Drexel University

Office of the Dean of the College of Engineering

ENGR 232 – Dynamic Engineering Systems

Recitation Guidelines for Week 9

Solutions to Systems of Differential Equations Using Laplace Transforms (Matrix Methods)

Example 1: Non-Homogeneous Differential System

Consider the following system of differential equations represented in matrix form:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 2e^{-2t}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since this equation is not homogenous or autonomous, we cannot find the solution using eigenvalues and eigenvectors the way we learnt, but we can use Laplace transforms on the matrix equation:

This system can be represented in matrix form, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the state vector and $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{B}f \qquad \text{where} \qquad f(t) = 2e^{-2t}$$

Laplace transform of both sides:

$$\mathcal{L}\{\dot{\mathbf{x}}\} = \mathcal{L}\{A\mathbf{x} + \mathbf{B}f\}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = A\mathbf{X}(s) + \mathbf{B}F(s)$$

Which can be rearranged to give:

$$\mathbf{X}(\boldsymbol{s}) = \boldsymbol{(sI - A)^{-1}\mathbf{x}(0)} + \boldsymbol{(sI - A)^{-1}\mathbf{B}F(s)}$$

First find:

$$sI - A = \begin{bmatrix} s & -1 \\ -6 & s-1 \end{bmatrix}$$

Laplace domain solution will be:

$$\mathbf{X}(s) = (sI - A)^{-1}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{2}{s+2} \end{bmatrix}\right\}$$

$$\mathbf{X}(s) = \frac{1}{s^2 - s - 6}\begin{bmatrix} s-1 & 1 \\ 6 & s \end{bmatrix}\begin{bmatrix} 1 \\ \dfrac{2}{s+2} \end{bmatrix}$$

$$\mathbf{X}(s) = \begin{bmatrix} \dfrac{s^2 + s}{(s^2 - s - 6)(s+2)} \\ \dfrac{8s + 12}{(s+2)(s^2 - s - 6)} \end{bmatrix}$$

To the observant student: Notice that the determinant of $(sI - A)$ is the same as the characteristic polynomial – this gives rise to the dynamics of the system. Take a guess what the eigenvalues of matrix A are.

The solution in the time domain can be found by taking the inverse Laplace transform which is:

$$\mathbf{x}(t) = \begin{bmatrix} +\frac{13}{25}e^{-2t} + \frac{12}{25}e^{3t} - \frac{2}{5}te^{-2t} \\ -\frac{36}{25}e^{-2t} + \frac{36}{25}e^{3t} + \frac{4}{5}te^{-2t} \end{bmatrix}$$

This would require finding the partial fraction expansion of the terms if you wish to show the answer using hand calculations.

Example 2: Zero State and Zero Input Components

Consider the following second-order system:

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Here, let $f(t)$ be the input to the system, and $\mathbf{x}(t)$ be the state vector. In this example, $f(t) = e^{-2t}$

Solution via Laplace transform will have the form:

$$\mathbf{X}(s) = \underbrace{(sI - A)^{-1}\mathbf{x}(0)}_{\text{Zero Input Solution (homog.)}} + \underbrace{(sI - A)^{-1}\mathbf{B}F(s)}_{\text{Zero State Soln. (forcing function)}}$$

This solution is a superposition of two components: the forcing function solution or **zero state,** and the homogeneous solution or **zero input** components. These components are:

- Forcing function solution or Zero State: When the initial **state** of the system is set to zero (no initial conditions), and we only have the forcing function

$$X_{ZS}(s) = (sI - A)^{-1}BF(s)$$

- Homogeneous solution or Zero Input: When the **input** to the system is zero, that is, $F(s) = 0$.

$$X_{ZI}(s) = (sI - A)^{-1}x(0)$$

The solution of this problem can be obtained by solving for the zero input and the zero state individually and then adding the two

i. Forcing function solution: Zero State

$$\mathbf{X}_{\mathbf{ZS}}(s) = (sI - A)^{-1}\mathbf{B}F(s)$$

$$= \begin{bmatrix} s+1 & -3 \\ 0 & s-4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{s+2} \end{bmatrix} = \frac{1}{(s+1)(s-4)} \cdot \begin{bmatrix} s-4 & +3 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{s+2} \end{bmatrix}$$

$$= \frac{1}{(s+1)(s-4)} \cdot \begin{bmatrix} \frac{3}{(s+2)} \\ \frac{s+1}{(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{3}{(s+1)(s+2)(s-4)} \\ \frac{1}{(s+2)(s-4)} \end{bmatrix}$$

Solving for the inverse Laplace transform (after finding the partial fraction expansion for both terms) gives the Zero State Response:

$$\mathbf{x}_{ZS}(t) = \begin{bmatrix} \frac{1}{2}e^{-2t} - \frac{3}{5}e^{-t} + \frac{1}{10}e^{4t} \\ \frac{1}{6}e^{4t} - \frac{1}{6}e^{-2t} \end{bmatrix}$$

ii. Homogenous Solution: Zero Input

$$\mathbf{X}_{ZI}(s) = (sI - A)^{-1}\mathbf{x}(0)$$

$$= \begin{bmatrix} s+1 & -3 \\ 0 & s-4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{(s+1)(s-4)} \cdot \begin{bmatrix} s-4 & +3 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+1} + \frac{3}{(s+1)(s-4)} \\ \frac{1}{s-4} \end{bmatrix}$$

Taking the inverse Laplace transform gives:

$$x_{ZI}(t) = \begin{bmatrix} \frac{2}{5}e^{-t} + \frac{3}{5}e^{4t} \\ e^{4t} \end{bmatrix}$$

iii. Complete solution: Finally, the solution to the system equation will be given by adding these two components:

$$\mathbf{x}(t) = \mathbf{x}_{Zs} + \mathbf{x}_{ZI}$$

$$= \begin{bmatrix} \frac{1}{2}e^{-2t} - \frac{3}{5}e^{-t} + \frac{1}{10}e^{4t} \\ \frac{1}{6}e^{4t} - \frac{1}{6}e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{2}{5}e^{-t} + \frac{3}{5}e^{4t} \\ e^{4t} \end{bmatrix}$$

$$\mathbf{x}(t) = \mathbf{x}_{Zs} + \mathbf{x}_{ZI} = \begin{bmatrix} \frac{1}{2}e^{-2t} - \frac{1}{5}e^{-t} + \frac{7}{10}e^{4t} \\ \frac{7}{6}e^{4t} - \frac{1}{6}e^{-2t} \end{bmatrix}$$

Example 3: Homogenous Equation - Comparison of two Methods

This next example can also be solved by finding its eigenvalues/eigenvectors. We will use both Laplace transforms and eigenvectors to see that the relationships are equivalent. Consider the following system of coupled differential equations represented in matrix form:

$$\textbf{DE}: \quad \dot{\mathbf{x}} = \begin{bmatrix} -1 & 3 \\ 0 & -4 \end{bmatrix} \mathbf{x}, \qquad \textbf{IC}: \quad \mathbf{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Method 1: The eigenvectors and eigenvalues of the system matrix are:

$$\lambda_1 = -1, \ \ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = -4, \ \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The eigenvalues are the diagonal entrees, since the matrix is triangular.

Thus, we expect a solution of the form:

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Using the initial conditions, we get:

$$\mathbf{x}(t) = -e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Method 2: The solution obtained by first transforming, then inverting the Laplace transform will be:

$$\mathcal{L}^{-1}\{(sI - A)^{-1}\mathbf{x}(0)\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s+1 & -3 \\ 0 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+4)} \cdot \begin{bmatrix} s+4 & +3 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$$

$$\mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s+1} & \dfrac{3}{(s+1)(s+4)} \\ 0 & \dfrac{1}{s+4} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$$

$$\mathbf{x}(t) = \begin{bmatrix} -e^{-t} - e^{-4t} \\ e^{-4t} \end{bmatrix}$$

Notice however, we can split this expression up into its individual dynamic terms:

$$\mathbf{x}(t) = e^{-t} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + e^{-4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

This expression is identical to the one obtained using eigenvalues and eigenvectors above. Notice that in this formulation, the eigenvectors are $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which are the same eigenvectors from above scaled by the constants c_1 and c_2.

Non-Linear System Analysis

Methods to study linear systems were covered over the last few weeks, but the majority of systems in the real world are non-linear. We introduce a technique used to study the behavior of non-linear systems, and particularly investigate the stability of any equilibrium points through linearization.

Example 4: Consider the following coupled **non-linear** system of equations:

$$\dot{x_1} = x_1 - x_2$$
$$\dot{x_2} = x_1 + x_2^2 - x_2$$

This is non-linear due to square term x_2^2 in the second equation. We cannot write this as a matrix equation. Furthermore, we cannot use Laplace transforms in the manner introduced in this class. What we do know is how to find the equilibrium point(s) of the system. Set the derivatives to zero!

$$0 = x_1 - x_2$$

$$0 = x_1 + x_2^2 - x_2$$

From the first equation $x_1 = x_2$. Using this result with the second equation, we get $x_2 = 0$. Thus, thus system has 1 equilibrium point at $x_1 = 0, x_2 = 0$.

Is this equilibrium point stable or unstable? To answer this, we compute a **linearized** form of the system by finding the Jacobian.

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

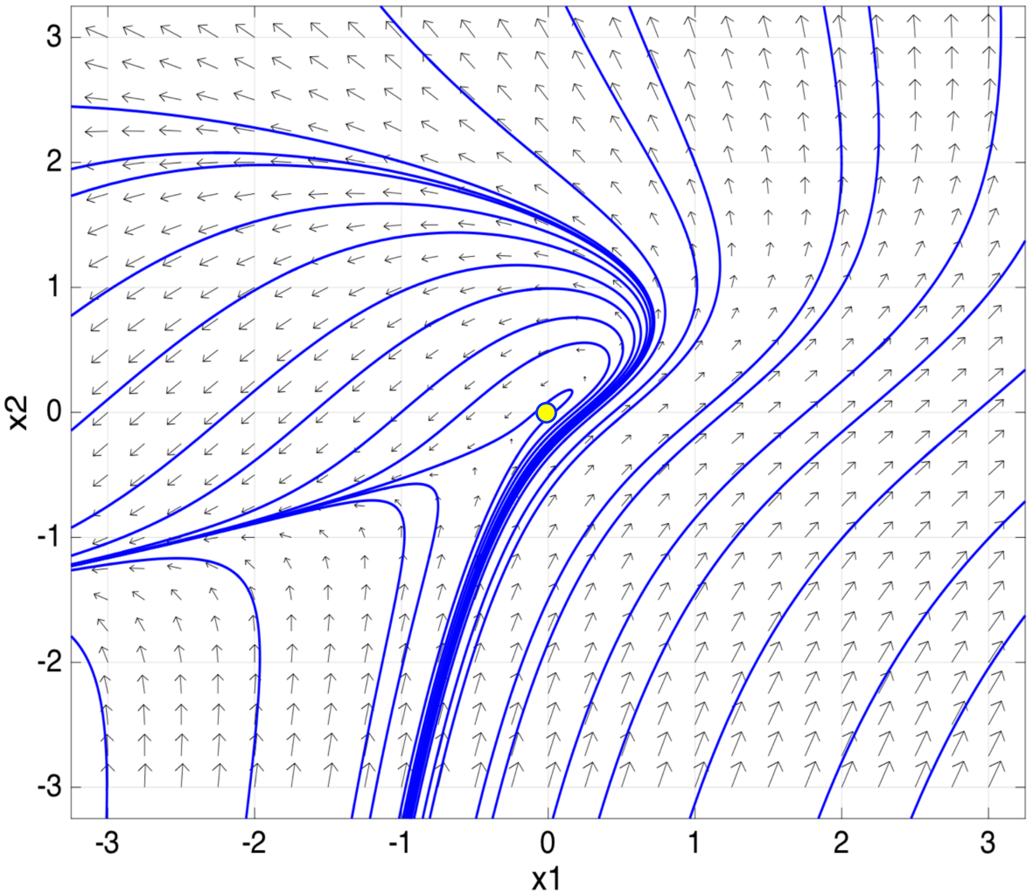
Here, $f_1 = \dot{x_1} = x_1 - x_2$ and $f_2 = \dot{x_2} = x_1 + x_2^2 - x_2$. The Jacobian is thus:

$$J = \begin{bmatrix} 1 & -1 \\ 1 & 2x_2 - 1 \end{bmatrix}$$

We then evaluate the Jacobian matrix at the equilibrium point $x_1 = 0, x_2 = 0$ and find:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

The eigenvalues of this matrix are: 0 and 0. Thus, this equilibrium point is **unstable**. We can check by plotting the phase portrait using PPLANE8.



The same rules apply for the values of λ obtained from the linearized system matrix in determining whether an equilibrium point is stable or not. (See class table).

It is important to also recognize as you will see in lecture that non-linear systems have multiple equilibrium points for which each must be evaluated individually to assess stability.

MATLAB Primer

This Matlab code, shows how to solve the non-homogeneous system:

$$\textbf{DE}: \quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot e^{-2t}, \qquad \textbf{IC}: \ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

using:

$$\mathbf{X}(s) = (sI - A)^{-1}\mathbf{x}(0) \ + \ (sI - A)^{-1}\mathbf{B}U(s)$$

```
%% Recitation 9
% ENGR 232 - symbolic solution of a forced system with initial conditions.
clear, close all, clc

% Define the Matrix and IC
% Set up the system

syms s t
A = [0 1; -6 -5]    % System matrix
B = [0; 1]          % The standard B column vector.
u = exp(-2*t)       % The given forcing function
x0 = [1;2]          % Initial conditions

% Now find transform X(s) of the solution x(t). (It's now a vector.)
Us = laplace(u);  % Transform of input function
X = inv(s*eye(2)-A)*x0+ inv(s*eye(2)-A)*B*Us   % Formula for Laplace solution

% Now get each component
X1 = simplify(X(1))
X2 = simplify(X(2))

% now invert to go back to the time domain.
x1t = ilaplace(X1)
x2t = ilaplace(X2)

% plot results
IC = x0;
ep = [0; 0]  % Because the driving term goes to zero.
x1 = matlabFunction(x1t)   % make it a function so we can plot it
x2 = matlabFunction(x2t)   % make it a function so we can plot it

t = 0:0.01:4;
subplot(2,2,1)
plot(t,x1(t),'r', 'linewidth', 2)
grid on; ylabel('x_1(t)')
title('Component Plots')

subplot(2,2,3)
plot(t,x2(t), 'linewidth', 2)
grid on; ylabel('x_2(t)')
xlabel('t')

subplot(2,2,[2 4])
plot(x1(t),x2(t),'g', 'linewidth', 2)
hold on, plot(IC(1),IC(2),'b*',ep(1),ep(2),'k*')
plot(ep(1),ep(2),'ko')
axis([-2 2 -2 2])
hold off
xlabel('x_1(t)'), ylabel('x_2(t)')
title('Phase Plot')
legend('Analytical Trajectory','Initial Condition',...
       'Eq Point','Location','best')
grid on
```

```
Here is a snapshot of the expected plot.
```