Drexel University

Office of the Dean of the College of Engineering

ENGR 232 – Dynamic Engineering Systems

Recitation Guidelines for Week 8 – Fall 2016

Topics for recitation this week:

- Laplace wrap up - Transfer functions, Initial Value Theorem, Final Value Theorem
- Using the Symbolic Toolbox to solve Laplace Transforms

Example 1: Transfer Function

a. Find the transfer function $G(s) = \frac{output}{input} = \frac{Y(s)}{F(s)}$ for the system governed by the following second-order differential equation with output $y(t)$ and input $f(t) = u(t)$, where $u(t)$ denotes the unit step function. Above, $F(s) = U(s) = \frac{1}{s}$ denotes the transform of the input term (the unit step function)

$$y'' - 7y' + 12y = u(t)$$

Recall the transfer function is defined for initial conditions set to zero. Taking the Laplace transform of both sides gives:

$$s^2Y(s) - 7sY(s) + 12Y(s) = U(s)$$

Now collect terms and solve for the transfer function:

$$Y(s)(s^2 - 7s + 12) = U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 - 7s + 12}$$

Notice the actual transform of the input was not used. The transfer function works for all inputs!

b. Now find the **step response** (in the time domain) for this second-order system. The step response is the output, when the input is the unit step function. Let's first find this in the *s*-domain.

$$Y(s) = G(s)U(s)$$

$$Output \; = G * Input$$

Since the input is the unit step function, we see $U(t) = L\{u(t)\} = \frac{1}{s}$

The output (in the *s*-domain) is therefore:

$$Y(s) = \frac{1}{s^2 - 7s + 12} \cdot \frac{1}{s} = \frac{1}{(s-3)(s-4)} \cdot \frac{1}{s}$$

Next, find the **partial fraction expansion**:

$$Y(s) = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-4} = \frac{1}{12s} - \frac{1}{3}\frac{1}{(s-3)} + \frac{1}{4}\frac{1}{s-4}$$

Check: Verify each of the coefficients *A*, *B* and *C* using Heaviside's cover-up method.

Taking the inverse Laplace transform of each term, we find the step response in the time domain:

$$y(t) = \frac{1}{12} - \frac{1}{3}e^{3t} + \frac{1}{4}e^{4t}$$

This is the step response of the system – note it is **unstable** since the roots of the characteristic equations $\{r_1 = 3, r_2 = 4\}$ are both positive. Check your lecture notes to see the classification of the various types of step responses as they depend on the roots of the characteristic equation – also known as the system poles. The terminology used was critically damped, underdamped and overdamped depending on the values of the roots.

Example 2: Initial Value Theorem and Final Value Theorem

Initial Value Theorem - always works

Given $L\{f(t)\} = F(s)$, then the initial value is given by $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s \cdot F(s)$

a. Find the initial value of the function $y(t)$ if its transform is: $Y(s) = \frac{3s+7}{s(s+3)}$

By the Initial Value Theorem:

$$\lim_{t\to 0} y(t) = \lim_{s\to\infty} s \cdot Y(s) = \lim_{s\to\infty} \frac{3s+7}{s+3}$$

To find the limit at infinity, just look at the highest powers in the numerator and denominator. They will dominate all the other terms.

Therefore:

$$y(0) = \lim_{s\to\infty} \frac{3s+7}{s+3} = 3$$

Final Value Theorem - Be careful! Won't work for sine or cosine terms or terms corresponding to roots with positive real parts.

Given the transform $L\{f(t)\} = F(s)$ **and** that the product $s \cdot F(s)$ has only negative real roots in the denominator, then the final value is given by:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s \cdot F(s)$$

b. Find the final value of the above function $y(t)$ with the transform: $Y(s) = \frac{3s+7}{s(s+3)}$

By the Final Value Theorem:

$$y(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0} s \cdot Y(s) = \lim_{s\to 0} \frac{3s+7}{s+3}$$

The final value theorem applies, since the only root of the denominator is $s = -3$ which is negative.
In this case, the limit can be found simply by substituting in $s = 0$.

$$y(\infty) = \lim_{s\to 0} \frac{3s+7}{s+3} = \frac{7}{3}$$

Check: As a check, let's invert the transform $Y(s)$ back to the time domain. First find the partial fraction expansion in the form:

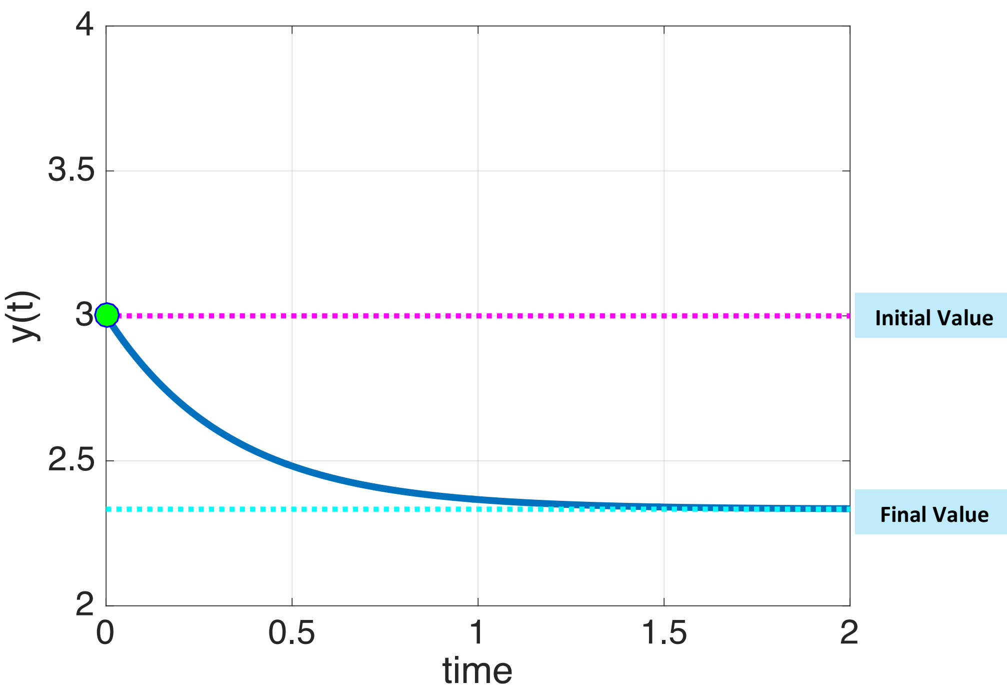
$$Y(s) = \frac{3s+7}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} = \frac{7/3}{s} + \frac{2/3}{s+3}$$

Check: Verify each of the coefficients A and B using Heaviside's cover-up method.

Thus the function in the time domain is:

$$y(t) = \frac{7}{3} + \frac{2}{3}e^{-3t}$$

You can see its graph has the predicted initial and final values.



MATLAB commands for Laplace and Inverse Laplace Transforms:

The symbolic toolbox can be used to solve the Laplace transform and inverse Laplace transform of symbolic expressions.

a. For example, find the Laplace transform of

$$y(t) = 3t^2 - 2\cos(t)\, e^{-4t}$$

b. Find the inverse Laplace transform of

$$Y(s) = \frac{1}{s-2} + \frac{4s^2-2}{s^2+4}$$

MATLAB Commands

```
syms s t

Y = laplace(3*t^2 - 2*cos(t)*exp(-4*t));      % Note use of capital

y = ilaplace(1/(s-2) + (4*s-2)/(s^2 +4))     % Note use of lower case
```

Command Window Output:

```
Y = 6/s^3 - (2*(s + 4))/((s + 4)^2 + 1)


y = 4*cos(2*t) + exp(2*t) - sin(2*t)
```

Recall that these functions are symbolic expressions and we would need to use `matlabFunction` to convert each to a function we can evaluate for plotting.

Example 3: Solve the following differential equation using Laplace transforms:

$$y'' + y' - 12y = e^{-2t}, \;\; y(0) = 1, \qquad y'(0) = 0$$

a. First, let's find the solution <u>without</u> using the Laplace transform. The code below shows an alternative way to enter differential equations as **<u>strings</u>**.

MATLAB Commands

```
clear, clc
syms y t s
y = dsolve('D2y+Dy-12*y= exp(-2*t)','y(0)=1','Dy(0)=0')
```

Command Window Output:

```
y = (3*exp(3*t))/5 - exp(-2*t)/10 + exp(-4*t)/2
```

b. Now let's solve the same differential equation using the Laplace Transform approach. This code is meant to mirror hand calculations for several of the steps.

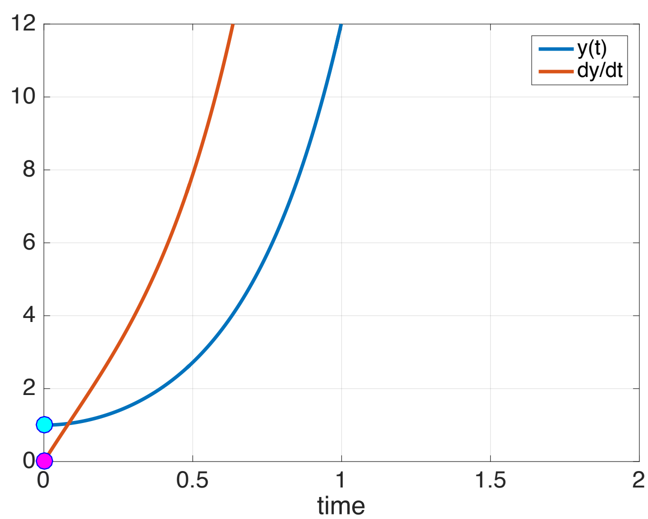
$$y'' + y' - 12y = e^{-2t}, \quad y(0) = 1, \quad y'(0) = 0$$

```
%% Laplace transform mirroring hand calculations
clear, clc
syms t s Y

% Now transform the first and second derivatives using the initial conditions
% y(0) = 1
Y1 = s*Y - 1
% y'(0) = 0
Y2 = s*Y1 - 0

% Find the Laplace transform F of the forcing term f(t) = exp(-2*t)
F = laplace( exp(-2*t) )

% Next, combine all of the terms to find the transform of the differential equation,
% which we will name LTofDE for Laplace Transform of DE.
LTofDE = Y2 + Y1 - 12*Y == F

Y = solve(LTofDE, Y)        % Solve for the transform Y
pretty(Y)
sol = ilaplace(Y)           % The solution in the time domain!
y = matlabFunction( sol )   % Express the solution as a function.
```

The solution in the time domain is found to be the same as the previous answer:

$$y(t) = \frac{3}{5}e^{3t} - \frac{1}{10}e^{-2t} + \frac{1}{2}e^{-4t}$$

The system is unstable and approaches $+\infty$ after a long time.



c. Investigate the **initial value theorem** and the **final value theorem** for this differential equation.

The above code (and the `collect` command) shows that the transform of the unknown function is:

$$Y(s) = \frac{s^2 + 3s + 3}{s^3 + 3s^2 - 10s - 24}$$

By the initial value theorem, the value at time 0 should be:

$$y(0) = \lim_{t \to 0} y(t) = \lim_{s \to \infty} s \cdot Y(s) = \lim_{s \to \infty} \frac{s^3}{s^3} = 1$$

Just look at the highest powers in the numerator and denominator.

This agrees with the given initial condition.

What about the final value? Our system is unstable and the final value theorem does not apply! Indeed, the roots of the denominator are $\{-4, -2, +3\}$. The positive root makes the system unstable.

Suppose we ignored this instability and blinding tried to apply the final value theorem anyways! What will happen??

If the final value theorem applied in this case, then we'd expect:

$$y(\infty) = \lim_{t \to \infty} y(t) = \lim_{s \to 0} s \cdot Y(s) = \lim_{s \to 0} s \cdot \frac{s^2 + 3s + 3}{s^3 + 3s^2 - 10s - 24} = 0$$

Wrong!

But this is wrong! The function instead diverges towards infinity!

Moral of the Story: Always double check the roots in the denominator before applying the final value theorem. If you don't, you may get an incorrect result. In contrast, the initial value theorem is always good.

Rest assured, MATLAB can find the correct results using its built-in `limit` command.

`>> limit(y(t), inf)` → returns `Inf` (which means infinity and is correct)

`>> limit(y(t), 0)` → returns `0` (which is correct)