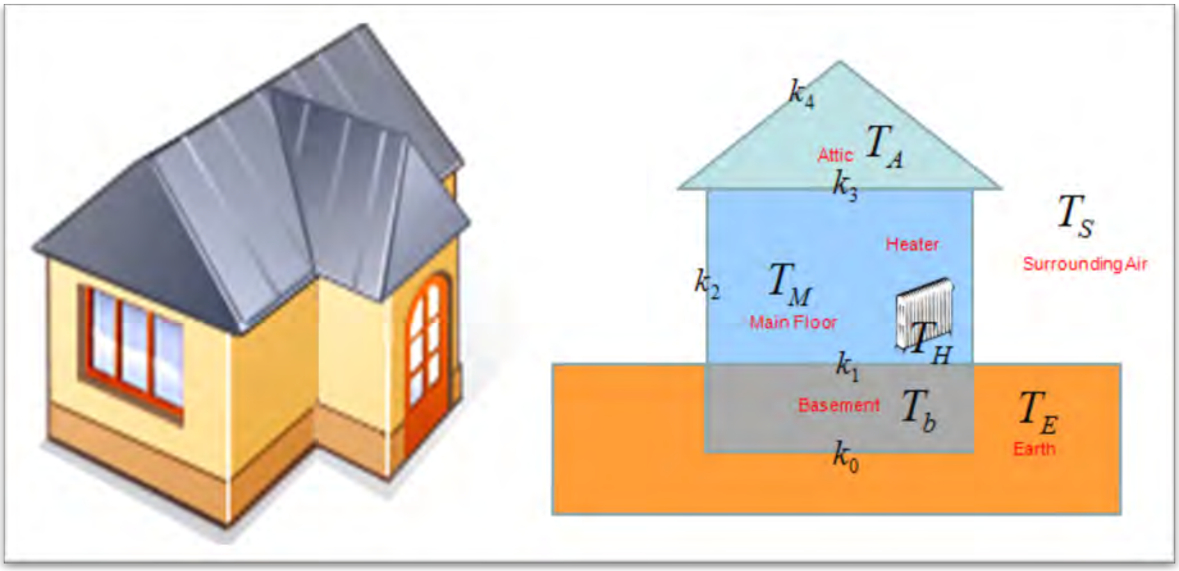
Lab 8 : Heating a Three-Room Home

Introduction: Consider a typical three-room home, having a basement B, main room M and attic A.



Assumptions: The temperature of the earth T_E around the basement is relatively constant and we will assume it has the fixed value $T_E = 40$ °F during the time interval of interest. The main room has fairly good insolation on all sides so the cooling constants k_1, (floor) k_2 (walls) and k_3 (ceiling) are smaller than the other cooling constants in the figure. The space between the attic and the roof is assumed to be poorly insolated, so k_4 is comparatively larger. There is no added insolation in the basement for its floor and walls, so k_0 is also comparatively larger. This term describes heat flow across the floor and walls of the basement. As a further simplification, we will assume the air outside remains constant at $T_S = 50$ °F. Finally, assume all three rooms start at 50°F at time 0.

Under these assumptions, applying Newton's Law of Cooling at all boundaries for each room, we can find the differential equation for the temperature in each room.

1. The DE for the temperature $T_B(t)$ in the **basement** is:

$$\frac{dT_B}{dt} = -k_0(T_B - T_E) - k_1(T_B - T_M)$$

Heat lost across the basement walls and basement floor.

Heat flow from basement up into the main room.

2. The DE for the temperature $T_M(t)$ in the **main** floor is:

Heat flow from the main room, through the walls to the outside.

H is the heat produced by the heater.

$$\frac{dT_M}{dt} = -\,k_1(T_M - T_B) - \;k_2(T_M - T_S) - \;k_3(T_M - T_A) + H$$

Heat lost from the main room to the basement.

Heat lost from the main room to the attic.

3. The DE for the temperature $T_A(t)$ in the **attic** is:

$$\frac{dT_A}{dt} = -k_3(T_A - T_M) - \;k_4(T_A - T_S)$$

Heat lost from the attic to the main room.

Heat lost from the attic to the outside.

Collecting these three equations together, we arrive at the following system of three first-order differential equations.

Model for Heating a 3-Room House

$$\frac{dT_B}{dt} = -k_0(T_B - T_E) - \;k_1(T_B - T_M)$$

$$\frac{dT_M}{dt} = -\,k_1(T_M - T_B) - \;k_2(T_M - T_S) - \;k_3(T_M - T_A) + H$$

$$\frac{dT_A}{dt} = -k_3(T_A - T_M) - \;k_4(T_A - T_S)$$

Part A: First, let's solve this equation using **dsolve**. There are some parts which are rather tricky, so some starter code is provided.

Questions 1-2: Set up the model and enter the three differential equations.

```
%% Heating for a Three-Room House - Newton's Law of Cooling.
clear, clc, close all

% Heating/cooling constants.
k1 = 0.15;    k2 = 0.15;    k3 = 0.15;  % Insulation is very good for the main room.
k0 = 0.5;   k4 =1;     % Poor insulation in the attic and basement. Attic is drafty.
```

Next, set the temperatures for the earth around the basement and the air outside. Give the heating power of the heater.

```
% Temperatures and the Heater
TE = 40;    % Temperature of the earth, is constant and cold.
TS = 50 ;   % The outside air is a cool 50 degrees Fahrenheit.
H = 10;   % The heater can heat the main room at 10 degrees per hour if there are no losses.
```

Now enter all three equations. Here is the first to get you started.

```
% The differential equations for the three rooms.
syms t TB(t) TM(t) TA(t)
% Temperature TB in the basement
EQ1 = diff(TB, t) == -k0*(TB-TE)  -k1*(TB-TM)
```

Questions 1 -2: Show your completed code for the equations for the temperature in the Main Room and in the Attic.

```
% The differential equations for the main room and the attic are:

EQ2 = diff(TM, t) == _ _ _ _ _ _ _ _ _ _

EQ3 = diff(TA, t) == _ _ _ _ _ _ _ _ _ _
```

Questions 3-5: Solve the system of differential equations and plot the solutions.

Now for the first tricky part. This system of differential equation can really bog down the solver. To speed things up, let's tell MATLAB to use **variable-precision arithmetic**.

```
% Tell MATLAB it's OK to use variable-precision arithmetic.
EQNS = vpa([EQ1, EQ2, EQ3], 4)
```

```
% Use variable-
precision arithmetic.
```

```
% Use at least 4
significant digits.
```

If you don't do this, MATLAB might churn for a very long time before you see any answers.
You can learn more about `vpa` and `vpasolve` here.

https://www.mathworks.com/help/symbolic/vpa.html

We are ready to solve the system of differential equations. Recall all three rooms are at 50°F at time 0.

```
% Assume these initial conditions.  TB(0)=50, TM(0)=50, TA(0)=50
sol = dsolve( EQNS, [TB(0)==50, TM(0)==50, TA(0)==50] )
```

The solutions for all three rooms are now in the structure named `sol`.
You can see them using commands like:

\>> `sol.TB`
\>> `sol.TM`
\>> `sol.TA`

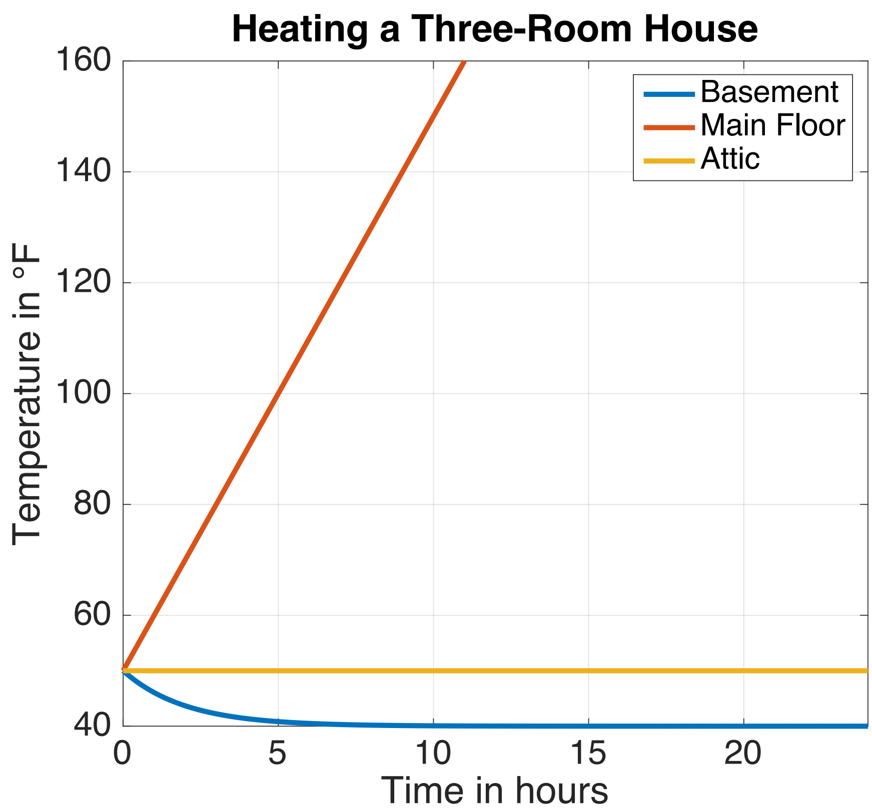
Make each solution into a function so you can plot it using `matlabFunction`.
Here's how to do that for the basement temperature. Use similar code for all three rooms.

```
TB = matlabFunction(sol.TB)
```

Plot all three temperatures on the same graph. Here's some starter code to set up your graph which includes only the basement temperature.

```
%% Plot the temperature in all three rooms.
figure
t_pts = 0: 0.01: 24;  % Model temperatures in the home for one day.
% Plot the temperature in the basement.
plot(t_pts, TB(t_pts), 'LineWidth', 3 )
hold on; grid on
set(gca, 'FontSize', 20)
% Add code here for the main room and attic. Add labels and a legend.
```

Questions 3-5: Replace the sample figure below with your completed figure. (3 points!)
Label both axes as shown, **title** your graph and add the **legend** as shown.
Your graph will look very different! For this sample, the main room was made perfectly insulated by setting $k_1 = 0,\ k_2 = 0,\ k_3 = 0$. You can see it has effectively become an **oven**!



Part B: Let's solve the same system using the **Laplace Transform**. We'll clean up the equations a little bit first. Let's rewrite the equations to highlight their linear form. The equations are linear in the three unknowns T_B, T_M and T_A. The other terms can be interpreted as forcing functions (see **red terms**).

Model for Heating a 3-Room House

$$\frac{dT_B}{dt} = -k_0(T_B - T_E) - k_1(T_B - T_M) = -(k_0 + k_1)T_B + k_1 T_M + k_0 T_E$$

$$\frac{dT_M}{dt} = +k_1 T_B - (k_1 + k_2 + k_3)T_M + k_3 T_A + k_2 T_S + H$$

$$\frac{dT_A}{dt} = +k_3 T_M - (k_3 + k_4)T_A + k_4 T_S$$

Define the state vector $\mathbf{T}(t) = \begin{bmatrix} T_B(t) \\ T_M(t) \\ T_A(t) \end{bmatrix}$, the forcing term $\mathbf{F} = \begin{bmatrix} k_0 T_E \\ k_2 T_S + H \\ k_4 T_S \end{bmatrix}$

and the matrix $A = \begin{bmatrix} -(k_0 + k_1) & k_1 & 0 \\ k_1 & -(k_1 + k_2 + k_3) & k_3 \\ 0 & k_3 & -(k_3 + k_4) \end{bmatrix}$.

Then our system can be written in the linear form:

Linear Model for Heating a 3-Room House

$$\frac{d\mathbf{T}}{dt} = \begin{bmatrix} \dot{T}_B \\ \dot{T}_M \\ \dot{T}_A \end{bmatrix} = A\,\mathbf{T} + \mathbf{F}$$

Taking the Laplace Transform of each term we find:

$$L\left\{\frac{d\mathbf{T}}{dt}\right\} = s\,L\{\mathbf{T}\} - \mathbf{T}(0) = A\,L\{\mathbf{T}\} + L\{\mathbf{F}\}$$

or:

$$(sI - A)\,L\{\mathbf{T}\} = \mathbf{T}(0) + L\{\mathbf{F}\}$$

where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the 3×3 identity matrix. Since **F** is constant in our case, we can write:

$$L\{\mathbf{F}\} = \frac{1}{s}\,\mathbf{F} = \frac{1}{s}\cdot\begin{bmatrix} k_0 T_E \\ k_2 T_S + H \\ k_4 T_S \end{bmatrix}$$

```
Later, the heater will break
and you will need to reenter
F and find its new transform
in the last part.
```

In your MATLAB script, you will need to compute the following Laplace transform:

$$L\{\mathbf{T}\} = (sI - A)^{-1}\left[\mathbf{T}(0) + \frac{1}{s}\cdot\mathbf{F}\right]$$

Here is some starter code to get you up and running. We will start over from scratch.

```
%% Part B - Using Laplace Transforms
% Questions 6 - 7
clear, clc, close all

% Heating/cooling constants.
k1 = 0.15;    k2 = 0.15; k3 = 0.15;  % Insulation is very good for the main room.
k0 = 0.5;   k4 =1;       % Poor insulation in the attic and basement. Attic is drafty.

% Temperatures and the Heater
TE = 40;     % Temperature of the earth, is constant and cold.
TS = 50 ;    % The outside air is a cool 50 degrees Fahrenheit.
H = 10;   % The heater can heat the main room at 10 degrees per hour if there are no losses.

T0 = [50 50 50]' % All three rooms start off at 50 degrees F.
```

Questions 6-7: Record the numerical values for the system matrix A and the forcing term **F.**

a. First define the 3x3 matrix A using $A = \begin{bmatrix} -(k_0+k_1) & k_1 & 0 \\ k_1 & -(k_1+k_2+k_3) & k_3 \\ 0 & k_3 & -(k_3+k_4) \end{bmatrix}$

b. Then define the forcing term $\mathbf{F} = \begin{bmatrix} k_0 T_E \\ k_2 T_S + H \\ k_4 T_S \end{bmatrix}$, and don't forget it is a **column** vector.

Questions 6-7: Record the numerical values for A and **F** here.

```
A =


F =
```

Question 8: Compute the Laplace Transform then use the final value theorem to find the limiting temperature in each room after a long time. Use **eye(3)** for the 3x3 identity matrix *I*. Use **inv()** for the inverse matrix of the matrix. Declare s to be symbolic.
Name your result Ts. This is the solution for the model in the *s*-domain.

$$L\{\mathbf{T}\} = (sI - A)^{-1}\left[\mathbf{T}(0) + \frac{1}{s}\cdot \boldsymbol{F}\right]$$

```
>> Tinf = vpa(limit(s*Ts, 0), 4)
```

To earn your point for credit, record the **final** temperature in each room, using the final value theorem.

Question 8: The **final** temperature in each room is:

$$\vec{\mathbf{T}}(\infty) = \begin{bmatrix} T_B(\infty) \\ T_M(\infty) \\ T_A(\infty) \end{bmatrix} = \begin{bmatrix} 47.47 \\ \ldots \\ \ldots \end{bmatrix}$$

```
The final temp in the basement is given for you.
```

Questions 9- 10: Broken Heater! For this part, assume the heater breaks down at time $t = 10$. Instead of outputting at the constant rate $H = 10$, the function that describes the heater is now:

$$H(t) = 10 \cdot (1 - u(t - 10))$$

Above, $u(t)$ denotes the unit step function which is known in MATLAB as **heaviside(t).**
You can define this in MATLAB, and take the transform of the new forcing term **F** using:

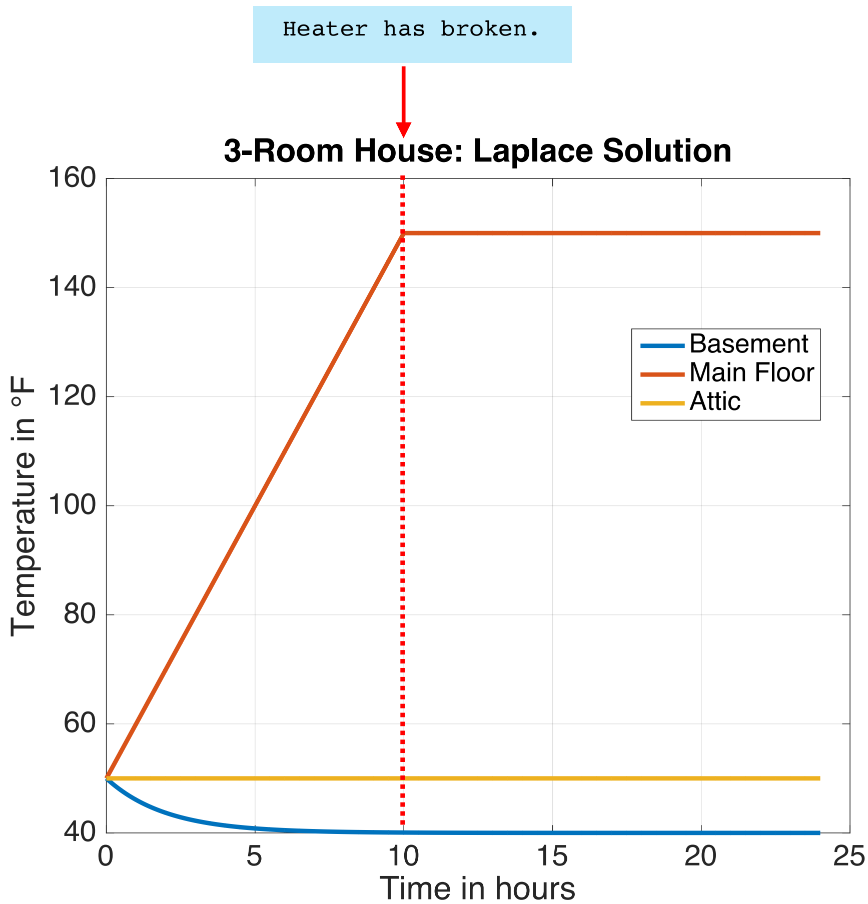
```
syms t
H = @(t) 10*(1-heaviside(t-10)) % Turn off the heater at time t = 10.
F = [k0*TE   k2*TS+H(t)  k4*TS ]'
Fs = laplace(F)
```

Recalculate the solution in the s-domain with the broken heater. Then invert back to the time domain and create a plot showing the new temperature solutions for each room.

And here is some code to help with the inverse transform **after** you have updated the transform Ts.

```
% Use variable-precision arithmetic to at least 4 digits.
Tt = vpa(ilaplace( Ts ), 4)  % The solution in the time domain.
T = matlabFunction(Tt) % Make it into a function.
```

Q 9-10: Replace the sample graph below for two points!
Alert: Sample uses perfect insulation for the main room. Your graph will look <u>very</u> different.



Questions 9-10: **<u>Replace</u>** the above sample with the correct graph showing how the **<u>broken heater</u>** effects the results. (Two points!)