This file contains the exercises, hints, and solutions for Chapter 3 of the book "Introduction to the Design and Analysis of Algorithms," 3rd edition, by A. Levitin. The problems that might be challenging for at least some students are marked by ▷; those that might be difficult for a majority of students are marked by ►.

Exercises 3.1

1. a. Give an example of an algorithm that should not be considered an application of the brute-force approach.

   b. Give an example of a problem that cannot be solved by a brute-force algorithm.

2. a. What is the efficiency of the brute-force algorithm for computing a^n as a function of n? As a function of the number of bits in the binary representation of n?

   b. If you are to compute $a^n \bmod m$ where $a > 1$ and n is a large positive integer, how would you circumvent the problem of a very large magnitude of a^n?

3. For each of the algorithms in Problems 4, 5, and 6 of Exercises 2.3, tell whether or not the algorithm is based on the brute-force approach.

4. a. Design a brute-force algorithm for computing the value of a polynomial

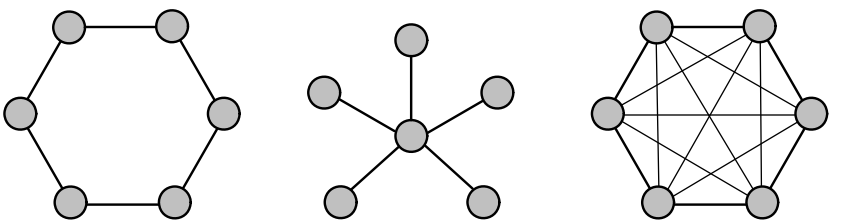
   $$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

   at a given point x_0 and determine its worst-case efficiency class.

   b. If the algorithm you designed is in $\Theta(n^2)$, design a linear algorithm for this problem.

   c. Is it possible to design an algorithm with a better-than-linear efficiency for this problem?

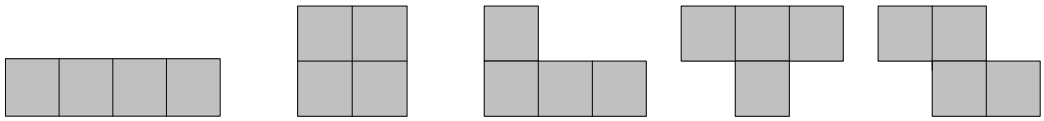
5. A network topology specifies how computers, printers, and other devices are connected over a network. The figure below illustrates three common topologies of networks: Ring, Star, and Fully Connected Mesh.



Ring Star Fully Connected Mesh

You are given a boolean matrix $A[0..n-1, 0..n-1]$, where $n > 3$, which is supposed to be the adjacency matrix of a graph modeling a network with one of these topologies. Your task is to determine which of these three topologies, if any, the matrix represents. Design a brute-force algorithm for this task and indicate its time efficiency class.

6. *Tetromino tilings* Tetrominoes are tiles made of four 1×1 squares There are five types of tetrominoes shown below:



straight tetromino square tetromino L-tetromino T-tetromino Z-tetromino

Is it possible to tile—i.e., cover exactly without overlaps—an 8×8 chessboard with

a. straight tetrominoes? b. square tetrominoes? c. L-tetrominoes?

d. T-tetrominoes? e. Z-tetrominoes?

7. *A stack of fake coins* There are n stacks of n identical-looking coins. All of the coins in one of these stacks are counterfeit, while all the coins in the other stacks are genuine. Every genuine coin weighs 10 grams; every fake weighs 11 grams. You have an analytical scale that can determine the exact weight of any number of coins.

   a. Devise a brute-force algorithm to identify the stack with the fake coins and determine its worst-case efficiency class.

   b. What is the minimum number of weighings needed to identify the stack with the fake coins?

8. Sort the list E, X, A, M, P, L, E in alphabetical order by selection sort.

9. Is selection sort stable? (The definition of a stable sorting algorithm was given in Section 1.3.)

10. Is it possible to implement selection sort for linked lists with the same $\Theta(n^2)$ efficiency as the array version?

11. Sort the list E, X, A, M, P, L, E in alphabetical order by bubble sort.

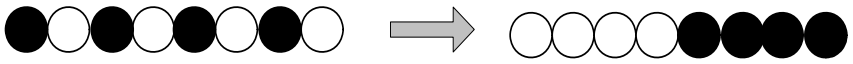
12. a. Prove that if bubble sort makes no exchanges on its pass through a list, the list is sorted and the algorithm can be stopped.

    b. Write pseudocode of the method that incorporates this improvement.

c. Prove that the worst-case efficiency of the improved version is quadratic.

13. Is bubble sort stable?

14. *Alternating disks* You have a row of $2n$ disks of two colors, n dark and n light. They alternate: dark, light, dark, light, and so on. You want to get all the dark disks to the right-hand end, and all the light disks to the left-hand end. The only moves you are allowed to make are those which interchange the positions of two neighboring disks.



Design an algorithm for solving this puzzle and determine the number of moves it makes. [Gar99]

Hints to Exercises 3.1

1. a. Think of algorithms that have impressed you with their efficiency and/or sophistication. Neither characteristic is indicative of a brute-force algorithm.

   b. Surprisingly, it is not a very easy question to answer. Mathematical problems (including those you have studied in your secondary school and college courses) are a good source of such examples.

2. a. The first question was all but answered in the section. Expressing the answer as a function of the number of bits can be done by using the formula relating the two metrics.

   b. How can we compute $(ab) \bmod m$?

3. It helps to have done the exercises in question.

4. a. The most straightforward algorithm, which is based on substituting x_0 into the formula, is quadratic.

   b. Analyzing what unnecessary computations the quadratic algorithm does should lead you to a better (linear) algorithm.

   c. How many coefficients does a polynomial of degree n have? Can one compute its value at an arbitrary point without processing all of them?

5. For each of the three network topologies, what properties of the matrix should the algorithm check ?

6. The answer to four of the questions is "yes".

7. a. Just apply the brute-force thinking to the problem in question.

   b. The problem can be solved in one weighing.

8. Just trace the algorithm on the input given. (It was done for another input in the section.)

9. Although the majority of elementary sorting algorithms are stable, do not rush with your answer. A general remark about stability made in Section 1.3, where the notion of stability is introduced, could be helpful, too.

10. Generally speaking, implementing an algorithm for a linked list poses problems if the algorithm requires accessing the list's elements not in a sequential order.

11. Just trace the algorithm on the input given. (See an example in the section.)

12. a. A list is sorted if and only if all its adjacent elements are in a correct order. Why?

    b. Add a boolean flag to register the presence or absence of switches.

    c. Identify worst-case inputs first.

13. Can bubble sort change the order of two equal elements in its input?

14. Thinking about the puzzle as a sorting-like problem may or may not lead you to the most simple and efficient solution.

Solutions to Exercises 3.1

1. a. Euclid's algorithm and the standard algorithm for finding the binary representation of an integer are examples from the algorithms previously mentioned in this book. There are, of course, many more examples in its other chapters.

   b. Solving nonlinear equations or computing definite integrals are examples of problems that cannot be solved exactly (except for special instances) by any algorithm.

2. a. $M(n) = n \approx 2^b$ where $M(n)$ is the number of multiplications made by the brute-force algorithm in computing a^n and b is the number of bits in the n's binary representation. Hence, the efficiency is linear as a function of n and exponential as a function of b.

   b. Perform all the multiplications modulo m, i.e.,

   $$a^i \bmod m = (a^{i-1} \bmod m \cdot a \bmod m) \bmod m \quad \text{for } i = 1, ..., n.$$

3. Problem 4 (computes $\sum_1^n i^2$): yes

   Problem 5 (computes the range of an array's values): yes

   Problem 6 (checks whether a matrix is symmetric): yes

4. a. Here is a pseudocode of the most straightforward version:

   Algorithm *BruteForcePolynomialEvaluation*($P[0..n]$, x)
   //The algorithm computes the value of polynomial P at a given point x
   //by the "highest-to-lowest term" brute-force algorithm
   //Input: Array $P[0..n]$ of the coefficients of a polynomial of degree n,
   // stored from the lowest to the highest and a number x
   //Output: The value of the polynomial at the point x
   $p \leftarrow 0.0$
   for $i \leftarrow n$ **downto** 0 **do**
   $power \leftarrow 1$
   for $j \leftarrow 1$ **to** i **do**
   $power \leftarrow power * x$
   $p \leftarrow p + P[i] * power$
   return p

   We will measure the input's size by the polynomial's degree n. The basic operation of this algorithm is a multiplication of two numbers; the number of multiplications $M(n)$ depends on the polynomial's degree only.

Although it is not difficult to find the total number of multiplications in this algorithm, we can count just the number of multiplications in the algorithm's inner-most loop to find the algorithm's efficiency class:

$$M(n) = \sum_{i=0}^{n} \sum_{j=1}^{i} 1 = \sum_{i=0}^{n} i == \frac{n(n+1)}{2} \in \Theta(n^2).$$

b. The above algorithm is very inefficient: we recompute powers of x again and again as if there were no relationship among them. Thus, the obvious improvement is based on computing consecutive powers more efficiently. If we proceed from the highest term to the lowest, we could compute x^{i-1} by using x^i but this would require a division and hence a special treatment for $x = 0$. Alternatively, we can move from the lowest term to the highest and compute x^i by using x^{i-1}. Since the second alternative uses multiplications instead of divisions and does not require any special treatment for $x = 0$, it is both more efficient and cleaner. It leads to the following algorithm:

Algorithm *BetterBruteForcePolynomialEvaluation*($P[0..n]$, x)
//The algorithm computes the value of polynomial P at a given point x
//by the "lowest-to-highest term" algorithm
//Input: Array $P[0..n]$ of the coefficients of a polynomial of degree n,
// from the lowest to the highest, and a number x
//Output: The value of the polynomial at the point x
$p \leftarrow P[0]$; $power \leftarrow 1$
for $i \leftarrow 1$ **to** n **do**
  $power \leftarrow power * x$
  $p \leftarrow p + P[i] * power$
return p

The number of multiplications here is

$$M(n) = \sum_{i=1}^{n} 2 = 2n$$

(while the number of additions is n), i.e., we have a linear algorithm.

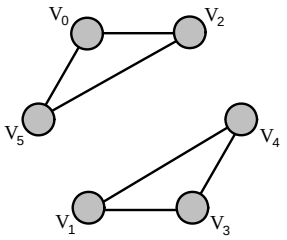
Note: Horner's Rule discussed in Section 6.5 needs only n multiplications (and n additions) to solve this problem.

c. No, because any algorithm for evaluating an arbitrary polynomial of degree n at an arbitrary point x must process all its $n + 1$ coefficients. (Note that even when $x = 1$, $p(x) = a_n + a_{n-1} + ... + a_1 + a_0$, which needs at least n additions to be computed correctly for arbitrary $a_n, a_{n-1}, ..., a_0$.)

5. For simplicity, we check each of the three topologies separately.

The adjacency matrix of a graph with the ring topology must be, of course, symmetric, and each of its rows must have exactly two 1's, both not on the main diagonal to avoid loops. These necessary conditions are not sufficient because they do not guarantee connectivity of the graph, as the following example demonstrates.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



The following brute-force algorithm follows the 1's in a given matrix indicating two edges incident with a vertex: one for the edge entering it and the other leaving the vertex, making sure the cycle closes at the starting vertex only after visiting all the other vertices of the graph.

Start by scanning row 0 to verify that it has exactly two 1's in columns we denote j_1 and j_{n-1} so that $0 < j_1 < j_{n-1}$. If it is not the case, stop: the matrix is not the adjacency matrix of a graph with the ring topology. If it is the case, mark column 0 as that of a visited vertex and proceed to row j_1. ($0 - j_1$ is the first edge of the cycle being traversed.) Check that $A[j_1, 0] = 1$ and find the unmarked column $j_2 \neq j_1$ with the only other 1 in this row. If this is impossible to do, stop; otherwise, mark column j_1 as that of a visited vertex and proceed to row j_2. ($j_1 - j_2$ is the second edge of the cycle being traversed.) Continue in this fashion until the row corresponding to the only remaining unmarked vertex needs to be processed. This must be row j_{n-1}, where j_{n-1} was found on the first iteration of the algorithm. The two 1's in row j_{n-1} must be in columns j_{n-2} (of the vertex from which vertex j_{n-1} was reached) and 0 (to close the cycle).

The time efficiency of the algorithm is $O(n^2)$, because it checks all the elements of an $n \times n$ matrix in the worst case.

Note: It is not difficult to prove that a graph has the ring topology if and only if all its vertices have degree 2 while having no loops, and it is connected. Hence the problem can also be solved by the obvious checking of the first condition and checking the graph's connectivity by one of the two standard algorithms for doing that: depth-first search or breadth-first search, which are discussed in Section 3.5 of the book. The above algorithm mimics, in fact, the first of these alternatives.

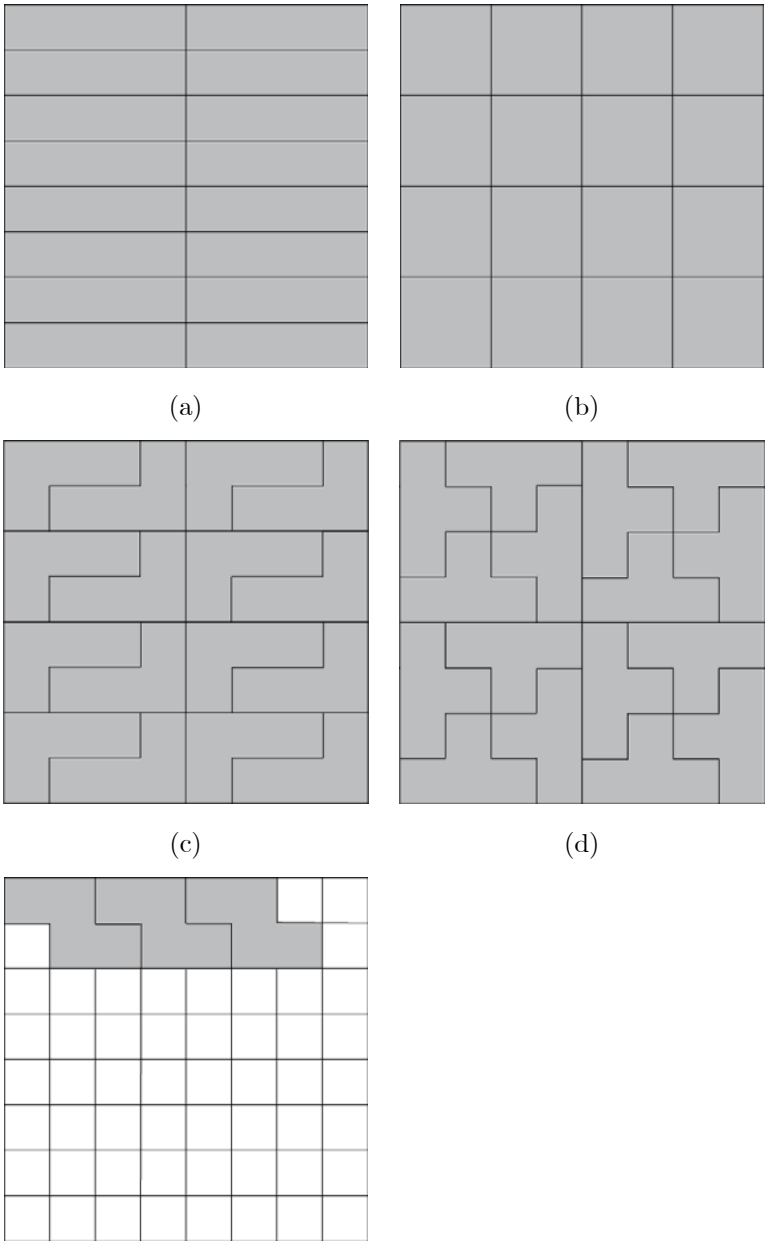
The adjacency matrix of a graph with the star topology contains only 0's except some row j_0 and column j_0 with all 1's except $A[j_0, j_0]$, the element on the main diagonal. Hence a brute-force algorithm can start by scanning row 0

to check whether it contains all 1's except in column 0 or it contains a single 1 in some column $j_0 > 0$. In the former case, every subsequent row $i = 1, 2, ..., n-1$ must contain a single 1 in column 0 (i.e., $A[i, 0] = 1$ and $A[i, j] = 0$ for $j = 1, ..., n-1$). In the latter case, every subsequent row $i = 1, 2, ..., n-1$ except $i = j_0$ is checked for the same properties as row 0 (i.e., $A[i, j_0] = 1$ and $A[i, j] = 0$ for $j = 0, 1, ..., n-1, j \neq j_0$), whereas row j_0 must contain only 1's except the element on its main diagonal (i.e., $A[j_0, , j_0] = 0$ and $A[j_0, j] = 1$ for $j = 0, 1, ..., n-1, j \neq j_0$). Obviously, the algorithm's time efficiency is also $O(n^2)$.

Finally, the adjacency matrix of a graph with the fully connected mesh topology contains only 1's except 0's on its main diagonal. Checking this property by scanning rows (or columns) of the matrix requires $O(n^2)$ time as well.

6. Tilings of an 8×8 board with straight tetrominoes, square tetrominoes, L-tetrominoes, and T-tetrominoes are shown below. It is impossible to cover an 8×8 board with Z-tetrominoes. Indeed, putting such a tile to cover a corner of the board makes it necessary to continue putting two more tiles along the boarder with no possibility to cover the two remaining squares

in the first row.



(a) (b)

(c) (d)

(e)

Tiling a chessboard with (a) straight tetrominoes; (b) square tetrominoes; (c) L-tetrominoes; (d) T-tetrominoes; (e) Z-tetrominoes (failed)

Note: Tiling with tetrominoes is discussed, among other types of polyominoes, by their inventor S. W. Golomb in his monograph *Polyominoes: Puzzles, Patterns, Problems, and Packings*. Revised and expanded second edition, Princeton University Press, Princeton, NJ, 1994.

7. a. Number the coin stacks from 1 to n. Starting with the first stack, repeat the following. If the current stack is not the last one, take any coin from the stack and weigh it: if it weighs 11 grams, this stack contains the fake coins and algorithm stops; if the coin weighs 10 grams, proceed to the next stack. If the last stack is reached, it must be the one with the fake coins and none of its coins need to be weighed. In the worst case (the stack of the fake coins is the last one), the algorithm makes $n-1$ weighings, which puts it in the $\Theta(n)$ class.

   b. The problem can be solved in one weighing. Number the coin stacks from 1 to n. Take one coin from the first stack, two coins from the second, and so on until all n coins are taken from the last stack. Weigh all these coins together. The difference between this weight and $10n(n+1)/2 = 5n(n+1)$, the weight of $(1+2+...+n) = n(n+1)/2$ genuine coins, indicates the number of the fake coins weighed, which is equal to the number of the stack with the fake coins. For example, if $n = 10$ and the selected coins weigh 553 grams, 3 coins are fake and hence it is the third stack that contains the fake coins.

   Note: Finding a stack of fake coins in one weighing is a well-known puzzle, which has been included in many collections of brain teasers.

8.
$$
\begin{array}{|ccccccc}
E & X & \mathbf{A} & M & P & L & E \\
A & X & \mathbf{E} & M & P & L & E \\
A & E & X & M & P & L & \mathbf{E} \\
A & E & E & M & P & \mathbf{L} & X \\
A & E & E & L & P & \mathbf{M} & X \\
A & E & E & L & M & \mathbf{P} & X \\
A & E & E & L & M & P & X
\end{array}
$$

9. Selection sort is not stable: In the process of exchanging elements that are not adjacent to each other, the algorithm can reverse an ordering of equal elements. The list $2'$, $2''$, 1 is such an example.

10. Yes. Both operations—finding the smallest element and swapping it—can be done as efficiently with a linked list as with an array.

11. E, X, A, M, P, L, E

$$
\begin{array}{llllllllllll}
E & \overset{?}{\leftrightarrow} & X & \overset{?}{\leftrightarrow} & A & & M & & P & & L & & E \\
E & & A & & X & \overset{?}{\leftrightarrow} & M & & P & & L & & E \\
E & & A & & M & & X & \overset{?}{\leftrightarrow} & P & & L & & E \\
E & & A & & M & & P & & X & \overset{?}{\leftrightarrow} & L & & E \\
E & & A & & M & & P & & L & & X & \overset{?}{\leftrightarrow} & E \\
E & & A & & M & & P & & L & & E & & |X \\
E & \overset{?}{\leftrightarrow} & A & & M & & P & & L & & E & & \\
A & & E & \overset{?}{\leftrightarrow} & M & \overset{?}{\leftrightarrow} & P & \overset{?}{\leftrightarrow} & L & & E & & \\
A & & E & & M & & L & & P & \overset{?}{\leftrightarrow} & E & & \\
A & & E & & M & & L & & E & & |P & & \\
A & \overset{?}{\leftrightarrow} & E & \overset{?}{\leftrightarrow} & M & \overset{?}{\leftrightarrow} & L & & E & & & & \\
A & & E & & L & & M & \overset{?}{\leftrightarrow} & E & & & & \\
A & & E & & L & & E & & |M & & & & \\
A & \overset{?}{\leftrightarrow} & E & \overset{?}{\leftrightarrow} & L & \overset{?}{\leftrightarrow} & E & & & & & & \\
A & & E & & E & & |L & & & & & & \\
A & \overset{?}{\leftrightarrow} & E & \overset{?}{\leftrightarrow} & E & \overset{?}{\leftrightarrow} & L & & & & & &
\end{array}
$$

The algorithm can be stopped here (see the next question).

12. a. Pass i $(0 \le i \le n-2)$ of bubble sort can be represented by the following diagram:

$$A_0, ..., A_j \overset{?}{\leftrightarrow} A_{j+1}, ..., A_{n-i-1} \le | \underbrace{A_{n-i} \le \cdots \le A_{n-1}}_{\text{in their final positions}}$$

If there are no swaps during this pass, then

$$A_0 \le A_1 \le \cdots \le A_j \le A_{j+1} \le \cdots \le A_{n-i-1},$$

with the larger (more accurately, not smaller) elements in positions $n-i$ through $n-1$ being sorted during the previous iterations.

b. Here is a pseudocode for the improved version of bubble sort:

Algorithm *BetterBubbleSort*$(A[0..n-1])$
//The algorithm sorts array $A[0..n-1]$ by improved bubble sort
//Input: An array $A[0..n-1]$ of orderable elements
//Output: Array $A[0..n-1]$ sorted in ascending order
count $\leftarrow n-1$ //number of adjacent pairs to be compared
sflag $\leftarrow$ **true** //swap flag
while *sflag* **do**

```
sflag ← false
for j ← 0 to count−1 do
    if A[j + 1] < A[j]
        swap A[j] and A[j + 1]
        sflag ← true
count ← count −1
```

c. The worst-case inputs will be strictly decreasing arrays. For them, the improved version will make the same comparisons as the original version, which was shown in the text to be quadratic.

13. Bubble sort is stable. It follows from the fact that it swaps adjacent elements only, provided $A[j+1] < A[j]$.

14. Here is a simple and efficient (in fact, optimal) algorithm for this problem: Starting with the first and ending with the last light disk, swap it with each of the i ($1 \le i \le n$) dark disks to the left of it. The ith iteration of the algorithm can be illustrated by the following diagram, in which 1s and 0s correspond to the dark and light disks, respectively.

$$\underbrace{00..0}_{i-1}\underbrace{11..1}_{i-1}\mathbf{1}0\mathbf{1}0..10 \quad \Rightarrow \quad \underbrace{00..0}_{i}\mathbf{0}\underbrace{11..11}_{i}0..10$$

The total number of swaps made is equal to $\sum_{i=1}^{n} i = n(n+1)/2$.

The problem can also be solved by mimicking the swaps made by bubble sort in sorting the array of 1's and 0's representing the dark and light disks, respectively.

Exercises 3.2

1. Find the number of comparisons made by the sentinel version of sequential search

   a. in the worst case.

   b. in the average case if the probability of a successful search is p ($0 \leq p \leq 1$).

2. As shown in Section 2.1, the average number of key comparisons made by sequential search (without a sentinel, under standard assumptions about its inputs) is given by the formula

$$C_{avg}(n) = \frac{p(n+1)}{2} + n(1-p),$$

   where p is the probability of a successful search. Determine, for a fixed n, the values of p ($0 \leq p \leq 1$) for which this formula yields the largest value of $C_{avg}(n)$ and the smallest value of $C_{avg}(n)$.

3. *Gadget testing* A firm wants to determine the highest floor of its n-story headquarters from which a gadget can fall without breaking. The firm has two identical gadgets to experiment with. If one of them gets broken, it can not be repaired, and the experiment will have to be completed with the remaining gadget. Design an algorithm in the best efficiency class you can to solve this problem.

4. Determine the number of character comparisons made by the brute-force algorithm in searching for the pattern `GANDHI` in the text

   `THERE_IS_MORE_TO_LIFE_THAN_INCREASING_ITS_SPEED`

   (Assume that the length of the text—it is 47 characters long—is known before the search starts.)

5. How many comparisons (both successful and unsuccessful) are made by the brute-force string-matching algorithm in searching for each of the following patterns in the binary text of 1000 zeros?

   a. 00001 b. 10000 c. 01010

6. Give an example of a text of length n and a pattern of length m that constitutes the worst-case input for the brute-force string-matching algorithm. Exactly how many character comparisons are made for such input?

7. In solving the string-matching problem, would there be any advantage in comparing pattern and text characters right-to-left instead of left-to-right?

8. Consider the problem of counting, in a given text, the number of substrings that start with an A and end with a B. (For example, there are four such substrings in CABAAXBYA.)

   (a) Design a brute-force algorithm for this problem and determine its efficiency class.

   (b) Design a more efficient algorithm for this problem [Gin04].

9. Write a visualization program for the brute-force string-matching algorithm.

10. *Word Find* A popular diversion in the United States, "word find" (or "word search") puzzles ask the player to find each of a given set of words in a square table filled with single letters. A word can read horizontally (left or right), vertically (up or down), or along a 45 degree diagonal (in any of the four directions), formed by consecutively adjacent cells of the table; it may wrap around the table's boundaries but it must read in the same direction with no zigzagging. The same cell of the table may be used in different words, but, in a given word, the same cell may be used no more than once. Write a computer program for solving this puzzle.

11. *Battleship game* Write a program for playing Battleship (a classic strategy game) on the computer which is based on a version of brute-force pattern matching. The rules of the game are as follows. There are two opponents in the game (in this case, a human player and the computer). The game is played on two identical boards (10-by-10 tables of squares) on which each opponent places his or her ships, not seen by the opponent. Each player has five ships, each of which occupies a certain number of squares on the board: a destroyer (2 squares), a submarine (3 squares), a cruiser (3 squares), a battleship (4 squares), and an aircraft carrier (5 squares). Each ship is placed either horizontally or vertically, with no two ships touching each other. The game is played by the opponents taking turns "shooting" at each other's ships. A result of every shot is displayed as either a hit or a miss. In case of a hit, the player gets to go again and keeps playing until this player misses. The goal is to sink all the opponent's ships before the opponent succeeds in doing it first. (To sink a ship, all squares occupied by the ship must be hit.)

Hints to Exercises 3.2

1. Modify the analysis of the algorithm's version in Section 2.1.
2. As a function of p, what kind of function is C_{avg}?
3. Solve a simpler problem with a single gadget first. Then design a better than linear algorithm for the problem with two gadgets.
4. The content of this quote from Mahatma Gandhi is more thought provoking than this drill.
5. For each input, one iteration of the algorithm yields all the information you need to answer the question.
6. It will suffice to limit your search for an example to binary texts and patterns.
7. The answer, surprisingly, is yes.
8. a. For a given occurrence of A in the text, what are the substrings you need to count?

   b. For a given occurrence of B in the text, what are the substrings you need to count?
9. You may use either bit strings or a natural language text for the visualization program. It would be a good idea to implement, as an option, a search for all occurrences of a given pattern in a given text.
10. Test your program thoroughly. Be especially careful about the possibility of words read diagonally with wrapping around the table's border.
11. A (very) brute-force algorithm can simply shoot at adjacent feasible cells starting at, say, one of the corners of the board. Can you suggest a better strategy? (You can investigate relative efficiencies of different strategies by making two programs implementing them play each other.) Is your strategy better than the one that shoots at randomly generated cells of the opponent's board?

Solutions to Exercises 3.2

1. a. $C_{worst}(n) = n + 1$.

   b. $C_{avg}(n) = \frac{(2-p)(n+1)}{2}$. In the manner almost identical to the analysis in Section 2.1, we obtain

$$\begin{aligned} C_{avg}(n) &= [1 \cdot \frac{p}{n} + 2 \cdot \frac{p}{n} + \cdots + i \cdot \frac{p}{n} + \cdots + n \cdot \frac{p}{n}] + (n+1) \cdot (1-p) \\ &= \frac{p}{n}[1 + 2 + ... + i + \cdots + n] + (n+1)(1-p) \\ &= \frac{p}{n}\frac{n(n+1)}{2} + (n+1)(1-p) = \frac{(2-p)(n+1)}{2}. \end{aligned}$$

2. The expression

$$\frac{p(n+1)}{2} + n(1-p) = p\frac{n+1}{2} + n - np = n - p(n - \frac{n+1}{2}) = n - \frac{n-1}{2}p$$

   is a linear function of p. Since the p's coefficient is negative for $n > 1$, the function is strictly decreasing on the interval $0 \le p \le 1$ from n to $(n+1)/2$. Hence $p = 0$ and $p = 1$ are its maximum and minimum points, respectively, on this interval. (Of course, this is the answer we should expect: The average number of comparisons should be the largest when the probability of a successful search is 0, and it should be the smallest when the probability of a successful search is 1.)

3. Drop the first gadget from floors $\lceil\sqrt{n}\rceil$, $2\lceil\sqrt{n}\rceil$, and so on until either the floor $i\lceil\sqrt{n}\rceil$ a drop from which makes the gadget malfunction is reached or no such floor in this sequence is encountered before the top of the building is reached. In the former case, the floor to be found is higher than $(i-1)\lceil\sqrt{n}\rceil$ and lower than $i\lceil\sqrt{n}\rceil$. So, drop the second gadget from floors $(i-1)\lceil\sqrt{n}\rceil + 1$, $(i-1)\lceil\sqrt{n}\rceil + 2$, and so on until the first floor a drop from which makes the gadget malfunction is reached. The floor immediately preceding that floor is the floor in question. If no drop in the first-pass sequence resulted in the gadget's failure, the floor in question is higher than $i\lceil\sqrt{n}\rceil$, the last tried floor of that sequence. Hence, continue the successive examination of floors $i\lceil\sqrt{n}\rceil + 1$, $i\lceil\sqrt{n}\rceil + 2$, and so on until either a failure is registered or the last floor is reached. The number of times the two gadgets are dropped doesn't exceed $\lceil\sqrt{n}\rceil + \lceil\sqrt{n}\rceil$, which puts it in $O(\sqrt{n})$.

4. 43 comparisons.
   The algorithm will make $47 - 6 + 1 = 42$ trials: In the first one, the G of the pattern will be aligned against the first T of the text; in the last one, it will be aligned against the last space. On each but one trial, the algorithm will make one unsuccessful comparison; on one trial—when the G of the pattern is aligned against the G of the text —it will make two

comparisons. Thus, the total number of character comparisons will be $41 \cdot 1 + 1 \cdot 2 = 43$.

5. a. For the pattern `00001`, the algorithm will make four successful and one unsuccessful comparison on each of its trials and then shift the pattern one position to the right:

```
0 0 0 0 0 0                                   0 0 0 0 0
0 0 0 0 1
  0 0 0 0 1
                  etc.
                                              0 0 0 0 1
```

The total number of character comparisons will be $C = 5 \cdot 996 = 4980$.

b. For the pattern `10000`, the algorithm will make one unsuccessful comparison on each of its trials and then shift the pattern one position to the right:

```
0 0 0 0 0 0                                   0 0 0 0 0
1 0 0 0 0
  1 0 0 0 0
                  etc.
                                              1 0 0 0 0
```

The total number of character comparisons will be $C = 1 \cdot 996 = 996$.

c. For the pattern `01010`, the algorithm will make one successful and one unsuccessful comparison on each of its trials and then shift the pattern one position to the right:

```
0 0 0 0 0 0                                   0 0 0 0 0
0 1 0 1 0
  0 1 0 1 0
                  etc.
                                              0 1 0 1 0
```

The total number of character comparisons will be $C = 2 \cdot 996 = 1,992$.

6. The text composed of n zeros and the pattern $\underbrace{0 \ldots 0}_{m-1}1$ is an example of the worst-case input. The algorithm will make $m(n - m + 1)$ character comparisons on such input.

7. Comparing pairs of the pattern and text characters righ-to-left can allow farther pattern shifts after a mismatch. This is the main insight the two string matching algorithms discussed in Section 7.2 are based on. (As a specific example, consider searching for the pattern 11111 in the text of one thousand zeros.)

8. a. Note that the number of desired substrings that starts with an A at a given position i $(0 \le i < n-1)$ in the text is equal to the number of B's to the right of that position. This leads to the following simple algorithm:

   Initialize the count of the desired substrings to 0. Scan the text left to right doing the following for every character except the last one: If an A is encountered, count the number of all the B's following it and add this number to the count of desired substrings. After the scan ends, return the last value of the count.

   For the worst case of the text composed of n A's, the total number of character comparisons is

$$n + (n-1) + \cdots + 2 = n(n+1)/2 - 1 \in \Theta(n^2).$$

   b. Note that the number of desired substrings that ends with a B at a given position i $(0 < i \le n-1)$ in the text is equal to the number of A's to the left of that position. This leads to the following algorithm:

   Initialize the count of the desired substrings and the count of A's encountered to 0. Scan the text left to right until the text is exhausted and do the following. If an A is encountered, increment the A's count; if a B is encountered, add the current value of the A's count to the desired substring count. After the text is exhausted, return the last value of the desired substring count.

   Since the algorithm makes a single pass through a given text spending constant time on each of its characters, the algorithm is linear.

9. n/a

10. n/a

11. n/a

Exercises 3.3

1. Assuming that *sqrt* takes about ten times longer than each of the other operations in the innermost loop of *BruteForceClosestPoints*, which are assumed to take the same amount of time, estimate how much faster will the algorithm run after the improvement discussed in Section 3.3.

2. Can you design a more efficient algorithm than the one based on the brute-force strategy to solve the closest-pair problem for n points $x_1, ..., x_n$ on the real line?

3. Let $x_1 < x_2 < ... < x_n$ be real numbers representing coordinates of n villages located along a straight road. A post office needs to be built in one of these villages.

   a. Design an efficient algorithm to find the post-office location minimizing the average distance between the villages and the post office.

   b. Design an efficient algorithm to find the post-office location minimizing the maximum distance from a village to the post office.

4. a.▷ There are several alternative ways to define a distance between two points $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$. In particular, the ***Manhattan distance*** is defined as
   $$d_M(p_1, p_2) = |x_1 - x_2| + |y_1 - y_2|.$$
   Prove that d_M satisfies the following axioms that every distance function must satisfy:

   (i) $d_M(p_1, p_2) \geq 0$ for any two points p_1 and p_2 and $d_M(p_1, p_2) = 0$ if and only if $p_1 = p_2$;

   (ii) $d_M(p_1, p_2) = d_M(p_2, p_1)$;

   (iii) $d_M(p_1, p_2) \leq d_M(p_1, p_3) + d_M(p_3, p_2)$ for any p_1, p_2, and p_3.

   b. Sketch all the points in the x, y coordinate plane whose Manhattan distance to the origin (0,0) is equal to 1. Do the same for the Euclidean distance.

   c.▷ True or false: A solution to the closest-pair problem does not depend on which of the two metrics—d_E (Euclidean) or d_M (Manhattan)—is used.

5. The ***Hamming distance*** between two strings of equal length is defined as the number of positions at which the corresponding symbols are different. It is named after Richard Hamming (1915–1998), a prominent American scientist and engineer, who introduced it in his seminal paper on error-detecting and error-correcting codes.

(a) Does the Hamming distance satisfy the three axioms of a distance metric listed in Problem 4?

(b) What is the time efficiency class of the brute-force algorithm for the closest-pair problem if the points in question are strings of m symbols long and the distance between two of them is measured by the Hamming distance?

6. ▷ *Odd pie fight* There are $n \geq 3$ people positioned in a field (Euclidean plane) so that each has a unique nearest neighbor. Each person has a cream pie. At a signal, everybody hurls his or her pie at the nearest neighbor. Assuming that n is odd and that nobody can miss his or her target, true or false: There always remains at least one person not hit by a pie? [Car79].

7. The closest-pair problem can be posed in k-dimensional space in which the Euclidean distance between two points $p'(x'_1, ..., x'_k)$ and $p''(x''_1, ..., x''_k)$ is defined as

$$d(p', p'') = \sqrt{\sum_{s=1}^{k} (x'_s - x''_s)^2}.$$

What is the time-efficiency class of the brute-force algorithm for the k-dimensional closest-pair problem?

8. Find the convex hulls of the following sets and identify their extreme points (if they have any).
a. a line segment
b. a square
c. the boundary of a square
d. a straight line

9. Design a linear-time algorithm to determine two extreme points of the convex hull of a given set of $n > 1$ points in the plane.

10. What modification needs to be made in the brute-force algorithm for the convex-hull problem to handle more than two points on the same straight line?

11. Write a program implementing the brute-force algorithm for the convex-hull problem.

12. Consider the following small instance of the linear programming problem:

$$\begin{array}{rl} \text{maximize} & 3x + 5y \\ \text{subject to} & x + y \leq 4 \\ & x + 3y \leq 6 \\ & x \geq 0, \;\; y \geq 0 \end{array}$$

a. Sketch, in the Cartesian plane, the problem's ***feasible region*** defined as the set of points satisfying all the problem's constraints.

b. Identify the region's extreme points.

c. Solve this optimization problem by using the following theorem: A linear programming problem with a nonempty bounded feasible region always has a solution, which can be found at one of the extreme points of its feasible region.

Hints to Exercises 3.3

1. You may want to consider two versions of the answer: without taking into account the comparison and assignments in the algorithm's innermost loop and with them.

2. Sorting n real numbers can be done in $O(n \log n)$ time.

3. a. Solving the problem for $n = 2$ and $n = 3$ should lead you to the critical insight.

   b. Where would you put the post office if it would not have to be at one of the village locations?

4. a. Check requirements (i)–(iii) by using basic properties of absolute values.

   b. For the Manhattan distance, the points in question are defined by the equation $|x-0|+|y-0| = 1$. You can start by sketching the points in the positive quadrant of the coordinate system (i.e., the points for which $x, y \geq 0$) and then sketch the rest by using the symmetries.

   c. The assertion is false. You can choose, say, $p_1(0, 0)$ and $p_2(1, 0)$ and find p_3 to complete a counterexample.

5. a. Prove that the Hamming distance does satisfy the three axioms of a distance metric.

   b. Your answer should include two parameters.

6. True; prove it by mathematical induction.

7. Your answer should be a function of two parameters: n and k. A special case of this problem (for $k = 2$) was solved in the text.

8. Review the examples given in the section.

9. Some of the extreme points of a convex hull are easier to find than others.

10. If there are other points of a given set on the straight line through p_i and p_j, which of all these points need to be preserved for further processing?

11. Your program should work for any set of n distinct points, including sets with many colinear points.

12. a. The set of points satisfying inequality $ax + by \leq c$ is the half-plane of the points on one side of the straight line $ax + by = c$, including all the points on the line itself. Sketch such a half-plane for each of the inequalities and find their intersection.

    b. The extreme points are the vertices of the polygon obtained in part (a).

c. Compute and compare the values of the objective function at the extreme points.

Solutions to Exercises 3.3

1. If we take into account only the arithmetical operations involved into computing the Euclidean distance between two points versus computing its square, we will end up with the following estimate of the time ratio (both for the algorithm's innermost loop and the entire algorithm): $(2 + 3 + 10)/(2 + 3) = 3$.

If we also take into account the comparison and assignment, we get the time ratio estimate as $(2 + 3 + 10 + 2)/(2 + 3 + 2) \approx 2.4$.

2. Sort the numbers in ascending order, compute the differences between adjacent numbers in the sorted list, and find the smallest such difference. If sorting is done in $O(n \log n)$ time, the running time of the entire algorithm will be in
$$O(n \log n) + \Theta(n) + \Theta(n) = O(n \log n).$$

3. a. If we put the post office at location x_i, the average distance between it and all the points $x_1 < x_2 < ... < x_n$ is given by the formula $\frac{1}{n} \sum_{j=1}^{n} |x_j - x_i|$. Since the number of points n stays the same, we can ignore the multiple $\frac{1}{n}$ and minimize $\sum_{j=1}^{n} |x_j - x_i|$. We'll have to consider the cases of even and odd n separately.

Let n be even. Consider first the case of $n = 2$. The sum $|x_1 - x| + |x_2 - x|$ is equal to $x_2 - x_1$, the length of the interval with the endpoints at x_1 and x_2, for any point x of this interval (including the endpoints), and it is larger than $x_2 - x_1$ for any point x outside of this interval. This implies that for any even n, the sum

$$\sum_{j=1}^{n} |x_j - x| = [|x_1 - x| + |x_n - x|] + [|x_2 - x| + |x_{n-1} - x|] + \cdots + [|x_{n/2} - x| + |x_{n/2+1} - x|]$$

is minimized when x belongs to each of the intervals $[x_1, x_n] \supset [x_2, x_{n-1}] \supset ... \supset [x_{n/2}, x_{n/2+1}]$. If x must be one of the points given, either $x_{n/2}$ or $x_{n/2+1}$ solves the problem.

Let $n > 1$ be odd. Then, the sum $\sum_{j=1}^{n} |x_j - x|$ is minimized when $x = x_{\lceil n/2 \rceil}$, the point for which the number of the given points to the left of it is equal to the number of the given points to the right of it.

Note that the point $x_{\lceil n/2 \rceil}$—the $\lceil n/2 \rceil$th smallest called the *median*—solves the problem for even n's as well. For a sorted list implemented as an array, the median can be found in $\Theta(1)$ time by simply returning the $\lceil n/2 \rceil$th element of the array. (Section 5.6 provides a more general discussion of algorithms for computing the median.)

b. Assuming that the points $x_1, x_2, ..., x_n$ are given in increasing order, the answer is the point x_i that is the closest to $m = (x_1 + x_n)/2$, the middle point between x_1 and x_n. (The middle point would be the obvious solution if the

post-post office didn't have to be at one of the given locations.) Indeed, if we put the post office at any location x_i to the left of m, the longest distance from a village to the post office would be $x_n - x_i$; this distance is minimal for the rightmost among such points. If we put the post office at any location x_i to the right of m, the longest distance from a village to the post office would be $x_i - x_1$; this distance is minimal for the leftmost among such points.

Algorithm *PostOffice1*(P)
//Input: List P of n $(n \geq 2)$ points $x_1, x_2, ..., x_n$ in increasing order
//Output: Point x_i that minimizes $\max\limits_{1 \leq j \leq n} |x_j - x_i|$ among all $x_1, x_2, ..., x_n$
$m \leftarrow (x_1 + x_n)/2$
$i \leftarrow 1$
while $x_i < m$ **do**
$\quad i \leftarrow i + 1$
if $x_i - x_1 < x_n - x_{i-1}$
$\quad$ **return** x_i
else return x_{i-1}

The time efficiency of this algorithm is $O(n)$.

4. a. For $d_M(p_1, p_2) = |x_1 - x_2| + |y_1 - y_2|$, we have the following:

(i) $d_M(p_1, p_2) = |x_1 - x_2| + |y_1 - y_2| \geq 0$ and $d_M(p_1, p_2) = 0$ if and only if both $x_1 = x_2$ and $y_1 = y_2$, i.e., P_1 and P_2 coincide.

(ii) $d_M(p_1, p_2) = |x_1 - x_2| + |y_1 - y_2| = |x_2 - x_1| + |y_2 - y_1|$
$= d_M(p_2, p_1)$.

(iii) $d_M(p_1, p_2) = |x_1 - x_2| + |y_1 - y_2|$
$= |(x_1 - x_3) + (x_3 - x_2)| + |(y_1 - y_3) + (y_3 - y_2)|$
$\leq |x_1 - x_3| + |x_3 - x_2| + |y_1 - y_3| + |y_3 - y_2| = d(p_1, p_3) + d(p_3, p_2)$.

b. For the Manhattan distance, the points in question are defined by the equation

$$|x - 0| + |y - 0| = 1, \text{ i.e., } |x| + |y| = 1.$$

The graph of this equation is the boundary of the square with its vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(-1, -1)$.

For the Euclidean distance, the points in question are defined by the equation

$$\sqrt{(x-0)^2 + (y-0)^2} = 1, \text{ i.e., } x^2 + y^2 = 1.$$

The graph of this equation is the circumference of radius 1 and the center at $(0, 0)$.

c. False. Consider points $p_1(0, 0)$, $p_2(1, 0)$, and, say, $p_3(\frac{1}{2}, \frac{3}{4})$. Then

$$d_E(p_1, p_2) = 1 \text{ and } d_E(p_3, p_1) = d_E(p_3, p_2) = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2} < 1.$$

Therefore, for the Euclidean distance, the two closest points are either p_1 and p_3 or p_2 and p_3. For the Manhattan distance, we have

$$d_M(p_1, p_2) = 1 \text{ and } d_M(p_3, p_1) = d_M(p_3, p_2) = \frac{1}{2} + \frac{3}{4} = \frac{5}{4} > 1.$$

Therefore, for the Manhattan distance, the two closest points are p_1 and p_2.

5. a. Since the first two axioms of a metric is obviously satisfied for the Hamming distance, only the third one—the triangle inequality—needs a proof. It can be obtained by mathematical induction on the string length m. If $m = 1$, the inequality $d_H(S_1, S_2) \le d_H(S_1, S_3) + d_H(S_3, S_2)$ holds for any one-character strings S_1, S_2, and S_3: if $S_1 = S_2$, the left-hand side is equal to 0; if $S_1 \neq S_2$, the left-hand side is equal to 1 and the right-hand side is greater than or equal to 1 because S_3 cannot be the same as both S_1 and S_2. For the inductive step, assume that the triangle inequality holds for any three strings of length m and consider three arbitrary strings $S_i = S_i^{'}c_i$, where $i = 1, 2, 3$ and $S_i^{'}$'s are strings of length m and c_i's are their last characters. Then

$$\begin{aligned} d_H(S_1, S_2) &= d_H(S_1^{'}, S_2^{'}) + d_H(c_1, c_2) \\ &\le d_H(S_1^{'}, S_3^{'}) + d_H(S_3^{'}, S_2^{'}) + d_H(c_1, c_3) + d_H(c_3, c_2) \\ &= [d_H(S_1^{'}, S_3^{'}) + d_H(c_1, c_3)] + [d_H(S_3^{'}, S_2^{'}) + d_H(c_3, c_2)] \\ &= d_H(S_1, S_3) + d_H(S_3, S_2). \end{aligned}$$

b. Since the basic operation of the algorithm is comparing two characters in the strings of length m, the worst-case time efficiency class will be $\Theta(mn^2)$.

6. We'll prove by induction that there will always remain at least one person not hit by a pie. The basis step is easy: If $n = 3$, the two persons with the smallest pairwise distance between them throw at each other, while the third person throws at one of them (whoever is closer). Therefore, this third person remains "unharmed".
For the inductive step, assume that the assertion is true for odd $n \ge 3$, and consider $n + 2$ persons. Again, the two persons with the smallest pairwise distance between them (the closest pair) throw at each other.

Consider two possible cases as follows. If the remaining n persons all throw at one another, at least one of them remains "unharmed" by the inductive assumption. If at least one of the remaining n persons throws at one of the closest pair, among the remaining $n-1$ persons, at most $n-1$ pies are thrown at one another, and hence at least one person must remain "unharmed" because there is not enough pies to hit everybody in that group. This completes the proof.

Note: The problem is from the paper by L. Carmony titled "Odd pie fights," *Mathematics Teacher*, vol. 72, no. 1, 1979, 61–64.

7. The number of squarings will be

$$\begin{aligned} C(n,k) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{s=1}^{k} 1 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} k = k \sum_{i=1}^{n-1} (n-i) \\ &= k[(n-1) + (n-2) + \cdots + 1] = \frac{k(n-1)n}{2} \in \Theta(kn^2). \end{aligned}$$

8. a. The convex hull of a line segment is the line segment itself; its extreme points are the endpoints of the segment.

   b. The convex hull of a square is the square itself; its extreme points are the four vertices of the square.

   c. The convex hull of the boundary of a square is the region comprised of the points within that boundary and on the boundary itself; its extreme points are the four vertices of the square.

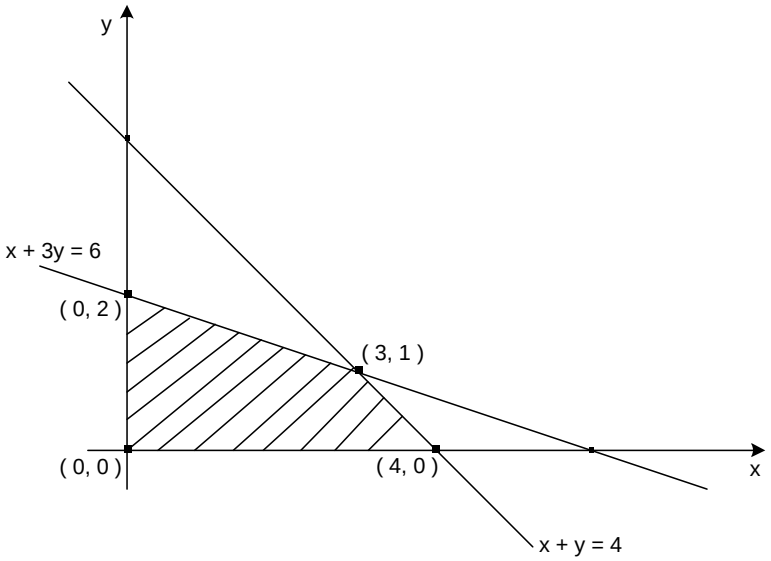
   d. The convex hull of a straight line is the straight line itself. It doesn't have any extreme points.

9. Find the point with the smallest x coordinate; if there are several such points, find the one with the smallest y coordinate among them. Similarly, find the point with the largest x coordinate; if there are several such points, find the one with the largest y coordinate among them. (Note that it's more efficient to look for the smallest and largest x coordinates on the same pass through the list of the points given. While it does not change the linear efficiency class of the algorithm, it can reduce the total number of comparisons to about $1.5n$.)

10. If there are other points of a given set on the straight line through p_i and p_j (while all the other points of the set lie on the same side of the line), a line segment of the convex hull's boundary will have its end points at the two farthest set points on the line. All the other points on the line can be eliminated from further processing.

11. n/a

12. a. Here is a sketch of the feasible region in question:



b. The extreme points are: $(0, 0)$, (4,0), (3,1), and (0,2).

c.

Extreme point	Value of $3x + 5y$
(0,0)	0
(4,0)	12
(3,1)	14
(0,2)	10

So, the optimal solution is (3, 1), with the maximum value of $3x + 5y$ equal to 14. (Note: This instance of the linear programming problem is discussed further in Section 10.1.)

Exercises 3.4

1. a. Assuming that each tour can be generated in constant time, what will be the efficiency class of the exhaustive-search algorithm outlined in the text for the traveling salesman problem?

   b. If this algorithm is programmed on a computer that makes 10 billion additions per second, estimate the maximum number of cities for which the problem can be solved in (i) one hour. (ii) 24-hours. (iii) one year. (iv) one century.

2. Outline an exhaustive-search algorithm for the Hamiltonian circuit problem.

3. Outline an algorithm to determine whether a connected graph represented by its adjacency matrix has a Eulerian circuit. What is the efficiency class of your algorithm?

4. Complete the application of exhaustive search to the instance of the assignment problem started in the text.

5. Give an example of the assignment problem whose optimal solution does not include the smallest element of its cost matrix.

6. Consider the ***partition problem***: given n positive integers, partition them into two disjoint subsets with the same sum of their elements. (Of course, the problem does not always have a solution.) Design an exhaustive search algorithm for this problem. Try to minimize the number of subsets the algorithm needs to generate.

7. Consider the ***clique problem***: given a graph G and a positive integer k, determine whether the graph contains a ***clique*** of size k, i.e., a complete subgraph of k vertices. Design an exhaustive-search algorithm for this problem.

8. Explain how exhaustive search can be applied to the sorting problem and determine the efficiency class of such an algorithm.

9. *Eight-queens problem* Consider the classic puzzle of placing eight queens on an 8×8 chessboard so that no two queens are in the same row or in the same column or on the same diagonal. How many ways are there so that

   a. no two queens are on the same square?

   b. no two queens are in the same row?

   c. no two queens are in the same row or in the same column?

Also estimate how long it would take to find all the solutions to the problem by exhaustive search based on each of these approaches on a computer capable of checking 10 billion positions per second.

10. A magic square of order n is an arrangement of the integers from 1 to n^2 in an $n \times n$ matrix, with each number occurring exactly once, so that each row, each column, and each main diagonal has the same sum.

    a. Prove that if a magic square of order n exists, the sum in question must be equal to $n(n^2 + 1)/2$.

    b. Design an exhaustive search algorithm for generating all magic squares of order n.

    c. Go to the Internet or your library and find a better algorithm for generating magic squares.

    d. Implement the two algorithms—the exhaustive search and the one you have found—and run an experiment to determine the largest value of n for which each of the algorithms is able to find a magic square of order n in less than 1 minute of your computer's time.

11. *Famous alphametic* A puzzle in which the digits in a correct mathematical expression, such as a sum, are replaced by letters is called ***cryptarithm***; if, in addition, the puzzle's words make sense, it is said to be an ***alphametic***. The most well-known alphametic was published by the renowned British puzzlist Henry E. Dudeney (1857-1930):

$$\begin{array}{r} \text{S E N D} \\ +\ \text{M O R E} \\ \hline \text{M O N E Y} \end{array}$$

    Two conditions are assumed: First, the correspondence between letters and digits is one-to-one, that is each letter represents one digit only and different letters represent different digits. Second, the digit zero does not appear as the left-most digit in any of the numbers. To solve an alphametic means to find which digit each letter represents. Note that a solution's uniqueness cannot be assumed and has to be verified by the solver.

    a. Write a program for solving cryptarithms by exhaustive search. Assume that a given cryptarithm is a sum of two words.

    b. Solve Dudeney'e puzzle the way it was expected to be solved when it was first published in 1924.

Hints to Exercises 3.4

1. a. Identify the algorithm's basic operation and count the number of times it will be executed.

   b. For each of the time amounts given, find the largest value of n for which this limit will not be exceeded.

2. How different is the traveling salesman problem from the problem of finding a Hamiltonian circuit?

3. Your algorithm should check the well-known conditions that are both necessary and sufficient for the existence of a Eulerian circuit in a connected graph.

4. Generate the remaining $4! - 6 = 18$ possible assignments, compute their costs, and find the one with the smallest cost.

5. Make the size of your counterexample as small as possible.

6. Rephrase the problem so that the sum of elements in one subset, rather than two, needs to be checked on each try of a possible partition.

7. Follow the definitions of a clique and of an exhaustive-search algorithm.

8. Try all possible orderings of the elements given.

9. Use common formulas of elementary combinatorics.

10. a. Add all the elements in the magic square in two different ways.

    b. What combinatorial objects do you have to generate here?

11. a. For testing, you may use alphametic collections available on the Internet.

    b. Given the absence of electronic computers in 1924, you must refrain here from using the Internet.

Solutions to Exercises 3.4

1. a. $\Theta(n!)$
   For each tour (a sequence of $n+1$ cities), one needs n additions to compute the tour's length. Hence, the total number of additions $A(n)$ will be n times the total number of tours considered, i.e., $n*\frac{1}{2}(n-1)! = \frac{1}{2}n! \in \Theta(n!)$.

   b. (i) $n_{\max} = 16$; (ii) $n_{\max} = 17$; (iii) $n_{\max} = 19$; (iv) $n_{\max} = 21$.
   Given the answer to part a, we have to find the largest value of n such that
   $$\frac{1}{2}n!10^{-10} \leq t$$
   where t is the time available (in seconds). Thus, for $t = 1\text{hr} = 3.6*10^3\text{sec}$, we get the inequality
   $$n! \leq 2 * 10^{10}t = 7.2 * 10^{13}.$$
   The largest value of n for which this inequality holds is 16 (since $16! \approx 2.1 * 10^{13}$ and $17! \approx 3.6 * 10^{14}$).

   For the other given values of t, the answers can be obtained in the same manner.

2. The problem of finding a Hamiltonian circuit is very similar to the traveling salesman problem. Generate permutations of n vertices that start and end with, say, the first vertex, and check whether every pair of successive vertices in a current permutation are connected by an edge. If it's the case, the current permutation represents a Hamiltonian circuit, otherwise, a next permutation needs to be generated.

3. A connected graph has a Eulerian circuit if and only if all its vertices have even degrees. An algorithm should check this condition until either an odd vertex is encountered (then a Eulerian circuit doesn't exist) or all the vertices turn out to be even (then a Eulerian circuit must exist). For a graph (with no loops) represented by its $n \times n$ adjacency matrix, the degree of a vertex is the number of ones in the vertex's row. Thus, computing its degree will take the $\Theta(n)$ time, checking whether it's even will take $\Theta(1)$ time, and it will be done between 1 and n times. Hence, the algorithm's efficiency will be in $O(n^2)$.

4. The following assignments were generated in the chapter's text:
   $$C = \begin{bmatrix} 9 & 2 & 7 & 8 \\ 6 & 4 & 3 & 7 \\ 5 & 8 & 1 & 8 \\ 7 & 6 & 9 & 4 \end{bmatrix}$$
1, 2, 3, 4	cost = 9+4+1+4 = 18
1, 2, 4, 3	cost = 9+4+8+9 = 30
1, 3, 2, 4	cost = 9+3+8+4 = 24
1, 3, 4, 2	cost = 9+3+8+6 = 26
1, 4, 2, 3	cost = 9+7+8+9 = 33
1, 4, 3, 2	cost = 9+7+1+6 = 23

   etc.

The remaining ones are

2, 1, 3, 4	cost = 2+6+1+4 = 13
2, 1, 4, 3	cost = 2+6+8+9 = 25
2, 3, 1, 4	cost = 2+3+5+4 = 14
2, 3, 4, 1	cost = 2+3+8+7 = 20
2, 4, 1, 3	cost = 2+7+5+9 = 23
2, 4, 3, 1	cost = 2+7+1+7 = 17
3, 1, 2, 4	cost = 7+6+8+4 = 25
3, 1, 4, 2	cost = 7+6+8+6 = 27
3, 2, 1, 4	cost = 7+4+5+4 = 20
3, 2, 4, 1	cost = 7+4+8+7 = 26
3, 4, 1, 2	cost = 7+7+5+6 = 25
3, 4, 2, 1	cost = 7+7+8+7 = 29
4, 1, 2, 3	cost = 8+6+8+9 = 31
4, 1, 3, 2	cost = 8+6+1+6 = 21
4, 2, 1, 3	cost = 8+4+5+9 = 26
4, 2, 3, 1	cost = 8+4+1+7 = 20
4, 3, 1, 2	cost = 8+3+5+6 = 22
4, 3, 2, 1	cost = 8+3+8+7 = 26

The optimal solution is: Person 1 to Job 2, Person 2 to Job 1, Person 3 to Job 3, and Person 4 to Job 4, with the total (minimal) cost of the assignment being 13.

5. Here is a very simple example:

$$\begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}$$

6. Start by computing the sum S of the numbers given. If S is odd, stop because the problem doesn't have a solution. If S is even, generate the subsets until either a subset whose elements' sum is $S/2$ is encountered or no more subsets are left. Note that it will suffice to generate only subsets with no more than $n/2$ elements.

7. Generate a subset of k vertices and check whether every pair of vertices in the subset is connected by an edge. If it's true, stop (the subset is a clique); otherwise, generate the next subset.

8. Generate a permutation of the elements given and check whether they are ordered as required by comparing values of its consecutive elements. If they are, stop; otherwise, generate the next permutation. Since the

number of permutations of n items is equal to $n!$ and checking a permutation requires up to $n-1$ comparisons, the algorithm's efficiency class is in $O(n!(n-1)) = O((n+1)!)$.

9. The number of different positions of eight queens on the 8×8 board is equal to

   a. $C(64, 8) = 4,426,165,368$ if no two queens are on the same square.

   b. $8^8 = 16,777,216$ if no two queens are in the same row.

   c. $8! = 40,320$ if no two queens are in the same row or in the same column.

10. a. Let s be the sum of the numbers in each row of an $n \times n$ magic square. Let us add all the numbers in rows 1 through n. We will get the following equality:

$$sn = 1+2+\cdots+n^2, \text{ i.e., } sn = \frac{n^2(n^2+1)}{2}, \text{ which implies } s = \frac{n(n^2+1)}{2}.$$

    b. Number positions in an $n \times n$ matrix from 1 through n^2. Generate a permutation of the numbers 1 through n^2, put them in the corresponding positions of the matrix, and check the magic-square equality (proved in part (a)) for every row, every column, and each of the two main diagonals of the matrix.

    c. n/a

    d. n/a

11. a. Since the letter-digit correspondence must be one-to-one and there are only ten distinct decimal digits, the exhaustive search needs to check $P(10, k) = 10!/(10-k)!$ possible substitutions, where k is the number of distinct letters in the input. (The requirement that the first letter of a word cannot represent 0 can be used to reduce this number further.) Thus a program should run in a quite reasonable amount of time on today's computers. Note that rather than checking two cases—with and without a "1-carry"—for each of the decimal positions, the program can check just one equality, which stems from the definition of the decimal number system. For Dudeney's alphametic, for example, this equality is 1000(S+M) + 100(E+O) + 10(N+R) + (D+E) = 10000M + 1000O + 100N + 10E + Y

    b. Here is a "computerless" solution to this classic problem. First, notice that M must be 1. (Since both S and M are not larger than 9, their

sum, even if increased by 1 because of the carry from the hundred column, must be less than 20.) We will have to rely on some further insights into specifics of the problem. The leftmost digits of the addends imply one of the two possibilities: either S + M = 10 + O (if there was no carry from the hundred column) or 1 + S + M = 10 + O (if there was such a carry). First, let us pursue the former of the two possibilities. Since M = 1, S $\leq$ 9 and O $\geq$ 0, the equation S + 1 = 10 + O has only one solution: S = 9 and O = 0. This leaves us with

$$\begin{array}{r} \text{E N D} \\ +\ \text{O R E} \\ \hline \text{N E Y} \end{array}$$

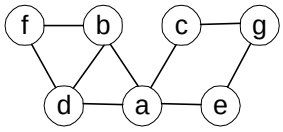
Since we deal here with the case of no carry from the hundreds and E and N must be distinct, the only possibility is a carry from the tens: 1 + E = N and either N + R = 10 + E (if there was no carry from the rightmost column) or 1 + N + R = 10 + E (if there was such a carry). The first combination leads to a contradiction: Substituting 1 + E for N into N + R = 10 + E, we obtain R = 9, which is incompatible with the same digit already represented by S. The second combination of 1 + E = N and 1 + N + R = 10 + E implies, after substituting the first of these equations into the second one, R = 8. Note that the only remaining digit values still unassigned are 2, 3, 4, 5, 6, and 7. Finally, for the righmost column, we have the equation D + E = 10 + Y. But 10 + Y $\geq$ 12, because the smallest unassigned digit value is 2 while D + E $\leq$ 12 because the two largest unassigned digit values are 6 and 7 and E < N. Thus, D + E = 10 + Y = 12. Hence Y = 2 and D + E = 12. The only pair of still unassigned digit values that add up to 12, 5 and 7, must be assigned to E and D, respectively, since doing this the other way (E = 7, D = 5) would imply N = E + 1 = 8, which is already represented by R. Thus, we found the following solution to the puzzle:

$$\begin{array}{r} 9\ 5\ 6\ 7 \\ +\ 1\ 0\ 8\ 5 \\ \hline 1\ 0\ 6\ 5\ 2 \end{array}$$

Is this the only solution? To answer this question, we should pursue the carry possibility from the hundred column to the thousand column (see above). Then 1 + S + M = 10 + O or, since M =1, S = 8 + O. But S $\leq$ 9, while 8 + O $\geq$ 10 since O $\geq$ 2. Hence the last equation has no solutions in our domain. This proves that the puzzle has no other solutions.

Exercises 3.5

1. Consider the following graph.



   a. Write down the adjacency matrix and adjacency lists specifying this graph. (Assume that the matrix rows and columns and vertices in the adjacency lists follow in the alphabetical order of the vertex labels.)

   b. Starting at vertex a and resolving ties by the vertex alphabetical order, traverse the graph by depth-first search and construct the corresponding depth-first search tree. Give the order in which the vertices were reached for the first time (pushed onto the traversal stack) and the order in which the vertices became dead ends (popped off the stack).

2. If we define sparse graphs as graphs for which $|E| \in O(|V|)$, which implementation of DFS will have a better time efficiency for such graphs, the one that uses the adjacency matrix or the one that uses the adjacency lists?

3. Let G be a graph with n vertices and m edges.

   a. True or false: All its DFS forests (for traversals starting at different vertices) will have the same number of trees?

   b. True or false: All its DFS forests will have the same number of tree edges and the same number of back edges?

4. Traverse the graph of Problem 1 by breadth-first search and construct the corresponding breadth-first search tree. Start the traversal at vertex a and resolve ties by the vertex alphabetical order.

5. Prove that a cross edge in a BFS tree of an undirected graph can connect vertices only on either the same level or on two adjacent levels of a BFS tree.

6. a. Explain how one can check a graph's acyclicity by using breadth-first search.

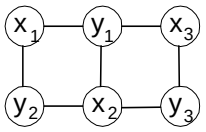
   b. Does either of the two traversals—DFS or BFS—always find a cycle faster than the other? If you answer yes, indicate which of them is better and explain why it is the case; if you answer no, give two examples supporting your answer.

7. Explain how one can identify connected components of a graph by using

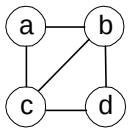
   a. a depth-first search.

   b. a breadth-first search.

8. A graph is said to be ***bipartite*** if all its vertices can be partitioned into two disjoint subsets X and Y so that every edge connects a vertex in X with a vertex in Y. (One can also say that a graph is bipartite if its vertices can be colored in two colors so that every edge has its vertices colored in different colors; such graphs are also called ***2-colorable***). For example, graph (i) is bipartite whereas graph (ii) is not.



(i)



(ii)

   a. Design a DFS-based algorithm for checking whether a graph is bipartite.
   b. Design a BFS-based algorithm for checking whether a graph is bipartite.

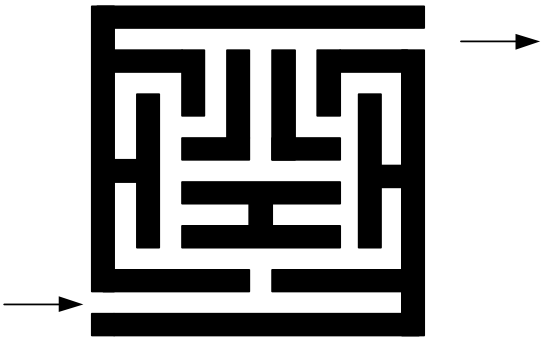
9. Write a program that, for a given graph, outputs

   (a) vertices of each connected component.

   (b) its cycle or a message that the graph is acyclic.

10. One can model a maze by having a vertex for a starting point, a finishing point, dead ends, and all the points in the maze where more than one path can be taken, and then connecting the vertices according to the paths in the maze.

    a. Construct such a graph for the following maze.



b. Which traversal— DFS or BFS— would you use if you found yourself in a maze and why?

11. *Three Jugs* Siméon Denis Poisson (1781–1840), a famous French mathematician and physicist, is said to have become interested in mathematics after encountering some version of the following old puzzle. Given an 8-pint jug full of water and two empty jugs of 5- and 3-pint capacity, get exactly 4 pints of water in one of the jugs by completely filling up and/or emptying jugs into others. Solve this puzzle by using breadth-first search.

Hints to Exercises 3.5

1. a. Use the definitions of the adjacency matrix and adjacency lists given in Section 1.4.

   b. Perform the DFS traversal the same way it is done for another graph in the text (see Fig. 3.10).

2. Compare the efficiency classes of the two versions of DFS for sparse graphs.

3. a. What is the number of such trees equal to?

   b. Answer this question for connected graphs first.

4. Perform the BFS traversal the same way it is done in the text (see Fig. 3.11).

5. You may use the fact that the level of a vertex in a BFS tree indicates the number of edges in the shortest (minimum-edge) path from the root to that vertex.

6. a. What property of a BFS forest indicates a cycle's presence? (The answer is similar to the one for a DFS forest.)

   b. The answer is "no". Find two examples supporting this answer.

7. Given the fact that both traversals can reach a new vertex if and only if it is adjacent to one of the previously visited vertices, which vertices will be visited by the time either traversal halts (i.e., its stack or queue becomes empty)?

8. Use a DFS forest and a BFS forest for parts (a) and (b), respectively.

9. Use either DFS or BFS.

10. a. Follow the instructions of the problem's statement.

    b. Trying both traversals should lead you to a correct answer very fast.

11. You can apply BFS without an explicit sketch of a graph representing the states of the puzzle.

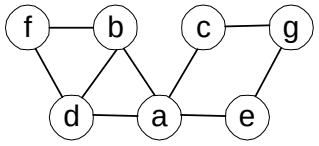
Solutions to Exercises 3.5

1. a. Here are the adjacency matrix and adjacency lists for the graph in question:

	a	b	c	d	e	f	g
a	0	1	1	1	1	0	0
b	1	0	0	1	0	1	0
c	1	0	0	0	0	0	1
d	1	1	0	0	0	1	0
e	1	0	0	0	0	0	1
f	0	1	0	1	0	0	0
g	0	0	1	0	1	0	0

$a \rightarrow b \rightarrow c \rightarrow d \rightarrow e$
$b \rightarrow a \rightarrow d \rightarrow f$
$c \rightarrow a \rightarrow g$
$d \rightarrow a \rightarrow b \rightarrow f$
$e \rightarrow a \rightarrow g$
$f \rightarrow b \rightarrow d$
$g \rightarrow c \rightarrow e$

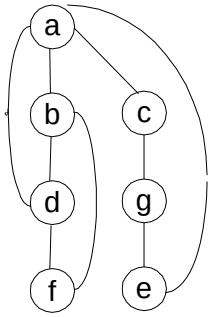
b. See below: (i) the graph; (ii) the traversal's stack (the first subscript number indicates the order in which the vertex was visited, i.e., pushed onto the stack, the second one indicates the order in which it became a dead-end, i.e., popped off the stack); (iii) the DFS tree (with the tree edges shown with solid lines and the back edges shown with dashed lines).



(i)

$f_{4,1}$ $e_{7,4}$
$d_{3,2}$ $g_{6,5}$
$b_{2,3}$ $c_{5,6}$
$a_{1,7}$

(ii)



(iii)

2. The time efficiency of DFS is $\Theta(|V|^2)$ for the adjacency matrix representation and $\Theta(|V| + |E|)$ for the adjacency lists representation, respectively. If $|E| \in O(|V|)$, the former remains $\Theta(|V|^2)$ while the latter becomes $\Theta(|V|)$. Hence, for sparse graphs, the adjacency lists version of DFS is more efficient than the adjacency matrix version.

3. a. The number of DFS trees is equal to the number of connected components of the graph. Hence, it will be the same for all DFS traversals of the graph.

b. For a connected (undirected) graph with $|V|$ vertices, the number of

tree edges $|E^{(tree)}|$ in a DFS tree will be $|V|-1$ and, hence, the number of back edges $|E^{(back)}|$ will be the total number of edges minus the number of tree edges: $|E|-(|V|-1) = |E|-|V|+1$. Therefore, it will be independent from a particular DFS traversal of the same graph. This observation can be extended to an arbitrary graph with $|C|$ connected components by applying this reasoning to each of its connected components:

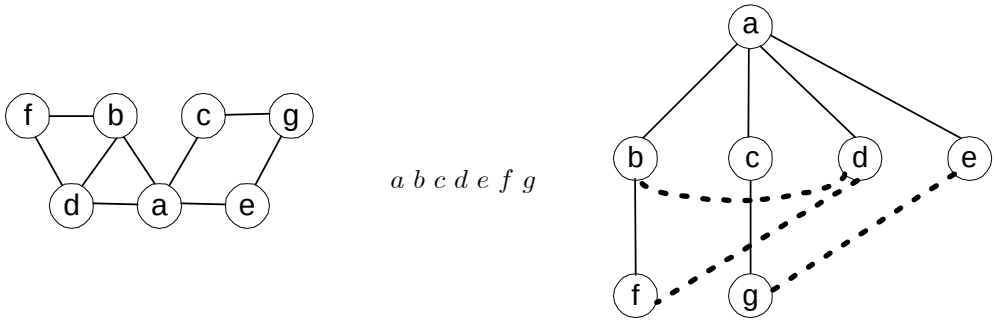
$$|E^{(tree)}| = \sum_{c=1}^{|C|} |E_c^{(tree)}| = \sum_{c=1}^{|C|} (|V_c| - 1) = \sum_{c=1}^{|C|} |V_c| - \sum_{c=1}^{|C|} 1 = |V| - |C|$$

and

$$|E^{(back)}| = |E| - |E^{(tree)}| = |E| - (|V| - |C|) = |E| - |V| + |C|,$$

where $|E_c^{(tree)}|$ and $|V_c|$ are the numbers of tree edges and vertices in the cth connected component, respectively.

4. Here is the result of the BFS traversal of the graph of Problem 1:

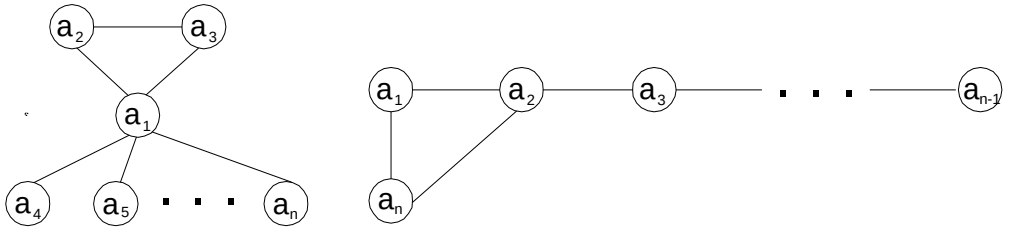


(i) the graph; (ii) the traversal's queue; (iii) the tree (the tree and cross edges are shown with solid and dotted lines, respectively).

5. We'll prove the assertion in question by contradiction. Assume that a BFS tree of some undirected graph has a cross edge connecting two vertices u and v such that $level[u] \geq level[v] + 2$. But $level[u] = d[u]$ and $level[v] = d[v]$, where $d[u]$ and $d[v]$ are the lengths of the minimum-edge paths from the root to vertices u and v, respectively. Hence, we have $d[u] \geq d[v] + 2$. The last inequality contradicts the fact that $d[u]$ is the length of the minimum-edge path from the root to vertex u because the minimum-edge path of length $d[v]$ from the root to vertex v followed by edge (v, u) has fewer edges than $d[u]$.

6. a. A graph has a cycle if and only if its BFS forest has a cross edge.

   b. Both traversals, DFS and BFS, can be used for checking a graph's acyclicity. For some graphs, a DFS traversal discovers a back edge in its DFS forest sooner than a BFS traversal discovers a cross edge (see example (i) below); for others the exactly opposite is the case (see example (ii) below).



(i) (ii)

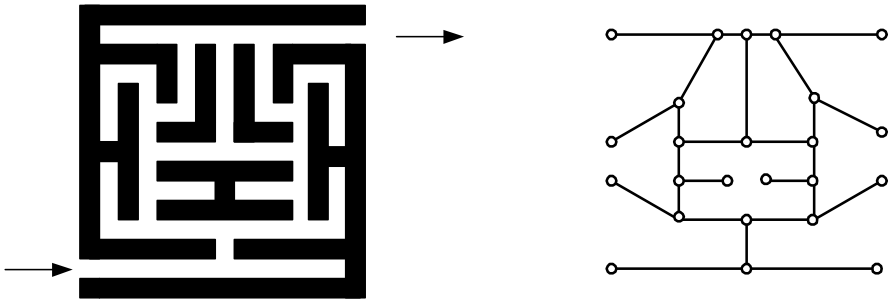
7. Start a DFS (or BFS) traversal at an arbitrary vertex and mark the visited vertices with 1. By the time the traversal's stack (queue) becomes empty, all the vertices in the same connected component as the starting vertex, and only they, will have been marked with 1. If there are unvisited vertices left, restart the traversal at one of them and mark all the vertices being visited with 2, and so on until no unvisited vertices are left.

8. a. Let F be a DFS forest of a graph. It is not difficult to see that F is 2-colorable if and only if there is no back edge connecting two vertices both on odd levels or both on even levels. It is this property that a DFS traversal needs to verify. Note that a DFS traversal can mark vertices as even or odd when it reaches them for the first time.

   b. Similarly to part (a), a graph is 2-colorable if and only if its BFS forest has no cross edge connecting vertices on the same level. Use a BFS traversal to check whether or not such a cross edge exists.

9. n/a

10. a. Here is the maze and a graph representing it:



b. DFS is much more convenient for going through a maze than BFS. When DFS moves to a next vertex, it is connected to a current vertex by an edge (i.e., "close nearby" in the physical maze), which is not generally the case for BFS. In fact, DFS can be considered a generalization of an ancient right-hand rule for maze traversal: go through the maze in such a way so that your right hand is always touching a wall.

11. The sequence shown in the figure below solves the puzzle in six steps, which is the minimum.

Step#	8-pint jug	5-pint jug	3-pint jug
	8	0	0
1	3	5	0
2	3	2	3
3	6	2	0
4	6	0	2
5	1	5	2
6	1	4	3

Solution to the Three Jugs puzzle

Although the solution can be obtained by trial and error, there is a systematic way of getting to it. We can represent a state of the jars by a triple of nonnegative integers indicating the amount of water in the 3-pint, 5-pint, and 8-pint jugs, respectively. Thus, we start with the triple 008. We will consider all legal transformations from a current state of the jugs to new possible states in the BFS manner. We initialize the queue with the initial state triple 008 and repeat the following until a desired state—a triple containing a 4—is encountered for the first time. For the state at the front of the queue, label all the *new*

states reachable from it by the triple of the front state, add them to the queue, and then delete the front state from the queue. After a desired state is reached for the first time, follow the labels backwards to get the shortest sequence of transformations that solve the puzzle.

The application of this algorithm to the puzzle's data yields the following sequence of the queue states, where the aforementioned labels are shown as subscripts the first time the new triples are added to the queue:

$$008 \mid 305_{008}, 053_{008} \mid 053,\ 035_{305}, 350_{305} \mid 035,\ 350,\ 323_{053} \mid 350,\ 323,\ 332_{035} \mid 323,\ 332 \mid$$
$$332,\ 026_{323} \mid 026,\ 152_{332} \mid 152,\ 206_{026} \mid 206,\ 107_{152} \mid 107,\ 251_{206} \mid 251,\ 017_{107} \mid 017,\ 341_{251}$$

Tracing the labels from that of 341 backwards, we get the following transformation sequence that solves the puzzle in the minimum number of six steps:

$$008 \to 053 \to 323 \to 026 \to 206 \to 251 \to 341.$$