Drexel University

Office of the Dean of the College of Engineering

ENGR 232 – Dynamic Engineering Systems

Recitation Guidelines for Week 4

Topics for recitation this week:

- Second-order linear homogenous differential equations

Second-order linear homogeneous differential equations with constant coefficients

General form: $$ay'' + by' + cy = 0$$

i. Second order → highest derivative is $y'' = \frac{d^2y}{dt^2}$

ii. Linear → Satisfies superposition principle.

iii. Homogeneous → only terms in y'', y', y. All terms without the dependent variable or one of its derivatives = 0.

iv. Constant coefficient → a, b, c are constants (with $a \neq 0$) since has to be second order.

The **characteristic equation (auxiliary equation)** for this differential equation is:

$$ar^2 + br + c = 0$$

Where r_1 and r_2 can be found by solving the characteristic equation:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Depending on the values of r_1 and r_2 the equation will have one of the following solutions:

1. Real distinct roots if $D = b^2 - 4ac > 0$. D is called the **discriminant**.

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

2. One real root if $b^2 = 4ac$. That is, $D = 0$.

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}$$

3. Complex roots if $b^2 - 4ac < 0$

$$y(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Here, $r_{1,2} = \alpha \pm i\beta$

This works because we can check that e^{rt} is a solution for some r.

Example 1: Case of Real Distinct Roots

$$y'' - y' - 6y = 0 \quad , \quad y(0) = 1, y'(0) = 0$$

Characteristic Equation:

$$r^2 - r - 6 = 0$$

Solving this equation for r gives:

$$(r-3)(r+2) = 0 \quad \rightarrow \quad r_1 = 3, r_2 = -2$$

General solution is given by:

$$y(t) = c_1 e^{3t} + c_2 e^{-2t}$$

The constants c_1 and c_2 can be found by substituting the initial conditions for y and y'.

Substituting initial conditions for $y(0) = 1$ gives:

$$1 = c_1 + c_2$$

Substituting the initial conditions for $y'(0) = 0$ gives:

$$0 = 3c_1 - 2c_2$$

(Note, here we substituted the values of $t = 0$ and $y' = 0$ in the expression $y'(t) = 3c_1 e^{3t} - 2c_2 e^{-2t}$

Solving these equations simultaneously gives: $c_1 = 2/5$ and $c_2 = 3/5$.

Final solution is:

$$y(t) = \frac{2}{5} e^{3t} + \frac{3}{5} e^{-2t}$$

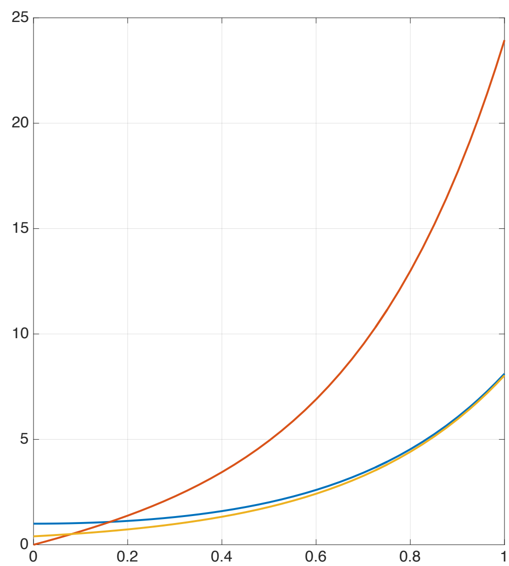
Observe that the values of r_1 and r_2 determine if the solution will be **stable** or **unstable**.

After reading the section on converting a 2nd-order differential equation to a 1st-order system, come back and see if you can reproduce this matrix form:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Solution shown in **blue**. Derivative in **red.**
The positive exponential term dominates.
The yellow curve shows just the positive exponential term: $\frac{2}{5} e^{3t}$

Or solve the equation exactly using **dsolve**.

```
syms y(t); Dy = diff(y);  D2y = diff(y,2);
DE = D2y - Dy -6*y == 0
Y = dsolve(DE, y(0) == 1, Dy(0) == 0)
Y = (3*exp(-2*t))/5 + (2*exp(3*t))/5
```

Example 2: Case of Real Equal Roots

$$4y'' - 12y' + 9y = 0 \quad , \quad y(0) = 1, y'(0) = 2$$

Characteristic Equation:

$$4r^2 - 12r + 9 = 0$$

Solving this equation for r gives:

$$(2r-3)^2 = 0 \quad \rightarrow \quad r_1 = r_2 = 3/2$$

General solution is given by:

$$y(t) = c_1 e^{\frac{3}{2}t} + c_2\, t\, e^{\frac{3}{2}t}$$

(Note how we deal with repeated roots)

The constants r_1 and r_2 can be found by substituting the initial conditions for y and y'.

Substituting initial conditions for $y(0) = 1$ gives:

$$1 = c_1$$

Substituting the initial conditions for $y'(0) = 0$ gives:

$$2 = \frac{3}{2}c_1 + c_2$$

(Note, here we substituted the values of $t = 0$ and $y' = 0$ in the expression $y'(t) = \frac{3}{2}c_1 e^{\frac{3}{2}t} + c_2\left(t\frac{3}{2}e^{\frac{3}{2}t} + e^{\frac{3}{2}t}\right)$

Solving these equations simultaneously gives: $c_1 = 1$ and $c_2 = \frac{1}{2}$

Final solution is: $y(t) = e^{\frac{3}{2}t} + \frac{1}{2}te^{\frac{3}{2}t}$

Observe that the values of r determines if the solution will be stable or unstable.

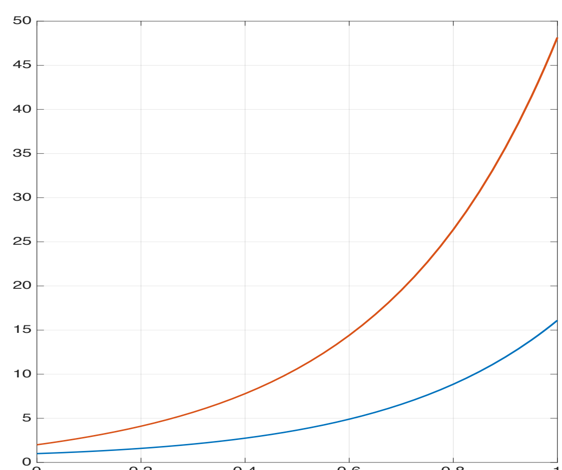
After reading the section on converting a 2nd-order differential equation to a 1st-order system, come back and see if you can reproduce this matrix form:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9/4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution shown in **blue**. Derivative in **red**.



Or solve the equation exactly using **dsolve**.

```
syms y(t); Dy = diff(y);  D2y = diff(y,2);
DE = 4*D2y -12*Dy + 9*y == 0
Y = dsolve(DE, y(0) == 1, Dy(0) == 2)
Y = exp((3*t)/2) + (t*exp((3*t)/2))/2
```

Example 3: Case of Complex Roots

$$5y'' - 2y' + y = 0 \quad , \quad y(0) = 1, y'(0) = 1$$

Characteristic Equation:

$$5r^2 - 2r + 1 = 0$$

Solving this equation for r gives:

$$r_{1,2} = \frac{2 \pm \sqrt{2^2 - 4(5)(1)}}{10}$$

$$r_1 = \frac{1}{5} + i\frac{2}{5} \qquad r_2 = \frac{1}{5} - i\frac{2}{5}$$

Note, since we have complex roots, we define: $\alpha = \frac{1}{5}$ and $\beta = \frac{2}{5}$

General solution is given by:

$$y(t) = e^{\frac{1}{5}t}\left[c_1 \cos\left(\frac{2}{5}t\right) + c_2 \sin\left(\frac{2}{5}t\right)\right]$$

<u>(Note how we deal with complex roots)</u>

The constants r_1 and r_2 can be found by substituting the initial conditions for y and y'.

Solving these equations simultaneously gives: $c_1 = 1$ and $c_2 = 2$. Solving these equations to get the constants are algebraically cumbersome. I used wolframalpha, but it can also be done in MATLAB as we will learn later on. For now, just know that you use the initial conditions to get c_1 and c_2. Final solution is:

$$y(t) = e^{\frac{1}{5}t}\left\{\cos\left(\frac{2}{5}t\right) + 2\sin\left(\frac{2}{5}t\right)\right\}$$

Observe again that the values of r determines if the solution will be stable or unstable.

<u>The most common question that will be asked:</u>

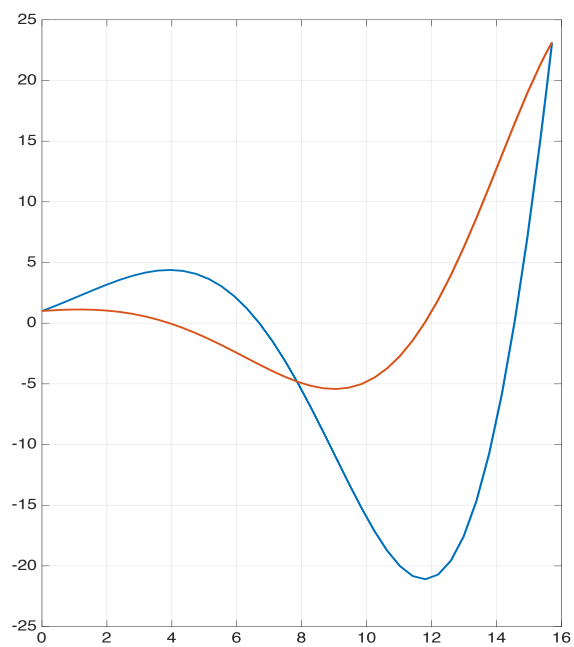
Do we have to remember those equations? That is, the form of the solution based on what the roots are to the characteristic equation. The answer is **<u>yes</u>**. The only thing that will be given to them is the differential equation.

After reading the section on converting a 2nd-order differential equation to a 1st-order system, come back and see if you can reproduce this matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/5 & 2/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Solution shown in **blue**. Derivative in **red.**

Plotted over the interval from 0 to 5π.

Or solve the equation exactly using **dsolve**.

```
syms y(t); Dy = diff(y);  D2y = diff(y,2);
DE = 5*D2y -2*Dy + y == 0
Y = dsolve(DE, y(0) == 1, Dy(0) == 1)
Y = cos((2*t)/5)*exp(t/5) + 2*sin((2*t)/5)*exp(t/5)
```

Example 4: Real roots with one of them equal to zero.

$$y'' + 2y' = 0, \quad y(0) = 1, \qquad y'(0) = -1$$

The roots of the characteristic equation $r^2 + 2r = 0$ are:

$$r_1 = 0, r_2 = \ -2$$

Using the general formula for real distinct roots we get:

$$y(t) = c_1 e^{0t} + c_2 e^{-2t}$$

$$y(t) = c_1 + c_2 e^{-2t}$$

Using the initial conditions we get:

$$1 = c_1 + c_2$$
$$-1 = -2c_2 \quad \rightarrow \quad c_2 = 1/2, c_1 = 1/2$$

Thus, the solution is:

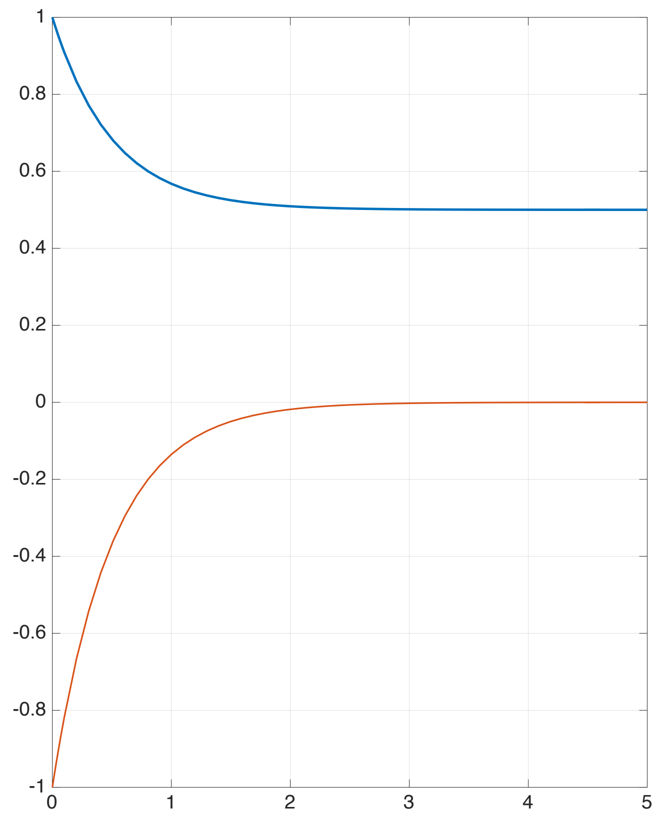
$$y(t) = \frac{1}{2} + \frac{1}{2} e^{-2t}$$

A little later, you will learn how to make plots like this using **ode45**.

After reading the section on converting a 2nd-order differential equation to a 1st-order system, come back and see if you can reproduce this matrix form:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution shown in **blue**. Derivative in **red**.



Or solve the equation exactly using **dsolve**.

```
syms y(t); Dy = diff(y);  D2y = diff(y,2);
DE = D2y + 2*Dy == 0
Y = dsolve(DE, y(0) == 1, Dy(0) == -1)
Y = exp(-2*t)/2 + 1/2
```

Decoupling Second Order Linear Differential Equations for ode45

Consider a 2^{nd}-order, non-homogeneous differential equation of the form:

$$ay'' + by' + cy = g(t) \qquad y(0) = y_0,\;\; y'(0) = v_0$$

We know we can solve this using method of undetermined coefficients which will be completed next week.

These differential equations can be decoupled into a system of first-order differential equations by considering the following variables:

$$x_1(t) = y(t)$$

$$x_2(t) = y'(t)$$

Notice, based on this definition:

$$x_1'(t) = y'(t) = x_2(t)$$

$$x_2'(t) = y''(t)$$

From the original differential equation, we have:

$$y''(t) = \frac{1}{a}(g(t) - by' - cy)$$

$$\rightarrow x_2'(t) = \frac{1}{a}g(t) - \frac{b}{a}x_2(t) - \frac{c}{a}x_1(t)$$

Now we have 2 coupled differential equations that represent the same system, now in terms of variables x_1 and x_2. Solving for variable $x_1 = y$ gives the solution to the system.

$$x_1'(t) = x_2$$

$$x_2'(t) = -\frac{b}{a}x_2 - \frac{c}{a}x_1 + \frac{1}{a}g(t)$$

The initial conditions will apply to each equation separately, with $x_1(0) = y(0) = x_0$, and $x_2(0) = y'(0) = v_0$.

The variables x_1 and x_2 are called **states** of the system.

Now we have a system of coupled differential equations, which we can find the solution to using **ode45** in MATLAB. Remember, in matrix notation this can be expressed by:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/a \end{bmatrix} g(t)$$

This formulation has the representation:

$$\dot{X} = AX + BU$$

Here, $U \equiv g(t)$

Example: Converting a 2nd-order differential equation to a 1st-order system.

Convert the following 2nd-order DE, with the given initial conditions.

$$3y'' - 2y' + 6y = \sin(t) \qquad y(0) = 2,\ \ y'(0) = 1$$

Let $x_1 = y$ and $x_2 = y'$

Thus, we get:

$$x_1' = x_2$$

$$x_2' = \frac{2}{3}x_2 - 2x_1 + \frac{\sin(t)}{3}$$

$$x_1(0) = 2, \quad x_2(0) = 1$$

Matrix notation:

$$\dot{X} = AX + BU$$

where:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ A = \begin{bmatrix} 0 & 1 \\ -2 & 2/3 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}, \quad U = \sin(t)$$

We can make use of this formulation to implement a solution in MATLAB using ode45. Remember, although **ode45** is only for first-order equations, it can also handle **systems** of first order equations.

MATLAB Example

1. Define the function in MATLAB in the same way we did for 1st-order systems:

```
function  [x_dot]  = diff_sys(t,x)
    x1 = x(1,1);
    x2 = x(2,1);

    x_dot(1,1) = x2;
    x_dot(2,1) = (2/3)*x2 - 2*x1 + sin(t)/3;
end
```

2. Our initial conditions go in a column vector, just like the states go in a column vector as well.

```
x10 = 2;
x20 = 1;

x_0 = [x10; x20];
```

3. Now we use the same syntax as we did for solving first-order differential equations as follows:

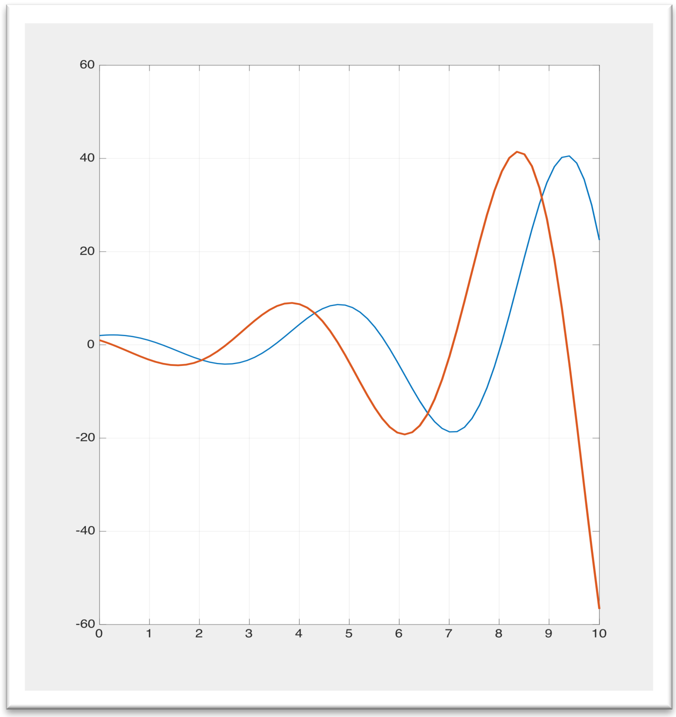
```
tSpan = [0 10];

[t_out, y_out] = ode45(@diff_sys, tSpan, x_0)
```

Note that the variable y_out now will have 2 columns of data.
The first column will contain the numerical solution to the variable x_1 which corresponds to the solution $y(t)$ of the original differential equations. The second column will contain the data from the numerical solution of the variable x_2 which corresponds to $y'(t)$.

You can plot both curves $x(t)$ and $x'(t)$ simply using:

```
>> plot(t_out, y_out)
>> grid on
```



Derivative shown in **red**. Solution in **blue**. Note the derivative equals zero whenever $x(t)$ achieves a max or min.

Alternatively, you could plot one curve at a time using matrix indexing. Here is just $x(t)$.

```
>> figure
>> plot(t_out, y_out(:,1))
>> grid on
```

Just the first column of y_out.