CS 205: Homework Set 2

16:198:205 (Sections 4 - 6)

Complete each of the following problems to the best of your ability. Remember, you can't be graded on what you don't write down so (unless you are just making stuff up) something is better than nothing. Discussing problems between each other is fine, but your final writeup and work must be your own.

- **Sets**

  1) (Ch. 2 Supplementary Exercises, 1) Let A be the set of English words that contain the letter x, and let B be the set of English words that contain the letter q. Express each of these as a combination of A and B:
     a) The set of English words that do not contain the letter x.
     b) The set of english words that contain both an x and a q.
     c) The set of English words that contain an x but not a q.
     d) The set of English words that do not contain either an x or a q.
     e) The set of English words that contain an x or a q, but not both.
  2) (Ch. 2 Supplementary Exercises, 2) Show that if A is a subset of B, then the power set of A is a subset of the power set of B.
  3) (Ch. 2 Supplementary Exercises, 4) Let E denote the set of even integers, let O denote the set of odd integers. As usual, let Z denote the set of all integers. Determine each of these sets:
     a) $E \cup O$
     b) $E \cap O$
     c) $Z - E$
     d) $Z - O$
  4) Show that the even and odd integers (E and O) form a **partition** of the integers Z.
  5) (Ch. 2 Supplementary Exercises, 6) Let A and B be sets. Show that $A \subset B$ **if and only if** $A \cap B = A$.
  6) (Ch. 2 Supplementary Exercises, 8) Suppose that A, B, and C are sets. Prove or disprove that $(A - B) - C = (A - C) - B$.

- **Functions** For these problems, let S_N be the set of numbers $\{1, 2, 3, \ldots, N\}$.

  1) How many functions are there that map S_N to S_N?
  2) Argue that if a map $f : S_N \mapsto S_N$ is surjective, then f is a bijection.
  3) Argue that if a map $f : S_N \mapsto S_N$ is injective, then f is a bijection.
  4) How many bijections are there that map S_N to S_N?
  5) Suppose f is a map from a set S to itself, $f : S \mapsto S$.
     * If S is a finite set, argue that $|f(S)| = |S|$ **if and only if** f is a bijection.
     * Give an example of an infinite S and an f such that
       · f is not a bijection, but $|f(S)| = |S|$.
       · f is a bijection, but $f(S)$ is a **proper subset** of S.

- **Cardinality**

1) (Ch. 2 Supplementary Exercises, 12) Let A and B be subsets of the finite universal set U. Show that

$$|A^c \bigcap B^c| = |U| - |A| - |B| + |A \bigcap B|. \tag{1}$$

2) (Ch. 2 Supplementary Exercises, 14) Suppose that f is a function from A to B where A and B are finite sets. Explain why $|f(S)| \leq |S|$ for all subsets S of A.

3) Argue that the set of irrational numbers is uncountable.

- **Computability**

1) Argue that the set of computer programs in any language must be at most **countably** infinite. *What do we know about computer programs?*

2) Argue that for a given programming language, **most real numbers cannot be computed as the (potentially infinite) output of any program.**

3) Different programming languages have different abilities. Is it possible that between **ProgrammingLanguage 1** and **ProgrammingLanguage 2**, they could, taken together, compute all real numbers? Why or why not.

4) Is it possible that the set of programming languages is uncountably infinite? Why or why not. Would it help us compute more things if it were?

5) Argue that there are functions from the natural numbers to the natural numbers that are uncomputable by any computer program. What do you need to show in order to prove this?