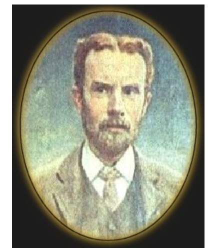
Drexel University
Office of the Dean of the College of Engineering
ENGR 232 – Dynamic Engineering Systems

Recitation Guidelines for Week 7 – Laplace Workshop

Topics for recitation this week:

- **Partial fraction expansion**
- **Laplace and inverse Laplace transforms using tables**



Oliver Heaviside
(1850-1925)

Partial Fraction Expansion Examples

1. Find the partial fraction expansion for the following rational expression:

$$\frac{10(s+3)}{(s-2)(s+1)(s-5)}$$

This fraction can be expressed in the following form:

$$\frac{10(s+3)}{(s-2)(s+1)(s-5)} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{s-5}$$

The simplest way to solve for the unknown coefficients is to use the Heaviside **cover-up method**.

Firstly, to solve for A, you simple choose $s = 2$ since that makes the denominator of the A term equal to zero. You then cover up that factor from the left hand side. This gives:

$$A = \frac{10(s+3)}{(s+1)(s-5)} \qquad @s = 2$$

Plugging in $s = 2$ gives: $A = -\frac{50}{9}$

To solve for B, we notice that the denominator in the B term is zero when $s = -1$. We then cover up that factor in the LHS and substitute $s = -1$ to get B.

$$B = \frac{10(s+3)}{(s-2)(s-5)} \qquad @\ s = -1$$

Plugging in $s = -1$ gives: $B = \frac{20}{18} = \frac{10}{9}$

Finally, we solve for C by setting $s = 5$ and covering up the factor $(s-5)$ from the left hand side.

$$C = \frac{10(s+3)}{(s+1)(s-2)} \qquad @\ s = 5$$

```
% MATLAB check for example 1!
syms s
F = 10*(s+3) / ((s-2) * (s+1) * (s-5))
partfrac(F, s)
```

Plugging in $s = 5$ gives us: $C = \frac{80}{18} = \frac{40}{9}$

```
ans =
10/(9*(s + 1)) - 50/(9*(s - 2)) + 40/(9*(s - 5))
```

The partial fraction expansion is therefore: $-\frac{50}{9(s-2)} + \frac{10}{9(s+1)} + \frac{40}{9(s-5)}$

2. Find the partial fraction expansion for this second example with a **quadratic term** in the denominator:

$$\frac{4\,s^2}{(s-3)(s^2-4s+13)}$$

The partial fraction representation of this will have the form:

$$\frac{A}{s-3}+\frac{Bs+C}{s^2-4s+13}$$

Notice we didn't factorize the quadratic term which has complex roots, although it is possible to solve the problem using these quadratic roots as well. It is more desirable to complete the square of the quadratic term.

For completing the square, we do the following:

$$s^2-4s+13=s^2-4s+(4-4)+13=(s^2-4s+4)+13-4=(s-2)^2+9$$

This term now looks like something in the Laplace table that we can look up.

Our partial fraction expansion will then look like:

$$\frac{4\,s^2}{(s-3)(s^2-4s+13)}=\frac{A}{s-3}+\frac{Bs+C}{(s-2)^2+3^2}$$

Now we need to solve for A, B and C. The cover-up method will work for A, and we set $s = 3$, cover up (s−3) from the original expression to get:

$$A=\frac{4\,s^2}{(s^2-4s+13)}\qquad @\,s=3$$

This gives: $A=\frac{36}{10}=\frac{18}{5}$

To solve for B and C we need to expand the expression:

$$4\,s^2=\frac{18}{5}(s^2-4s+13)+(Bs+C)(s-3)$$

One can expand and compare coefficients to show:

$$4=\frac{18}{5}+B\rightarrow B=\frac{2}{5}$$ Coefficients for s^2

$$0=\frac{18\times 13}{5}-3C\rightarrow C=\frac{78}{5}$$ Coefficients for s^0

```
% MATLAB check for example 2!
syms s
F = 4*s^2 / (s-3) / (s^2 - 4*s + 13)
partfrac(F)

ans =
18/(5*(s - 3)) + ((2*s)/5 + 78/5)/(s^2 - 4*s + 13)
```

---

3. Now an example with **repeated roots**. Find the partial fraction expansion for:

$$\frac{3s^2 - 2}{(s^2-2)^2(s+1)(s^3)}$$

An example like this will have a partial fraction expansion of the form:

$$\frac{(As+B)}{s^2-2} + \frac{Cs+D}{(s^2-2)^2} + \frac{E}{s+1} + \frac{F}{s} + \frac{G}{s^2} + \frac{H}{s^3}$$

It is worth noting that the coefficients: E and H can be solved using cover up. The others will require the full algebraic expansion. Let's find those coefficients using MATLAB instead!

```
% MATLAB check for example 3!
syms s
F = (3*s^2-2) / ( (s^2-2)^2 * (s+1) * (s^3)  )
partfrac(F, s)
```

```
ans =

((5*s)/4 – 3/2)/(s^2 – 2) – 1/(s + 1) – (s – 2)/(s^2 – 2)^2 – 1/(4*s) + 1/(2*s^2) – 1/(2*s^3)
```

See if you can match each coefficient in our expansion.

Laplace and Inverse Laplace Transforms

We make use of tables to find the Laplace transform of expressions, as well as the inverse transform. See the table posted to the class website inside this week's Lab folder.

Determine the Laplace transform of the following functions using the provided table. (Online)

a. $f(t) = \sin 2t + 3\cos 4t - e^{t}$

$$F(s) = \frac{2}{s^2+4} + \frac{3s}{s^2+16} - \frac{1}{s-1}$$

```
% MATLAB check  - but be able to do this using the Table!
syms t
f = sin(2*t) + 3*cos(4*t) - exp(t)
F = laplace(f)
```

```
F = (3*s)/(s^2 + 16) - 1/(s - 1) + 2/(s^2 + 4)
```

b. $f(t) = e^{-2t}t^2$

$$F(s) = \frac{2}{(s+2)^3}$$

```
% MATLAB check  - but be able to do this using the Table!
syms t
f = exp(-2*t) * t^2
F = laplace(f)
```

```
F = 2/(s + 2)^3
```

c. $f(t) = e^{3t}cos(4t)$

$$F(s) = \frac{s-3}{(s-3)^2+16}$$

Note: We are making use of the linearity property of the Laplace transform for these.

```
% MATLAB check  - but be able to do this using the Table!
syms t
f = exp(3*t) * cos(4*t)
F = laplace(f,s)
```

```
F = (s - 3)/((s - 3)^2 + 16)
```

Example from Boyce and DiPrima

Find the solution of the differential equation

$$y'' + y = \sin 2t, \tag{19}$$

satisfying the initial conditions

$$y(0) = 2, \qquad y'(0) = 1. \tag{20}$$

We assume that this initial value problem has a solution $y = \phi(t)$, which with its first two derivatives satisfies the conditions of Corollary 6.2.2. Then, taking the Laplace transform of the differential equation, we have

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 2/(s^2 + 4),$$

where the transform of $\sin 2t$ has been obtained from line 5 of Table 6.2.1. Substituting for $y(0)$ and $y'(0)$ from the initial conditions and solving for $Y(s)$, we obtain

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}. \tag{21}$$

Using partial fractions we can write $Y(s)$ in the form

$$Y(s) = \frac{as + b}{s^2 + 1} + \frac{cs + d}{s^2 + 4} = \frac{(as + b)(s^2 + 4) + (cs + d)(s^2 + 1)}{(s^2 + 1)(s^2 + 4)}. \tag{22}$$

By expanding the numerator on the right side of Eq. (22) and equating it to the numerator in Eq. (21) we find that

$$2s^3 + s^2 + 8s + 6 = (a + c)s^3 + (b + d)s^2 + (4a + c)s + (4b + d)$$

for all s. Then, comparing coefficients of like powers of s, we have

$$\begin{aligned} a + c &= 2, & b + d &= 1, \\ 4a + c &= 8, & 4b + d &= 6. \end{aligned}$$

Consequently, $a = 2$, $c = 0$, $b = \frac{5}{3}$, and $d = -\frac{2}{3}$, from which it follows that

$$Y(s) = \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1} - \frac{2/3}{s^2 + 4}. \tag{23}$$

From lines 5 and 6 of Table 6.2.1, the solution of the given initial value problem is

$$y = \phi(t) = 2\cos t + \tfrac{5}{3}\sin t - \tfrac{1}{3}\sin 2t. \tag{24}$$

Example from Schaum's Outline

22.8. Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-2s+9}\right\}$.

No function of this form appears in Appendix A. But, by completing the square, we obtain

$$s^2-2s+9=(s^2-2s+1)+(9-1)=(s-1)^2+(\sqrt{8})^2$$

Hence,
$$\frac{1}{s^2-2s+9}=\frac{1}{(s-1)^2+(\sqrt{8})^2}=\left(\frac{1}{\sqrt{8}}\right)\frac{\sqrt{8}}{(s-1)^2+(\sqrt{8})^2}$$

Then, using Property 22.1 and entry 15 of Appendix A with $a=\sqrt{8}$ and $b=1$, we find that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+9}\right\}=\frac{1}{\sqrt{8}}\mathcal{L}^{-1}\left\{\frac{\sqrt{8}}{(s-1)^2+(\sqrt{8})^2}\right\}=\frac{1}{\sqrt{8}}e^x\sin\sqrt{8}x$$

22.9. Find $\mathcal{L}^{-1}\left\{\dfrac{s+4}{s^2+4s+8}\right\}$.

No function of this form appears in Appendix A. Completing the square in the denominator, we have

$$s^2+4s+8=(s^2+4s+4)+(8-4)=(s+2)^2+(2)^2$$

Hence,
$$\frac{s+4}{s^2+4s+8}=\frac{s+4}{(s+2)^2+(2)^2}$$

This expression also is not found in Appendix A. However, if we rewrite the numerator as $s+4=(s+2)+2$ and then decompose the fraction, we have

$$\frac{s+4}{s^2+4s+8}=\frac{s+2}{(s+2)^2+(2)^2}+\frac{2}{(s+2)^2+(2)^2}$$

Then, from entries 15 and 16 of Appendix A,

$$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s+8}\right\}=\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+(2)^2}\right\}+\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+(2)^2}\right\}$$
$$=e^{-2x}\cos 2x+e^{-2x}\sin 2x$$