Dov Kassai Discrete Final Question 3

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1. Compute the first 10 terms of this sequence.
   1. x0=1
   2. x1=2
   3. x2=8
   4. x3=28
   5. x4=100
   6. x5=356
   7. x6=1268
   8. x7=4514
   9. x8=16084
   10. x9=57284
2. Prove that this sequence is strictly increasing, i.e., ∀n ≥ 0 : Xn+1 > Xn. (2 points)
   1. We can say that $X_k$ >0 for the base case when k=1.
   2. Now that we have this we can then try to prove the rest by induction.
   3. we then have $3X_{k+1}$=$3X_k$+$2X_{x-1}$, and we know from the base case that the right side of the equation must be greater than zero because of the base case. This only proves that it is only greater than 0.
   4. The next step we need to take is proving that it is strictly increasing. If we set K to be a natural number in the set from {0…k}, this then gives us $X_k$>$X_{k-1}$. Next we say that $X_{k+1}$-$X_k$ = (3$X_k$+2$X_{k-2}$)-$X_k$>0. This is because we proved that they are all greater than 0 so it must be that it is an increasing sequence by induction.
3. Prove that ∀n ≥ 0 : X^n ≤ 4^n. What are the base cases? What is the inductive step?
   1. In this case there would be 2 base cases
      1. when n=0; 1<=4^0
      2. and n=1; 2<4
   2. The inductive step here would be looking that k+1 is going to be true
   3. $X_{k+1}$=3$X_k$+2$X_{k-1}$=3*4^k+2*4^(k-1).
   4. this can then show that $X_{k+1}$=3*(4^k)+.5*4^(k-1).
   5. this shows that $X_{k+1}$=3.5*4^k<=4*4^k=4^(k+1).
4. The above result suggest that this sequence grows in the worst case exponentially. Consider trying to tighten this bound in the following way: what are the smaller values of alpha and beta such that the promo still works for all n>= 0: $X_n$<=is (alpha)*(beta^n). Give the tightest bounds on $X_n$ you can. (for the purpose of this problem å=alpha and ß=beta)
   1. First we need to have the base case where n= 0, so this would be that å>=$X_0$=1

    2. The nest step would be saying that 3å$_k$<3åß^(k) and 2$X_{k-1}$≤3åß^(k-1), putting these together we get $X_n$<3åß^k+2åß^(k-1) and the right side of the equation is goin to be equal to åß^(k+1)+[((3-ß)ß+2)åß(n+1)] which is going to be ≤ åß^(n+1).
    3. now that we have åß^(k+1)+[((3-ß)ß+2)åß(n+1)], this must mean that [((3-ß)ß+2)åß(n+1)]≤0.
    4. Since åß(n+1) is not a negative number we must have that ((3-ß)ß+2)≤0.
    5. Now we must solve the quadratic equation, which would give us that ß≥(3+√17)/2
    6. Now if we plug in that å=1 and ß=(3+√17)/2, we can find out that the condition is satisfied, that means that these must be the lowest values of å and ß.
    7. This shows that the tightest upper bounds on this would be $X_k$=O(1(3+√17)/2)^k).
5. What is the tightest *lower* bound on $X_n$ you can achieve this way? Characterize the long term/ asymptotic behavior of $X_n$ as best as you can/
    1. For this we would essentially need to flip around the inequality to find out what the tightest lower bounds are, we need to prove that åß^(k)≤$X_k$ for all k.
    2. The base case for this would be when k=1. This would give us that åß<$X_1$=2, and this can say that å=2/ß.
    3. Since in the previous problem we had the quadratic ((3-ß)ß+2)≤0, now we need to prove that it is greater than or equal to zero.
    4. Again from the previous problem we know that ß in this problem must be ≤(3+√17)/2. Since we have the inequality that å≤2/ß this must be that å≤4/(3+√17)
    5. Since the two values for å and ß satisfy the conditions this would mean that they are the greatest that they could be.
    6. So these two values we find the sequence x to not work in the case where $X_n$=ø(((3+√17)/2)^2), which would be the asymptotic behavior of the sequence.
6. Compute the first 10 terms of this sequence (out to ten decimal places). (3 points)
    1. x0=1
    2. x1=2
    3. x3=1.25
    4. x4=1.8125
    5. x5=1.390625
    6. x6=1.70703125
    7. x7=1.4697265625
    8. x8=1.64770507813
    9. x9=1.51422119141
    10. x10=1.6143341064
7. Prove that this sequence satisfies ∀n≥0 : 1≤$Y_n$≤ 2.

1. We can say from the previous problem that $1 \leq Y_{n-1} \leq 2$, and $1 \leq Y_{n-2} \leq 2$.
2. If we add the two together we would get $2 \leq Y_{n-1}+Y_{n-2} \leq 4$.
3. Next we can divide both sides by 2 and this would be that
    1. $1 \leq [(Y_{n-1})/2] + [(Y_{n-2})/2] \leq 2$
4. This would then prove from the values that we have gotten before that: $1 \leq Y_n \leq 2$.

8. Prove that the following holds for all $n \geq 0$: …
    1. If we have $Y_0=1$ and $Y_1=2$ and $Y_n=(1/4)\ Y_{n-1}+(3/4)\ Y_{n-2}$
    2. We are trying to prove that $Y_{n+1}+ Y_n=(-3/4)$^n
    3. The base case would be for when n=0
        1. this would then make the equation 2-1=1, which is true
    4. The inductive step would be this:
        1. assuming that the given statement is true for n=1,2,3,…k
        2. Then $Y_{k+1}+ Y_k=(-3/4)$^k, for all the k=0,1,2,3…k-1,k
        3. Now we prove that the statement n=k+1, which would be that
        4. $Y_{k+2}+ Y_{k+1}=(-3/4)$^(k+1)
        5. If we take the left hand side of the problem and then plug it into the previous:
            1. $[\{(1/4)Y_{(k+1)}+(3/4)Y_k\}-\{(1/4)Y_{(k)}+(3/4)Y_{(k-1)}\}]$
            2. rewriting this it will become$\{(1/4)[Y_{(k+1)}-Y_k]+(3/4)[Y_k+ Y_{k-1}]$
            3. again rewriting this it will become (1/4)[(-3/4)^k]+(3/4)(-3/4)^(k-1) (using the problem in step 3).
            4. rearranging the number is would be (1/4)[(-3/4)(-3/4)^(k-1)]+(3/4)(-3/4)^(k-1)
            5. which is the same as (-3/4)^(k-1){(-3/16)+(3/4)}={(-3/4)^(k-1)}(-3/4)^2
            6. Finally showing that (-3/4)^(k+1)
        6. Given that the statement n=k+1 is true, then by using induction we proved that the statement is true.

9. Argue that $Y_n=1+$……
    1. We have to prove that $Y_n-Y_{n-1}=(-3/4)$^n, this is the same as $Y_n=(-3/4)$^(n-1)—$Y_{n-1}$
    2. The right hand side can be broken down to:
        1. {(-3/4)^(n-1)] +[(-3/4)^(n-2)+$Y_{n-1}$]…+(-3/4)+(-3/4)^0} OR {(-3/4)^(n-1)+(-3/4)^(n-2)…+(-3/4)+(-3/4)^0 +$Y_0$}
        2. This would mean that it would be 1+(n-1)∑(k=0)[(-3/4)^k]
    3. This would then prove that the statement is true.

10. What is the limit of $Y_n$ as $n \to \infty$.
    1. We have the equation $Y_n$=1+(n-1)∑(k=0)[(-3/4)^k].
    2. This can be written as $Y_n$=1(1-(-3/4)^k) / (1-(-3/4)), which is the same as writing 1+(4/7)(1-(-3/4)^k)

3. Now as $n \to \infty$ this would mean that $(-3/4)^k$ would go to zero and then we would have the equation of $1+(4/7)= 11/7$, this would be the final limit.