Dov Kassai Discrete Final Question 1

dk778

1. Prove that the function f(x)=(8x+5) mod 17 is injective on the set{0,1,2,3,…,15,16}.
    1. here we can prove every single point that there exists a unique element in f(x). This implies that the function f(x) is injective on the set {0,1,2,3,…,15,16}.
    2. Here is the proof
        1. x=0, f(0)=(5)mod 17, which means that it would be 5
        2. x=1, f(1)=(13)mod 17, which means that it would be 13
        3. x=2, f(2)=(21)mod 17, which means that it would be 4
        4. x=3, f(3)=(29)mod 17, which means that it would be 12
        5. x=4, f(4)=(37)mod 17, which means that it would be 3
        6. x=5, f(5)=(45)mod 17, which means that it would be 11
        7. x=6, f(6)=(53)mod 17, which means that it would be 2
        8. x=7, f(7)=(61)mod 17, which means that it would be 10
        9. x=8, f(8)=(69)mod 17, which means that it would be 1
        10. x=9, f(9)=(77)mod 17, which means that it would be 9
        11. x=10, f(10)=(85)mod 17, which means that it would be 0
        12. x=11, f(11)=(93)mod 17, which means that it would be 8
        13. x=12, f(12)=(101)mod 17, which means that it would be 16
        14. x=13, f(13)=(109)mod 17, which means that it would be 7
        15. x=14, f(14)=(117)mod 17, which means that it would be 15
        16. x=15, f(15)=(125)mod 17, which means that it would be 6
        17. x=16, f(16)=(133)mod 17, which means that it would be 14
    3. Since we see that each number from 0..16 there exists a unique element. If we have the contrapositive that say that f(x1)=f(x2) this would then imply that x1=x2. And we can see in this table that it is not true so it would mean that it is injective because x1≠x2.
2. Is f(x) = (8x + 5) mod 17 invertible on this set? If so, give its inverse function
    1. we first need to prove that the function is injective, meaning that each element in the function must map to something.
        1. if x=0,1, then f(0,1)=8x+5 (mod 17)= {5,13}
        2. if x=2,3, then f(2,3)=8x+5(mod 17)= {4,12}
        3. if x=4,5 then f(4,5)=8x+5(mod 17)= {3,11}
        4. if x=6,7 then f(6,7)=8x+5(mod 17)= {2,10}
        5. if x=8,9 then f(8,9)=8x+5(mod 17)= {1,9}
        6. if x=10,11 then f(10,11)=8x+5(mod 17)= {0,8}
        7. if x=12,13 then f(12,13)=8x+5(mod 17)= {16,7}

        8. if $x=14,15$ then $f(14,15)=8x+5 \pmod{17}= \{15,6\}$
    2. So from this table we can see that every element in that is in f(n) is mapped to an element in n, by definition this means that it is subjective. Since the function is both injective and surjective this means that it would be invertible.
    3. Next we need to find the inverse of the functions
        1. if $x=5,13$, then $f^{(-1)}(5,13)=(x-5)/8=\{0,1\}$
        2. if $x=4,12$, then $f(4,12)=(x+12)/8= \{2,3\}$
        3. if $x=3,11$, then $f(3,11)=(x+29)/8= \{4,5\}$
        4. if $x=2,10$, then $f(2,10)=(x+46)/8= \{6,7\}$
        5. if $x=1,9$ then $f(1,9)=(x+63)/8= \{8,9\}$
        6. if $x=0,8$, then $f(0,8)=(x+80)/8= \{10,11\}$
        7. if $x=16,7$, then $f(16,7)=(x+97)/8= \{12,13\}$
        8. if $x=15,6$, then $f(15,6)=(x+114)/8= \{14,15\}$
    4. so we can see that it is invertible as well.
3. Is $f(x) = (8x + 5) \bmod 18$ invertible on this set? If so, give its inverse function (5 points).
    1. so we start off with the function $f(x)=8x+5 \pmod{18}$
    2. we can rewrite this as saying $8x+5=y$, which becomes $x=(y-5)/8$.
    3. we then can say that $f(x)=y$, which implies the inverse of $f(y)=x$.
        1. so the inverse of of $f(y)=(y-5)/8$.
        2. this would then further imply that the inverse of $f(x)=(x-5)/8$
        3. this can be rewritten as $(x-5)(8^{(-1)}) \pmod{18}$.
    4. But we can see that the GCD of 8 and 18 is not equal to one but rather it is 2, also 8 has no multiplicative inverse mod 18. Therefore this proves that there is no inverse for the function $(x) = (8x + 5) \bmod 18$.
4. True or false - give a mathematical justification
    1. $n = O(n^2)$ → this would be **true**
        1. The reason why it is true is because we can have $f(n)=n$ and $g(n)=n^2$.
        2. we can then set up a table that if $n=1,2,3\ldots$ then $f(n)=1,2,3\ldots$, then $g(n)=1,4,9\ldots$
        3. we can then see for these three numbers that $f(n)<=g(n)$
        4. Since we know $f(n)<=g(n)$, we can then say that $n=O(n^2)$
    2. $n(n+1)(n+2)-n^3 =O(n^3)$ → **true**
        1. If we say that $f(n)= n(n+1)(n+2)-n^3$, then after factoring everything out the highest power in this entire equation would be n^3 since it dominates everything else. Then we say that $g(n)=n^3$. This would be true because we would have $n^3=n^3$.
        2. We can fully write this out by saying
            1. $n^3+2n^2+n^2+2n-n^3= O(n^3)$ of

        2. $3n^2+2n=O(n^3)$.
    3. If we divide the two functions we would see that as n goes to infinity according to l'Hopitals rule that the function n^3, which would be on the bottom, would dominate that function.
    4. hence proving that $n(n+1)(n+2)-n^3 = O(n^3)$.
3. $n(n+1)(n+2)-n^3 = O(n^2)$→ **True**
    1. This is very similar to the previous one in the sense that it can be written as
        1. $3n^2+2n=O(n^2)$.
    2. And as n grows we can say that it would essentially look like $n^2=O(n^2)$
    3. which would be true.
4. $n(\ln n) = O(n^2)$→ **true**
    1. we know that $\log(n)\leq n$ for all $n \geq 1$
    2. Since we are now just multiplying each see by n this would look like
        1. $n\log(n)\leq n^2$ and this would continue to be true
    3. therefore $\log(n)=O(n^2)$
5. $n^2 = O(n\ln n)$→ **False**
    1. as proved in the problem before $\log(n)\leq n$.
    2. Since n^2 is much larger than $n\log(n)$, then n^2 can never be equal to big O of $\log(n)$
    3. so this would be false
6. $1/n = O(1)$ → **True**
    1. this one again we can set up using a table $f(n)=1/n$ and $g(n)=1$
    2. if n=1,2,3 … then f(n)=1,.5,.333…. then g(n)=1.
    3. this can show that as n increases then $f(n)<= g(n)=1$.
    4. this proves that $f(n)=O(g(n))=1/n=O(1)$.
7. $1000000n = O(n)$ → **true**
    1. since we can say that n is a number then the limit of n would be a constant time, this would mean that even if the number is incredibly large the big oh would still be in constant time so $1000000n = O(n)$
8. $2^n = O(3^n)$ → **true**
    1. this is true because it is clear to say that $2^n<=3^n$, for any sufficiently large n.
    2. As 2 increases 3 increases that much faster as n grows.
9. $3^n = O(2^n)$ → **false**.
    1. again we can see from inspection that $3^n >=2^n$ and this would never be equal to big oh big 2^n is smaller than 3^n.

10. $i(i + 1)(i + 2) = O(n^4)$ →**True**
    1. here we can actually take the limits as n goes to infinity
    2. so if we have f(n)/g(n)=(n(n + 1)(n + 2))/(n^4)
    3. we can then take lhopitals rule of this function, since l'hopital follows the highest order we would have n^3/n^4. this would mean that n^4 would dominate n^3 so this would mean that the statement is true.
5. The 'double factorial' is a variant on the factorial function where every other number is multiplied together: for instance, $9!! = 9*7*5*3*1 = 945$, $10!! = 10*8*6*4*2 = 3840$. In general, n!! is much smaller than n! (20 points).
    1. Argue that n!!/n! goes to 0 as $n \to \infty$, i.e., n! dominates n!! severely. Hint: Be wary of cases.
        1. We have that n!=n(n-1)(n-2)…2*1.
        2. n!! has two different cases if it is odd or even
            1. if it is even then it would be:n(n-2)(n-4)…4*2.
            2. if it is odd then it would be:n(n-2)(n-4)…3*1
        3. If it is even
            1. it would be (n!!)/(n!), which would be the same as
                1. (n(n-2)(n-4)..4*2)/(n(n-1)(n-2)…2*1)
                2. this would divide down to 1/((n-1)(n-3)…3*2*1
            2. Then as the limit goes to infinity then it would just go to zero. This is because every multiplication of n!! it will be divided by something in n! leaving a 1 on top and many things still on the bottom, which would make it go to zero.
        4. If it is odd
            1. it would be (n!!)/(n!), which would be the same as
                1. (n(n-2)(n-4)..3*1)/(n(n-1)(n-2)…2*1)
            2. This again would reduce down to 1/((n-1)(n-3)…4*3*2*1
            3. And we can say as n goes to infinity this would go to zero for the same reason since the denominator will get so exponentially big and the numerator will just stay 1, which would make it 0.
    2. Argue that in the limit for even n, ln(n!!)/(n ln(n)) is bound above and below by some constants
        1. Since n is even this would be (ln[n(n-2)(n-4)…4*2])/(n ln(n))
            1. which is the same as [ln(n)+(ln(n)+ln(n)…..)/(n ln (n))
                1. the tope here would have n/2 terms.
            2. we can then say that if we are saying that ln(m)≤ln(n), this implies that m≤n.
        2. so the original problem of ln(n!!)/(n ln(n)) has become

        1. (ln(n!!)/(n ln (n)) ≤ (n/2) ln(n)/(n ln(n)), this would mean that ln(n!!)/(n ln(n)) is bounded from above by 1/2.
    3. Also ln(n!!) is greater than zero so this would be lower bounded by zero. The final answer would be
        1. 0≤ ln(n!!)/(n ln(n))≤(1/2)
3. Argue that for odd n, ln(n!!)/(n ln(n)) is bound from above and below by some constants as well
    1. this proof is identical to the one as before since it would be following the same patterns
    2. n is odd so it would become (ln[n(n-2)(n-4)…3*1])/(n ln(n))
        1. the ln would factor in for it to become which is the same as [ln(n)+(ln(n)+ln(n) …..ln(3)+ln(1))/(n ln (n))
        2. and since on the top there would be the same amount of n/2 terms
            1. following that ln(m)≤ln(n), this would imply that m≤n
        3. (ln(n!!)/(n ln (n)) ≤ (n/2) ln(n)/(n ln(n)) this would mean that ln(n!!)/(n ln(n)) is bounded from above by 1/2.
    3. Again since we know that ln(n!!) is going to be greater than 0 and ln(n) ≥ 0 for n is a number greater than 0 this would mean that ln(n!!)/(n ln(n)) is bounded below by o
    4. This would mean the bounds are: 0≤ ln(n!!)/(n ln(n))≤(1/2)
4. Conclude there that while n! is a lot large than n!!, we have that ln n!!= ø(ln n!).
    1. here we can say that we know that since when it is in the lowest case which is when n=2, this would be the same and then as n grows then they would essentially be within the same range for any sufficiently large n. As they both grow exponentially then it would essentially be bounded by on both sides (ln n!). Meaning that the big theta notation of this function would be ln n!! = ø(ln n!).