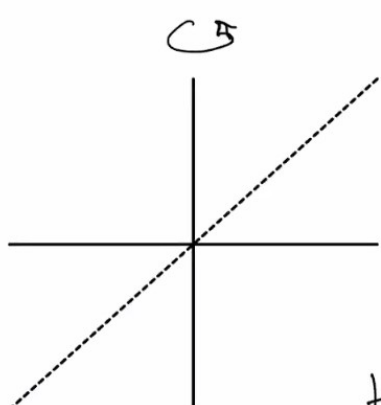


Lec 4

Coset!

- H is normal if
 - $\forall x \in G, \exists y, xH = Hy$
 - But $x \in xH$ then $x \in Hx$, thus it has to be $xH = Hx$
 - i.e. $xH = Hx$
 - i.e. $xHx^{-1} = H$
 - $H \triangleleft G$ iff $\forall x, xHx^{-1} = H$
- Consider the group of rotations of \mathbb{R}^3 , $SO(3)$
 - every rotation is described by an orthogonal 3×3 matrix (i.e. a 3×3 matrix with real entries which, when multiplied by its transpose, results in the identity matrix) with determinant 1

Consider the group of rotations of \mathbb{R}^3 , $SO(3)$



Let $H =$ rotations about the vertical axis $SO(2)$

$$\begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$H < SO(3)$ not normal.

- Now consider $G/H = SO(3)/SO(2)$, which is not a group, just a coset space
- Suppose Ω takes N to a fixed P
- then $\Omega H = \Omega SO(2)$ all takes N to the P
- Given any P on the unit sphere, S , can rotate N to P .
 - i.e. $\exists \Omega \in SO(3)$, s.t. $\Omega N = P$, thus $SO(3)/SO(2) \cong S$, which is the unit sphere
- go through this bijection yourselves!

Definition: direct product

- If G, K are groups
 - denote $G \times K := \{(g, k) : g \in G, k \in K\}$
 - $(g, k) \cdot (g', k') = (g \cdot g', k \cdot k')$

- identity is of course (e_G, e_K)

- $(\mathbb{R}^2, +) = (\mathbb{R}, +) \times (\mathbb{R}, +)$

- In $G \times K$, have subgroups $\bar{G} = G \times \{e_K\}$, $\bar{K} := \{e\} \times K$
 - They are normal!
 - $(g, k) \bar{G} (g^{-1}, k^{-1}) = \{(g' g^{-1}, e)\} \subseteq \bar{G}$
 - the other subseteq is trivial
 - $G \times K / \bar{K}$

The screenshot shows a Zoom meeting interface. The main window displays a whiteboard with handwritten mathematical derivations. The derivations are as follows:

$$\begin{aligned}
 G \times K / \bar{K} &= \{(g, k) \bar{K}\} \\
 &= \{(g, k)(e, k')\} \\
 &= \{(g, kk')\} \\
 &= \{(g, e)(e, k')\} = \{(g, e) \bar{K}\} \\
 \text{So } G \times K / \bar{K} \text{ "is" } \bar{G} &= \{(g, e)\}
 \end{aligned}$$

On the right side, there is a Zoom Group Chat window with several messages. The messages are:

- From Taimur to Everyone: whats after comp. wise
- From David Boyd to Everyone: Multiplication
- From Tessa LeBlanc Doucet to Everyone: i just asked to clarify gg' and kk' were using the groups respective operations. idk if y'all can hear when I ask questions I sit in th effort today.
- From David LeBlanc to Everyone: Unfortunately still no
- From Joshua De Jesus to Everyone: Nope we cannot hear your questions
- From Tessa LeBlanc Doucet to Everyone: I'll remind him to repeat next time, and if he doesn't remember I can type in chat
- From Meghan Gordon to Everyone: doesn't this only show subset?
- From Zehua Sun to Everyone: what is a bar?
- From Taimur to Everyone: the line over G
- From Joshua De Jesus to Everyone: please scroll up
- From Danya Lette to Everyone: it's just notation for the set he defined
- From Yun-Chi Tang to Everyone: is there a bar over the K here?
- From Amir Jahangir to Everyone: There should be I guess he identifies the two
- From Venia Veselovsky to Everyone: why are elements in K of the form (e,k)?
- From Taimur to Everyone: that's how K bar was defined
- From Venia Veselovsky to Everyone: oh yes. K bar makes sense.

- geometrically speaking
 - in \mathbb{R}^2 , \mathbb{R} cosets are $(k, 0) + \text{vertical axis} = \text{vertical lines}$

Addition of cosets
is the same as addition of
representatives in the horizontal
axis.

■

Definition: homomorphism

- A map $\phi : G \rightarrow K$ is a homomorphism
 - iff $\phi(g \cdot_G g') = \phi(g) \cdot_K \phi(g')$
- if in addition ϕ is a bijection (1-1 and onto)
 - then ϕ is isomorphism
 - if we define $\phi : G \times K \rightarrow G$, by $\phi := (g, k) \mapsto g$
 - it's sometimes called projection of $G \times K$ onto G

- isomorphism is basically what equality means

Definition: Symmetric Group

- S_n symmetric group of order $n!$
 - = set of permutation of n symbols usually $(1, \dots, n)$
 - interestingly S_3 = exactly those reflections and rotations
- it is easy to see that $|S_n| = n!$
- Notation: if we write (a, b, c) with $a, b, c \in \mathbb{N}$
 - this is a "cycle", the permutation (the function) that takes $a \mapsto b, b \mapsto c, c \mapsto a$
 - (a_1, \dots, a_k) is a k -cycle
 - interestingly, this is also a cyclic group of order k
 - $(1\ 2\ 3\ 4)$ and $(1\ 2\ 4\ 3)$ both generate 4 elements
 - $(3\ 4)$ is a 2-cycle
 - $a_i \mapsto a_{i+1}, a_k \mapsto a_1$
 - (a, b) two cycle = transportation
 - $(1\ 2\ 3)(3\ 4) = (1\ 2\ 3\ 4)$
 - $(3\ 4)(1\ 2\ 3) = (1\ 2\ 3\ 4)$
 - not disjoint, don't expect commute
 - Any permutation can be written as a product of disjoint cycles (unique except for rotational and order of disjoint cycles)
 - $(1\ 2\ 3) = (3\ 2\ 1)$
 - not unique for non-disjoint cycles
 - same permutation can be written in more than 1 way
 - a k -cycle has order k

Definition:

- if $\phi : G \rightarrow K$ is a homomorphism
 - then $\{g \in G : \phi(g) = e\}$ is called kernel of ϕ , $\ker(\phi)$
- Easy: $\ker(\phi) < G$, after all $e \in \ker(\phi)$
- in linear algebra, a linear transformation $\phi : V \rightarrow W$, groups with addition, has $\ker(\phi) = \text{nullspace}(\phi)$
- Also $\text{image}(\phi) = \{\phi(g) : g \in G\} = \text{cokernel}(\phi)$ also a subgroup, but of K
- In fact, $\ker(\phi)$ is a normal subgroup

- $\ker((g, k) \mapsto g) = \bar{K}$
- $\phi : \{e\} \times K \rightarrow K$ is an isomorphism
- thus $\ker(\text{proj}) \cong K$
 - on \mathbb{R}^2 , kernel of projection to x -axis is y -axis
- $(\mathbb{Z} < \mathbb{R})$, normal subgroup due to abelian
 - what is \mathbb{R} / \mathbb{Z} ? all the number with same number after point in decimal
 - every coset has exactly one element in $[0, 1)$
 - it is actually a bit misleading, actually it corresponds to the closed circle
 - $\mathbb{R} / \mathbb{Z} \cong \text{circle}$
 - $\phi : t \mapsto e^{2\pi i t} : \mathbb{R} \rightarrow \text{circle}$

- $\ker(\phi) = \mathbb{Z}$, thus $e^{2\pi i t} = e^{2\pi i (t+n)}$ for $n \in \mathbb{Z}$
- $\mathbb{Z}^2 \subset \mathbb{R}^2$
- $\mathbb{R}^2 / \mathbb{Z}^2 \cong \text{torus} \cong \text{circle} \times \text{circle}$

Zoom Meeting

Ende Jin Patrice Moisan... Mik yuchongzhang... Asha McMullin

Microsoft Whiteboard

$$\sigma^N = 1 \Rightarrow \sigma_1^N \dots \sigma_{m-1}^N = \sigma_m^{-N}$$

$$\langle \sigma_1 \dots \sigma_{m-1} \rangle \cap \langle \sigma_m \rangle = \{1\} \text{ (disjointness)}$$

$$\Rightarrow \sigma_1^N \dots \sigma_{m-1}^N = \sigma_m^N = 1$$

recurrence

$$\Rightarrow \sigma_1^N = \dots = \sigma_m^N = 1$$

$$\Rightarrow \text{ord}(s) = \text{lcm}(k_1, \dots, k_m)$$

Zoom Group Chat

From Me to Everyone: transpose I think I think $\phi|_{\text{circ } \phi} = \text{id}$ thus we can see ϕ is bijective on set

From Mik to Everyone: how did you get that composition is multiplication here? yeah second last line to last line oh I see thanks!

From Me to Everyone: can you use the fact that cardinality is the same here? e aren't they disjoint? lom

From Yun-chi Tang to Everyone: can you scroll up a bit? how did you get the statement right before you start stating the orders of the 2 cycles? yes okay I see

From yuchongzhang@mail.utoronto.ca to Everyone: what does n refer to here? ok

From Me to Everyone: why this is Q7 here in my handout and you solved Q8 first? I think $k+m \leq n$ is necessary otherwise the domain is weird I mean (1 2 3) has to work on over S_3 , cannot work on S_2 right? Thanks :) See you next week!

From Yun-chi Tang to Everyone: thank you!

From Mik to Everyone: thanks!

From yuchongzhang@mail.utoronto.ca to Everyone: thank you

To: Everyone

Type message here...

1:00 PM 2020-09-22