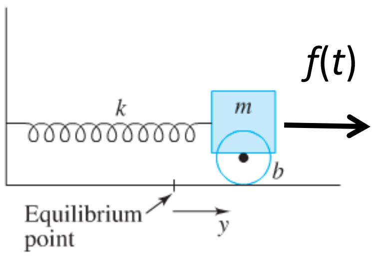
Lab 4: Model of a Spring-Mass-Damper

Consider this Spring-Mass-Damper System (SMD) and its differential equation with forcing function *f*(*t*).



$$\frac{d^2y}{dt^2} + \left(\frac{b}{m}\right)\frac{dy}{dt} + \left(\frac{k}{m}\right)y = f(t)$$

Initial Conditions: $y(0) = 10,\ y'(0) = 0$

Constants: `m = 1, b = 2, k = 10, f(t) = 0`

Part A: The exact solution

1. The values for the constants *m*, *b* and *k* have been assigned in the code below.

```
%% 1: Constants for the spring-mass-damper system.
% Prepare the workspace.
clc, clear

% First look at the homogeneous case where the forcing term is zero.
% That is, f(t) = 0.
% Constants entered for you.
m = 1; b =2; k = 10;

% Find the roots of the characteristic equation:
char_poly = [0, 0, 0]  % <- Fix this stub.
```

a. Fix the above stub for the characteristic polynomial. Notice it is a row vector.

b. Find the roots of the characteristic equation, using the roots command. >>> `help roots`

Paste your answers in the box below for credit.

Question 1: The roots are:

Note the roots are complex conjugates with a negative real part.

2. What is the discriminant $D = b^2 - 4ac$ of the characteristic equation?
Note the usual notation requires $a \rightarrow 1$, $b \rightarrow (b/m)$ and $c \rightarrow (k/m)$

Paste your answer in the box below for credit.

Question 2: The discriminant is *D* = ______

3. Find the exact solution for our IVP (initial value problem) using dsolve.
Initial Conditions: $y(0) = 10,\ y'(0) = 0$. Here's some starter code:

```
%% Question 3: Find the exact solution.
syms y(t)
Dy = diff(y,t); D2y = diff(y,t,t);
m = 1; b = 2; k = 10;
DE = D2y + (b/m) * Dy + (k/m) * y == 0
```

Now use dsolve to find the exact solution satisfying the initial conditions: $y(0) = 10,\ y'(0) = 0$
You will need to use two equal signs == inside dsolve to specify each initial condition.
Does the solution have the expected form: $y(t) = e^{\alpha t} \cdot (A\cos\beta t + B\sin\beta t)$ when the roots are complex conjugates $r = \alpha \pm \beta i$?

Paste your answer in the box below for credit.

Question 3: The exact solution is: $y(t) =$

Question 4: Use matlabFunction and diff do define the exact solution $y(t)$ and its derivative $y'(t)$.
Record the exact expression for the derivative $y'(t)$ below.
If you named the solution returned by dsolve as say sol, you can do this as follows.

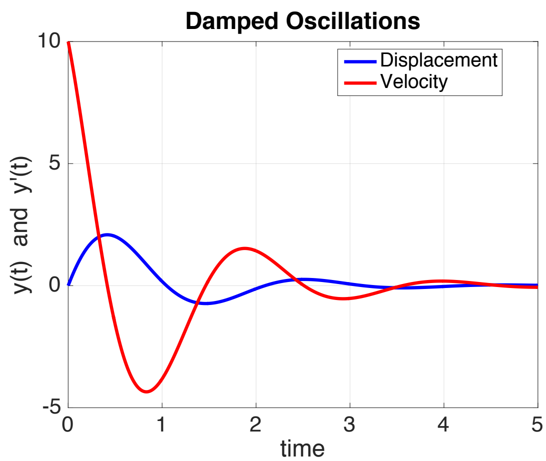
```
%% Exact solution and its derivative.
y = matlabFunction(sol)
Dy = matlabFunction(diff(y,t))
```

Question 4: The exact solution for the derivative is: $y'(t) =$

Question 5: Plot the exact solution $y(t)$ and its derivative $y'(t)$ over the interval from 0 to 5.
Plot the exact solution in blue with a line thickness of 3, and plot the derivative in red. Label both axes, and apply a title such as "Damped Oscillations.

Replace this graph with your completed graph. This graph uses different initial conditions, so your graph will look different.

Sample:



Part B. State Space Representation

Let us now rewrite the **spring-mass-damper** system in **state space** form. Instead of the given 2nd-Order DE for the unknown displacement $y(t)$, we can convert to a 1st-order **system** by making the following substitutions. Let $x_1 = y$ and let $x_2 = \frac{dy}{dt}$. Collect these two new variables into a **column** vector, which is called the **state vector**.

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State vector

Differentiating the state vector, and using the original 2nd-Order DE to eliminate $\frac{dx_2}{dt} = \frac{d^2y}{dt^2}$ we find:

$$\frac{d}{dt}\vec{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ f(t) - \frac{k}{m}x_1 - \frac{b}{m}x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

Thus if we define:

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The state space equations are:

$$\frac{d}{dt}\vec{\mathbf{x}} = A\,\vec{\mathbf{x}} + \boldsymbol{B}\,\boldsymbol{f}$$

6. Write a function in a file named **smd.m** (short for spring-mass-damper) which returns the vector:

xdot = $\frac{d}{dt}\vec{\mathbf{x}}$

Your header will be:

```
function xdot = smd(t,x) % Note, x will be a column vector.

        % The last two lines will be:
        xdot = A*x + B*f;
end
```

In between you must:

i. Specify the values for *m*, *b* and *k*. (Use `m = 1, b = 2, k = 10 for all of the lab.)`

ii. Set *f* to zero.

iii. Define the 2x2 matrix $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}$.

iv. Define the column vector $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Tip: Use a semi-colon to start a new row in a matrix or column vector.

Verify your **smd** function is working by evaluating it. First recall the initial conditions.

y(0) = 10, y'(0) = 0, so that $x0 = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$. Enter the column vector x0, then evaluate:

```
>> smd(0, x0)
```

6. Record your answer here for credit. Just write in the column vector returned by: >> smd(0, x0)

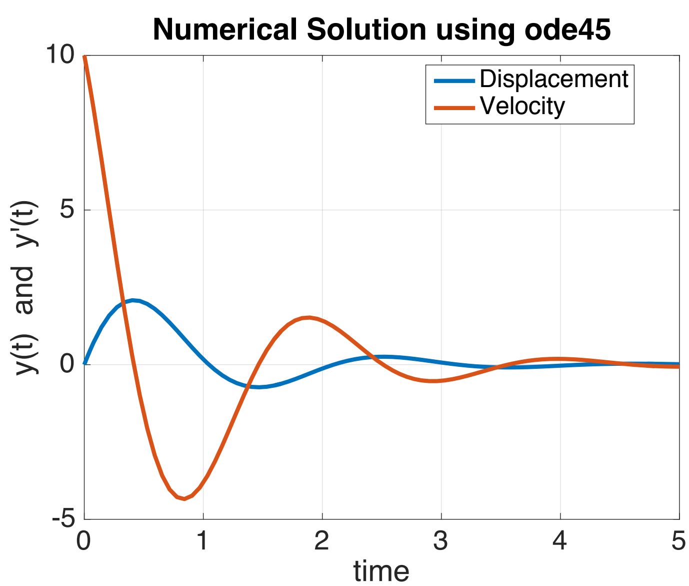
Question 6: The value of smd(0, x0) is:

Question 7. Now that **smd.m** is working, you are ready to use ode45 to solve the 1st-order system.
Produce a second simultaneous plot of y and y', but now use ode45.
Use the initial conditions y(0) = 10, y'(0) = 0, so that $x0 = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$.
Add a title: `Numerical Solution using ode45`
Add a legend which includes Displacement and Velocity as shown in the sample plot.
Increase the font size to 20.
Sample code to get you started using ode45.

```
x0 = [10; 0]    % Initial condition.
tf = 5;
[t,x] = ode45(@smd, [0,tf], x0);
```

7. **Replace** this sample plot with your completed graph. This plot exchanged the initial conditions so your plot will **not** look the same.

Sample simultaneous plot using **different** initial conditions.



Points 8 – 10: Tiled Plots

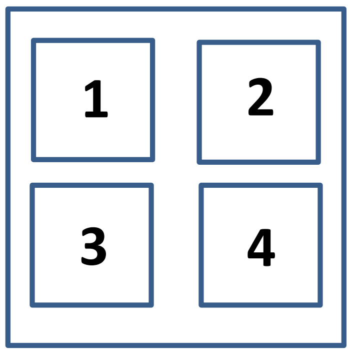
The rest of the points will be earned by submitting a **tiled plot** of the system meeting all the following requirements. Note the sample plot below uses different initial conditions, so your tiled plot will look a little different. Now we will present the above solution in a different fashion using tiled plots to highlight different features. Enter the following command to review how tiled plots work.

>> help subplot

Consider the command: `subplot(2, 2, 1)`

This will break your figure window into a 2 x 2 grid yielding four sets of smaller axes labelled as follows.

Figure



a. We will plot just the displacement y(t) in tile #1. Use: subplot(2, 2, 1)

b. We will plot just the velocity y'(t) in tile #3. Use: subplot(2, 2, 3)

c. We will create a phase plot using both tiles on the right. Use: subplot(2, 2, [2,4])

In tile #1, plot just the displacement y(t) in blue with a line width of 3. There is no need to recalculate it. The displacement data is in the first column of **x,** which was already computed using ode45. Set the title to 'Displacement' and label both axes. Set the xlabel to `'Time in seconds'` and the `ylabel` to `'y(t)'` Turn the `grid on`.

In tile #3 (bottom left), plot just the velocity y'(t) in red with a line width of 3. There is no need to recalculate it. The velocity data is in the second column of **x,** which was already computed using ode45. Set the title to 'Velocity' and label both axes. Set the xlabel to `'Time in seconds'` and the `ylabel` to `y''(t).` You may need to use the "two-quotes trick" mentioned already. Turn the `grid on`.

Now to earn the last three points.

8. Now create the **phase-plot** on the right. It will occupy the entire right-half, and thus use both tiles 2 and 4. That is why the sample code uses: subplot(2, 2, [2,4])

The phase plot uses y for the horizontal axis and y' for the vertical axis. Note these are just the first and second columns of x. Plot the phase curve now in **black** (k) using a line width of 3. The xlabel should be set to y and the ylabel to dy/dt

9. Turn **hold on** immediately after plotting the phase curve. We will decorate the curve with some salient points. The initial point (in phase space) is (y(0), y'(0)) = (10,0). Show this point now using a **green** circle. The equilibrium point is (0,0). Show the equilibrium point using a **red** circle. Green for start, red for stop.

10. Add a legend to your phase plot similar to that shown in the sample tiled plot below.

Points 8-10

Note the sample plot below uses **<u>different</u>** initial conditions, so your tiled plot will look a little different.

You **<u>must</u>** use the initial conditions y(0) = 10, y'(0) = 0 to receive credit.

<u>Replace</u> this sample plot with your own tiled plot for points 8-10. Set an appropriate font size.