Demo-variant of the module 4 control work

VARIANT 0

1. (16) Prove that for any set $G \in \mathbb{R}^2$
$$G \cup G_l = G \cup \partial G.$$

2. (8) Describe the domain and range of the function $z = \ln(xy - 1)$. Describe its level curves and sketch the level curves for $C = 0,\ 2,\ 4$.

3. (8) For the function $u = \dfrac{1}{r}$ where $r = \sqrt{x^2 + y^2 + z^2}$ prove that $f_{xx} + f_{yy} + f_{zz} = 0$. Compute the length and the direction of the gradient of $u(x, y, z)$ at a point $M(x_0, y_0, z_0)$ (in terms of r).

4. (8) Prove that if the function $f(u, v)$ is differentiable, then $\varphi(x, y, z) = f\left(\dfrac{x}{y},\ x^2 + y - z^2\right)$ satisfies the following equation:
$$2xz\frac{\partial \varphi}{\partial x} + 2yz\frac{\partial \varphi}{\partial y} + (2x^2 + y)\frac{\partial \varphi}{\partial z} = 0.$$

5. (16) Prove that if partial derivatives $f'_x(x, y)$, $f'_y(x, y)$, $f''_{xy}(x, y)$ and $f''_{yx}(x, y)$ are defined at a neighborhood of (x_0, y_0) and continuous at (x_0, y_0) then
$$f''_{xy}(x_0, y_0) = f''_{yx}(x_0, y_0).$$

6. (8) Find the critical points of the functions $f = e^{2x+3y}\,(8x^2 - 6xy + 3y^2)$. Test the nature of the critical points.

7. (8) Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 + 4x - 3y$ on the region $R = \{(x, y) |\, x^2 + y^2 \leqslant 25\}$.

To get the final grade all the marks will be summed up and divided by 5 (so, maximum possible grade is 16).

List of theoretical questions (with proof).

Lecture 04.04.
Lemma. An open disc is an open set.

Lecture 10.04.
Theorems 2, 3, 4; Lemma 1.

Lecture 17.04.
Theorems 1, 2, 3.

Lecture 24.04.
Theorems 1, 2; Lemma 1; prove that the maximum of $\dfrac{\partial f}{\partial v}(a)$ can be archieved in the direction of the gradient $\dfrac{\partial f}{\partial v}(a)$.

Lecture 18.05.
Theorems 1, 3.

Lecture 22.05.
Theorems 1 from 'Advanced Calculus' by A. Friedman, part two, 'Differentiation' (Ch 7), 'Implicite functions theorem' (par 7).

Lecture 24.05.

Theorems 1, 3; prove that

- the gradient vector ∇F of the function $F(x, y, z)$ at a point P is perpendicular to the tangent vector to any curve C on level surface $F(x, y, z) = k$ that passes through P.
- If a point $P(x_0, y_0, z_0)$ is an extremum of a function $f(x, y, z)$ on a set $S \in \mathbb{R}^3$ given by the constraint $g(x, y, z) = 0$ and if $\nabla g(P) \neq 0$ then there is a number $\lambda \in \mathbb{R}$ such that $\nabla f(P) = \lambda \nabla g(P)$.

Lecture 29.05.

Theorem 1