Dov Kassai Discrete Final Question 4

dk778

1. Prove that for an integer X that is not a multiple of 5, X^n ≡ X^(n mod 4) (mod 5).
    1. So we know that for n mod 4 this would mean that x^(n mod 4) can only be the numbers 0,1,2,3, since when it would be 4mod 4 it would go back to 0.
    2. Now we can look at the whole congruence and can prove it by cases for the 4 cases that we have
        1. if n=1, then x^1 ≡x^1(mod 5): this is because 1 mod 4 is still 1
        2. if n=2, then x^2 ≡x^2(mod 5): this is because 2 mod 4 is still 2
        3. if n=3, then x^3 ≡x^3(mod 5): this is because 3 mod 4 is still 3
        4. if n=4 then x^4 ≡x^4(mod 5):this would be 0 because 4 mod 4 is 0.
    3. So as this continues that means for any number greater than or equal to 5 it would be repeating the sequence and it can be broken down into one of the 4 base cases/ it can be traced back to one of the 4 base cases.
2. Use this to compute, as efficiently as you can 123^4567 (mod 5).
    1. This we can break down to more simple steps, since we know that 123 is 120+3, and that 120 can be mod by 5 to become zero the congruence would become 3^4567 (mod 5).
    2. We don't really want to compute such a large number like 3^4567, so we can break it down to look like (3^2)^2283*3, which is the same as 9^2283*3.
        1. the *3 at the end comes from the fact that 4567 can not be divided by 2 cleanly buy 4566 can so we just move a 3 out to get (9^2283)*3.
    3. If we deal with 9 mod 5 we can say that it equals 4 mod 5, OR we can say that it is -1 mod 5.
    4. Now we have ((-1)^2283)*3. which is the same as (-1)*3 because (-1) to the power of an odd number will still be (-1). So the congruence we now have is -3 mod 5
    5. Finally we can say that -3 mod 5 is similar as saying 9 mod 5 is -1. So -3 mod 5 would just be 2
    6. The final answer is 2 mod 5.
3. A fast way to generate large numbers is with power towers. The following are the first few power towers of 2:
    1. How many digits (base-10) does X_5 have? Give an exact value. (1 points)
        1. Here they way that we would have to find the we have to find 2^(X_4)=2^(65536)
        2. The way to find out how many digits are in this number are to take n*Log_2+1
        3. This would give you the number 19729 digits.
    2. Give a recursive formulation for Xk+1 in terms of X^k. (2 points)

    1. we can rewrite this as $X_{k+1}=2^{\wedge}X_k$
    2. From the power tower that was given that we can see that $X_1=2$, and when we have $X_{1+1}=2^{\wedge}(X_1)=2^{\wedge}2=4$
    3. From this step which is the inductive step because we are proving that $X_{k+1}=2^{\wedge}X_k$, which is correct and this would be true.
3. Prove that for all sufficiently large k, X^k ≡ 1 (mod 5).
    1. First we need to state that this would only work when k>=3 this is because 2^1=2 and mod 5 that would still be 2. 2^2 would be 4 and mod 5 that would be 4. So we can establish that this will only work when k>=3.
    2. Now we can look at the power tower for when it is greater than or equal to 3.
    3. We can see that when it is k=3 the power tower is 2^2^2=16. This is because it can be rewritten as 4^2. This is is for X_K.
    4. For the inductive step we can now say that X_{K+1} is 4^2^2 which is also written as 4^4 which is is 256. The pattern seems to be showing that for whenever 4 is going to the power of an even number it will be 6. Since for whatever K it can be rewritten as 4 to the power of some even number, then the last digit will always be a 6.
    5. This is makes the statement true that for any X_k that for when k>=3 when mod by 5 it will always be 1.
4. Prove that for all sufficiently large k X_K ≡ 0 (mod 2)
        1. This can be proved from the previous problem. Since we showed that for any value of k the answer will end in a 6 because of the pattern that it follows. This would mean that it would be an even number and since it is an even number mod by 2 it would always be 0. This is only true when k>=1, since X^0=1 and that can not be mod by 2.
        2. You can also say that since in the power tower it is always getting raised to 2 which would take an even number to the power of an even which would still be an even then it would also be 0 when mod by 2.
            1. the proof by example can be done in the first couple of cases when 2^1=2 which is 0 mod 2. And 2^2=4 when mod by 2 is zero.
    2. Based on the previous results, compute the 1s digit for the kth power tower for all sufficiently large k
        1. this can be proved from when it is mod by 5. As long as it can be manipulated that it would be 4 to the power of something even then the 1s digit will always be 6.

    2. Since it was always going to be mod 1 in the case when it was 4 to the power of something that means that the last number would be a 6, since when mod by 5 it would be a 1.
4. Prove that for any integer value of D, the equation 27x + 14y = D has integer solutions for x and y.
    1. We can say that for the equation 27x + 14y = D, this would mean that for it to have integer solutions that x and y would have to have a GCD of D.
    2. here this equation the GCD between 27 and 14 is 1. This means that any integer value of D would be divided by 1.
    3. This means that for any value of D there will be an integer value.
5. Consider the equation 27x + 14y + 10z = 1. Give parameterized solutions for all integer solutions x, y, z. How many parameters do you need? Hint: What does this equation represent, geometrically?
    1. 27x + 14y + 10z = 1, Since a parametric equation needs only two variables, and in the one we are given we have three we have to represent the last one in terms of the first two.
    2. this would mean that we have x,y, and (1-27x+14y)/10. This would give us two variables and it would solve the equation.
    3. This is also tells us that it would be 3D.
6. Consider the following system of equations:… Are there any integer solutions to this system of equations? If so, what are they? Hint: What does the solution to this system of equations represent, geometrically?
    1. We are given the equations 27x+14y+10z=1 and 3x+5y+7z=1. Since there are two equations and 3 variables this would mean that there are infinite solutions.
        1. you can prove this from a matrix through gaussian elimination and find that there are only 2 pivot columns meaning that the last column is a free variable.
    2. So we can establish x=n this would then give us the two equations
        1. 14y+10z=1-27n
        2. 3y+5z=1-3n
    3. This after multiplying the second equation by 2 and subtracting it from the first one we get
        1. 8y=21n-1
        2. y=(21n-1)/8
    4. And we can now find out what z is since we have two of the variables and this would be
        1. z=((1-3n)+(5/8)(21n-1))/3
        2. This would represent a 3D line and the parametric equation would be

3. there are integer solutions to the system of equations and they are
   $\{n , (21n-1)/8, ((1-3n)+(5/8)(21n-1)/3)\}$