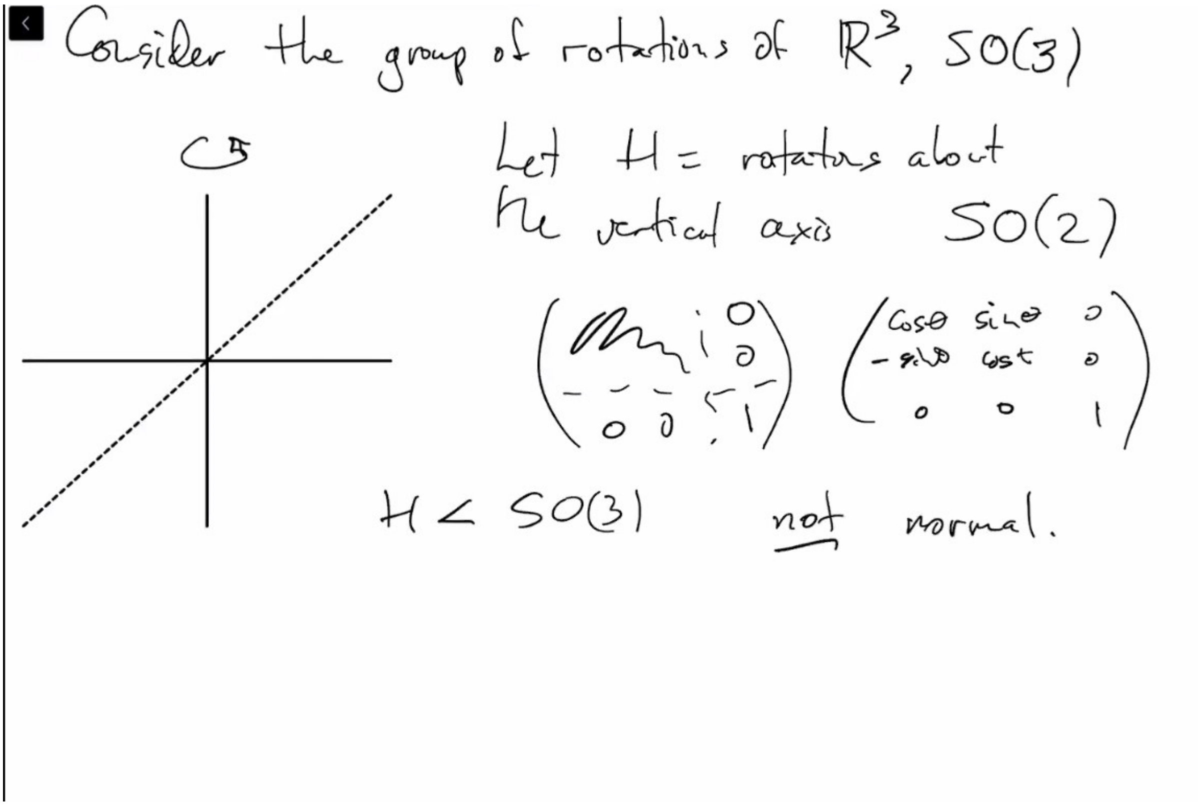
Lec 4

Coset!

- H is norrmal if
    - $\forall x \in G, \exists y, xH = Hy$
    - But $x \in xH$ then $x \in Hx$, thus it has to be Hx
    - i.e. $xH = Hx$
    - i.e. $xHx^{-1} = H$
    - $H <| G \iff \forall x, xHx^{-1} = H$

- Consider the group of ratations of $\reals^3$, $SO(3)$
    - every rotation is described by an orthogonal 3×3 matrix (i.e. a 3×3 matrix with real entries which, when multiplied by its transpose, results in the identity matrix) with determinant 1
    - 

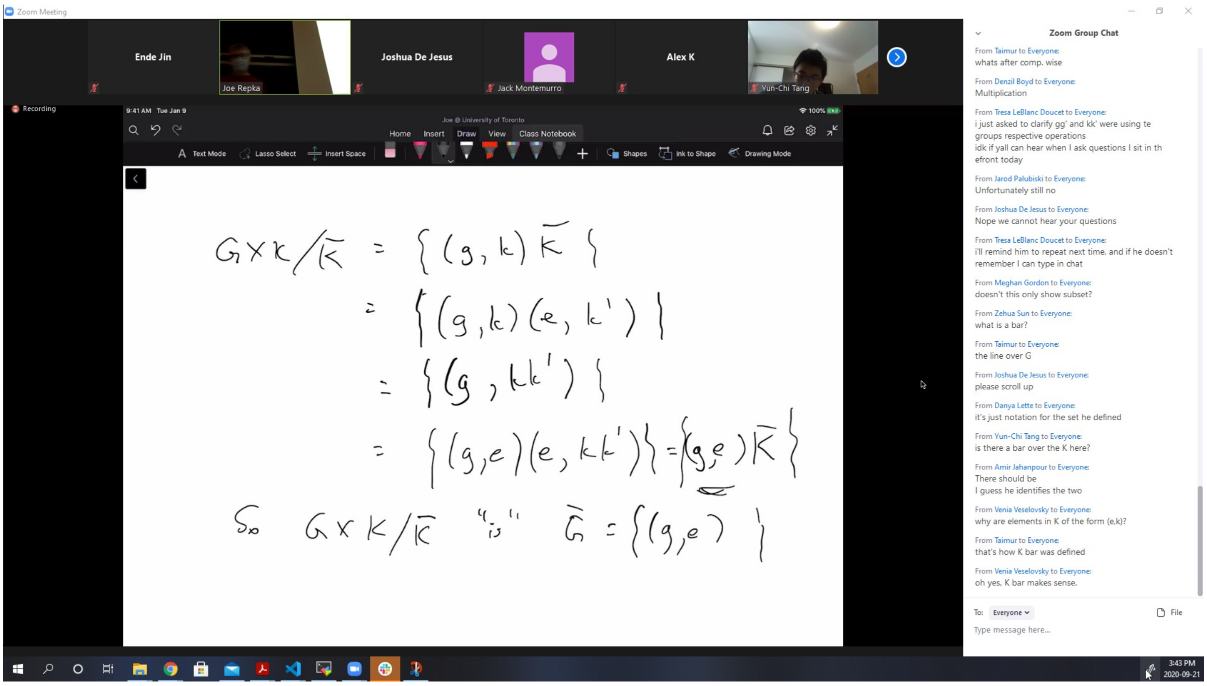
    - Now consider $G / H = SO(3) / SO(2)$, which is not a group, just a coset space
    - Suppose Ω takes N to a fixed P
    - then $\Omega H = \Omega SO(2)$ all takes N to the P
    - Given any P on the unit sphere, S, can rotate N to P .
        - i.e. $\exists \Omega \in SO(3), .s.t. \Omega N =P$, thus $SO(3) / SO(2) \cong S$, which is the unit sphere
    - go though this bijection yourselves!

Definition: direct product

- If G, K are groups
    - denote $G \times K := {(g,k) : g \in G, k \in K}$
    - $(g,k) \cdot (g', k') = (g \cdot g', k \cdot k')$

    - identity is of course (e_G, e_K)

- $(\Reals^2, +) = (\reals, +) \times (\reals, +)$

- In $G \times K$, have subgroups $\bar{G} = G \times {e_K}, \bar{K} := {e} \times K$
    - They are normal!
    - $(g,k) \bar{G} (g^{-1}, k^{-1}) ={(g g' g^{-1}, e)} \subseteq \bar{G}$
        - the other subseteq is trivial
    - $G \times K / \bar{K}$
    - 

    - geometrically speaking
        - in $\Reals^2, \reals$ coset are $(k, 0)$ + vertical axis = vertical lines
        - Addition of cosets is the same as addition of representatives in the horizontal axis.

Definition: homomorphism

- A map $\phi : G \rightarrow K$ ia a homomorphism
    - iff $\phi(g \cdot_G g') = \phi(g) \cdot_K \phi(g')$
- if in addition ϕ is a bijection (1-1 and onto)
    - then ϕ is isomorphism
    - if we define $\phi : G \times K \rightarrow G$, by $\phi := (g,k) \mapsto g$
        - it sometimes called projection of $G \times K$ onto G

    - isomorphism is basically what equality means

Definition: Symmetric Group

- $n \in \natnums, S_n =$ symmetric group of order$n$
    - = set of permutation of n symbols usually $(1, ..., n)$
    - interestingly $S_3 =$ exactly those reflections and rotations
- it is easy to see that $|S_n| = n!$
- Notation: if we write (a, b, c) with $a,b,c \in \natnums$
    - this is a "cycle", the permutation (the function) that takes $a \mapsto b, b \mapsto c, c \mapsto a$
    - $(a_1,..,a_k)$ is a k-cycle
        - interestingly, this is also a cyclic group of order k
            - (1 2 3 4) and (1 2 4 3) both generate 4 elements
            - (3 4) is a 2-cycle
        - $a_i \mapsto a_{i+1}, a_k \mapsto a_1$
        - (a,b) two cycle = transportation
        - (1 2 3)(3 4) = (1 2 3 4)
        - (3 4)(1 2 3) = (1 2 3 4)
        - not disjoint, don't expect commute
        - Any permutation can be written as a product of dijoint cycles (unqiue except for rotational and order of disjoint cycles)
        - (1 2 3) = (3 2 1)
        - not unique for non-disjoint cycles
        - same permutation can be written in more than 1 way
        - a k-cycle has order k

Definition:

- if $\phi : G \rightarrow K$ is a homomorphism
    - then {g : G : \phi(g) = e} is called kernel of ϕ, $ker(\phi)$
- Easy: $ker(\phi) < G$, after all $e \in ker(\phi)$
- in linear algebra, a linear transformation $\phi : V \rightarrow W$, groups with addition, has $ker(\phi) = nullspace(\phi)$
- Also $image(\phi) = {\phi(g) : g \in G} = cokernel(\phi)$ also a subgroup, but of K
- In fact, $ker(\phi)$ is a normal subgroup

- $ker((g,k) \mapsto g) = \bar{K}$
- $\phi : {e} \times K \rightarrow K$ is an isomorphism
- thus $ker(proj) \cong K$
    - on $\reals^2$, kernel of projection to x-axis is y -axis
- $(Z < \reals)$, normal subgroup due to abelion
    - what is $\real / Z$? all the number with same number after point in decimal
        - every coset has exactly one element in $[0, 1)$
            - it is actually a bit misleading, acutally it corresponds to the closed circle
            - $\reals / Z \cong circle$
        - $\phi : t \mapsto e^{2\pi i t} := \reals \rightarrow $ circle

- $ker(\phi) = Z$, thus $e^{2\pi i t} = e^{2 \pi i (t + n)}$ for $n \in Z$
- $Z^2 <| \reals^2$
- $\reals^2 / Z^2 \cong torus \cong circle \times circle$