Demo-variant of the exam on Calculus (2 hours or 120 min)

VARIANT 0

1. (12) (theoretical question with proof on lectures of the 3d module)
Prove that if $f(x) \in C[a,b]$ (is continuous on the segment $[a,b]$) there is a point $c \in [a,b]$ such that

$$\int\limits_a^b f(x)\,dx = f(c)(b-a).$$

2. (8) (problem on calculating an indefinite integral)
Find the following integral:

$$\int (x^2 - 2x + 3) \cdot e^{x/2}\,dx.$$

3. (8) (problem on calculating a definite integral)
Find the following integral

$$\int\limits_3^8 \frac{x+4}{\sqrt{x+1}}\,dx.$$

4. (12) (problem on finding the areas bounded by the curves, the volume of the body of revolution or length of a curve)
Find the length of the curve of astroid

$$x^{2/3} + y^{2/3} = 49.$$

5. (10) (problem on improper integrals) For any $\alpha > 0$ check the improper integral for convergence:

$$\int\limits_0^{+\infty} \frac{dx}{\sqrt{1+x^\alpha}}.$$

6. (12) (theoretical question with proof on lectures of the 4th module)
Prove that every bounded sequence of points in $\mathbb{R}^2$ has at least one limit point. As a result, derive that every bounded infinite set of points in $\mathbb{R}^2$ has at least one limit point.

7. (8) (problem on open and closed sets, domain and range, level curves, continuous FSV)
Is the function

$$f(x,y) = \begin{cases} \dfrac{4x^4y^6}{3x^4+2y^6}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

continuous at the point $(0,0)$? Justify your answer.

8. (8) (problem on partial derivatives, directional derivatives and derivatives of a composition)
Find the derivative of the function $f(x,y,z) = 7(x-5)^y \sin(z+y)$ in the direction of the gradient at the point $A(6,-5,5)$.

9. (12) (problem on relative extrema, constrained optimization or absolute max or min of FSV)
Find the absolute maximum and minimum values of $f(x,y) = x^2 + y^2 - 8x - 6y + 23$ on the closed rectangular region R with vertices $A(0,0)$, $B(12,0)$, $C(12,9)$ and $D(0,9)$.

10. (10) (theoretical question on definitions and formulations of the theorems with question on understanding the formulated results)
Give the definition of the Jacobian of the transformation $\mathbb{R}^n \to \mathbb{R}^n$ given by vector valued function $f = (f_1, f_2, \ldots,, f_n)$. Give the definition for the functions $f_1, f_2, \ldots,, f_n$ to be functionally dependent in

open set S (by means of F)? Formulate the theorem on $f_1, f_2, \dots, , f_n$ to be functionally dependent (in term of Jacobian). Show that the functions $u = \log(x - 2y)$ and $v = x^2 + 4y^2 - 4xy + 3$ are functionally dependent, and find explicitly the dependence.

To get the final grade all the marks will be summed up and divided by 10 (so, maximum possible grade for exam is 10).

List of theoretical questions (with proof).

Lecture 3.
Theorems (pages 1, 3); properties of s and S (1, 2, 4),

Lecture 4.
Theorems 1, 2; corollary 3.

Lecture 5.
Theorems 1, 3; theorem 1 (from paragraph "Methods of integration").

Lecture 6.
Theorem 1; formula for the volume of solid of revolution.

Lecture 7.
Formula for the length of a curve.

Lecture 8.
Trapezoidal rule (proof of approximating formula).

Lecture 04.04.
Lemma. An open disc is an open set.

Lecture 10.04.
Theorems 2, 3, 4; Lemma 1.

Lecture 17.04.
Theorems 1, 2, 3.
Lecture 24.04. Theorems 1, 2; Lemma 1; prove that the maximum of $\frac{\partial f}{\partial v}(a)$ can be archieved in the direction of the gradient $\frac{\partial f}{\partial v}(a)$.

Lecture 18.05.
Theorems 1, 3.

Lecture 22.05.
Theorems 1 from 'Advanced Calculus' by A. Friedman, part two, 'Differentiation' (Ch 7), 'Implicite functions theorem' (par 7).

Lecture 24.05. Theorems 1, 3; prove that
- the gradient vector ∇F of the function $F(x, y, z)$ at a point P is perpendicular to the tangent vector to any curve C on level surface $F(x, y, z) = k$ that passes through P.
- If a point $P(x_0, y_0, z_0)$ is an extremum of a function $f(x, y, z)$ on a set $S \in \mathbb{R}^3$ given by the constraint $g(x, y, z) = 0$ and if $\nabla g(P) \neq 0$ then there is a number λ such that $\nabla f(P) = \lambda \nabla g(P)$.

Lecture 29.05.
Theorem 1