Drexel University

Office of the Dean of the College of Engineering

ENGR 232 – Dynamic Engineering Systems

Recitation Guidelines for Week 5

Note to instructors:

Please discuss the notion of stability as it relates to the roots of the characteristic equation for these examples. This is discussed in the lectures. The class notes are posted to the website.

Eigenvalues or roots of C.E.	Stability Terminology	Response Description	Example
λ_1, λ_2 are real $\lambda_1, \lambda_2 < 0$	Stable	$y(t) \to 0$	$\lambda_1 = -1$ $\lambda_2 = -2$
λ_1, λ_2 are Real $\lambda_1, \lambda_2 > 0$	Unstable	$y(t) \to \infty$	$\lambda_1 = 1$ $\lambda_2 = 2$
λ_1, λ_2 are real $\lambda_1 < 0, \lambda_2 > 0$	Unstable	One component converges to equilibrium while the other components diverges from equilibrium. $y(t) \to \infty$	$\lambda_1 = 1$ $\lambda_2 = -2$
λ_1, λ_2 are real $\lambda_1 < 0, \lambda_2 = 0$	Stable	One component is a constant the other decays to zero	$\lambda_1 = -35$ $\lambda_2 = 0$
λ_1, λ_2 are Imaginary real part = 0	Stable	Both components oscillate sinusoidally about zero	$\lambda_1 = 2i$ $\lambda_2 = -2i$
λ_1, λ_2 are Complex real part <0	Stable	If the real part is < 0 the oscillations decay response converges to zero	$\lambda_1 = -1+2i$ $\lambda_2 = -1-2i$
λ_1, λ_2 are Complex real part > 0	Unstable	If the real part is >0 the oscillations grow without bound	$\lambda_1 = 1+2i$ $\lambda_2 = 1-2i$

Discuss stability of the homogenous system as it related to the roots of the characteristic equation.
Examples: (Any roots with positive real parts (which causes instability) are shown in **red**.)

a. $y'' - 25y = 0$ Roots: $-5, \mathbf{5}$ Eq. pt is unstable
b. $y'' + y' - 2y = 0$ Roots: $-1, \mathbf{1}$ Eq. pt is unstable
c. $y'' + 2y' + y = 0$ Roots: $-1, -1$ Eq. pt is stable
d. $y'' - 9y = 0$ Roots: $-3, \mathbf{3}$ Eq. pt is unstable
e. $y'' + y' = 0$ Roots: $0, -1$ Eq. pt is stable
f. $y'' + 4y = 0$ Roots: $\pm 2i$ Eq. pt is stable
g. $y'' - 4y' + 13y = 0$ Roots: $\mathbf{2} \pm 3i$ Eq. pt is unstable

Stability can easily be visualized with the help of pplane8.

We can rewrite this 2nd-order, homogeneous DE with constant coefficients: $a\,y'' + by' + cy = 0$

as a simple 1st-order system. Let $x_1 = y$ and let $x_2 = \frac{dy}{dt}$. Collect these two new variables into a **column** vector, which is called the **state vector**.

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The original DE is equivalent to the system:

$$\frac{d}{dt}\vec{\mathbf{x}} = A\,\vec{\mathbf{x}}$$

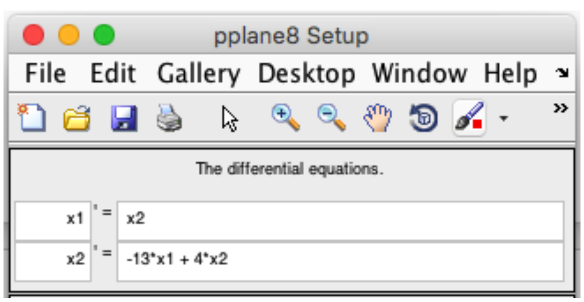
where $A = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}$. So in pplane8, enter the original 2nd-order, homogeneous DE as:

$$x'_1 = x_2 \qquad x'_2 = \left(-\frac{c}{a}\right) * x_1 + \left(-\frac{b}{a}\right) * x_2$$

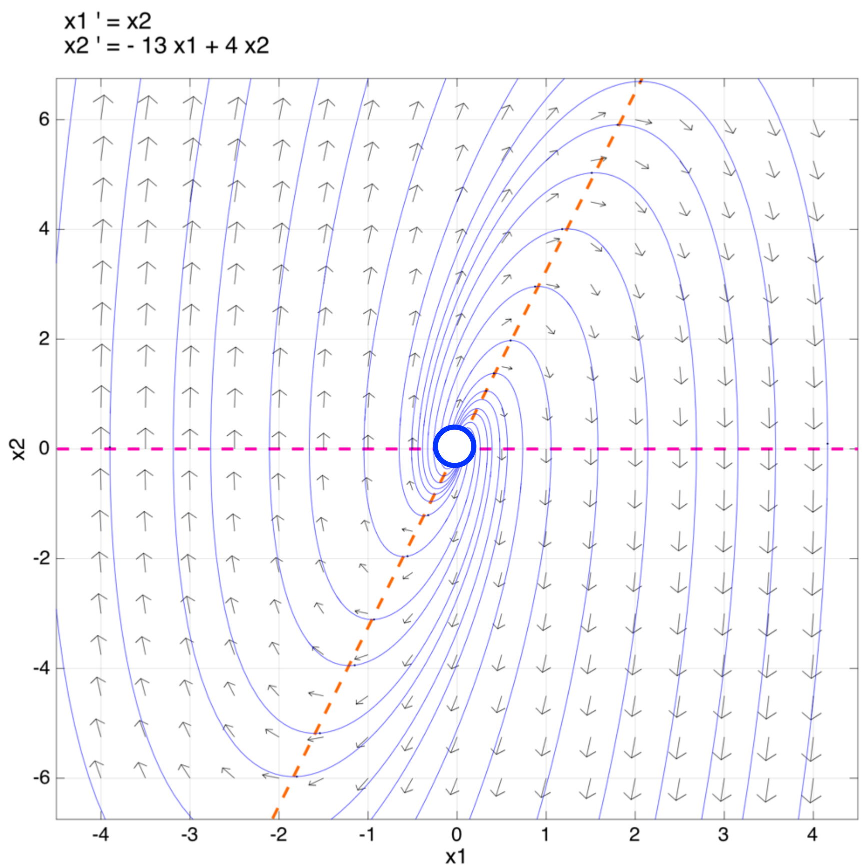
Let's do that for the last example above: $y'' - 4y' + 13y = 0$. The equivalent **state space equations** are:

$$x'_1 = x_2 \qquad x'_2 = -13 * x_1 + 4 * x_2$$

Here's a screenshot showing the Setup window for this DE in pplane8.



The next screenshot shows the system is **unstable** about its sole equilibrium point at the origin. All nearby solutions curves spiral off to infinity! That's due to the eigenvalues $\mathbf{2} \pm 3i$ having a positive real part. Using Euler's identity, we end up with an exponential with a positive argument, which grows exponentially towards infinity.



Unstable!

Solutions that start near (0,0) spiral away to infinity. That's due to the eigenvalues $\mathbf{2} \pm 3i$ having a <u>positive</u> real part.

1. **Method of Undetermined Coefficients**

The general non-homogeneous, second-order, linear differential equation has the form:

$$y'' + p(t)y' + q(t)y = g(t)$$

We studied the homogeneous case with $g(t) = 0$ last week. For the non-homogeneous case, the general solution is:

$$y(t) = y_p + y_h$$

Here, y_h is the solution to the homogeneous differential equation, found in the same way as before by setting $g(t) = 0$ and solving by first writing out the characteristic equation, finding the roots, and then writing the solution based on the nature of the roots.

y_p is called the particular solution, and is found by inspection of the form of $g(t)$, as well as the homogeneous solution.

Examples:

i. **Exponential Particular Solution**
$y'' - y' - 6y = 2e^{2t}$

The homogeneous equation is $y'' - y' - 6y = 0$, and this has a characteristic equation $\Delta(r) = r^2 - r - 6 = 0$. This gives the roots as $r_1 = 3$ and $r_2 = -2$. Thus, the homogeneous equation has the general solution:

$$y_h = c_1 e^{3t} + c_2 e^{-2t}$$

To find the particular solution, we observe that $g(t)$ has an exponential form, so we try a solution of the form: $y_p = Ae^{2t}$. If this is a solution, then it satisfies the differential equation, so we try substitution:

$$y_p'' = 4Ae^{2t} \qquad y_p{}' = 2Ae^{2t} \qquad y_p = Ae^{2t}$$

This gives:

$$4Ae^{2t} - 2Ae^{2t} - 6Ae^{2t} = 2e^{2t} \to -4A(e^{2t}) = 2e^{2t}$$

Which means that $A = -\frac{1}{2}$.
Therefore, this differential equation has a solution:

$$y(t) = y_h + y_p = c_1 e^{3t} + c_2 e^{-2t} - \frac{1}{2} e^{2t}$$

Note that one of the roots is > 0 so the system is **unstable**. Nevertheless, we could not assess BIBO stability as the input (forcing function) is not bounded.

Use dsolve to check your work.

```
%% Solve the DE using dsolve.
syms y(t)
Dy = diff(y,t); D2y = diff(y,t,t);
DE = D2y - Dy -6 * y == 2*exp(2*t)
dsolve(DE)
```

This gives the expected answer:

```
ans = C1*exp(3*t) + C2*exp(-2*t) - exp(2*t)/2
```

ii. **Polynomial Particular solution**
$y'' - 2y' + y = 2t^2 + 2$

The solution to the homogeneous equation can be shown to be:

$$y_h = c_1 e^t + c_2 t e^t$$

Note: The characteristic equation has equal, **positive** real roots.

For this problem, $g(t) = 2t^2 + 2$, thus we seek a particular solution that has a similar form to this.
Guess: $y_p = At^2 + Bt + C$

Substituting for $y_p'' = 2A$, $y_p' = 2At + B$ and $y_p = At^2 + Bt + C$ gives:

$$2A - 2(2At + B) + (At^2 + Bt + C) = 2t^2 + 2 \to At^2 - 4At + Bt + 2A + C - 2B$$

Comparing coefficients gives:
$A = 2$ and $B - 4A = 0$ which gives $B = 8$ and $2A + C - 2B = 2$ which gives: $C = 14$

Thus, the particular solution is: $y_p = 2t^2 + 8t + 14$

Therefore, the final solution is:

$$y(t) = y_h + y_p = c_1 e^t + c_2 t e^t + (2t^2 + 8t + 14)$$

Since the roots of the characteristic equation were positive, the homogenous system is unstable. The overall system is also unstable as the forcing function is not bounded.

Use dsolve to check your work.

```
%% Solve the DE using dsolve.
syms y(t)
Dy = diff(y,t); D2y = diff(y,t,t);
DE = D2y - 2*Dy + y == 2*t^2 + 2
dsolve(DE)
```

This gives the expected answer: (after some rearrangement)

```
ans = C1*exp(t) + C2*t*exp(t) + 2*t^2 + 8*t + 14
```

iii. Sinusoidal Particular Solution

$y'' + y = 6sin(2t)$

The homogeneous equation has a solution: $y_h = c_1 \cos(t) + c_2 \sin(t)$
Note: The roots of the characteristic equation are: $0 \pm i$

For this problem, $g(t) = 6\sin(2t)$ thus we seek a particular solution of a form similar to $g(t)$.
In this case, we try: $y_p = Asin(2t) + Bcos(2t)$

Substituting into the differential equation for:

$$y_p'' = -4A\sin(2t) - 4Bcos(2t) \quad y_p' = 2Acos(2t) - 2Bsin(2t) \quad y_p = Asin(2t) + Bcos(2t)$$

Gives

$$[-4Asin(2t) - 4Bcos(2t)] + [Asin(2t) + Bcos(2t)] = 6\sin(2t)$$

Comparing coefficients we find that:

$$-4A + A = 6 \text{ which gives } A = -2, \text{ and } B = 0.$$

Therefore, this differential equation has solution:

$$y(t) = y_h + y_p = c_1 \cos(t) + c_2 \sin(t) - 2\sin(2t)$$

Use dsolve to check your work.

```
%% Solve the DE using dsolve.
syms y(t)
Dy = diff(y,t); D2y = diff(y,t,t);
DE = D2y + y == 6 * sin(2*t)
dsolve(DE)
```

This gives the expected answer: (after some rearrangement)

```
ans = C1*cos(t) + C2*sin(t) - 2*sin(2*t)
```

iv. Particular Solution with a Product of Known Families of Solutions

$$y'' - 3y' + 2y = 3e^t \sin(3t)$$

Solution to homogeneous equation:

$$y_h = c_1 e^t + c_2 e^{2t}$$

In this problem, $g(t) = 3e^t \sin(3t)$ which is a product of two things we saw earlier. We try a solution of the form:

$$y_p = Ae^t \sin(3t) + Be^t \cos(3t)$$

Substituting into the differential equation for:

$$y_p' = (A - 3B)e^t \sin(3t) + (3A + B)e^t \cos(3t)$$
$$y_p'' = (-8A - 6B)e^t \sin(3t) + (6A - 8B)e^t \cos(3t)$$
$$y_p = Ae^t \sin(3t) + Be^t \cos(3t)$$

Substitution and algebraic manipulation gives:

$$(-9B - 3A)e^t \cos(3t) + (3B - 9A)e^t \sin(3t) = 3e^t \sin(3t)$$

Comparing coefficients gives:

$$3B - 9A = 3$$
$$-3A - 9B = 0$$

Solving these equations simultaneously gives:

$$A = -\frac{3}{10}$$
$$B = \frac{1}{10}$$

Thus the general form of the solution is given by:

$$y(t) = y_h + y_p = c_1 e^t + c_2 e^{2t} - \frac{3}{10} e^t \sin(3t) + \frac{1}{10} e^t \cos(3t)$$

Use dsolve to check your work.

```
%% Solve the DE using dsolve.
syms y(t)
Dy = diff(y,t); D2y = diff(y,t,t);
DE = D2y - 3* Dy + 2*y == 3 * exp(t) *sin(3*t)
dsolve(DE)
```

This gives the expected answer: (after simplification and some rearrangement). Use simplify.

```
ans = C1*exp(t) + C2*exp(2*t) +
      cos(3*t)*exp(t)/10 - 3*sin(3*t)*exp(t)/10
```

v. Particular solution has exponential root from homogeneous solution

$$y'' + y' - 6y = 4e^{-3t}$$

Solution to the homogeneous solution:

$$y_h = c_1 e^{-3t} + c_2 e^{2t}$$

In this problem, $g(t) = 4e^{-3t}$, so we may want to choose a particular solution that looks like this. But, the homogeneous solution already contains a term that has the exponential e^{-3t}, therefore we need to choose a different form of the particular solution. In cases like this, we try a particular solution of the form $y_p = Ate^{-3t}$, multiply by t.

Substituting into the original equation for

$$y_p' = -3Ate^{-3t} + Ae^{-3t}$$
$$y_p'' = 9Ate^{-3t} - 3Ae^{-3t} - 3Ae^{-3t} = 9Ate^{-3t} - 6Ae^{-3t}$$

$$[9Ate^{-3t} - 6Ae^{-3t}] + [-3Ate^{-3t} + Ae^{-3t}] - 6(Ate^{-3t}) = 4e^{-3t}$$
$$-5Ae^{-3t} = 4e^{-3t}$$

This gives $A = -\frac{4}{5}$.

Thus the solution is:

$$y(t) = y_h + y_p = c_1 e^{-3t} + c_2 e^{2t} - \frac{4}{5} te^{-3t}$$

Use dsolve to check your work.

```
%% Solve the DE using dsolve.
syms y(t)
Dy = diff(y,t); D2y = diff(y,t,t);
DE = D2y + Dy -6*y == 4 * exp(-3*t)
dsolve(DE)
```

This gives the expected answer: (after simplification and some rearrangement). Use simplify.

```
ans = C1*exp(-3*t) + C2*exp(2*t)  - (4*t*exp(-3*t))/5
```

Note the MATLAB solution may contain an extra exp(-3*t) term which can be absorbed into the constant C1.

Important Note

When given initial conditions, the unknown constants from the homogeneous solutions are found last, after the particular solution is found. You cannot find these constants from just the homogeneous solution.

How to choose the form of the particular solution?
See table following from the class text.

Table 4.4.1 Predicting forms of particular solutions

	Forcing Function $f(t)$ $\Rightarrow$	Particular Solution $y_p(t)$
(i)	k	A_0
(ii)	$P_n(t)$	$A_n(t)$
(iii)	Ce^{kt}	A_0e^{kt}
(iv)	$C\cos\omega t + D\sin\omega t$	$A_0\cos\omega t + B_0\sin\omega t$
(v)	$P_n(t)e^{kt}$	$A_n(t)e^{kt}$
(vi)	$P_n(t)\cos\omega t + Q_n(t)\sin\omega t$	$A_n(t)\cos\omega t + B_n(t)\sin\omega t$
(vii)	$Ce^{kt}\cos\omega t + De^{kt}\sin\omega t$	$A_0e^{kt}\cos\omega t + B_0e^{kt}\sin\omega t$
(viii)	$P_n(t)e^{kt}\cos\omega t + Q_n(t)e^{kt}\sin\omega t$	$A_n(t)e^{kt}\cos\omega t + B_n(t)e^{kt}\sin\omega t$

- $P_n(t), Q_n(t), A_n(t), B_n(t) \in \mathbb{P}_n$ (hence $A_0, B_0 \in \mathbb{P}_0 = \mathbb{R}$), and k, ω, C, and D are real constants.
- In (iv) and (vi)–(viii), both terms must be included in y_p, even if only one of the terms is present in $f(t)$.

If any term or terms of y_p are found in y_h (i.e., if such terms are solutions of $ay'' + by' + cy = 0$), multiply the expression for y_p by t (or, if necessary, by t^2) to eliminate the duplication.

Example where elimination of the duplication is necessary.

Consider the following second-order linear differential equation:

$$y'' + 3y' + 2y = te^{-t}$$

Roots of the characteristic equation: $r_1 = -1$ and $r_2 = -2$.
Homogeneous solution:

$$y_h(t) = c_1e^{-t} + c_2e^{-2t}$$

The particular solution is a product of a first-order polynomial and an exponential term, so a general form of the particular solution is: $(At + B)e^{-t}$, which can be expanded to give: $Ate^{-t} + Be^{-t}$

However, the term Be^{-t} is already a part of the homogeneous solution. We thus need to multiply the general form by t to avoid duplication. Thus, we choose a particular solution of the form:

$$y_p = (At^2 + Bt)e^{-t}$$

Substituting for $y_p'' = At^2e^{-t} - 4Ate^{-t} + 2Ae^{-t} - 2Be^{-t} + Bte^{-t}$ $\quad y_p' = 2Ate^{-t} - At^2e^{-t} + Be^{-t} - Bte^{-t}$, and then solving for A and B by comparing coefficients gives:

$$y_p(t) = \frac{1}{2}t^2e^{-t} - te^{-t}$$

Thus, the general solution is:

$$y(t) = c_2e^{-t} + c_2e^{-2t} + \frac{1}{2}t^2e^{-t} - te^{-t}$$

The constants can be found by substituting any given initial conditions.

Use dsolve to check your work.

```
%% Solve the DE using dsolve.
syms y(t)
Dy = diff(y,t); D2y = diff(y,t,t);
DE = D2y + 3*Dy +2*y == t * exp(-t)
dsolve(DE)
```

This gives the expected answer: (after simplification and some rearrangement).

```
ans = C1*exp(-t) + C2*exp(-2*t) + (t^2*exp(-t))/2 - t*exp(-t)
```

`Note the MATLAB solution may contain an extra exp(-t) term which can be absorbed into the constant C1.`

A second example where elimination of the duplication is necessary.

Consider the following second-order, linear differential equation:

$$y'' - y' = (t+1)^2 e^t + 4t + 2$$

The homogenous equation will have roots: $r = 0$ and $r = 1$, thus giving a homogeneous solution which looks like:

$$y_h(t) = c_1 e^{0t} + c_2 e^t = c_1 + c_2 e^t$$

The right hand side of the differential equation can be written as:

$$g(t) = (t^2 + 2t + 1)e^t + (4t + 2)$$

We can identify two parts of the RHS of this differential equation to make finding the solution easier:

$$g(t) = g_1 + g_2$$
$$g_1 = (t^2 + 2t + 1)e^t$$
$$g_2 = 4t + 2$$

Based on the form of $g_1(t)$ we may want to choose the particular solution that looks like a product of $(At^2 + Bt + C)$ and e^t. However, observe that the term e^t is part of the homogeneous solution, thus we multiply our initial guess by t to **avoid duplication**. Thus we choose the first part of our particular solution to be:

$$(At^3 + Bt^2 + Ct)e^t$$

Based on the form of g_2 we might want to choose a polynomial that looks like $Dt + E$, however, we observe the c_1 which is a constant is already part of the homogeneous solution. To **avoid repetition**, we multiply by t again so our guess would be $Dt^2 + Et$

Thus, the general form of the particular solution we need to choose would be:

$$y_p = (At^3 + Bt^2 + Ct)e^t + Dt^2 + Et$$

We go about the substitution of the terms y_p, y_p' and y_p'' in the differential equation and compare coefficients to get A, B, C, D and E and get a solution which can be written as:

$$[c_1 + c_2 e^t] + \left[\frac{1}{3}t^3 e^t + te^t - 2t^2 - 6t\right]$$

By knowing to multiply by t to account for terms in the homogeneous solution appearing in the particular solution, we are not surprised by the terms $\frac{1}{3}t^3 e^t$ and $-2t^2$ which appear in the solution.

Use dsolve to check your work.

```
%% Solve the DE using dsolve.
syms y(t)
Dy = diff(y,t); D2y = diff(y,t,t);
DE = D2y - Dy  == (t+1)^2 * exp(t) + 4*t +2
dsolve(DE)
```

This gives the expected answer: (after simplification and some rearrangement).

```
ans = C1 + C2*exp(t) + (t^3*exp(t))/3 + t*exp(t) - 2*t^2 - 6*t
```