Drexel University
Office of the Dean of the College of Engineering
ENGR 232 – Dynamic Engineering Systems

Recitation Guidelines for Week 1

Review ENGR 231 material – example with equilibrium points, d-field, stability, and solutions in MATLAB

In the prerequisite ENGR 231 we started the course with models of systems that could be modeled by **first-order differential equations**. These include:

1. Mixing models – Salt/Chemical in tank.
   Example:
$$\frac{dQ}{dt} = 3Q - 5$$

2. Population models – Logistic, Gompertz, Malthusian, Exponential
   Logistic equation model:
$$\frac{dP}{dt} = rP(K - P), \qquad r, K - constant$$
   Generalized Gompertz equation model:
$$\frac{dV}{dt} = aV^{\alpha} - bV^{\alpha}\ln V, \quad a, b, \alpha - constant$$

3. Radioactive decay
$$\frac{dR}{dt} = -kR, \quad k - constant$$

4. Newton's law of cooling
$$\frac{dT}{dt} = k(M - T), \quad k, M - constant$$

5. Compound interest / financial models
$$\frac{dS}{dt} = rS + K, \quad r, K - constant$$

These equations **need not be memorized**, we only need to recognize the many examples in the real world where first-order differential equations can be used to model systems.

All of these models could be written in the form:

$$\boxed{\frac{dy}{dt} = f(t, y)}$$

First-order models are characterized by first-order differential equations, identified by first derivatives.

Classification of differential equations:

1. First order: highest derivative is 1. See examples above.

2. N-th order: highest derivative is N

$$\frac{d^4y}{dt^4} - 3\frac{d^2y}{dt^2} = 4, \quad 4^{\text{th}} \text{ Order}$$

3. Ordinary differential equation: An equation containing a function of **one** independent variable and its derivatives.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \leftarrow \text{ Not ordinary} \qquad \frac{d^3z}{dx^3} = 3x^2 \sin(z) \quad \leftarrow \text{Ordinary}$$

4. Autonomous: Equation that does not depend explicitly on the independent variable.

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} = 4y \quad \leftarrow \quad \text{Autonomous} \qquad 4\frac{\mathrm{dy}}{\mathrm{dt}} = \sin(t) + 2y^2 \quad \leftarrow \quad \text{Not autonomous}$$

5. Linear: If the highest derivative can be written as a linear combination of the remaining derivatives.

$$\frac{dy}{dt} = 3y + 2t \quad \leftarrow \quad 1^{\text{st}} \text{ Order Linear} \qquad \frac{d^2y}{dt^2} = 3\frac{dy}{dt} - 2y \quad \leftarrow \quad 2^{\text{nd}} \text{ Order Linear}$$

$$\frac{d^3y}{dt^3} + 2\left(\frac{dy}{dt}\right)^2 = y \quad \leftarrow \quad 3^{\text{rd}} \text{ Order Non} - \text{linear}$$

6. Time-varying: If the differential equation is non-autonomous.

Examples

$$\frac{dy}{dt} = \sin(t)\, y$$

Separable, first-order, linear, time-varying, ordinary.

$$\frac{dy}{dt} + 3y\frac{d^2y}{dt^2} = 2y$$

Second-order, non-linear, autonomous, ordinary.

$$4\frac{d^2y}{dt^2} - t^2\, y = \sin(t)$$

Second-order, linear, time-varying, ordinary.

Equilibrium points of first-order differential equations:

Equilibrium points are the critical points of the system at which there is no change in the quantity being considered. Equilibrium points can be found by setting the derivative to zero.

Example: Logistic population growth

$$\frac{dP}{dt} = rP(K - P)$$

The equilibrium points are: $P = 0$, and $P = K$. In this model, K is the environmental carrying capacity and represents the size of the population that the current environment can sustain. Consider what these two equilibrium points imply: at $P = 0$ there is no population, therefore no inhabitants to reproduce and so the population will forever remain at 0; for any other value of P, the population will grow as they continue to reproduce, but to no greater value than the environment can support, thus when $P = K$, the overall size of the population will remain at K. One can apply this intuitive reasoning to any system model to understand the significance of the equilibrium point of a system model.

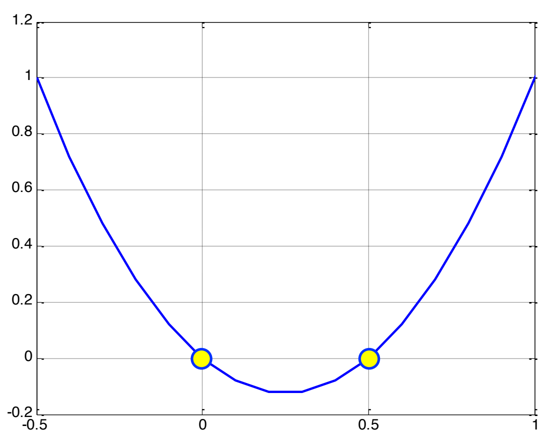
Stability of equilibrium points in first-order equations:

For first-order autonomous systems, the derivative does not change with time, so we can plot dy/dt vs. y and use this plot to determine the stability of the equilibrium points.

Recall that we covered: Stable, unstable and semi-stable equilibrium points in ENGR 231. You can review the notes posted to see how we determine these types of equilibrium points.

Example

$$\frac{dy}{dt} = 2y^2 - y$$



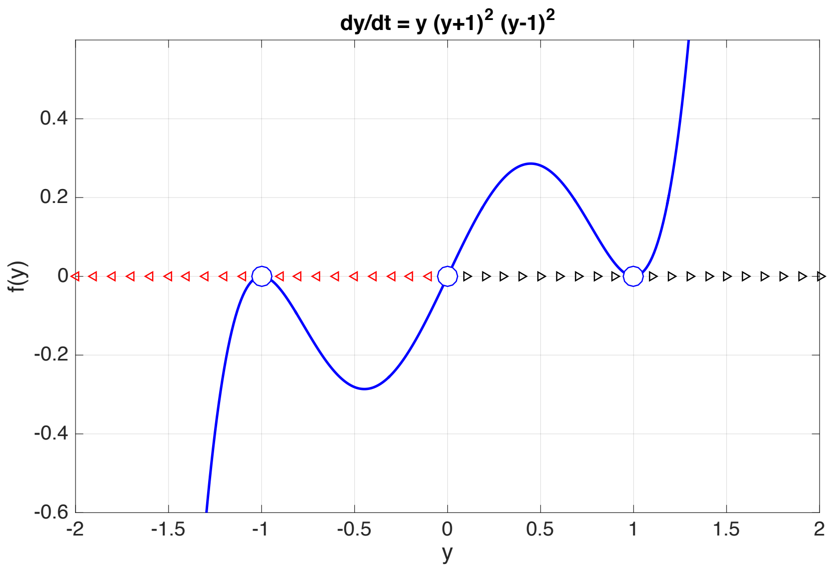
This is a graph of $\frac{dy}{dt}$ vs y. Here, we can clearly see the equilibrium points at 0 and $1/2$ which can be verified by setting $dy/dt = 0$ and solving for y. The equilibrium point 0 is stable and the equilibrium point ½ is unstable.

ROA: Region of attraction of the stable point is: $(-\infty, 0.5)$ and the ROA for the unstable point is [0.5]

Review: Find the ROA (region of attraction) for each critical point for the following differential equation.

$$\frac{dy}{dt} = y\,(y+1)^2(y-1)^2$$

Use the provided graph to help.



ROA(−1) = _____ ROA(0) = _____ ROA(+1) = _____

Direction Fields:

Consider the first-order ordinary differential equation:

$$\frac{dy}{dt} = f(t, y)$$

This gives an expression for the derivative of the variable for each value t and y. Recall that the derivative is the slope of the line tangent to the curve at that point. One can construct a figure on which the value of the derivative is used to draw an arrow of corresponding slope over a grid of points t, y.

For example:

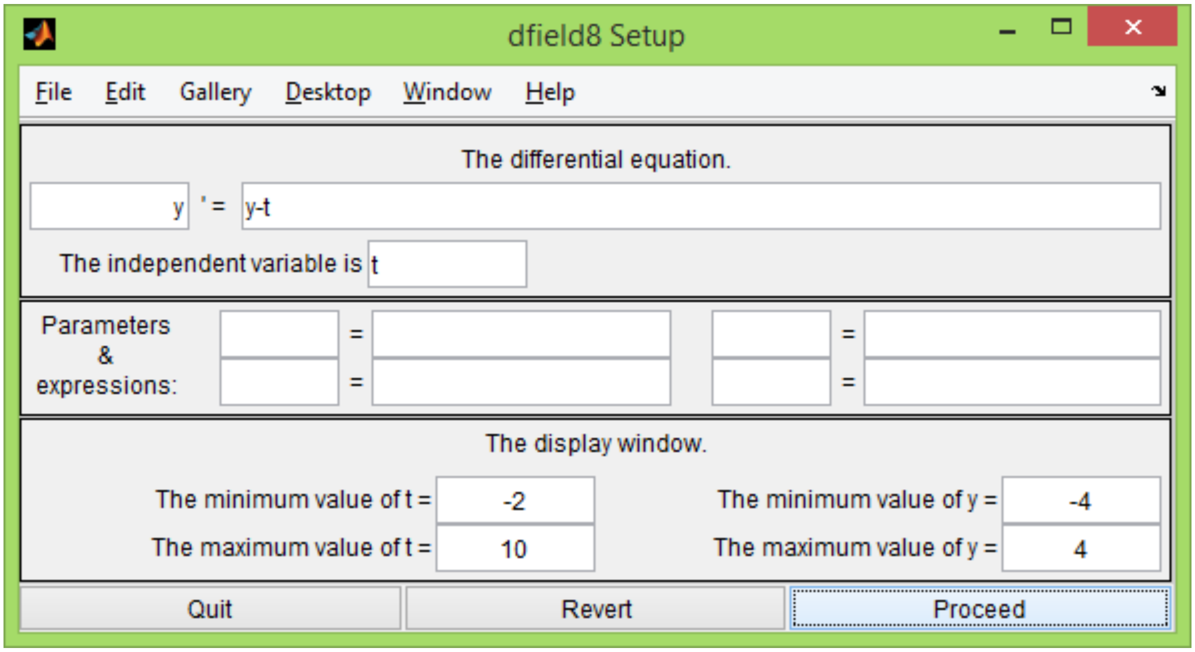
$$\frac{dy}{dt} = y - t$$

On a set of axes with t being the horizontal axis and y the vertical axis, the value of the derivative can be determined at each point by calculating dy/dt at that point. Representing this derivative as a directional arrow of slope = dy/dt yields a direction field plot.

We will not do this by hand, but rather using the Rice University's d-field tool.

You can download the m-file from the course website under the Software folder.

Instructors: Run the program and go through the interface

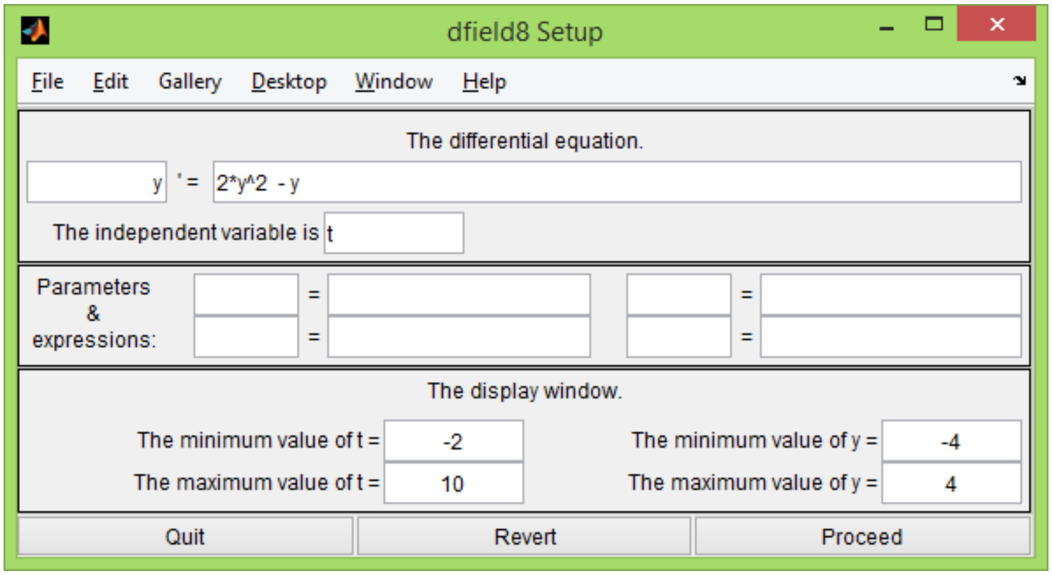


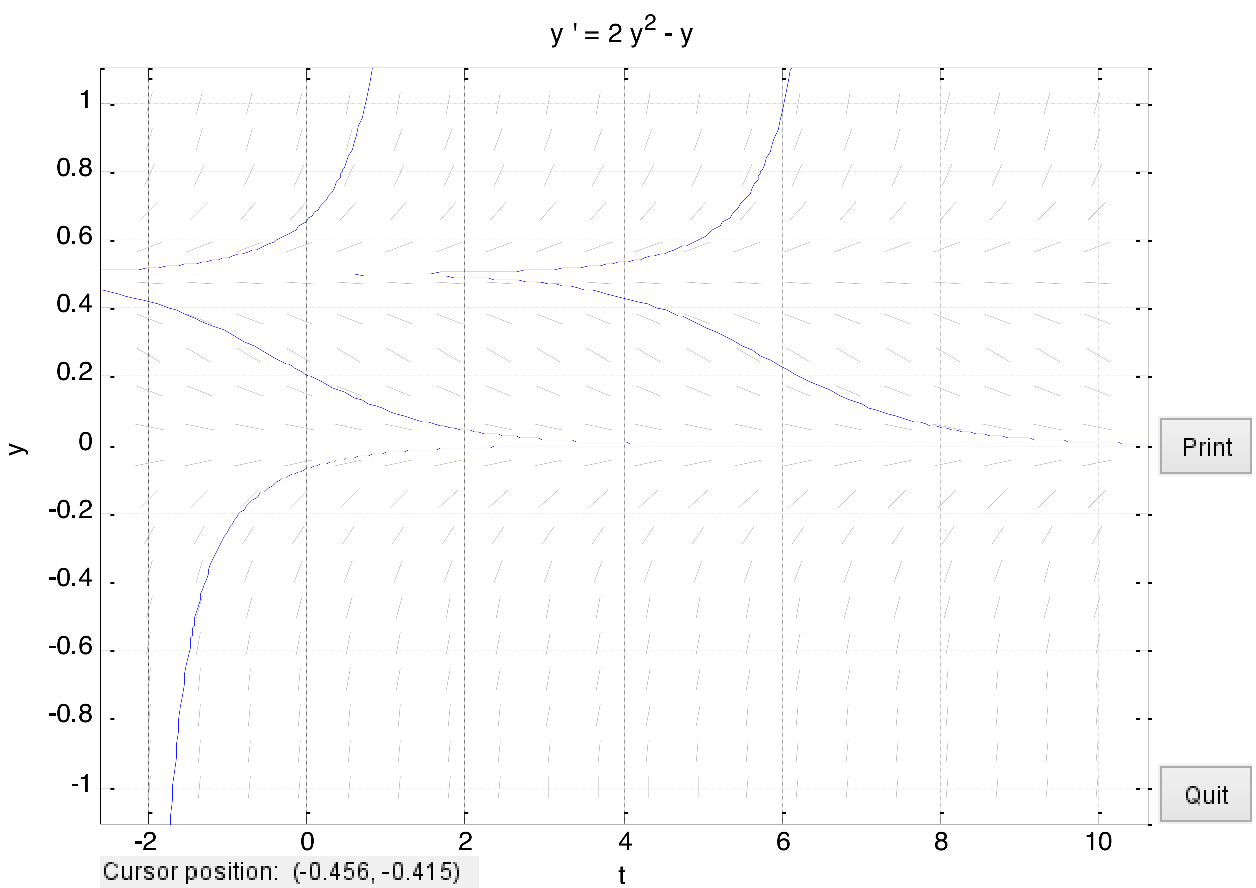
Note: You can type in the differential equation (MATLAB syntax needed) and set the time-scale and the limits for the y-axis.

The online applet uses arrows, the MATLAB version does not have arrows – we assume time is moving forward so all arrows point accordingly.

Previous Example:

$$\frac{dy}{dt} = 2y^2 - y$$





By clicking on the graph, you are asking MATLAB to generate a solution trajectory which will correspond to a unique set of initial conditions. Notice how the lines all move away from 0.5. The ROA of equilibrium point 0 is $(-\infty, 0.5)$ and so any lines that pass through this region will be attracted to the equilibrium point 0 – as seen.

Recognize that we have not solved the differential equation. We can see what the solution will look like simply based on knowing the function (derivative). This is a numerical solution and gives us quantitative values for a solution. While we cannot write down the equation for the solution based solely on this approach, we can certainly study the behaviour of the system by analyzing the solution trajectories.

Note: If you have any trouble using the MATLAB direction field script **dfield8.m**, it is perfectly fine to use the Java versions instead, at least for the first week. Links to these are also posted inside the Software folder.