Drexel University

Office of the Dean of the College of Engineering

ENGR 232 – Dynamic Engineering Systems

Recitation Guidelines for Week 3

Topics for recitation this week:

Applications of first-order differential equations

1. These examples are from the Schaum's Outlines series, Differential Equations. This week we will focus on the applications of first-order differential equations.

Example 1: Falling body

Underlying model: $m\frac{dv}{dt} = mg - kv$ or $\frac{dv}{dt} + \frac{k}{m}v = g$ so that limiting velocity is: $v_\infty = \frac{mg}{k}$ (called v_l below)

7.13. A body weighing 64 lb is dropped from a height of 100 ft with an initial velocity of 10 ft/sec. Assume that the air resistance is proportional to the velocity of the body. If the limiting velocity is known to be 128 ft/sec, find (*a*) an expression for the velocity of the body at any time *t* and (*b*) an expression for the position of the body at any time *t*.

(*a*) Locate the coordinate system as in Fig. 7-5. Here $w = 64$ lb. Since $w = mg$, it follows that $mg = 64$, or $m = 2$ slugs. Given that $v_1 = 128$ ft/sec, it follows from (*7.6*) that $128 = 64/k$, or $k = \frac{1}{2}$. Substituting these values into (*6.4*), we obtain the linear differential equation

$$\frac{dv}{dt} + \frac{1}{4}v = 32$$

which has the solution

$$v = ce^{-t/4} + 128 \tag{1}$$

At $t = 0$, we are given that $v = 10$. Substituting these values into (*1*), we have $10 = ce^0 + 128$, or $c = -118$. The velocity at any time *t* is given by

$$v = -118e^{-t/4} + 128 \tag{2}$$

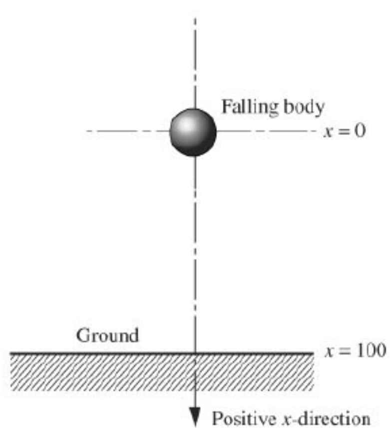


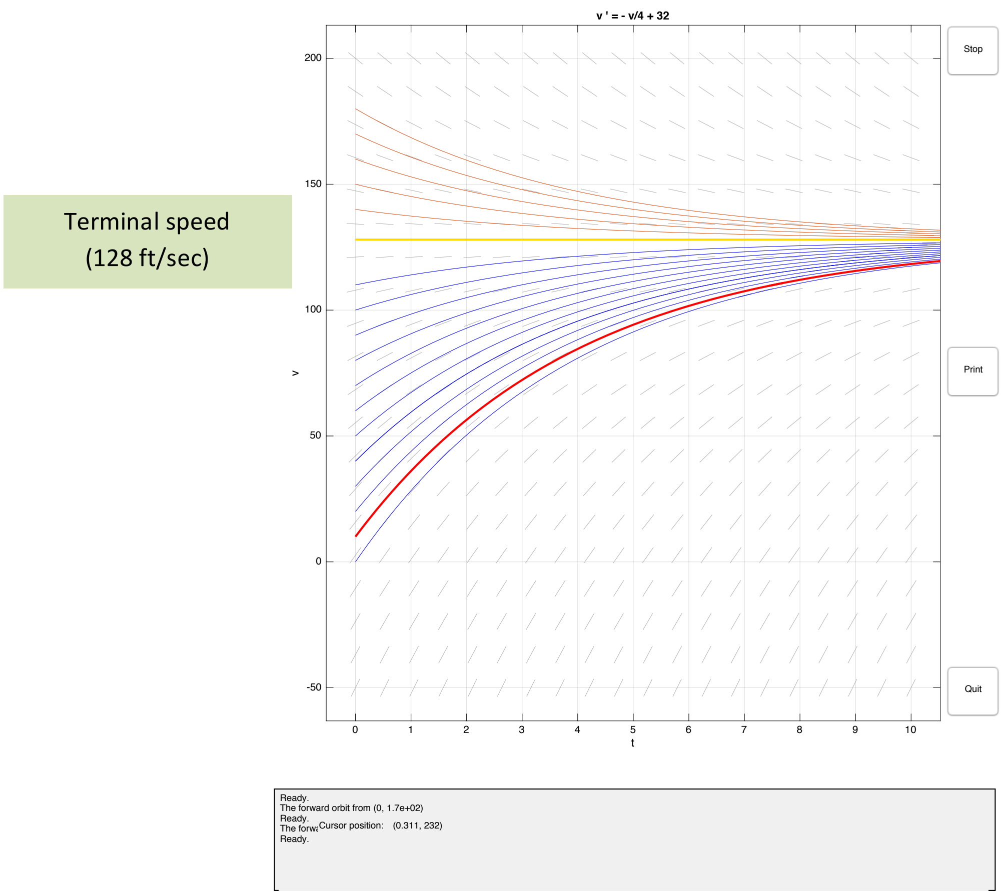
Fig. 7-5

The specific solution with $v(0) = 10$ is shown in **red** below. Image created using **dfield8.m** in MATLAB.

Notice all solutions approach the terminal velocity of 128 feet per second.

Can you spot the equilibrium solution? Is it stable?

If the object is ejected with a speed greater than the terminal velocity, it will slow down and approach the terminal velocity.



(*b*) Since $v = dx/dt$, where x is displacement, (*2*) can be rewritten as

$$\frac{dx}{dt} = -118e^{-t/4} + 128$$

This last equation, in differential form, is separable; its solution is

$$x = 472e^{-t/4} + 128t + c_1 \tag{3}$$

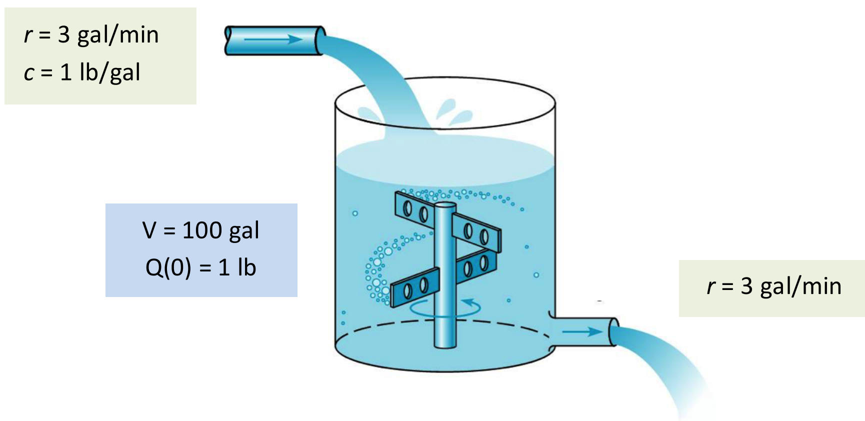
At $t = 0$, we have $x = 0$ (see Fig. 7-5). Thus, (*3*) gives

$$0 = 472e^0 + (128)(0) + c_1 \qquad \text{or} \qquad c_1 = -472$$

The displacement at any time t is then given by

$$x = 472e^{-t/4} + 128t - 472$$

Example 2: Tank Problem **Underlying model:** $\frac{dQ}{dt} = rate_{in} - rate_{out}$ with $rate_{in} = 3\ \frac{lb}{min}$ & $rate_{out} = \frac{3}{100}Q(t)\ \frac{lb}{min}$



7.17. A tank initially holds 100 gal of a brine solution containing 1 lb of salt. At $t = 0$ another brine solution containing 1 lb of salt per gallon is poured into the tank at the rate of 3 gal/min, while the well-stirred mixture leaves the tank at the same rate. Find (*a*) the amount of salt in the tank at any time t and (*b*) the time at which the mixture in the tank contains 2 lb of salt.

(*a*) Here $V_0 = 100$, $a = 1$, $b = 1$, and $e = f = 3$; hence, (*7.8*) becomes

$$\frac{dQ}{dt} + 0.03Q = 3$$

The solution to this linear differential equation is

$$Q = ce^{-0.03t} + 100 \tag{1}$$

At $t = 0$, $Q = a = 1$. Substituting these values into (*1*), we find $1 = ce^0 + 100$, or $c = -99$. Then (*1*) can be rewritten as

$$Q = -99e^{-0.03t} + 100 \tag{2}$$

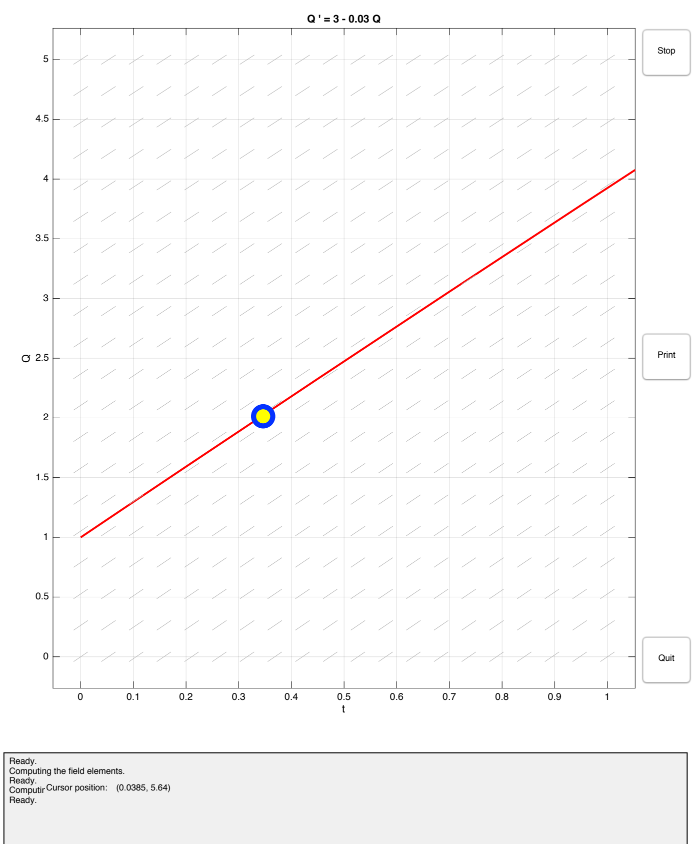
(*b*) We require t when $Q = 2$. Substituting $Q = 2$ into (2), we obtain

$$2 = -99e^{-0.03t} + 100 \quad \text{or} \quad e^{-0.03t} = \frac{98}{99}$$

from which

$$t = -\frac{1}{0.03}\ln\frac{98}{99} = 0.338 \text{ min}$$

In this first figure, we zoom in on the solution and note that $Q = 2$, at about 0.34 minutes as predicted above.



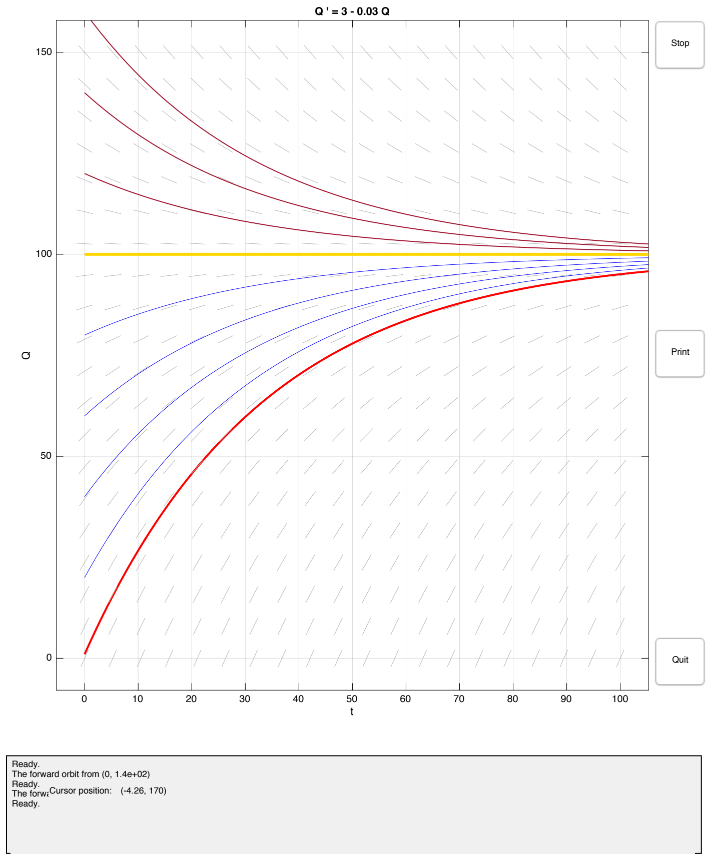
Below, numerous solutions to the linear DE are plotted (on a larger scale) using different initial values for $Q(0)$.

Can you spot the equilibrium solution? Is it stable?

Do all curves approach the same limit?

On what range are the solutions valid?

Why are some curves decreasing?



Example 3: Compound Interest

GROWTH AND DECAY PROBLEMS

Let $N(t)$ denote the amount of substance (or population) that is either growing or decaying. If we assume that dN/dt, the time rate of change of this amount of substance, is proportional to the amount of substance present, then $dN/dt = kN$, or

$$\frac{dN}{dt} - KN = 0 \tag{7.1}$$

where k is the constant of proportionality. (See Problems 7.1–7.7.)

We are assuming that $N(t)$ is a differentiable, hence continuous, function of time. For population problems, where $N(t)$ is actually discrete and integer-valued, this assumption is incorrect. Nonetheless, (7.1) still provides a good approximation to the physical laws governing such a system. (See Problem 7.5.)

7.3. **What constant interest rate is required if an initial deposit placed into an account that accrues interest compounded continuously is to double its value in six years?**

The balance $N(t)$ in the account at any time t is governed by (7.1)

$$\frac{dN}{dt} - kN = 0$$

which has as its solution

$$N(t) = ce^{kt} \tag{1}$$

We are not given an amount for the initial deposit, so we denote it as N_0. At $t = 0$, $N(0) = N_0$, which when substituted into (1) yields

$$N_0 = ce^{k(0)} = c$$

and (1) becomes

$$N(t) = N_0\, e^{kt} \tag{2}$$

We seek the value of k for which $N = 2N_0$ when $t = 6$. Substituting these values into (2) and solving for k, we find

$$2N_0 = N_0 e^{k(6)}$$

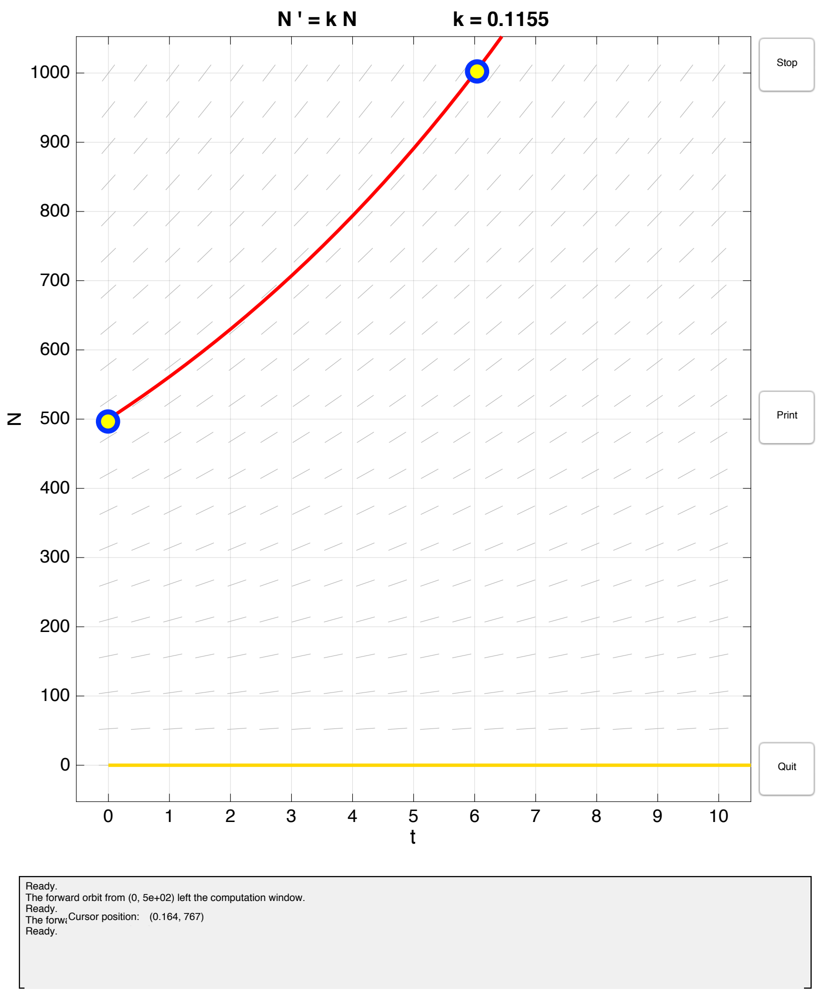
$$e^{6k} = 2$$

$$6k = \ln|2|$$

$$k = \frac{1}{6}\ln|2| = 0.1155$$

An interest rate of 11.55 percent is required.

Here we confirm an initial deposit of \$500 doubles in 6 years when $k = 0.1155$ or 11.55%



Example 4: Population Growth

7.5. The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled, and after three years the population is 20,000, estimate the number of people initially living in the country.

Let N denote the number of people living in the country at any time t, and let N_0 denote the number of people initially living in the country. Then, from (*7.1*),

$$\frac{dN}{dt} - kN = 0$$

which has the solution

$$N = ce^{kt} \tag{1}$$

At $t = 0$, $N = N_0$; hence, it follows from (*1*) that $N_0 = ce^{k(0)}$, or that $c = N_0$. Thus,

$$N = N_0\, e^{kt} \tag{2}$$

At $t = 2$, $N = 2N_0$. Substituting these values into (*2*), we have

$$2N_0 = N_0 e^{2k} \quad \text{from which} \quad k = \frac{1}{2}\ln 2 = 0.347$$

Substituting this value into (*2*) gives

$$N = N_0 e^{0.347t} \tag{3}$$

At $t = 3$, $N = 20{,}000$. Substituting these values into (*3*), we obtain

$$20{,}000 = N_0 e^{(0.347)(3)} = N_0(2.832) \qquad \text{or} \qquad N_0 = 7062$$

Let's first note that doubling in two years sounds more like rabbits than humans! Perhaps this is a fictional country inhabited by intelligent Leporidae.