispatched with the challenge. That's really impressive raw power! Now let's explore the solutions graphically to get a better intuition about the Gompertz model. We will now use the direction field tool to examine the solution. **Download** the Java JAR named **dfield** available in the **Software** folder. **Note:** Use only the Java version for this lab. If you use dfield8.m, you will have to work harder to prevent crashes as the system encounters logs of negative numbers.

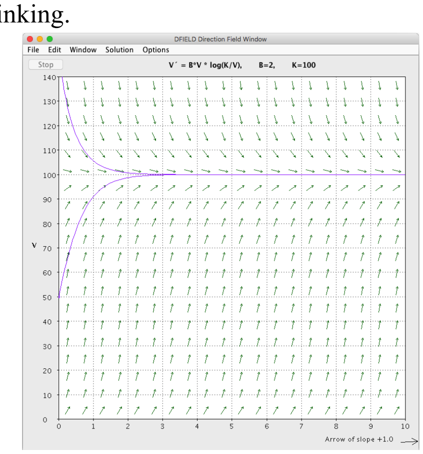
Plot the direction field for the Gompertz differential equation using the Java program named `dfield`. Set time from 0 to 10, and V from 0 to 140. Let $\beta = 2$ and $k = 100$ as before.

$$\frac{dV}{dt} = \beta V \ln\left(\frac{k}{V}\right)$$



The graph below shows two solutions starting at V(0) = 50 and V(0) = 150 obtained using the **keyboard input** option available under the **Solution** menu of the JAR. The curve approaching from above may correspond to a large tumor, which has lost some of its former blood supply due to necrosis and is now shrinking.





Question 5: Add 7 more solutions so that the volume at times 1, 2, 3, 4, 5, 6 and 7 is 50. These tumors are growing. Add solution curves so that the volume at times 1, 2, 3, 4, 5, 6 and 7 is 150. These tumors are shrinking. Note some of these curves may suddenly look chaotic, and will need to be recalculated using the "Recalculate All Solutions Using Current Settings" under the Solution menu. Paste your completed image here.

Solution curves for Question 5 with B = 2, K=100

Question 6: Estimate how long it takes a tumor of size $V = 20$ to double in size to $V = 40$. Zoom in! Use B = 2.

> **Question 6.** Keep B = 2, and K = 100. It takes a tumor starting at V = 20 about this long to double to V = 40:
>
> ANS: _ _ _ _ time units

Question 7: The critical values for this equation are $V = K = 100$ and $V = 0$. Classify each as stable or unstable in the box below.

> **Question 7: Stability**
> The critical value V = K = 100 is: _ _ _
> The critical value V = 0 is: _ _ _

(Don't worry about semi-stable. The DE is not even defined for negative volumes.)

Question 8: A More Aggressive Tumor! Simply change B (beta) from 2 to 4 and leave the carrying capacity as K=100. Obtain a new direction field, and include multiple solutions curves passing through V = 50 and 150 as before. Paste your completed plot below.

Solution curves for Question 7 with B = 4 and K = 100.

Ungraded Practice: Now B = 4. These tumors are growing much faster. Estimate how long it takes a tumor of size V = 20 to double in size to V = 40. Zoom in!

Ungraded: Zoom in on a solution curve that starts with $V = 20$ at time 0, and find when it reaches $V = 40$.
You should find a doubling time of about 0.14 time units.

Question 9: Fastest Growth

The tumors are growing the fastest at the maximum for:

$$\frac{dV}{dt} = \beta V \ln\left(\frac{k}{V}\right)$$

Let's find the maximum growth rate using Matlab's `diff` command. Start by clearing all variables in the Matlab window.
Then declare B, K and V(t) to be symbols.

```
syms V(t) B K
```

Enter the Gompertz equation again, but this time B and K are also symbolic.

```
eqn1 = diff(V,t) ==B*V*log(K/V);
```

To find the maximum, we need to set the double derivative to zero. (inflection point).

```
eqn2 = diff(eqn1, t)
eqn2(t) = diff(V(t), t, t) == B*log(K/V(t))*diff(V(t), t) - B*diff(V(t), t)
```

Simplifying by hand we see the inflection point is where: `log(K/V(t)) = 1`
These tumors are growing fastest at the inflection point: (answer is a simple fraction) Place your answer in the box below.

Question 9. V = ___

Question 10: Autonomous Property

The Gompertz equation does not depend on time explicitly. Such equations are called **autonomous**. All the solution curves for growing tumors (below the value *K*) can be obtained from just one by shifting it in the … direction. Enter **horizontal** or **vertical** below.

Question 10.

Be sure all ten questions are answered, then submit your Answer Template as a PDF. We only need to see your name and the answers labelled 1 – 10. Submission must be a single PDF file!