1.6 The Lagrange Equations of Motion

A single particle with no constraints is free to take up any position in the three-dimensional space. This is the reason we need three coordinates to describe its position. In the case of a system of p particles, there are $3p$ degrees of freedom. In general, if there are constraints imposed on the particles, then the number of the degrees of freedom is

$$n = 3p - \text{number of constraints}.$$

For example, if the system consists of only one particle in space and if the particle is restricted to move along a line, then $n = 3-2 = 1$. We can use many different coordinate systems to describe a location of a particle in space. We will use the letter q as a symbol for a coordinate regardless of its nature. A generalized coordinate may represent, for example, angular displacement or electric charge. This indicates that the results we are going to obtain will be of general nature and will not be restricted to mechanical systems. We refer to q as a generalized coordinate. Thus, if

$$\begin{aligned} x &= x(q_1, q_2, q_3), \\ y &= y(q_1, q_2, q_3), \\ z &= z(q_1, q_2, q_3), \end{aligned}$$

then

$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \frac{\partial x}{\partial q_3}\dot{q}_3,$$

and so on. We assume that the functions $x, y, z \in C^2$; that is, they are twice continuously differentiable, which implies that their second partial derivatives are equal. Suppose now that a particle is confined to move on a surface. Therefore, to describe the location of the particle, we need only two generalized coordinates. The position of the particle on a surface in a three-dimensional space is then described as

$$\begin{aligned} x &= x(q_1, q_2), \\ y &= y(q_1, q_2), \\ z &= z(q_1, q_2), \end{aligned}$$

and hence

$$\begin{aligned} \delta x &= \frac{\partial x}{\partial q_1}\delta q_1 + \frac{\partial x}{\partial q_2}\delta q_2, \\ \delta y &= \frac{\partial y}{\partial q_1}\delta q_1 + \frac{\partial y}{\partial q_2}\delta q_2, \\ \delta z &= \frac{\partial z}{\partial q_1}\delta q_1 + \frac{\partial z}{\partial q_2}\delta q_2. \end{aligned} \tag{1.16}$$

We know that the work δW done by the force $\boldsymbol{F}$ is $\delta W = F_x\delta x + F_y\delta y + F_z\delta z$. On the other hand, this work is equal to the change in the kinetic energy. Therefore,

$$\boxed{m(\ddot{x}\,\delta x + \ddot{y}\,\delta y + \ddot{z}\,\delta z) = F_x\,\delta x + F_y\,\delta y + F_z\,\delta z} \tag{1.17}$$

The above equation is known as *D'Alembert's* equation.

If we have a system of particles acted upon by forces $\boldsymbol{F}_1, \ldots, \boldsymbol{F}_p$ and if each of the particles undergoes displacements $\delta \boldsymbol{s}_1, \ldots, \delta \boldsymbol{s}_p$, then the total work done by the forces is

$$\delta W = \sum_{i=1}^{p} (F_{x_i}\,\delta x_i + F_{y_i}\,\delta y_i + F_{z_i}\,\delta z_i). \tag{1.18}$$

Substituting (1.16) into D'Alembert's equation and rearranging, we obtain

$$\begin{aligned} \delta W &= m\left(\ddot{x}\frac{\partial x}{\partial q_1} + \ddot{y}\frac{\partial y}{\partial q_1} + \ddot{z}\frac{\partial z}{\partial q_1}\right)\delta q_1 + m\left(\ddot{x}\frac{\partial x}{\partial q_2} + \ddot{y}\frac{\partial y}{\partial q_2} + \ddot{z}\frac{\partial z}{\partial q_2}\right)\delta q_2 \\ &= \left(F_x\frac{\partial x}{\partial q_1} + F_y\frac{\partial y}{\partial q_1} + F_z\frac{\partial z}{\partial q_1}\right)\delta q_1 + \left(F_x\frac{\partial x}{\partial q_2} + F_y\frac{\partial y}{\partial q_2} + F_z\frac{\partial z}{\partial q_2}\right)\delta q_2. \end{aligned} \tag{1.19}$$

Let

$$F_{q_1} = F_x\frac{\partial x}{\partial q_1} + F_y\frac{\partial y}{\partial q_1} + F_z\frac{\partial z}{\partial q_1} \quad \text{and} \quad F_{q_2} = F_x\frac{\partial x}{\partial q_2} + F_y\frac{\partial y}{\partial q_2} + F_z\frac{\partial z}{\partial q_2}.$$

Then, (1.19) can be represented as

$$\delta W = F_{q_1}\,\delta q_1 + F_{q_2}\,\delta q_2.$$

To proceed further, note that

$$\frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1}\right) = \ddot{x}\frac{\partial x}{\partial q_1} + \dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right).$$

Hence

$$\boxed{\ddot{x}\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1}\right) - \dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right)} \tag{1.20}$$

The time derivative of $x = x(q_1, q_2)$ is

$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2. \tag{1.21}$$

Differentiating the above partially with respect to $\dot{q}_1$ yields

$$\boxed{\frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial x}{\partial q_1}} \tag{1.22}$$

Finally, we will show that

$$\boxed{\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{\partial \dot{x}}{\partial q_1}} \tag{1.23}$$

Indeed, because $x = x(q_1, q_2)$ and the partial derivative $\partial x/\partial q_1$ is in general a function of q_1 and q_2, we have

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{\partial}{\partial q_1}\left(\frac{\partial x}{\partial q_1}\right)\dot{q}_1 + \frac{\partial}{\partial q_2}\left(\frac{\partial x}{\partial q_1}\right)\dot{q}_2. \tag{1.24}$$

Taking the partial derivative of (1.21) with respect to q_1 yields

$$\begin{aligned}\frac{\partial \dot{x}}{\partial q_1} &= \frac{\partial}{\partial q_1}\left(\frac{\partial x}{\partial q_1}\right)\dot{q}_1 + \frac{\partial}{\partial q_1}\left(\frac{\partial x}{\partial q_2}\right)\dot{q}_2\\ &= \frac{\partial}{\partial q_1}\left(\frac{\partial x}{\partial q_1}\right)\dot{q}_1 + \frac{\partial}{\partial q_2}\left(\frac{\partial x}{\partial q_1}\right)\dot{q}_2,\end{aligned} \tag{1.25}$$

because by assumption $x, y, z \in C^2$ and hence

$$\frac{\partial}{\partial q_1}\left(\frac{\partial x}{\partial q_2}\right) = \frac{\partial}{\partial q_2}\left(\frac{\partial x}{\partial q_1}\right).$$

With the above in mind, comparing the right-hand sides of (1.24) and (1.25), we obtain (1.23). We now use (1.20), (1.22), and (1.23) to arrive at the Lagrange equation of motion for a single particle. We first substitute (1.22) and (1.23) into (1.20) to get

$$\ddot{x}\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_1}\right) - \dot{x}\frac{\partial \dot{x}}{\partial q_1}. \tag{1.26}$$

Note that

$$\frac{\partial}{\partial \dot{q}_1}\left(\frac{\dot{x}^2}{2}\right) = \dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_1} \tag{1.27}$$

and

$$\frac{\partial}{\partial q_1}\left(\frac{\dot{x}^2}{2}\right) = \dot{x}\frac{\partial \dot{x}}{\partial q_1}. \tag{1.28}$$

Substituting (1.27) and (1.28) into (1.26) gives

$$\ddot{x}\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\frac{\partial(\dot{x}^2/2)}{\partial \dot{q}_1}\right) - \frac{\partial(\dot{x}^2/2)}{\partial q_1}.$$

We can obtain similar expressions for y and z. Taking this into account, when $\delta q_2 = 0$, equation (1.19) becomes

$$\begin{aligned}\delta W &= \left(\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_1}\frac{m(\dot{x}^2+\dot{y}^2+\dot{z}^2)}{2}\right) - \frac{\partial}{\partial q_1}\frac{m(\dot{x}^2+\dot{y}^2+\dot{z}^2)}{2}\right)\delta q_1\\ &= \left(F_x\frac{\partial x}{\partial q_1} + F_y\frac{\partial y}{\partial q_1} + F_z\frac{\partial z}{\partial q_1}\right)\delta q_1\\ &= F_{q_1}\delta q_1.\end{aligned} \tag{1.29}$$

Let $K = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ denote the kinetic energy of the particle. Then, we can represent (1.29) as

$$\boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_1}\right) - \frac{\partial K}{\partial q_1} = F_{q_1}} \tag{1.30}$$

The equation above is called the *Lagrange equation of motion* for the q_1 coordinate. Using the same arguments as above, we can derive the Lagrange equation of motion for the q_2 coordinate. In general there are as many the Lagrange equations of motion as there are degrees of freedom of the particle.

Recall that the expression $\delta W = F_x\delta x + F_y\delta y + F_z\delta z$ is a work done by a force $\boldsymbol{F} = [F_x\ F_y\ F_z]^T$ for a general displacement $\delta s = [\delta x\ \ \delta y\ \ \delta z]^T$. In terms of generalized coordinates and when, say, $\delta q_2 = 0$, we have

$$\begin{aligned}\delta W &= \left(F_x\frac{\partial x}{\partial q_1} + F_y\frac{\partial y}{\partial q_1} + F_z\frac{\partial z}{\partial q_1}\right)\delta q_1\\ &= F_{q_1}\delta q_1\\ &= \delta W_{q_1}.\end{aligned}$$

A generalized force F_{q_r} is of such nature that the product $F_{q_r}\delta q_r$ is the work done by driving forces when q_r alone is changed by δq_r. We mention that a generalized force does not have to be a force in the usual sense. For example, if q_r is an angle, then F_{q_r} must be a torque in order that $F_{q_r}\delta q_r$ be work.

We now derive a more general version of the Lagrange equations of motion for a single particle. In our derivation we will need the notion of *nonconservative* forces, which we discuss next. First recall that we interpret the kinetic energy of a particle as its ability to do work by virtue of its motion. This motion is the result of conservative and nonconservative forces acting upon the particle. A force is nonconservative if the work done by the force on a particle that moves through any round trip is not zero. Thus a force is nonconservative if the work done by that force on a particle that moves between two points depends on the path taken between these points.

Let us now suppose that in addition to the conservative forces, a single nonconservative force due to friction acts on the particle. Denote by $\sum W_c$ the sum of the work done by the conservative forces, and denote by W_f the work done by friction. We then have

$$W_f + \sum W_c = \Delta K. \tag{1.31}$$

Thus, if a nonconservative force, like friction, acts on a particle, then the total mechanical energy is not constant, but changes by the amount of work done by a nonconservative force according to the relation (1.31). Let us now denote by E the final mechanical energy of the particle and denote by E_0 its initial mechanical energy. Then, we can write

$$E - E_0 = \Delta E = W_f.$$

The work done by friction, the nonconservative force, is always negative. Hence, the final mechanical energy E is less than the initial mechanical energy. The "lost" mechanical energy is transformed into *internal energy* U_{int}. In other words, the loss of mechanical energy equals the gain in internal energy and we write

$$\Delta E + U_{int} = 0.$$

In general, we have

$$\Delta K + \sum \Delta U + U_{int} + \text{(change in other forms of energy)} = 0.$$

This means that the total energy, kinetic plus potential plus internal plus all other forms of energy, does not change. This is the *principle of the conservation of energy*. To proceed further, recall that any generalized force F_{q_i} acting on our particle can be expressed as

$$F_{q_i} = F_x \frac{\partial x}{\partial q_i} + F_y \frac{\partial y}{\partial q_i} + F_z \frac{\partial z}{\partial q_i}.$$

Assuming that the forces are conservative, we get

$$F_{q_i} = -\left(\frac{\partial U}{\partial x}\frac{\partial x}{\partial q_i} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial q_i} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial q_i} \right).$$

We thus have

$$F_{q_i} = -\frac{\partial U}{\partial q_i}.$$

Therefore, the Lagrange equations of motion for a single particle for conservative forces can be represented as

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} = -\frac{\partial U}{\partial q_i}.$$

We can rearrange the above equations as

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial}{\partial q_i}(K - U) = 0.$$

We now introduce the *Lagrangian function* L, defined as

$$L = K - U.$$

Note that because U is not a function of $\dot{q}_i$, we have

$$\frac{\partial K}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i}.$$

Taking the above into account, we can represent the Lagrange equations of motion for conservative forces as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0.$$

If some of the forces acting on the particle are nonconservative, the Lagrange equations of motion for our particle will take the form

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tilde{F}_{q_i}} \tag{1.32}$$

where $\tilde{F}_{q_i}$ are nonconservative forces acting on the particle.

◆ Example 1.2

Consider the simple pendulum shown in Figure 1.7. The simple pendulum is an idealized body consisting of a point mass M, suspended by a weightless inextensible cord of length l. The simple pendulum is an example of a one-degree-of-freedom system with a generalized coordinate being the angular displacement θ. The pendulum kinetic energy is

$$K = \tfrac{1}{2} M l^2 \dot{\theta}^2.$$

We assume that the potential energy of the pendulum is zero when the pendulum is at rest. Hence, its potential energy is

$$U = Mgl(1 - \cos\theta),$$

where $g = 10$ m/sec^2 is the acceleration due to gravity. The Lagrangian function is

$$\begin{aligned} L &= K - U \\ &= \tfrac{1}{2} M l^2 \dot{\theta}^2 - Mgl(1 - \cos\theta) \\ &= 6\dot{\theta}^2 - 60(1 - \cos\theta). \end{aligned}$$

The Lagrange equation describing the pendulum motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0.$$

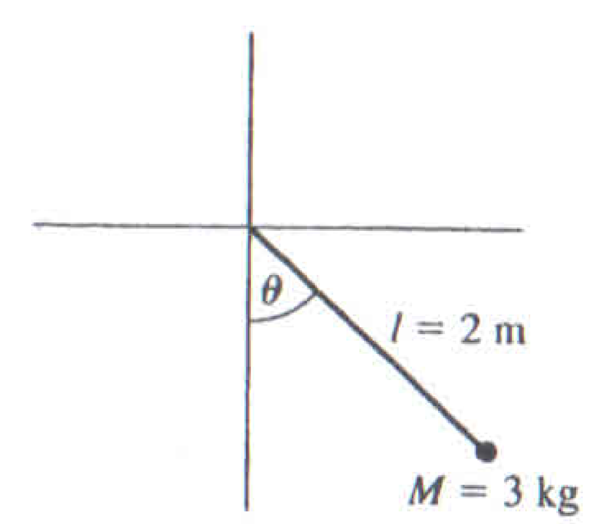

Figure 1.7 A simple pendulum.

In our example the above equation evaluates to

$$12\ddot{\theta} + 60\sin\theta = 0,$$

or, equivalently,

$$\ddot{\theta} = -5\sin\theta.$$

◆ Example 1.3

In Figure 1.8, the θ–r robot manipulator is shown. This example was adapted from Snyder [266, Chapter 10]. A lumped mass representation of the θ–r robot manipulator is shown in Figure 1.9. The mass $m_1 = 10$ kg represents the mass of the outer cylinder positioned at its center of mass. The constant distance $r_1 = 1$ m designates the fixed distance between the center of the mass of the outer cylinder and the center of rotation. The mass of the load is represented by $m_2 = 3$ kg and is assumed to be located at the end of a piston of a telescoping arm that is a variable radial distance r from the hub or center of rotation. The angle of rotation of the manipulator arm is θ. The inputs to the system are assumed to be (a) a torque T_θ

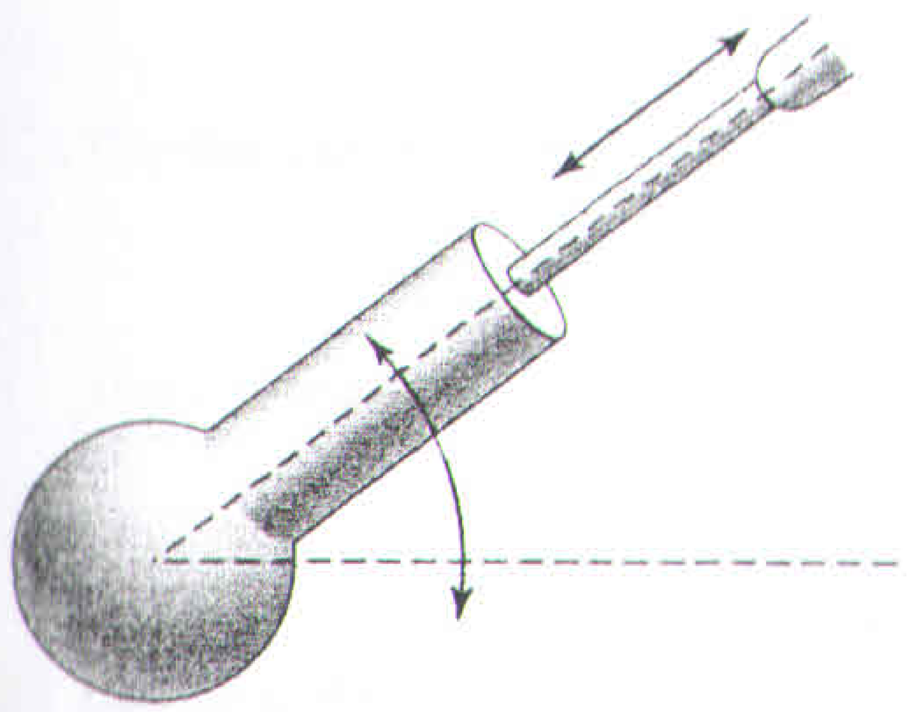

Figure 1.8 The θ–r robot manipulator.

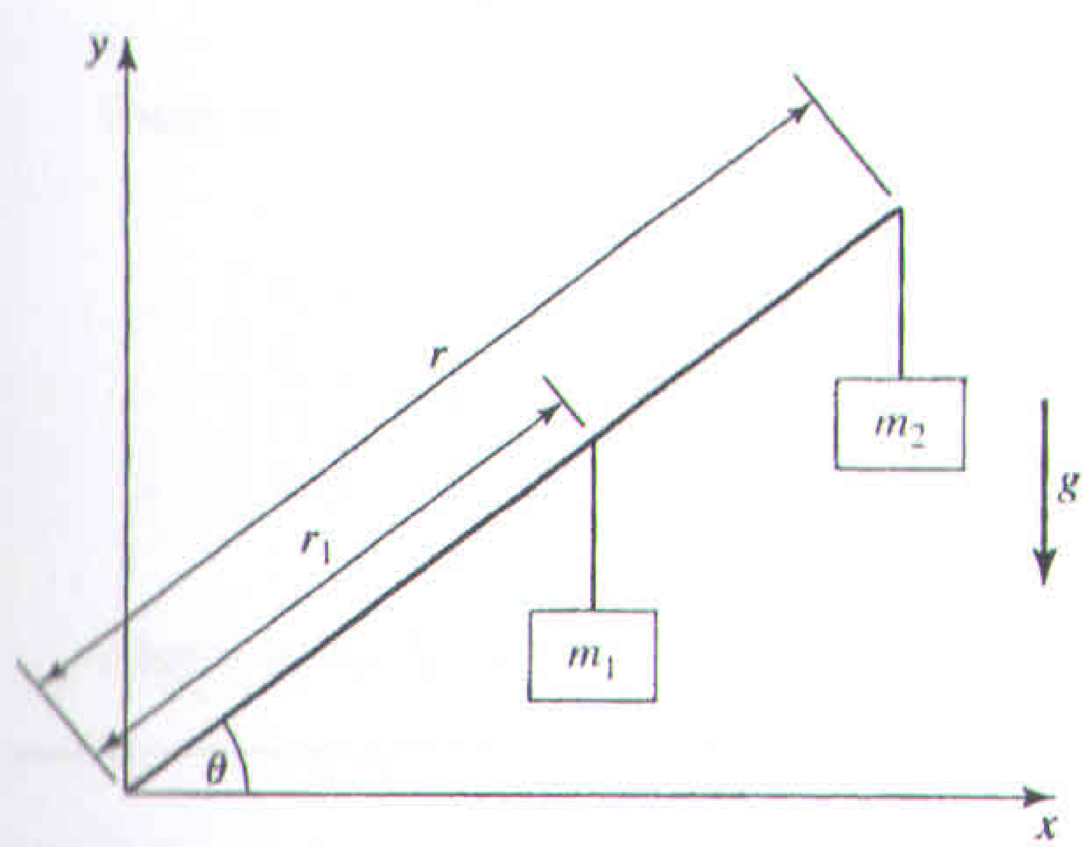

Figure 1.9 A lumped mass representation of the θ–r robot manipulator.