

Lecture 05

Loss Functions

(and probability models)

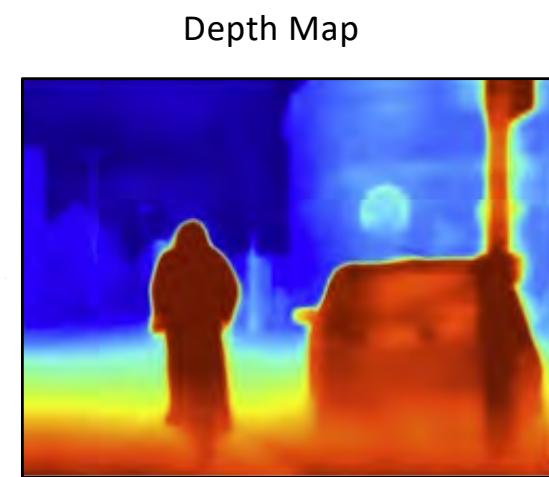
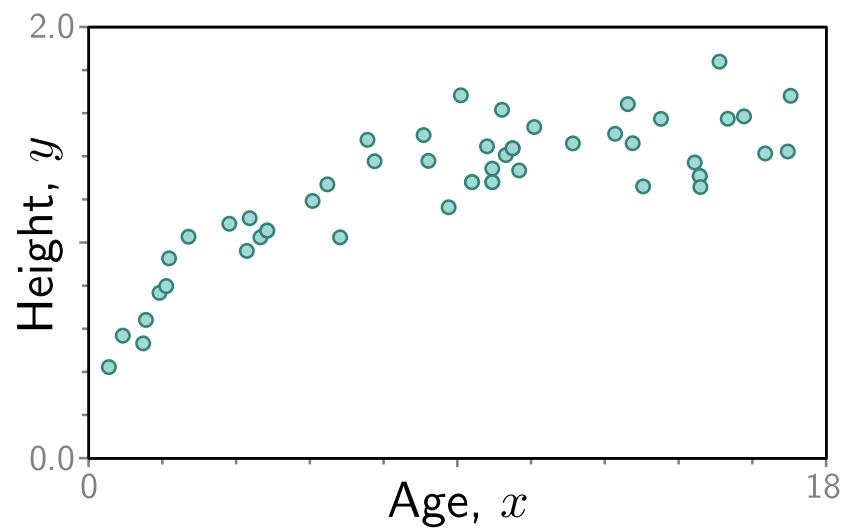
DL4DS – Spring 2025

Recap

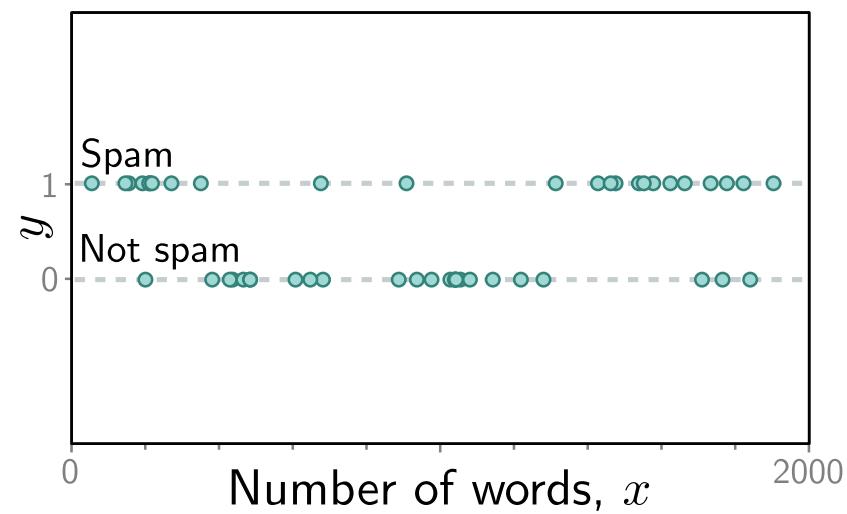
- So far, we talked about *linear regression*, *shallow neural networks* and *deep neural networks*
- Each have parameters, ϕ , that we want to choose for a *best possible mapping between input and output* training data
- A *loss function* or *cost function*, $L[\phi]$, returns a single number that describes a mismatch between $f[x_i, \phi]$ and the ground truth outputs, y_i .

We need to find a loss function
that works with...

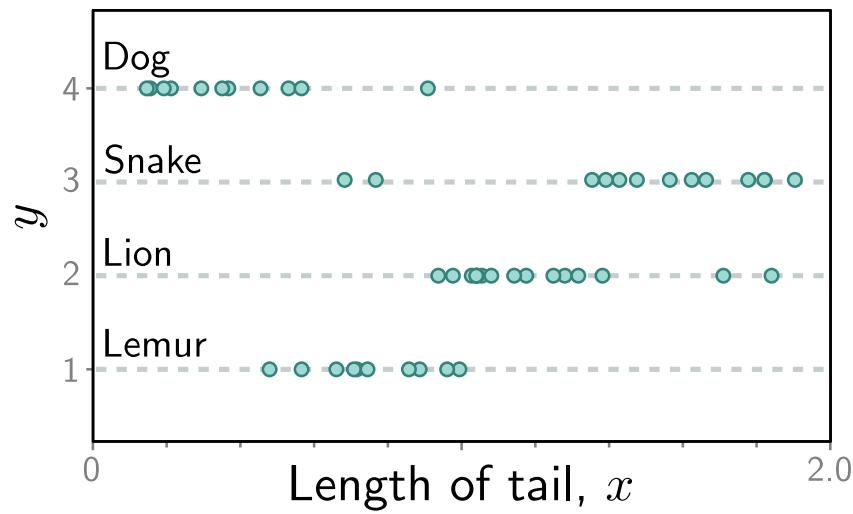
Univariate and Multivariate Regression

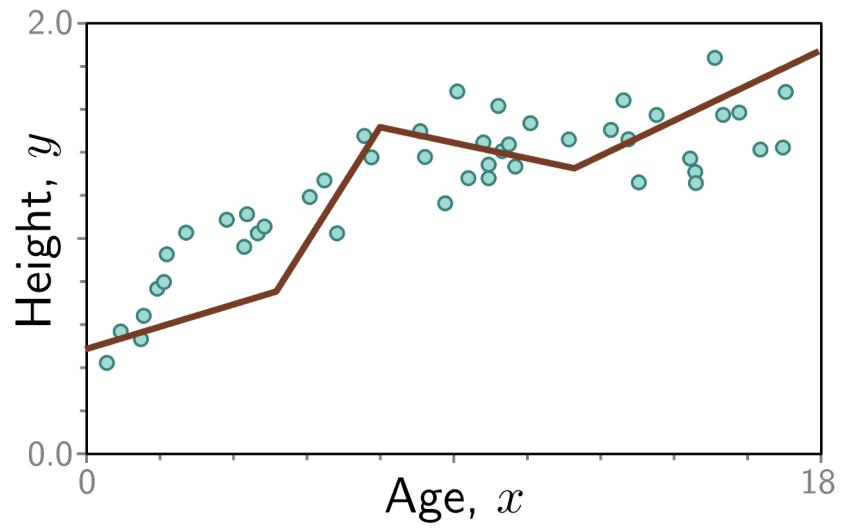


Binary Classification

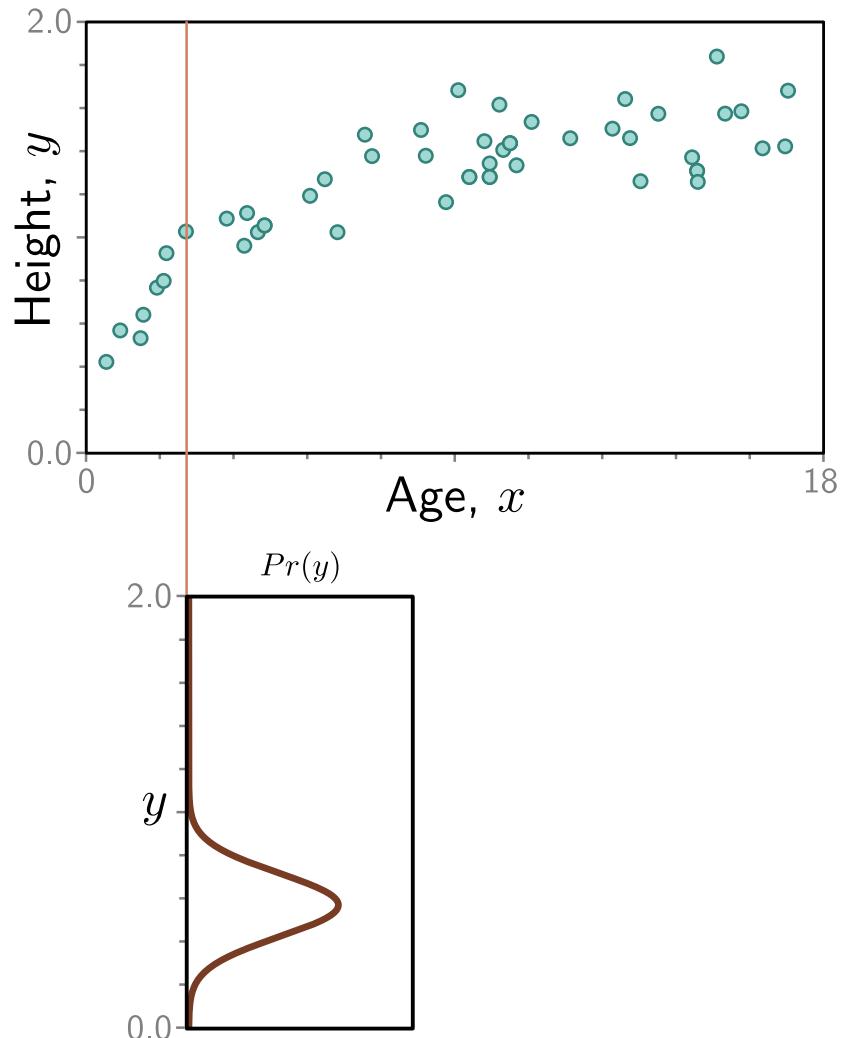


Multiclass Classification





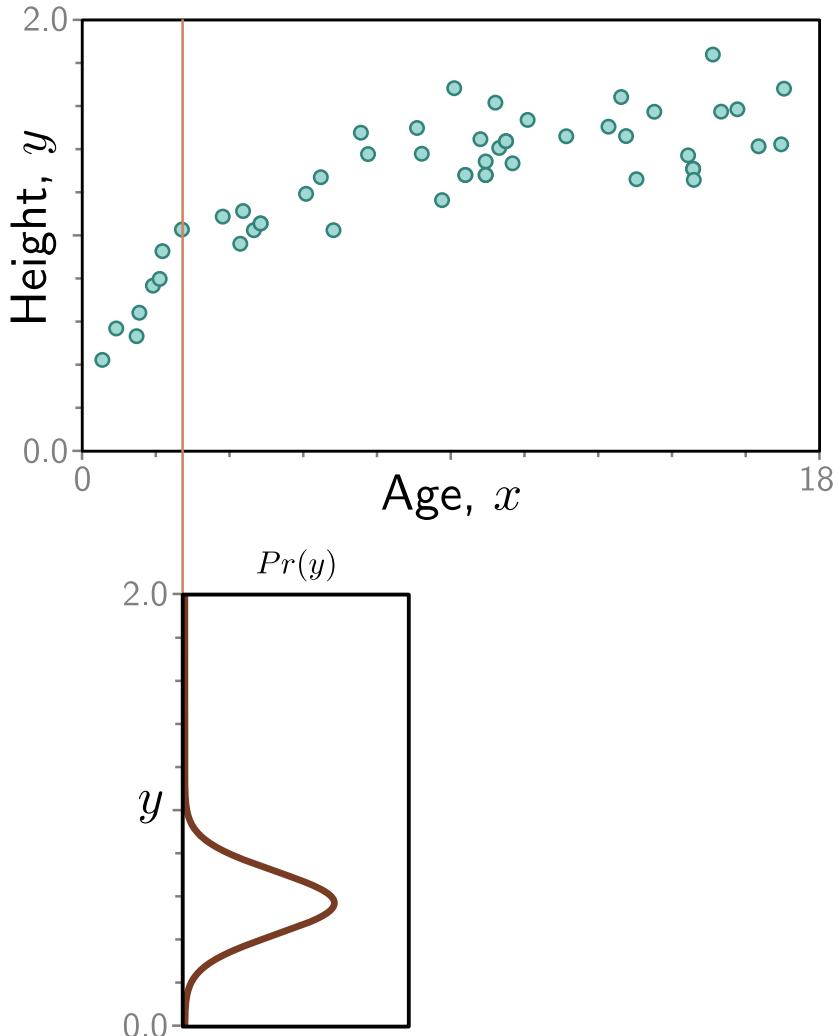
So far, we thought about
fitting a model to the data...



Alternatively, we can think about fitting a *probability model* to the data.

$$\Pr(y|x = 2.8)$$

Why?

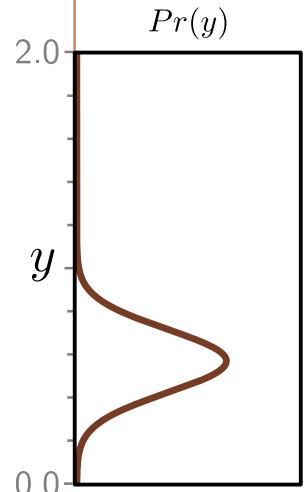
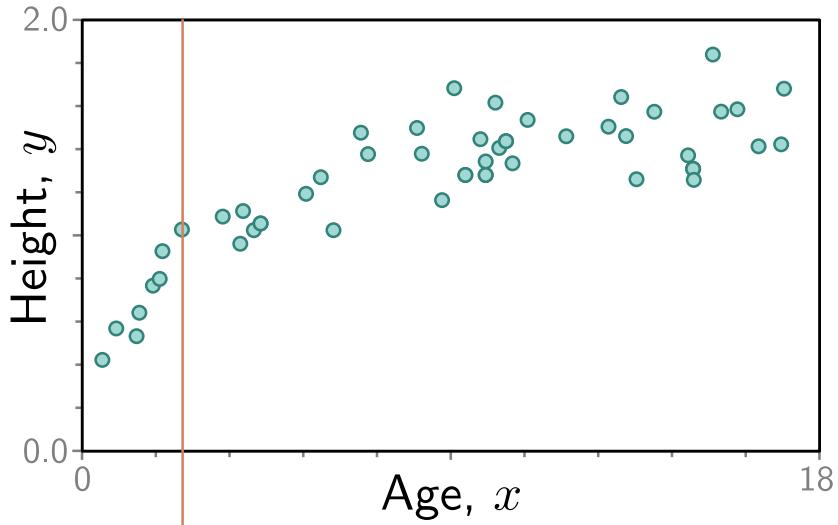


Alternatively, we can think about fitting a *probability model* to the data.

$$\Pr(y|x = 2.8)$$

Why?

Because this provides a *framework* to build loss functions for other prediction types...



Alternatively, we can think about fitting a *probability model* to the data.

$$\Pr(y|x = 2.8)$$

Why?

Because this provides a *framework* to build loss functions for other prediction types...

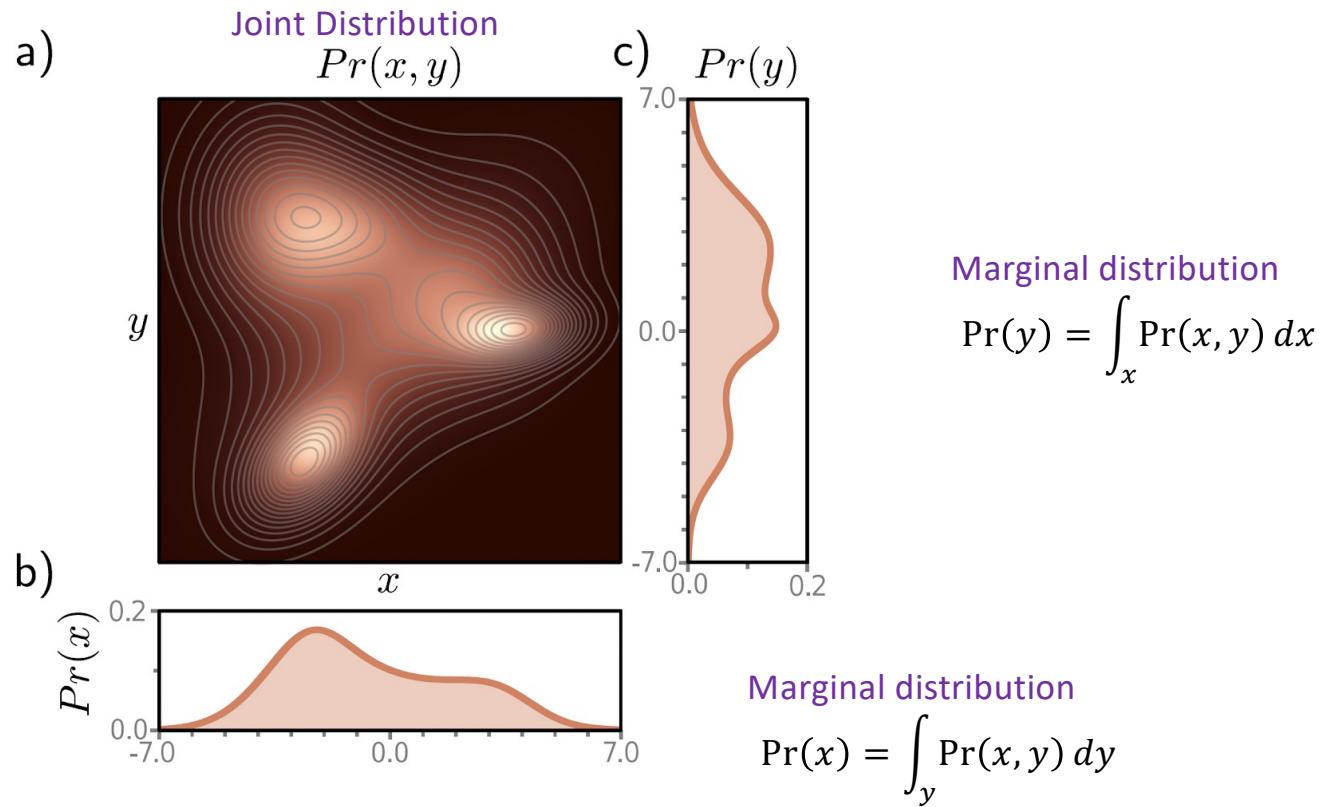
... and justifies least squares for real-valued regression models.

Brief Probability Review

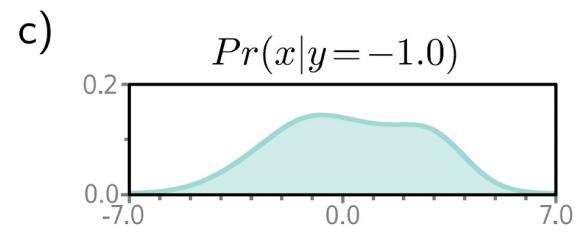
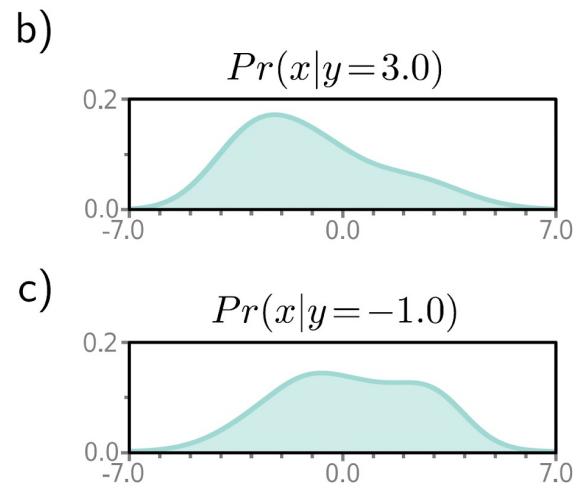
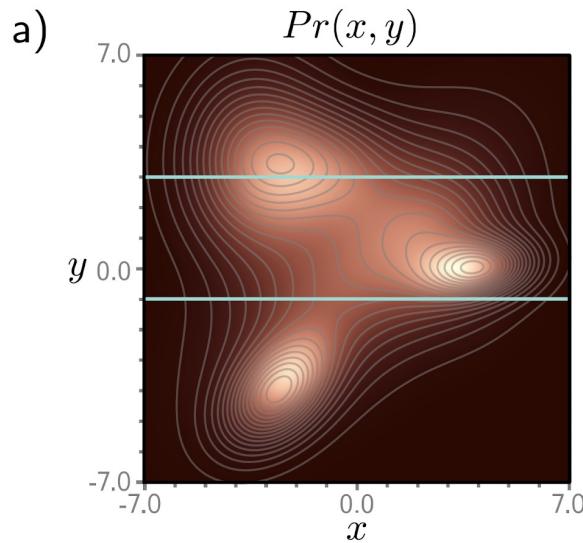
- Random variables, e.g. x and y
- $\Pr(x)$ is a probability distribution over x
- $0 \leq \Pr(x) \leq 1$
- $\int_x \Pr(x) dx = 1$ or $\sum_i \Pr(x_i) = 1$
- $\Pr(x, y) = \Pr(x) \cdot \Pr(y)$ when x and y are independent
- $\Pr(x | y) \Pr(y) = \Pr(x, y) = \Pr(y | x) \Pr(x)$
- And...

See [*Understanding Deep Learning*](#), Appendix C for more probability review

Joint and Marginal Probability Distributions

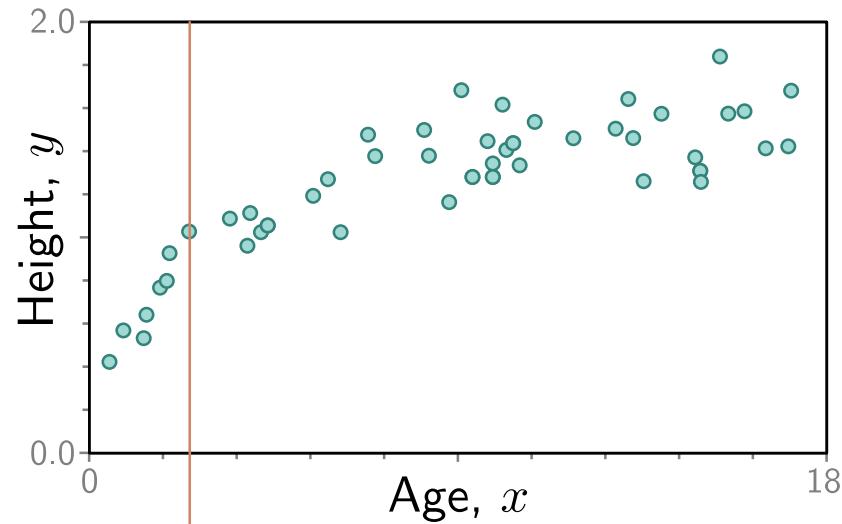


Conditional Probabilities

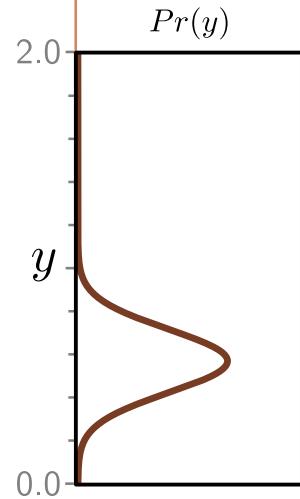


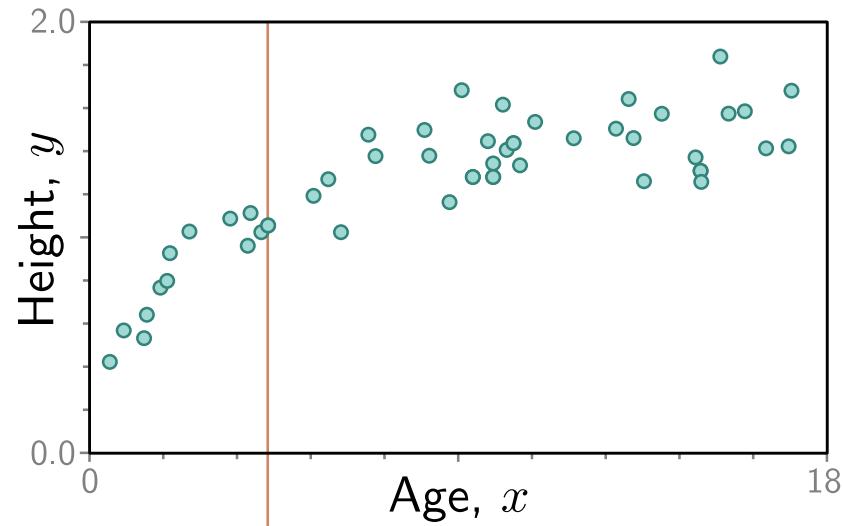
$$\int_x \Pr(x | y = 3.0) dx = 1$$

$$\int_x \Pr(x | y = -1.0) dx = 1$$

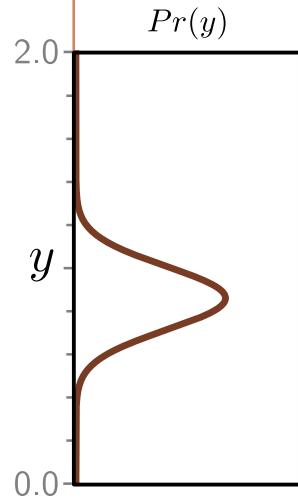


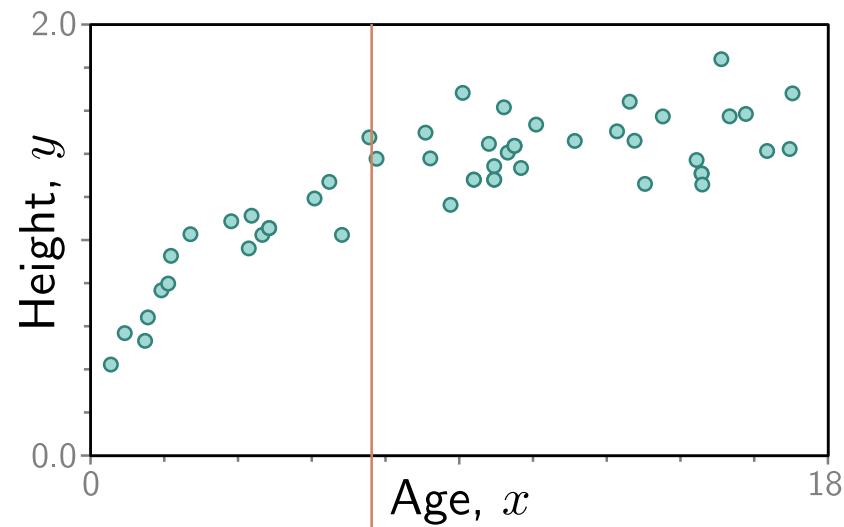
Continuous
 $\Pr(y|x = 2.5)$



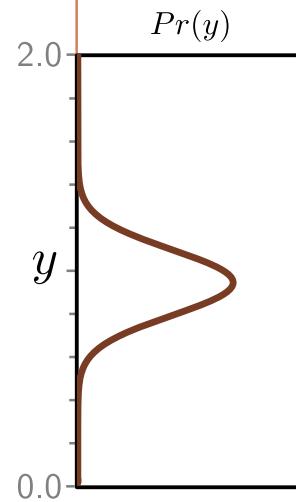


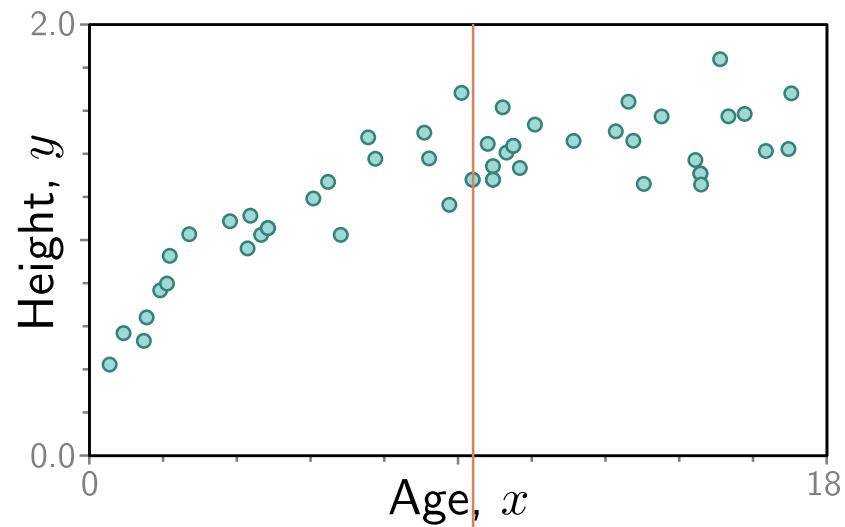
Continuous
 $\Pr(y|x = 4.8)$



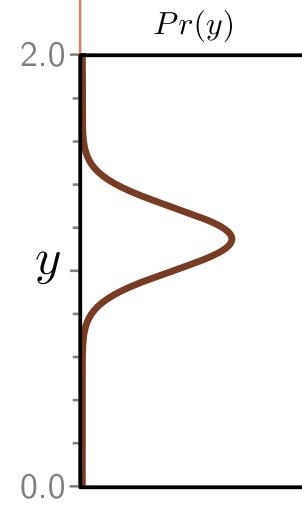


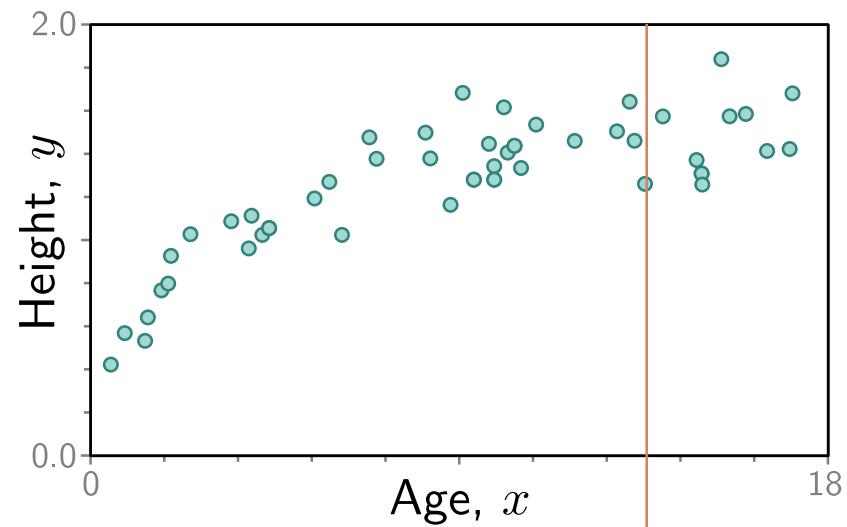
Continuous
 $\Pr(y|x = 7.5)$



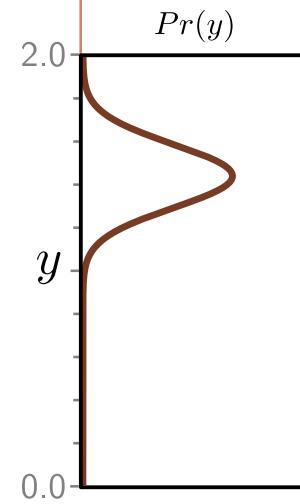


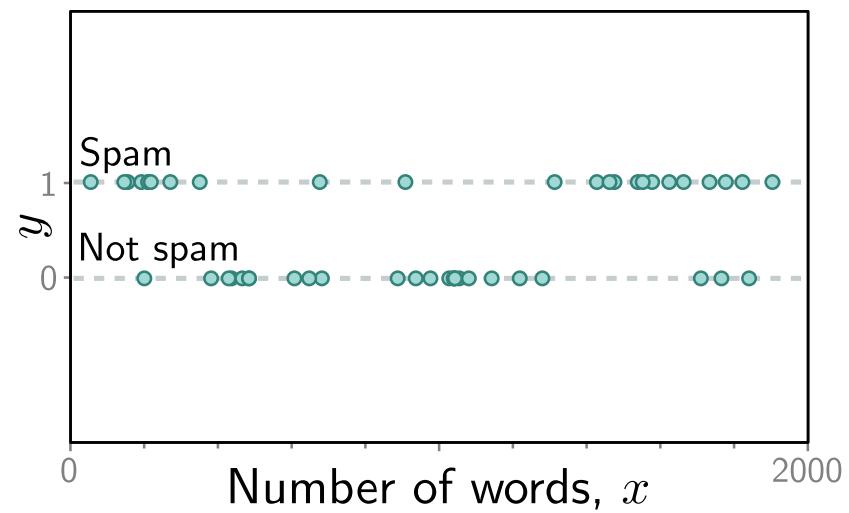
Continuous
 $\Pr(y|x = 10.5)$

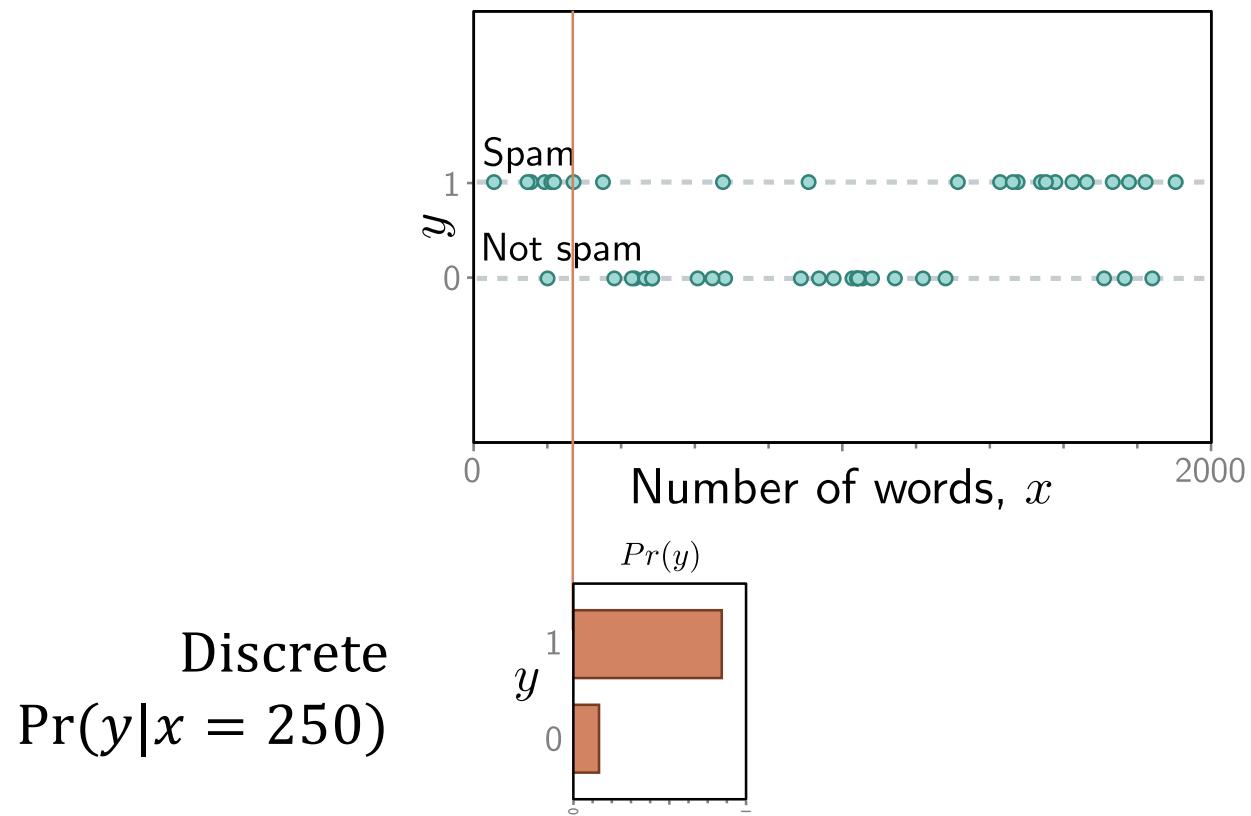


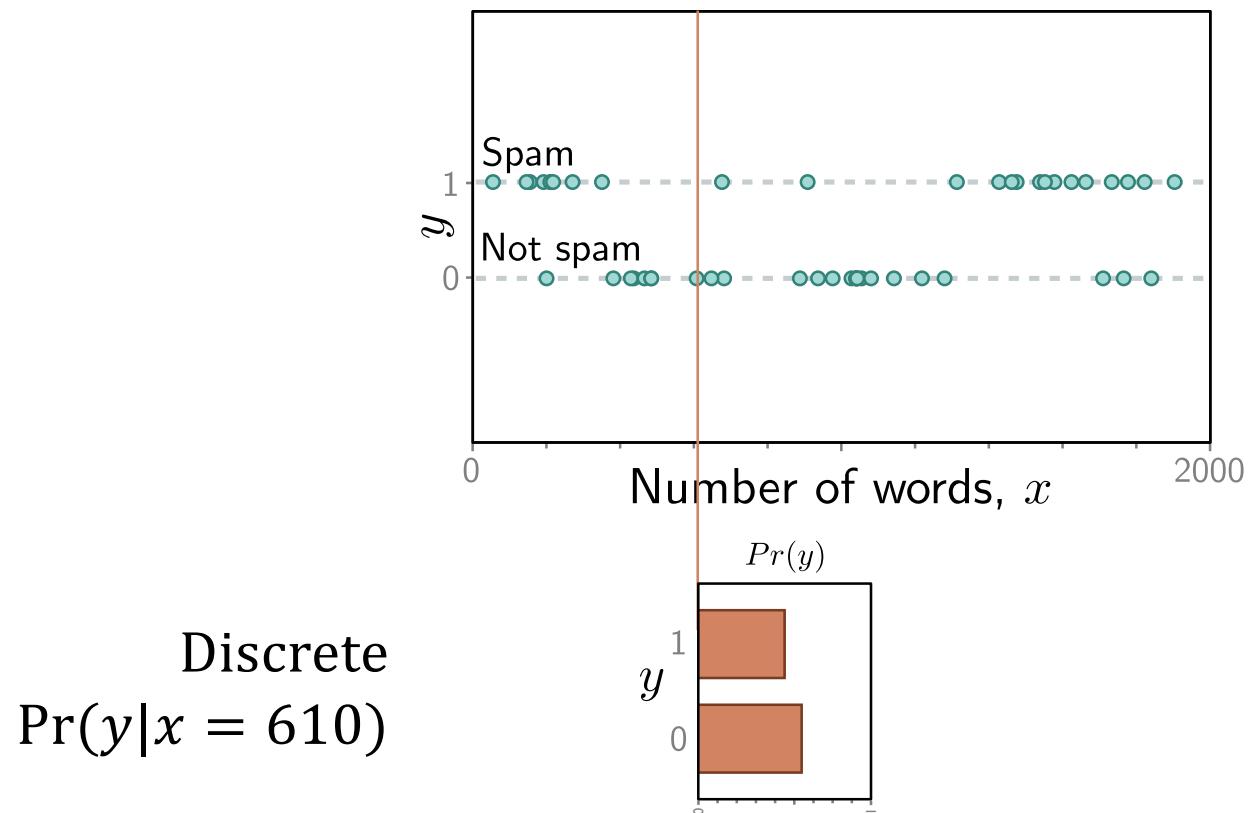


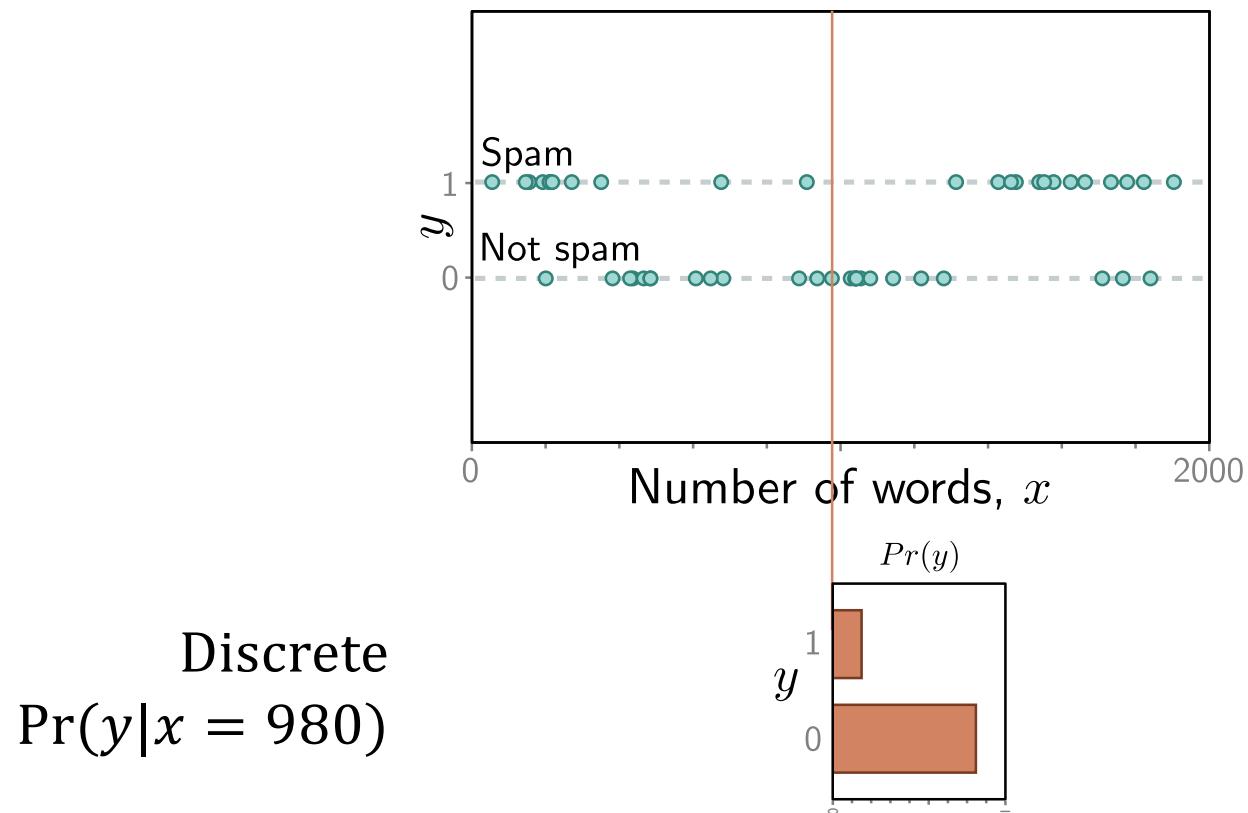
Continuous
 $\Pr(y|x = 13.2)$

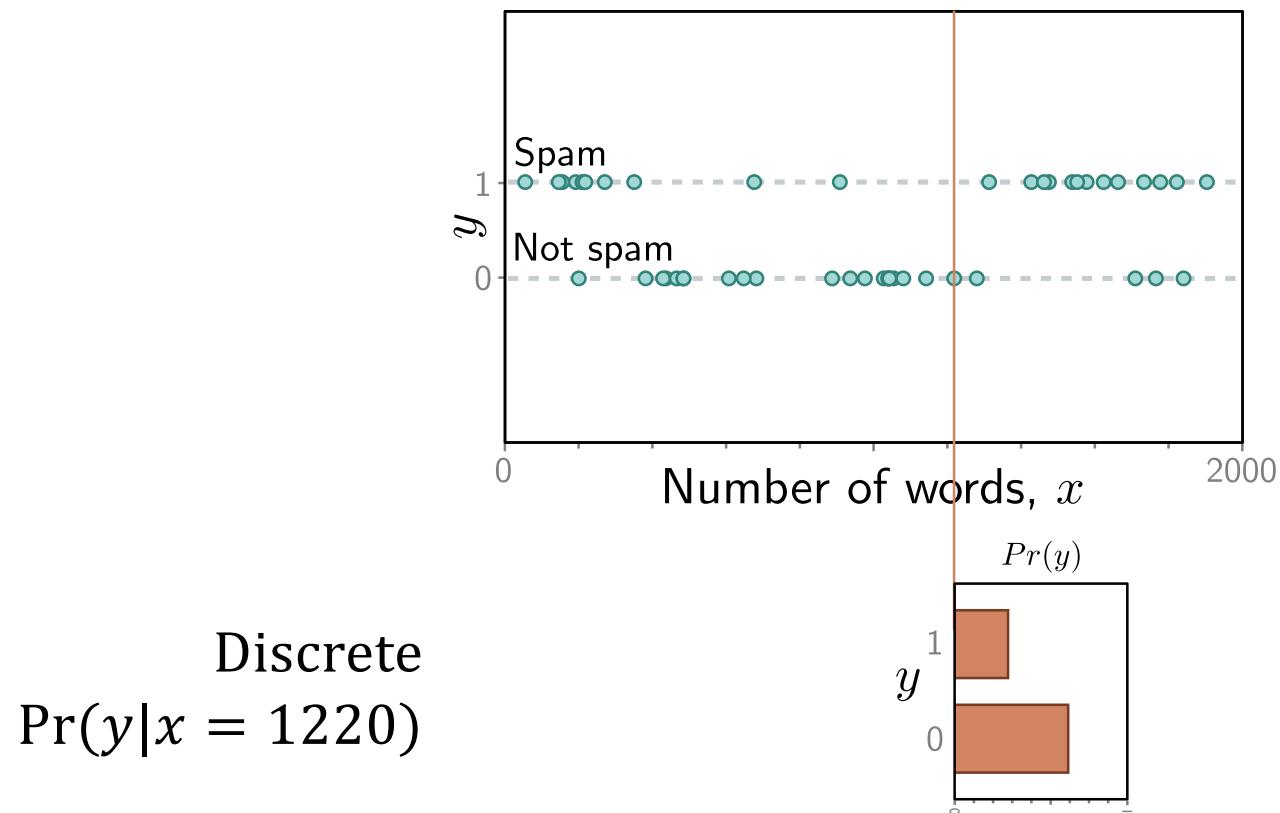




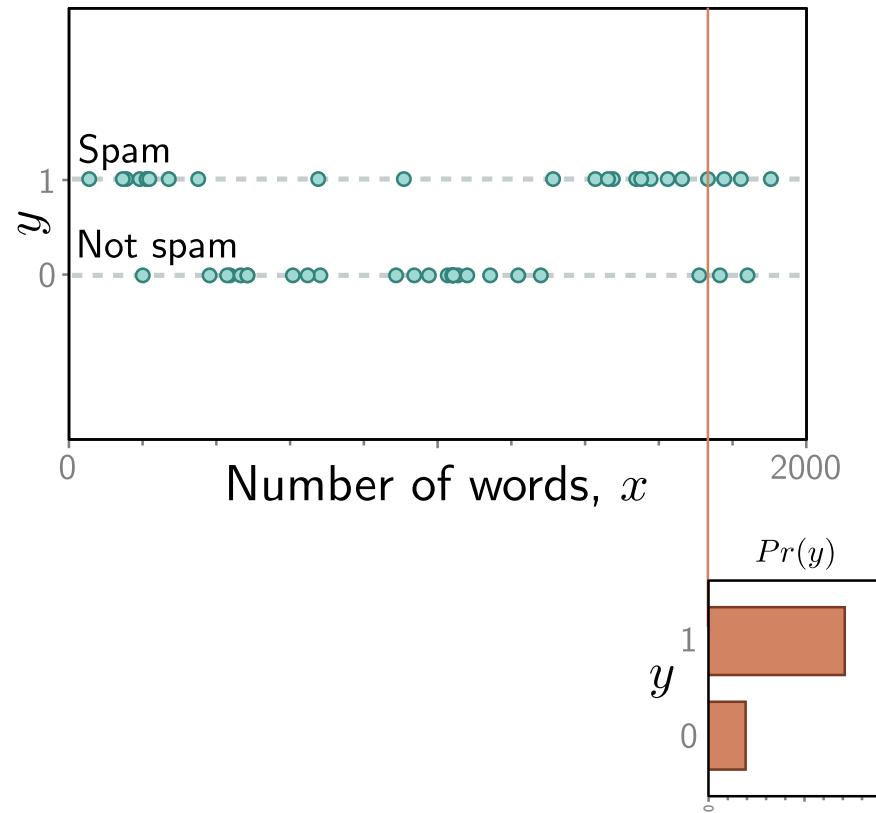


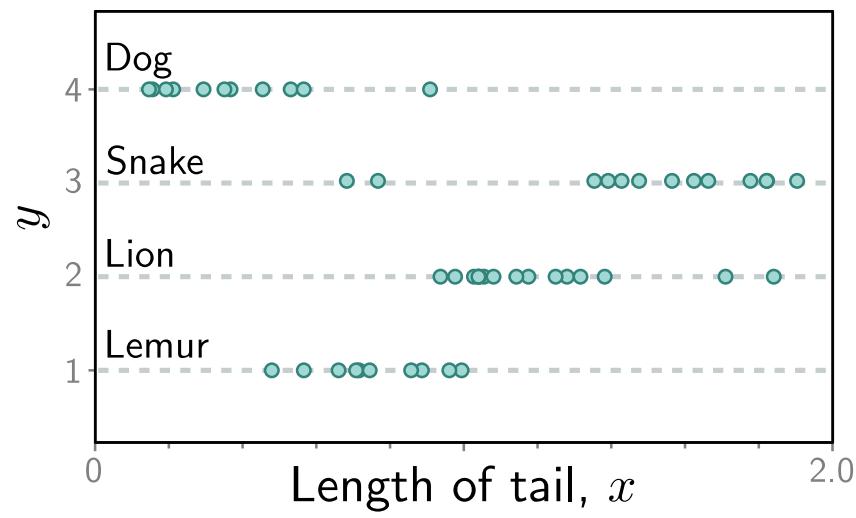


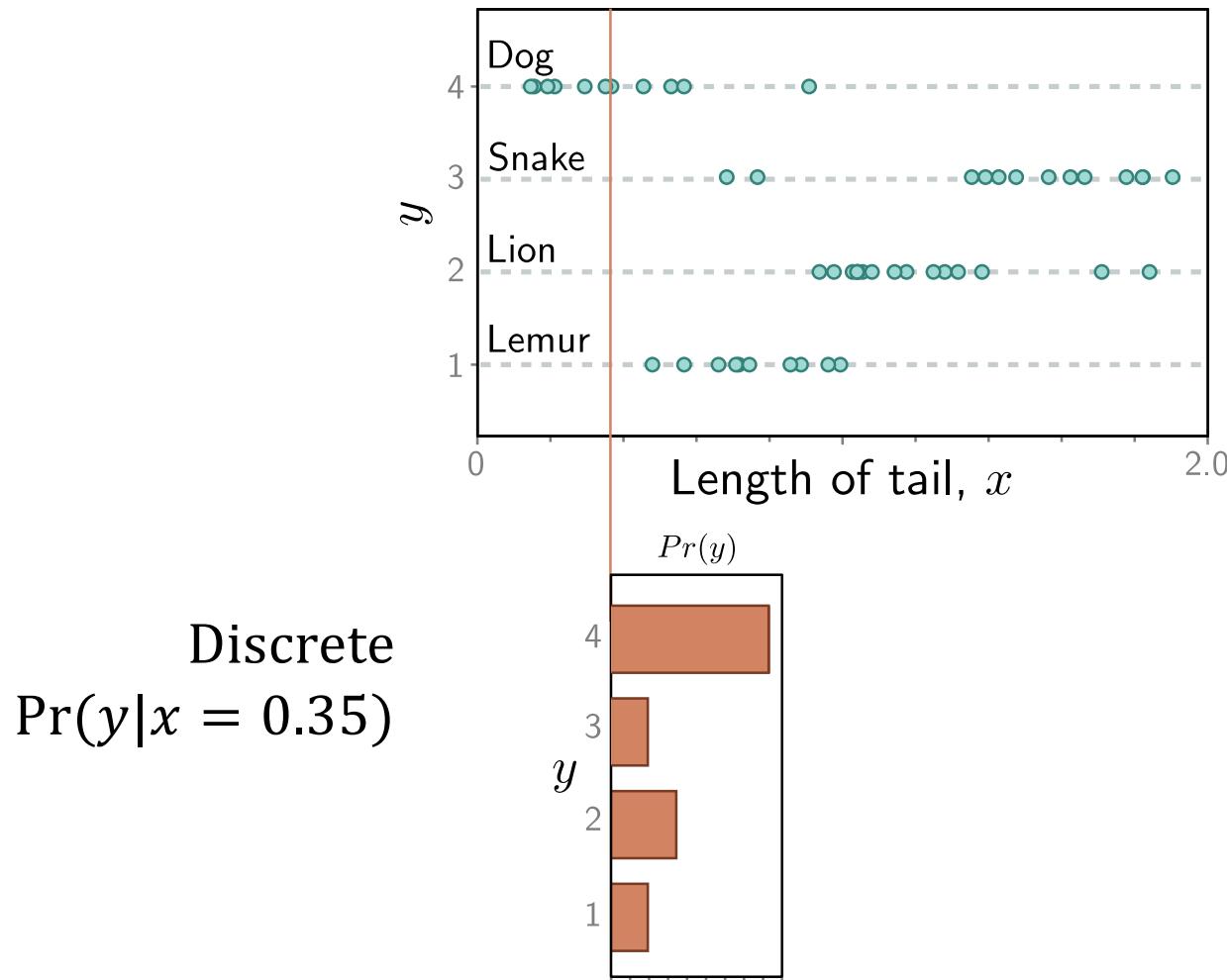


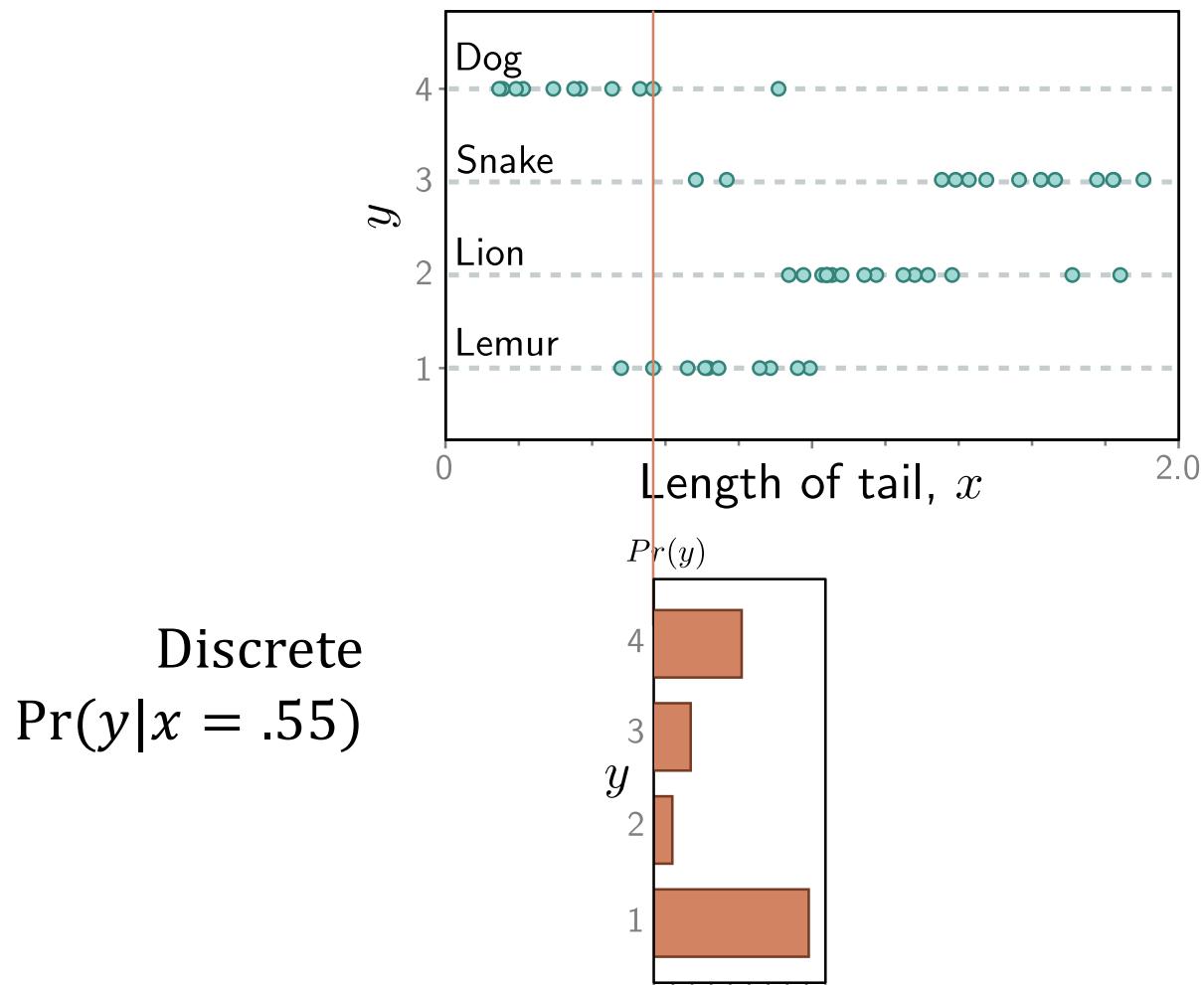


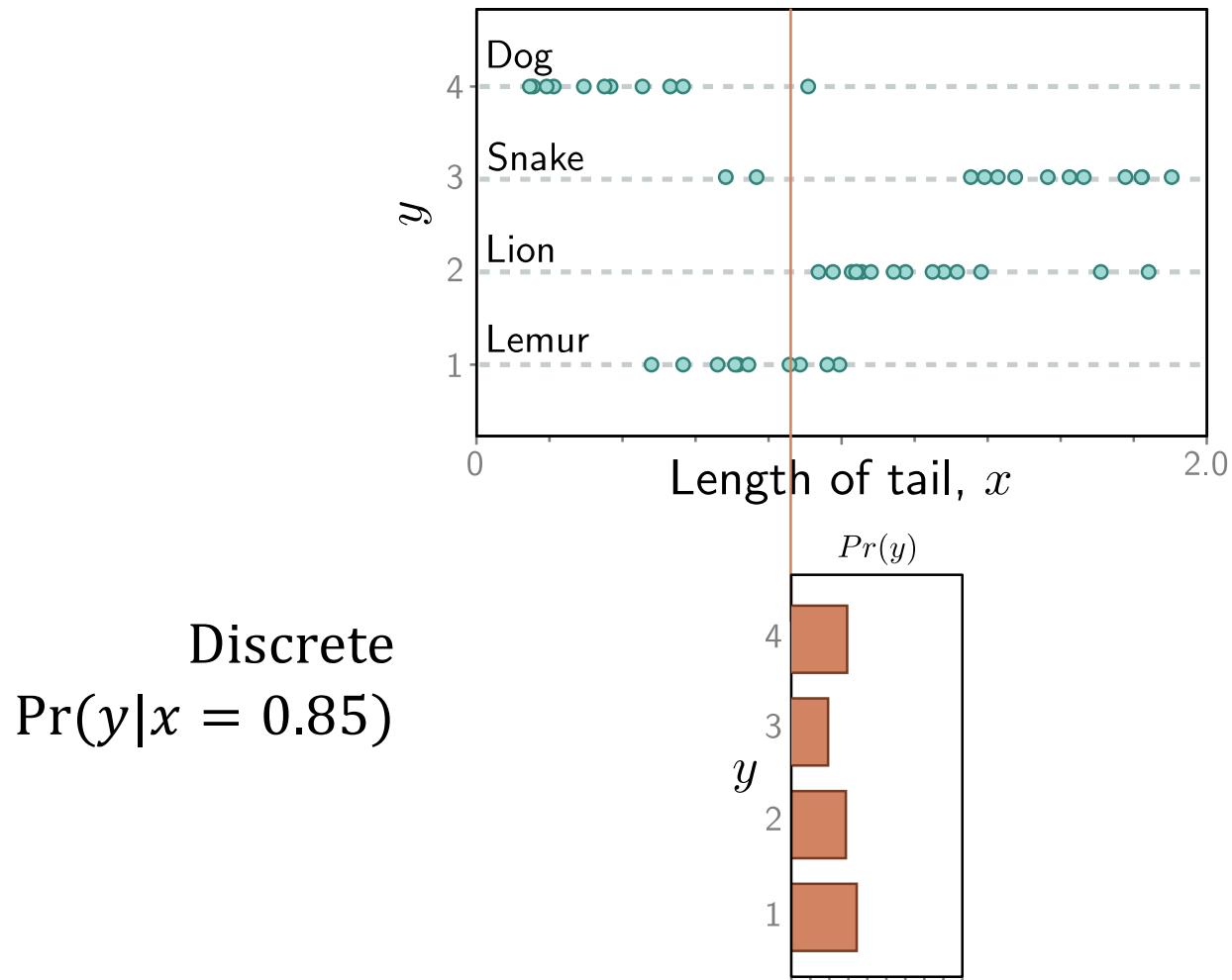
Discrete
 $\Pr(y|x = 1750)$

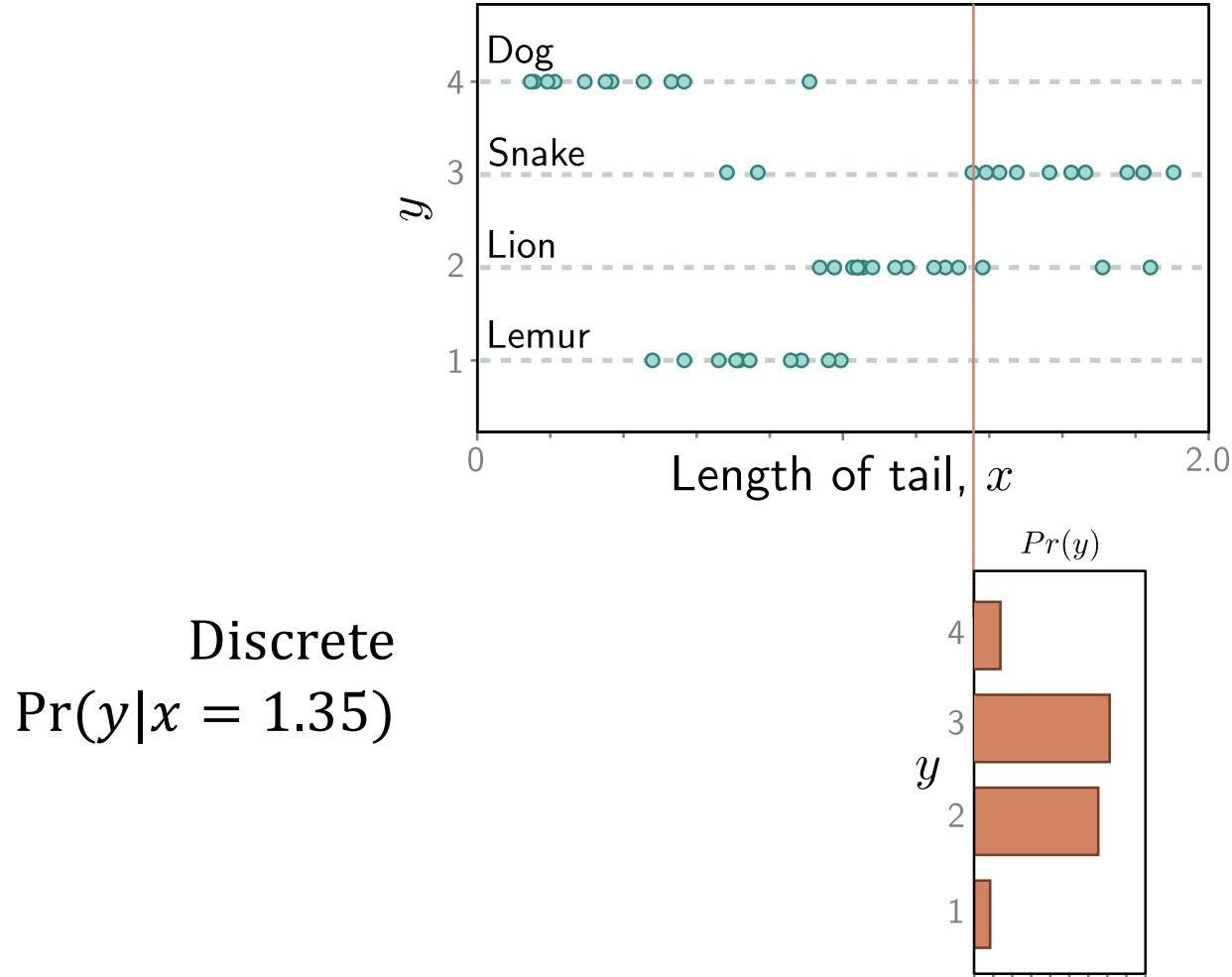


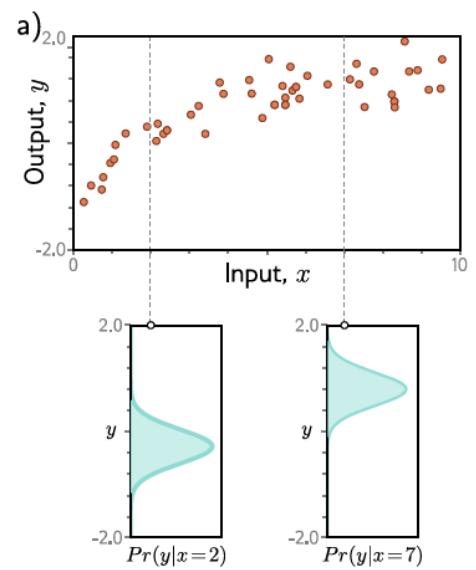


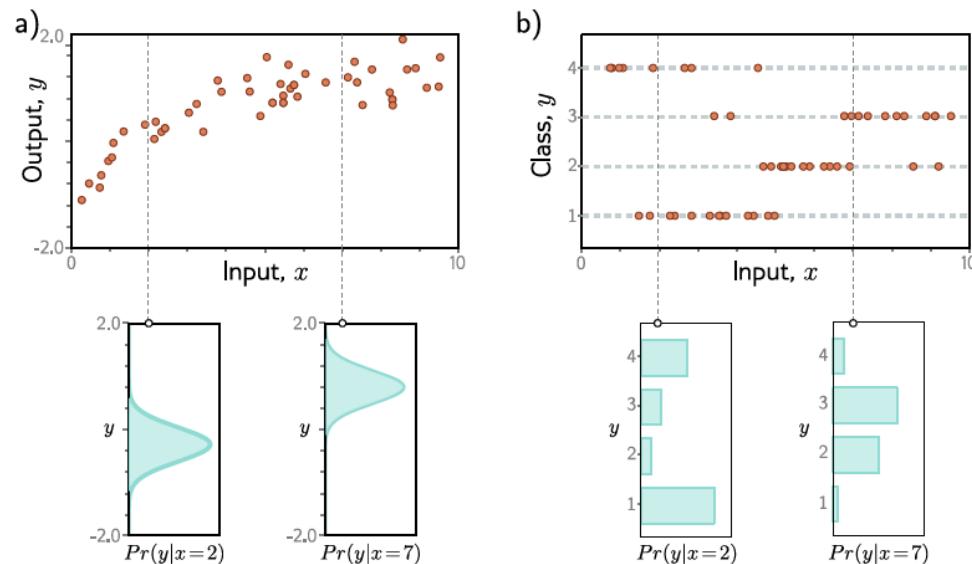


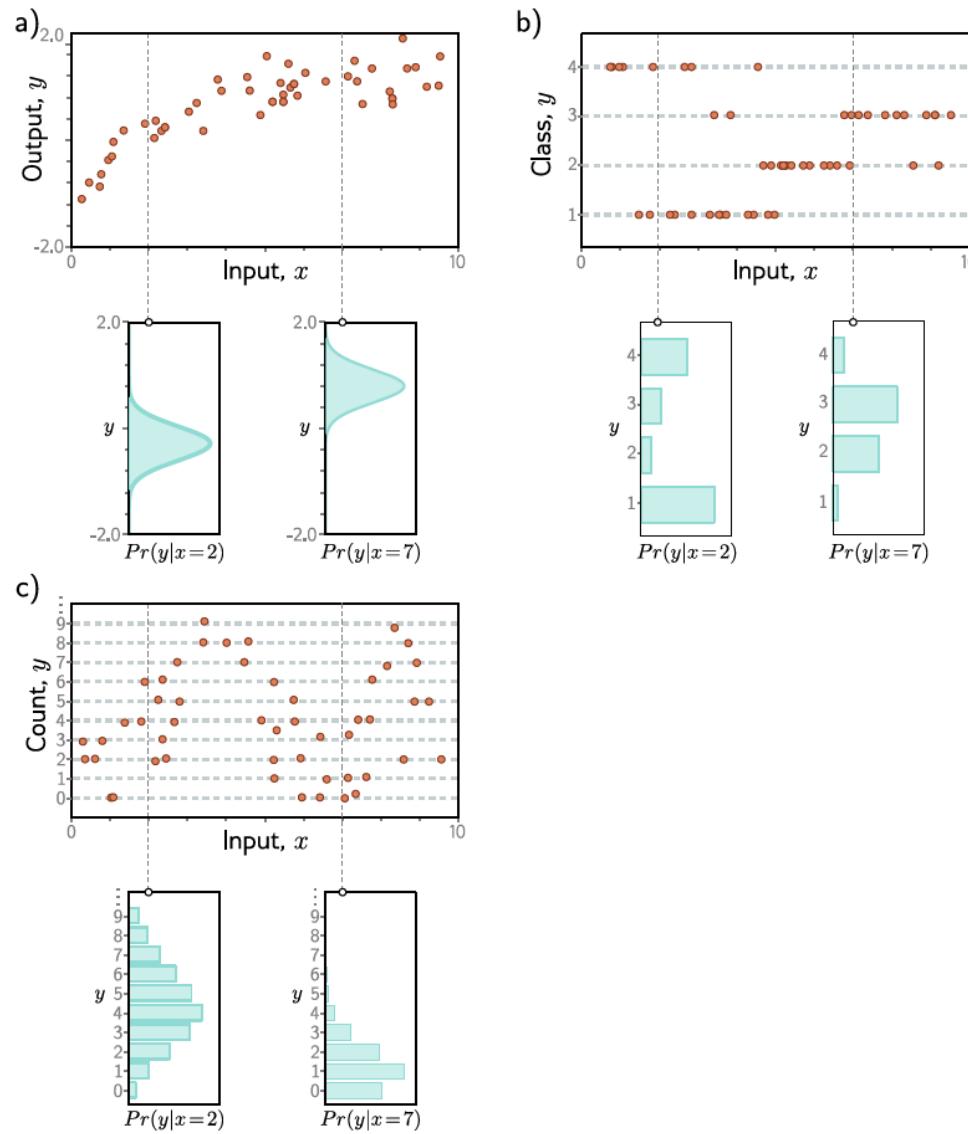


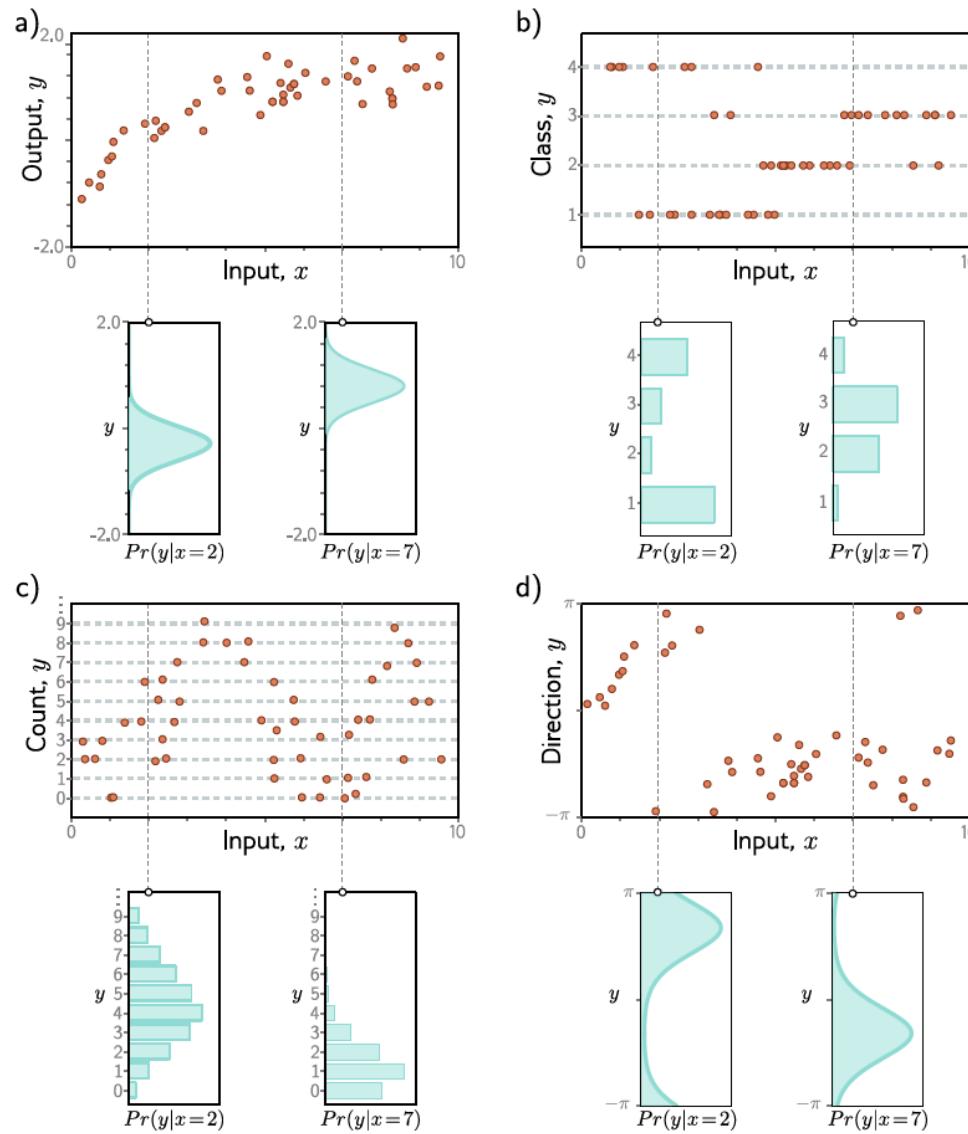












Loss function

- Training dataset of I pairs of input/output examples:

$$\{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^I$$

- Loss function or cost function measures how bad model is:

$$L\left[\phi, f[\mathbf{x}, \phi], \{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^I\right]$$



Loss function

- Training dataset of I pairs of input/output examples:

$$\{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^I$$

- Loss function or cost function measures how bad model is:

or for short:

$$L [\phi]$$

Returns a scalar that is smaller when model maps inputs to outputs better

Training

- Loss function:

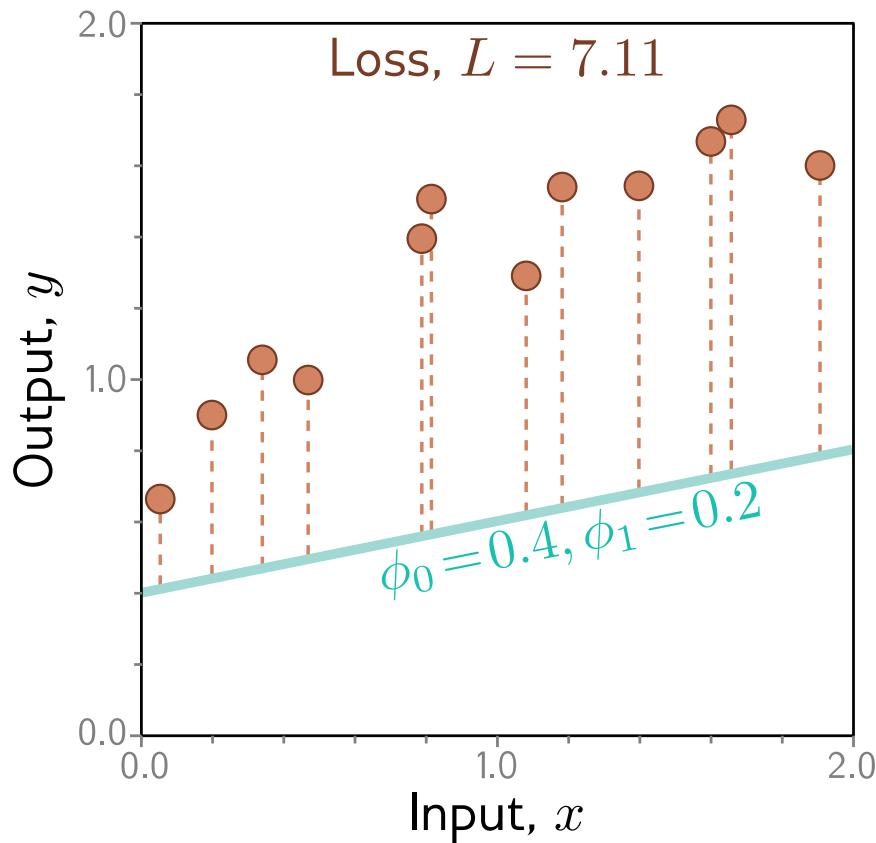
$$L [\phi]$$

>Returns a scalar that is smaller when model maps inputs to outputs better

- Find the parameters that minimize the loss:

$$\hat{\phi} = \operatorname{argmin}_{\phi} [L [\phi]]$$

Example: 1D Linear regression loss function

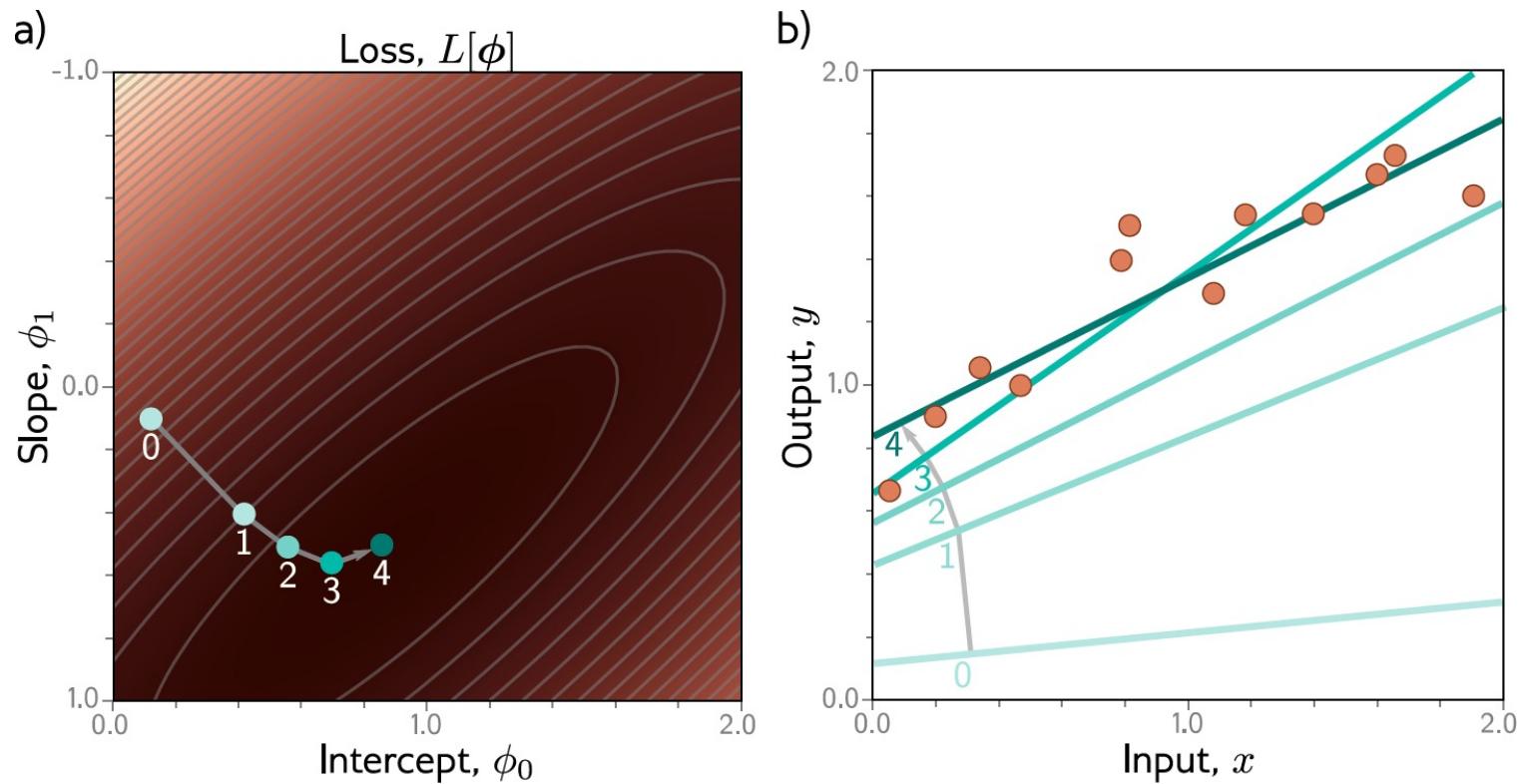


Loss function:

$$\begin{aligned} L[\phi] &= \sum_{i=1}^I (f[x_i, \phi] - y_i)^2 \\ &= \sum_{i=1}^I (\phi_0 + \phi_1 x_i - y_i)^2 \end{aligned}$$

“Least squares loss function”

Example: 1D Linear regression training



This technique is known as **gradient descent**

slido

Please download and install the Slido app on all computers you use



Why do we need a loss function in deep learning?

- ① Start presenting to display the poll results on this slide.

Loss functions

- Maximum likelihood
- Recipe for loss functions
- Example 1: univariate regression
- Example 2: binary classification
- Example 3: multiclass classification
- Other types of data
- Multiple outputs
- Cross entropy

Maximum Likelihood Estimation

- In statistics, *maximum likelihood estimation (MLE)* is a method of *estimating the parameters of an assumed probability distribution, given some observed data.*
- This is achieved by *maximizing a likelihood function* so that, under the assumed statistical model, *the observed data is most probable.*

How do we do this?

- Model predicts output y given input x

How do we do this?

- Model predicts output y given input x

How do we do this?

- Model predicts output y given input x
- Model predicts a conditional probability distribution:

$$Pr(y|x)$$

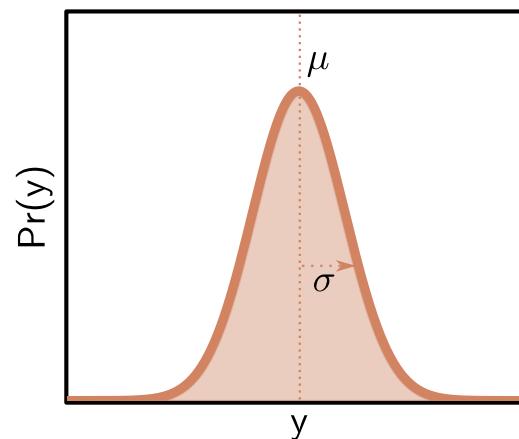
over outputs y given inputs x .

- Define and minimize a loss function that makes the outputs have high probability

How can a model predict a probability distribution? → Parametric Models

1. Pick a known distribution (e.g., normal distribution) to model output y with parameters θ

e.g., the normal distribution $\theta = \{\mu, \sigma^2\}$



2. Use model to predict parameters θ of probability distribution

Maximize the joint, conditional probability

- We know we picked a good model and the right parameters when the joint conditional probability is high for the observed (e.g. training) data.

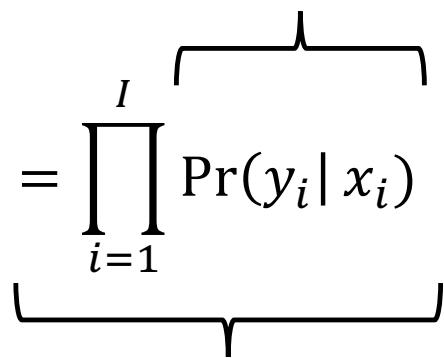
$$\Pr(y_1, y_2, \dots, y_I | x_1, x_2, \dots, x_I)$$

Two simplifying assumptions

Identically distributed (the form of the probably distribution is the same for each input/output pair)

$$\Pr(y_1, y_2, \dots, y_I | x_1, x_2, \dots, x_I) = \prod_{i=1}^I \Pr(y_i | x_i)$$

Independent



Independent and identically distributed (i.i.d)

Maximum likelihood criterion

$$\begin{aligned}\hat{\phi} &= \operatorname{argmax}_{\phi} \left[\prod_{i=1}^I Pr(\mathbf{y}_i | \mathbf{x}_i) \right] && \theta_i \text{ are the parameters of the probability distribution} \\ &= \operatorname{argmax}_{\phi} \left[\prod_{i=1}^I Pr(\mathbf{y}_i | \theta_i) \right] \\ &= \operatorname{argmax}_{\phi} \left[\prod_{i=1}^I Pr(\mathbf{y}_i | \mathbf{f}[\mathbf{x}_i, \phi]) \right] && \phi \text{ are the parameters of the neural network, e.g. } \theta_i = \mathbf{f}[\mathbf{x}_i, \phi]\end{aligned}$$

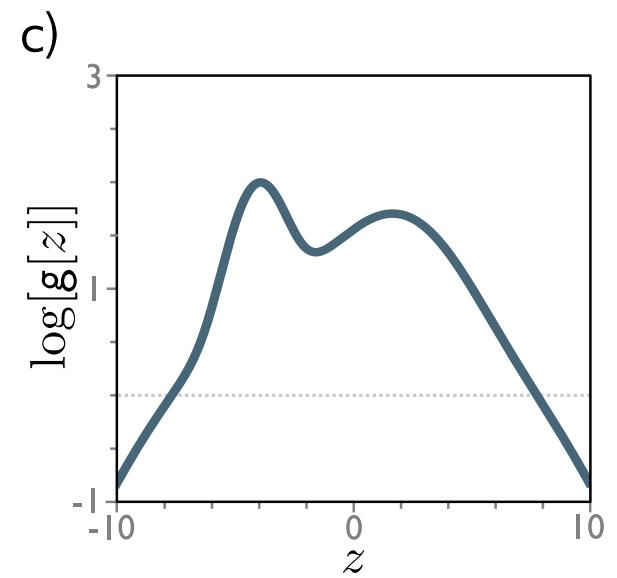
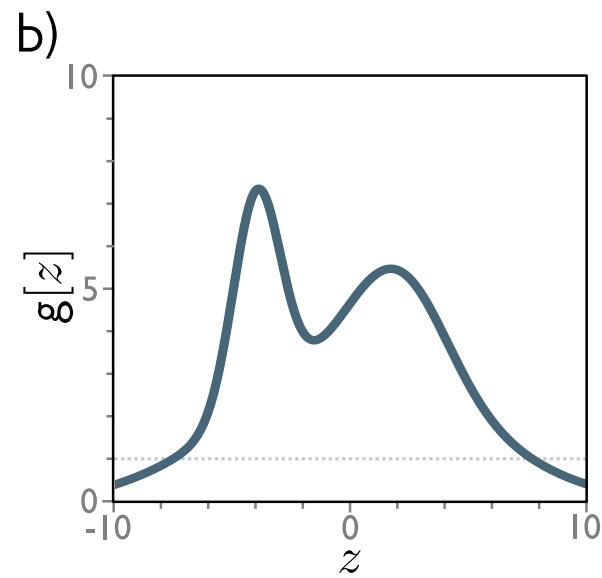
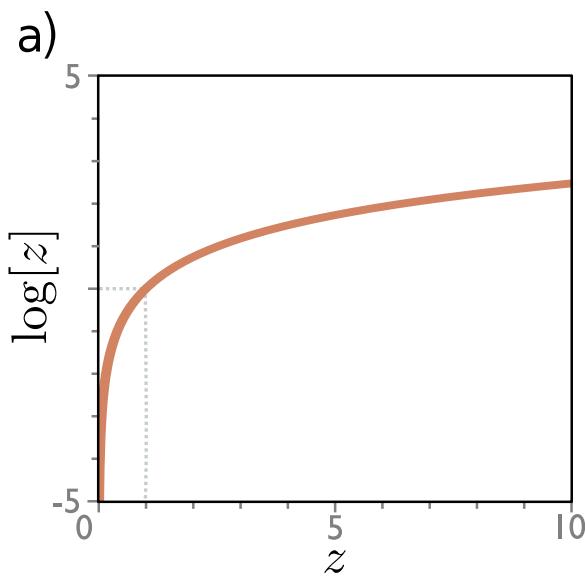
When we consider this probability as a function of the parameters ϕ , we call it a **likelihood**.

Problem:

$$\hat{\phi} = \operatorname{argmax}_{\phi} \left[\prod_{i=1}^I Pr(\mathbf{y}_i | \mathbf{f}[\mathbf{x}_i, \phi]) \right]$$

- The terms in this product might all be small, $0 \leq Pr(\cdot) \leq 1$
- The product might get so small that we *can't* easily represent it in fixed precision arithmetic

The log function is monotonic



Maximum of the logarithm of a function is in the same place as maximum of function

Maximum log likelihood

$$\begin{aligned}\hat{\phi} &= \operatorname{argmax}_{\phi} \left[\prod_{i=1}^I Pr(\mathbf{y}_i | \mathbf{f}[\mathbf{x}_i, \phi]) \right] \\ &= \operatorname{argmax}_{\phi} \left[\log \left[\prod_{i=1}^I Pr(\mathbf{y}_i | \mathbf{f}[\mathbf{x}_i, \phi]) \right] \right] \\ &= \operatorname{argmax}_{\phi} \left[\sum_{i=1}^I \log \left[Pr(\mathbf{y}_i | \mathbf{f}[\mathbf{x}_i, \phi]) \right] \right]\end{aligned}$$

Now it's a sum of terms, so doesn't matter so much if the terms are small

Minimizing negative log likelihood

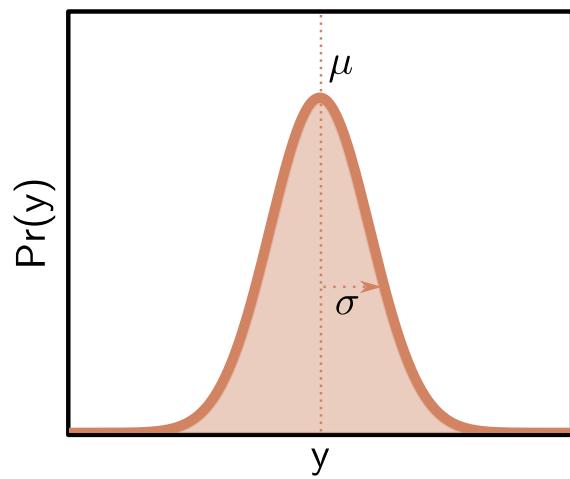
- By convention, we minimize things (i.e., a loss)

$$\begin{aligned}\hat{\phi} &= \operatorname{argmax}_{\phi} \left[\sum_{i=1}^I \log \left[Pr(\mathbf{y}_i | \mathbf{f}[\mathbf{x}_i, \phi]) \right] \right] \\ &= \underline{\operatorname{argmin}}_{\phi} \left[- \sum_{i=1}^I \log \left[Pr(\mathbf{y}_i | \mathbf{f}[\mathbf{x}_i, \phi]) \right] \right] \\ &= \underline{\operatorname{argmin}}_{\phi} \left[L[\phi] \right]\end{aligned}$$

Inference

- But now we predict a probability distribution
- We need an actual prediction (point estimate)
- Find the peak of the probability distribution (i.e., mean for normal)

$$\hat{y} = \hat{\mu} = \underset{y}{\operatorname{argmax}} [\Pr(y | \mathbf{f}[\mathbf{x}, \phi])]$$



slido

Please download and install the Slido app on all computers you use



Which of the following is true about MLE?

- ① Start presenting to display the poll results on this slide.

Loss functions

- Maximum likelihood
- Recipe for loss functions
- Example 1: univariate regression
- Example 2: binary classification
- Example 3: multiclass classification
- Other types of data
- Multiple outputs
- Cross entropy

Recipe for loss functions

1. Choose a suitable probability distribution $Pr(\mathbf{y}|\boldsymbol{\theta})$ that is defined over the domain of the predictions \mathbf{y} and has distribution parameters $\boldsymbol{\theta}$.

Recipe for loss functions

1. Choose a suitable probability distribution $Pr(\mathbf{y}|\boldsymbol{\theta})$ that is defined over the domain of the predictions \mathbf{y} and has distribution parameters $\boldsymbol{\theta}$.
2. Set the machine learning model $\mathbf{f}[\mathbf{x}, \boldsymbol{\phi}]$ to predict one or more of these parameters so $\boldsymbol{\theta} = \mathbf{f}[\mathbf{x}, \boldsymbol{\phi}]$ and $Pr(\mathbf{y}|\boldsymbol{\theta}) = Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}, \boldsymbol{\phi}])$.

Recipe for loss functions

1. Choose a suitable probability distribution $Pr(\mathbf{y}|\boldsymbol{\theta})$ that is defined over the domain of the predictions \mathbf{y} and has distribution parameters $\boldsymbol{\theta}$.
2. Set the machine learning model $\mathbf{f}[\mathbf{x}, \boldsymbol{\phi}]$ to predict one or more of these parameters so $\boldsymbol{\theta} = \mathbf{f}[\mathbf{x}, \boldsymbol{\phi}]$ and $Pr(\mathbf{y}|\boldsymbol{\theta}) = Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}, \boldsymbol{\phi}])$.
3. To train the model, find the network parameters $\hat{\boldsymbol{\phi}}$ that minimize the negative log-likelihood loss function over the training dataset pairs $\{\mathbf{x}_i, \mathbf{y}_i\}$:

$$\hat{\boldsymbol{\phi}} = \operatorname{argmin}_{\boldsymbol{\phi}} [L[\boldsymbol{\phi}]] = \operatorname{argmin}_{\boldsymbol{\phi}} \left[- \sum_{i=1}^I \log \left[Pr(\mathbf{y}_i | \mathbf{f}[\mathbf{x}_i, \boldsymbol{\phi}]) \right] \right]. \quad (5.7)$$

Recipe for loss functions

1. Choose a suitable probability distribution $Pr(\mathbf{y}|\boldsymbol{\theta})$ that is defined over the domain of the predictions \mathbf{y} and has distribution parameters $\boldsymbol{\theta}$.
2. Set the machine learning model $\mathbf{f}[\mathbf{x}, \boldsymbol{\phi}]$ to predict one or more of these parameters so $\boldsymbol{\theta} = \mathbf{f}[\mathbf{x}, \boldsymbol{\phi}]$ and $Pr(\mathbf{y}|\boldsymbol{\theta}) = Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}, \boldsymbol{\phi}])$.
3. To train the model, find the network parameters $\hat{\boldsymbol{\phi}}$ that minimize the negative log-likelihood loss function over the training dataset pairs $\{\mathbf{x}_i, \mathbf{y}_i\}$:

$$\hat{\boldsymbol{\phi}} = \operatorname{argmin}_{\boldsymbol{\phi}} [L[\boldsymbol{\phi}]] = \operatorname{argmin}_{\boldsymbol{\phi}} \left[- \sum_{i=1}^I \log \left[Pr(\mathbf{y}_i | \mathbf{f}[\mathbf{x}_i, \boldsymbol{\phi}]) \right] \right]. \quad (5.7)$$

4. To perform inference for a new test example \mathbf{x} , return either the full distribution $Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}, \hat{\boldsymbol{\phi}}])$ or the maximum of this distribution.

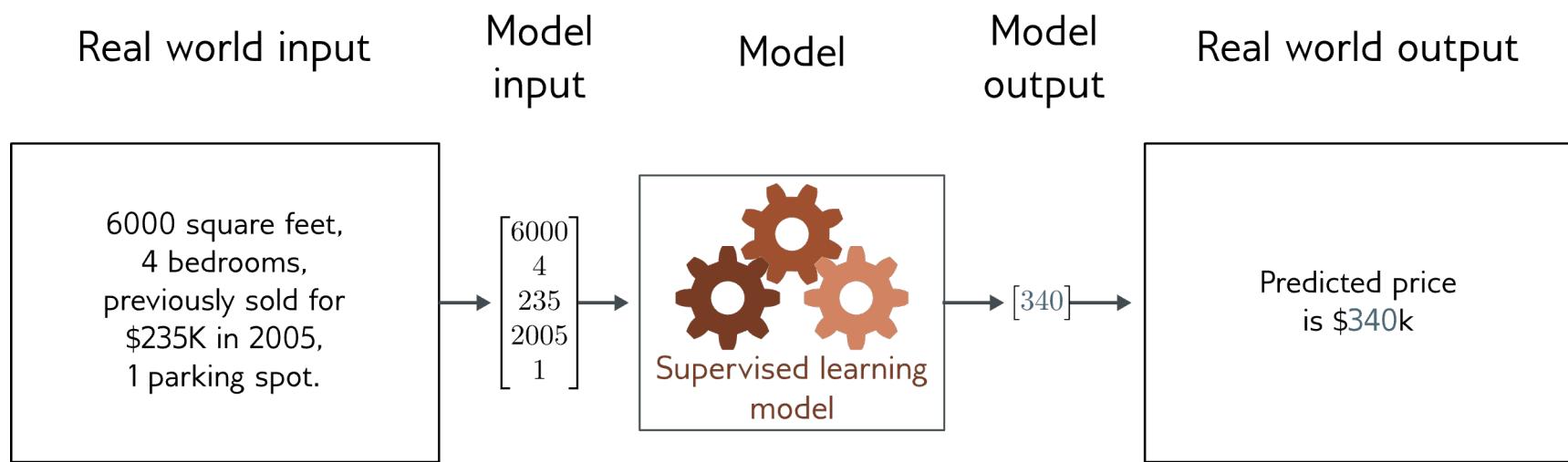
Let's apply this recipe to

- Example 1: Real valued univariate regression
- Example 2: Binary Classification
- Example 3: Multiclass Classification

Loss functions

- Maximum likelihood
- Recipe for loss functions
- Example 1: univariate regression
- Example 2: binary classification
- Example 3: multiclass classification
- Other types of data
- Multiple outputs
- Cross entropy

Example 1: univariate regression

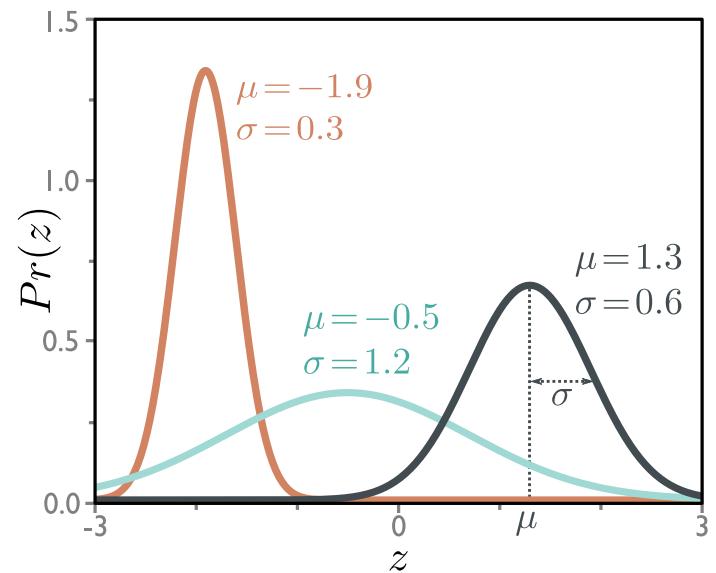


Example 1: univariate regression

1. Choose a suitable probability distribution $Pr(\mathbf{y}|\boldsymbol{\theta})$ that is defined over the domain of the predictions \mathbf{y} and has distribution parameters $\boldsymbol{\theta}$.

- Predict scalar output: $y \in \mathbb{R}$
- Sensible probability distribution:
 - Normal distribution

$$Pr(y|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y - \mu)^2}{2\sigma^2} \right]$$



Example 1: univariate regression

- Set the machine learning model $\mathbf{f}[\mathbf{x}, \phi]$ to predict one or more of these parameters so $\theta = \mathbf{f}[\mathbf{x}, \phi]$ and $Pr(\mathbf{y}|\theta) = Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}, \phi])$.

$$Pr(y|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y - \mu)^2}{2\sigma^2} \right]$$

In this case,
just the mean

$$Pr(y|\underbrace{\mathbf{f}[\mathbf{x}, \phi]}, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y - \mathbf{f}[\mathbf{x}, \phi])^2}{2\sigma^2} \right]$$

Just learn the mean, μ , and assume the variance is fixed.,.

Example 1: univariate regression

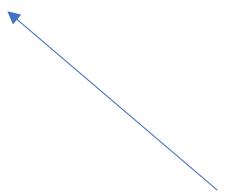
3. To train the model, find the network parameters $\hat{\phi}$ that minimize the negative log-likelihood loss function over the training dataset pairs $\{\mathbf{x}_i, \mathbf{y}_i\}$:

$$\begin{aligned} L[\phi] &= - \sum_{i=1}^I \log [Pr(y_i | f[\mathbf{x}_i, \phi], \sigma^2)] \\ &= - \sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \end{aligned}$$

$$\hat{\boldsymbol{\phi}} = \operatorname{argmin}_{\boldsymbol{\phi}} \left[- \sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[- \frac{(y_i - f[\mathbf{x}_i, \boldsymbol{\phi}])^2}{2\sigma^2} \right] \right] \right]$$

$$\begin{aligned}\hat{\phi} &= \operatorname{argmin}_{\phi} \left[-\sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \right] \\ &= \operatorname{argmin}_{\phi} \left[-\sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \right] + \log \left[\exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \right]\end{aligned}$$

$$\log[a \cdot b] = \log[a] + \log[b]$$



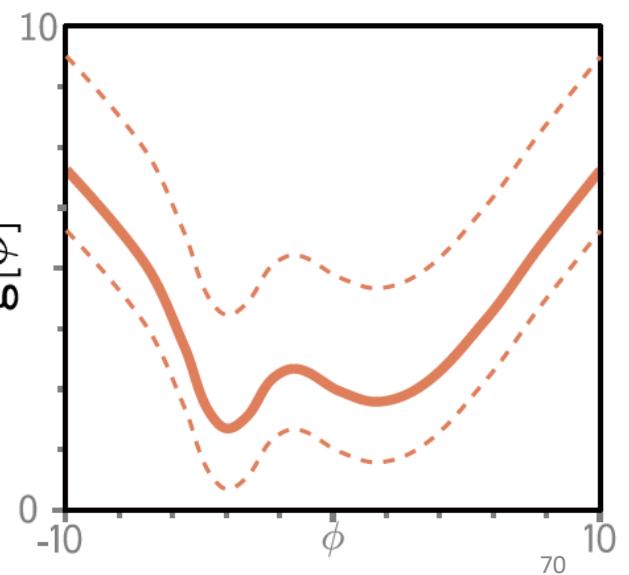
$$\begin{aligned}
\hat{\phi} &= \operatorname{argmin}_{\phi} \left[-\sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \right] \\
&= \operatorname{argmin}_{\phi} \left[-\sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \right] + \underbrace{\log \left[\exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right]}_{\log[\exp[x]] = x} \right] \\
&= \operatorname{argmin}_{\phi} \left[-\sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \right] - \frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right]
\end{aligned}$$

$\log[\exp[x]] = x$

$$\begin{aligned}
\hat{\phi} &= \operatorname{argmin}_{\phi} \left[-\sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \right] \\
&= \operatorname{argmin}_{\phi} \left[-\sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \right] + \log \left[\exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \right] \\
&= \operatorname{argmin}_{\phi} \left[-\sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \right] - \frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right]
\end{aligned}$$



Just a constant
offset

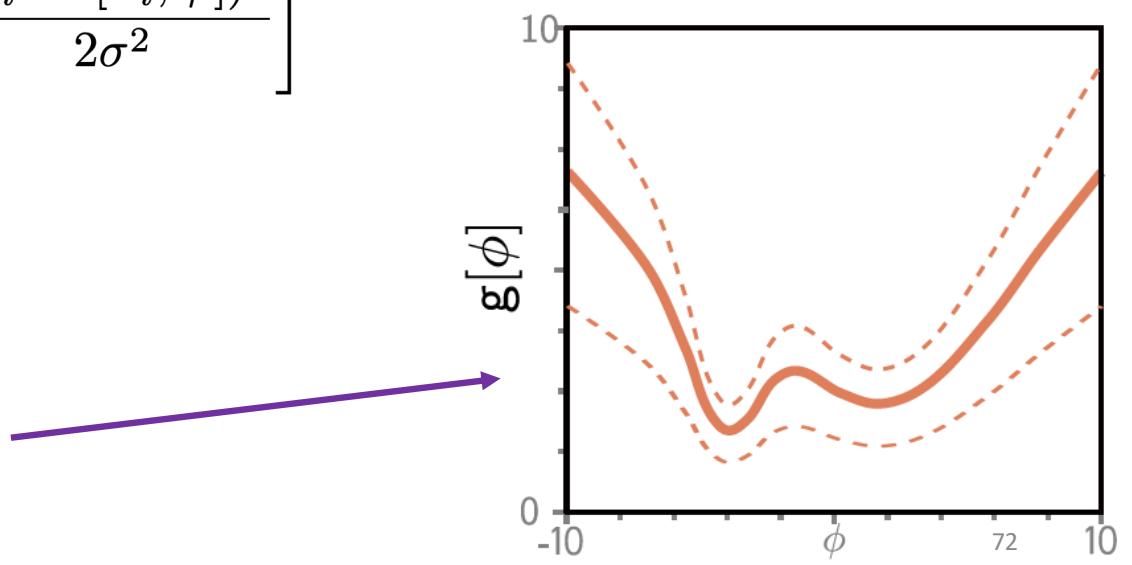


$$\begin{aligned}
\hat{\phi} &= \operatorname{argmin}_{\phi} \left[- \sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \right] \\
&= \operatorname{argmin}_{\phi} \left[- \sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \right] + \log \left[\exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \right] \\
&= \operatorname{argmin}_{\phi} \left[- \sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \right] - \frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \\
&= \operatorname{argmin}_{\phi} \left[- \sum_{i=1}^I -\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right]
\end{aligned}$$

$$\begin{aligned}
\hat{\phi} &= \operatorname{argmin}_{\phi} \left[-\sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \right] \\
&= \operatorname{argmin}_{\phi} \left[-\sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \right] + \log \left[\exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \right] \\
&= \operatorname{argmin}_{\phi} \left[-\sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \right] - \frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \\
&= \operatorname{argmin}_{\phi} \left[-\sum_{i=1}^I -\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right]
\end{aligned}$$

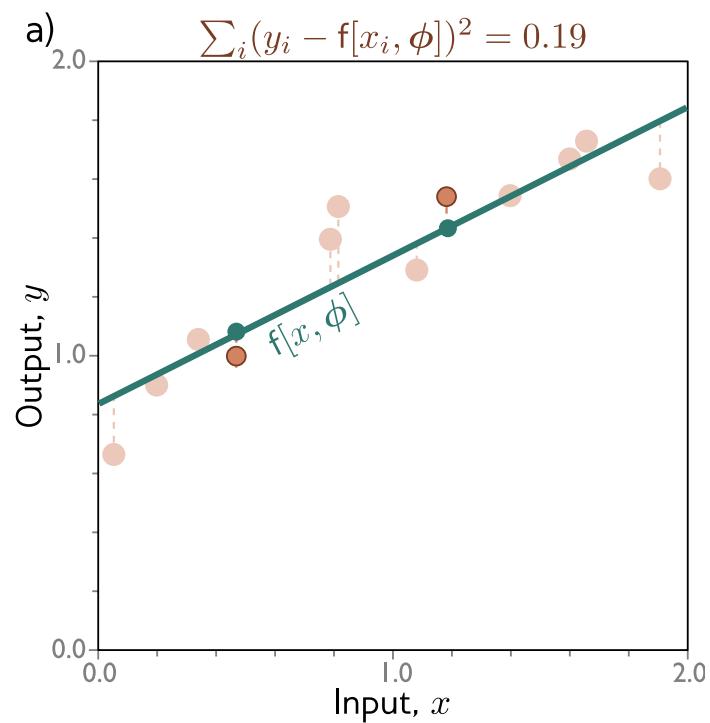


Just dividing by a positive constant

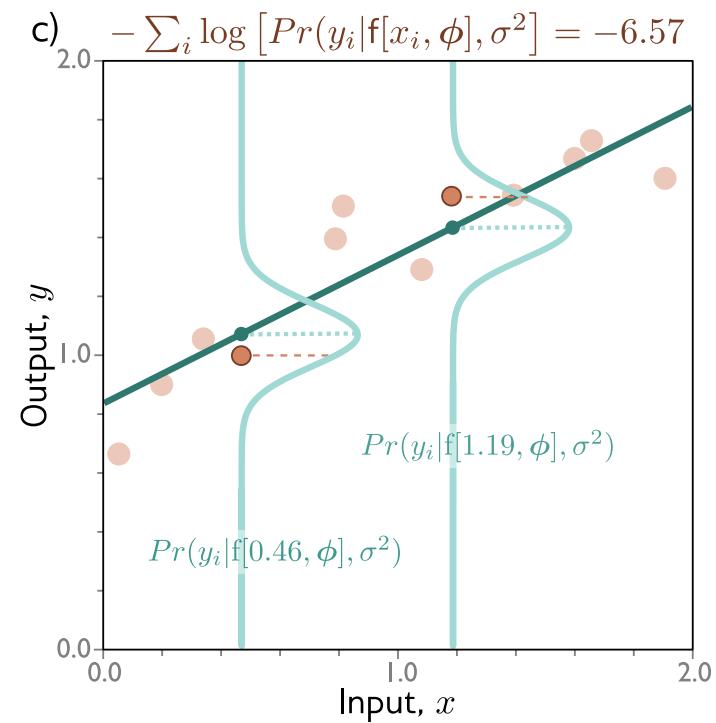


$$\begin{aligned}
\hat{\phi} &= \operatorname{argmin}_{\phi} \left[- \sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \right] \\
&= \operatorname{argmin}_{\phi} \left[- \sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \right] + \log \left[\exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \right] \\
&= \operatorname{argmin}_{\phi} \left[- \sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \right] - \frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \\
&= \operatorname{argmin}_{\phi} \left[- \sum_{i=1}^I -\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \\
&= \operatorname{argmin}_{\phi} \left[\sum_{i=1}^I (y_i - f[\mathbf{x}_i, \phi])^2 \right], \quad \xleftarrow{\text{Least squares!}}
\end{aligned}$$

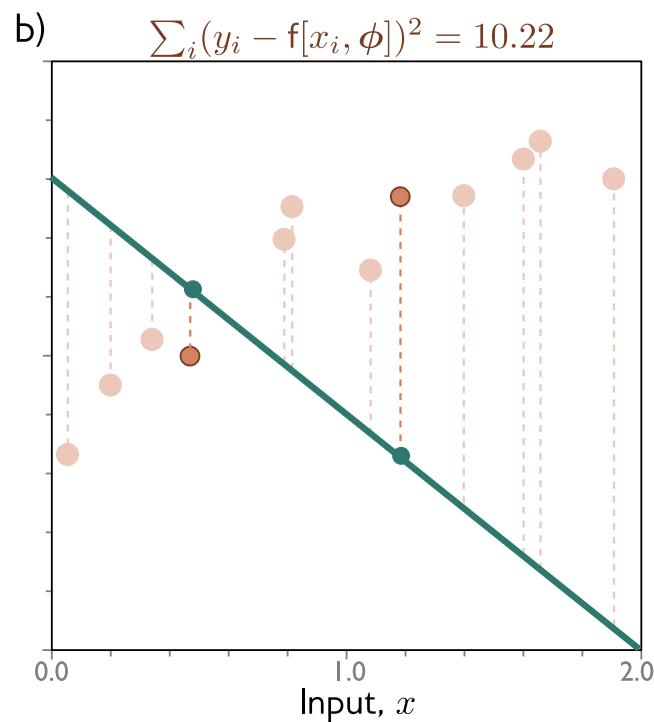
Least squares



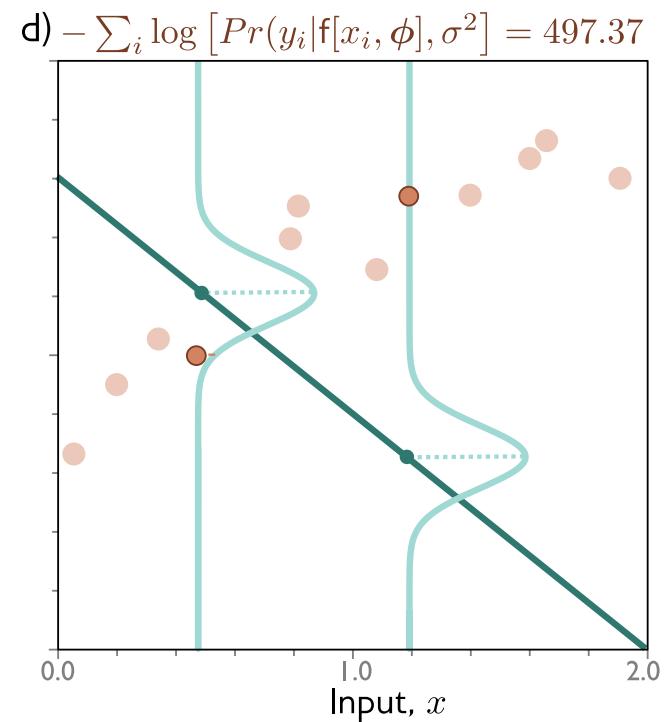
Negative log likelihood



Least squares



Maximum likelihood



Example 1: univariate regression

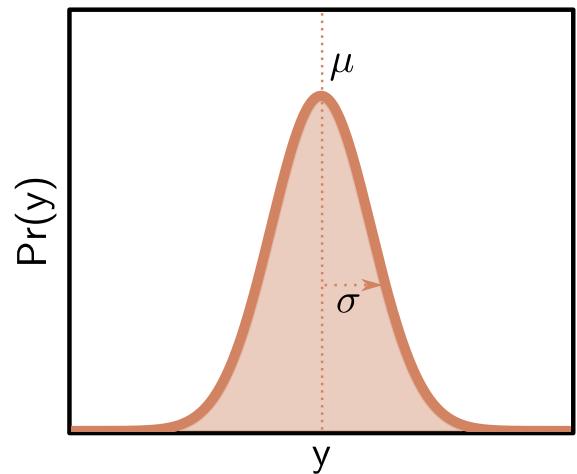
4. To perform inference for a new test example \mathbf{x} , return either the full distribution $Pr(y|\mathbf{f}[\mathbf{x}, \hat{\phi}])$ or the maximum of this distribution.

Full distribution:

$$Pr(y|\mathbf{f}[\mathbf{x}, \phi], \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y - \mathbf{f}[\mathbf{x}, \phi])^2}{2\sigma^2} \right]$$

Max probability:

$$\hat{y} = \hat{\mu} = \mathbf{f}[\mathbf{x} | \phi]$$



Estimating variance

- Perhaps surprisingly, the variance term disappeared:

$$\begin{aligned}\hat{\phi} &= \operatorname{argmin}_{\phi} \left[-\sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \right] \\ &= \operatorname{argmin}_{\phi} \left[\sum_{i=1}^I (y_i - f[\mathbf{x}_i, \phi])^2 \right]\end{aligned}$$

Estimating variance

- But we could learn it during training:

$$\hat{\phi}, \hat{\sigma}^2 = \operatorname{argmin}_{\phi, \sigma^2} \left[- \sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y_i - f[\mathbf{x}_i, \phi])^2}{2\sigma^2} \right] \right] \right]$$

- Do gradient descent on both model parameters, ϕ , and the variance, σ^2

$$\frac{\partial L}{\partial \phi} \text{ and } \frac{\partial L}{\partial \sigma^2}$$

Heteroscedastic regression

- We were assuming that the noise σ^2 is the same everywhere (homoscedastic).
- But we could make the noise a function of the data x .
- Build a model with two outputs:

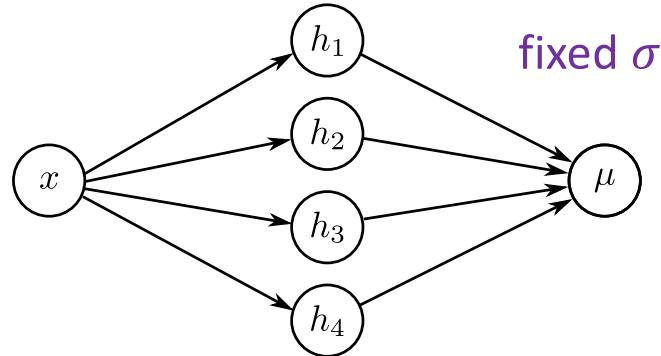
$$\begin{aligned}\mu &= f_1[\mathbf{x}, \boldsymbol{\phi}] \\ \sigma^2 &= f_2[\mathbf{x}, \boldsymbol{\phi}]^2\end{aligned}$$

← Squared to ensure it is positive

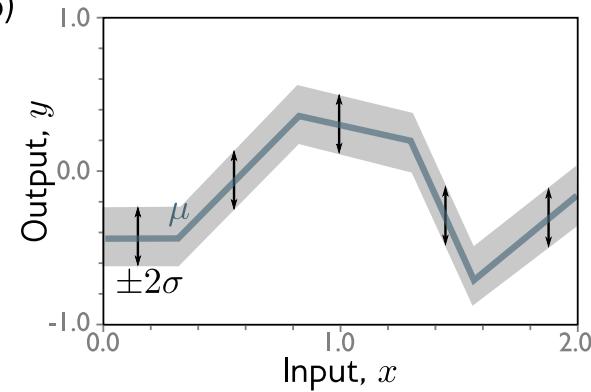
$$\hat{\boldsymbol{\phi}} = \underset{\boldsymbol{\phi}}{\operatorname{argmin}} \left[-\sum_{i=1}^I \log \left[\frac{1}{\sqrt{2\pi f_2[\mathbf{x}_i, \boldsymbol{\phi}]^2}} \right] - \frac{(y_i - f_1[\mathbf{x}_i, \boldsymbol{\phi}])^2}{2f_2[\mathbf{x}_i, \boldsymbol{\phi}]^2} \right]$$

Heteroscedastic regression

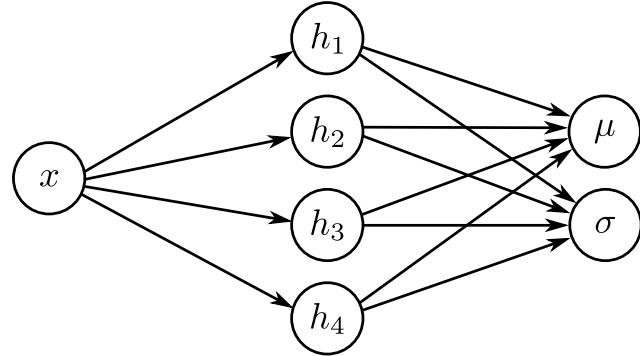
a)



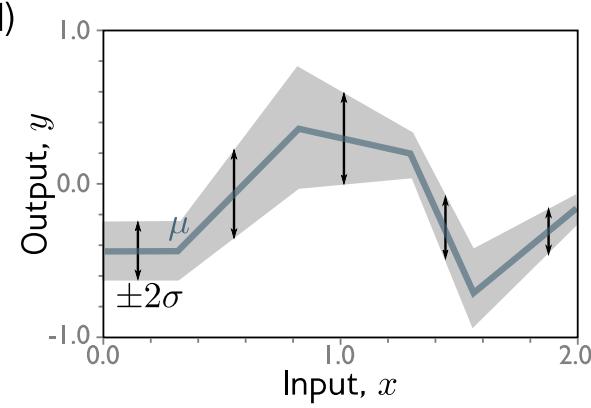
b)



c)



d)



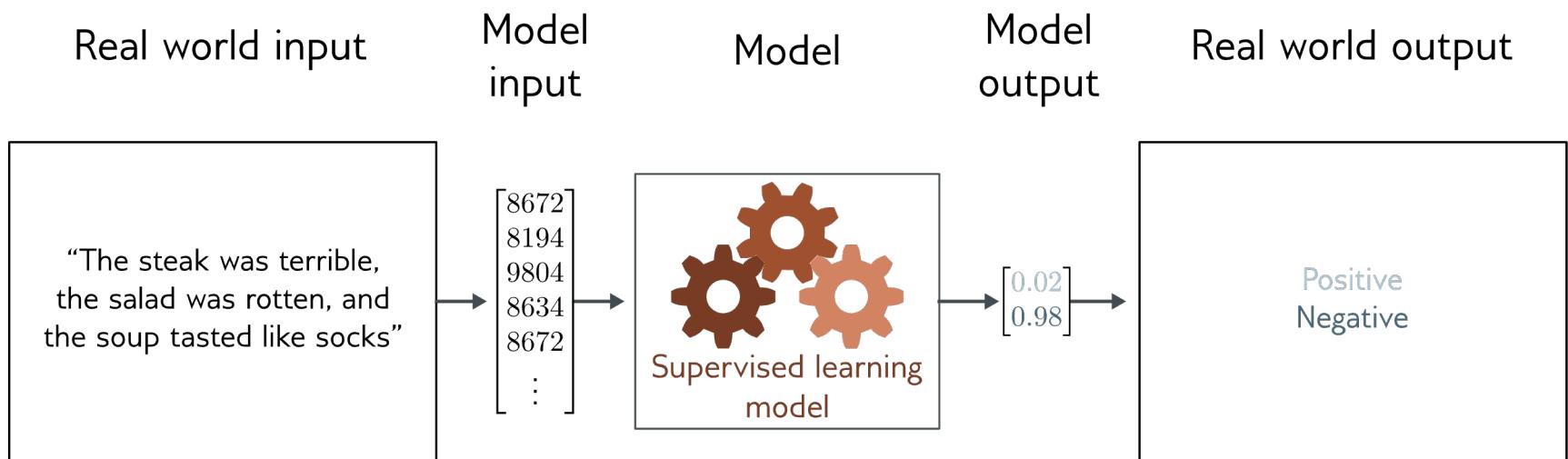
Example 1: Univariate Regression Takeaways

- *Least squares loss* is a good choice assuming normal distribution
- The best prediction is the predicted mean
- We can also estimate global or local variance

Loss functions

- Maximum likelihood
- Recipe for loss functions
- Example 1: univariate regression
- Example 2: binary classification
- Example 3: multiclass classification
- Other types of data
- Multiple outputs
- Cross entropy

Example 2: binary classification



- Goal: predict which of two classes $y \in \{0, 1\}$ the input x belongs to

Example 2: binary classification

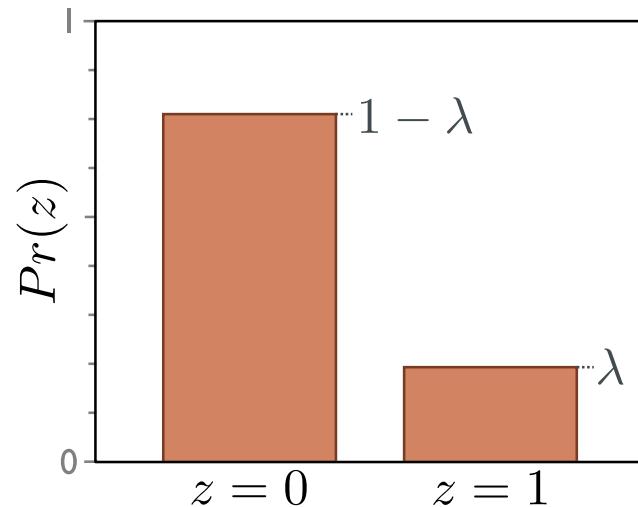
1. Choose a suitable probability distribution $Pr(\mathbf{y}|\boldsymbol{\theta})$ that is defined over the domain of the predictions \mathbf{y} and has distribution parameters $\boldsymbol{\theta}$.

- Domain: $y \in \{0, 1\}$
- Bernoulli distribution
- One parameter $\lambda \in [0, 1]$

$$Pr(y|\lambda) = \begin{cases} 1 - \lambda & y = 0 \\ \lambda & y = 1 \end{cases}$$

or

$$Pr(y|\lambda) = (1 - \lambda)^{1-y} \cdot \lambda^y$$



Example 2: binary classification

2. Set the machine learning model $\mathbf{f}[\mathbf{x}, \phi]$ to predict one or more of these parameters so $\boldsymbol{\theta} = \mathbf{f}[\mathbf{x}, \phi]$ and $Pr(\mathbf{y}|\boldsymbol{\theta}) = Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}, \phi])$.

Problem:

- Output of neural network can be anything
- Parameter $\lambda \in [0,1]$

Solution:

- Pass through function that maps “anything” to $[0,1]$

Example 2: binary classification

- Set the machine learning model $\mathbf{f}[\mathbf{x}, \phi]$ to predict one or more of these parameters so $\theta = \mathbf{f}[\mathbf{x}, \phi]$ and $Pr(\mathbf{y}|\theta) = Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}, \phi])$.

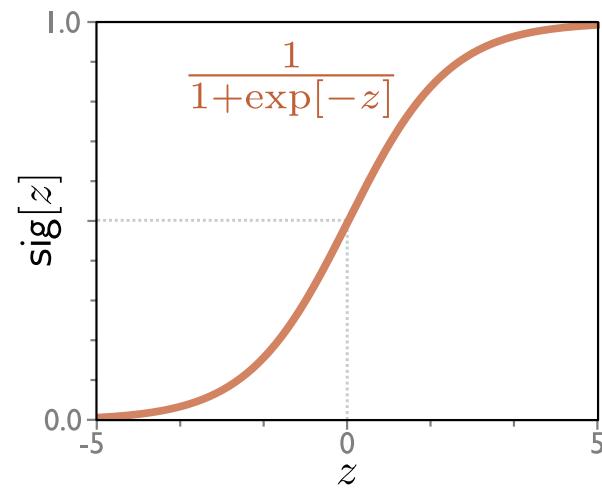
Problem:

- Output of neural network can be anything
- Parameter $\lambda \in [0,1]$

Solution:

- Pass through logistic sigmoid function that maps “anything to $[0,1]$:

$$\text{sig}[z] = \frac{1}{1 + \exp[-z]}$$



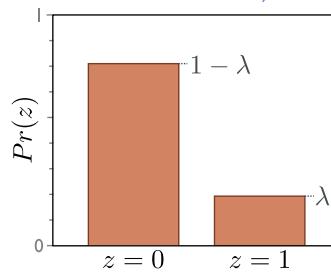
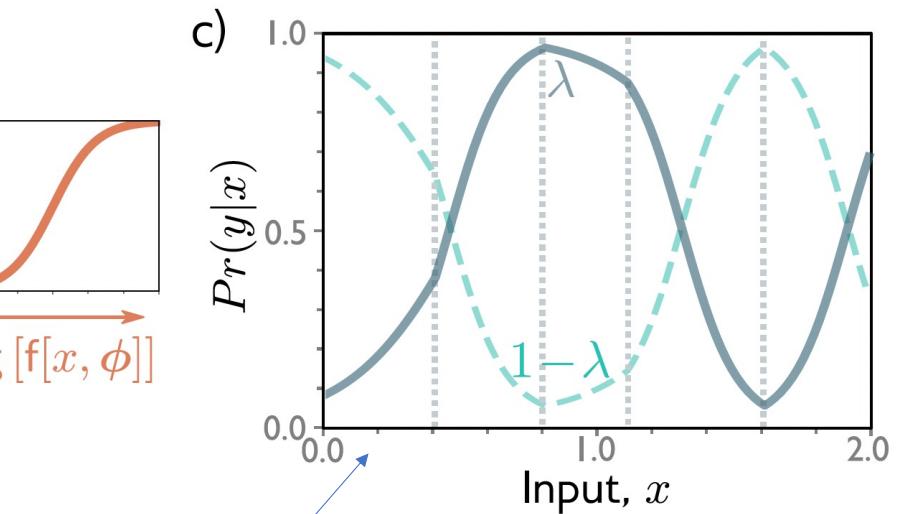
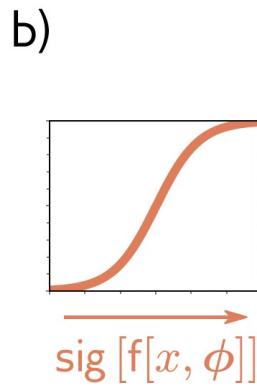
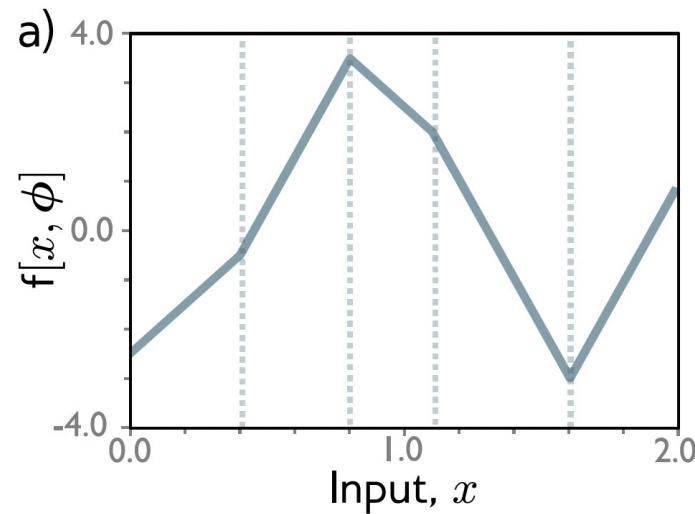
Example 2: binary classification

2. Set the machine learning model $\mathbf{f}[\mathbf{x}, \phi]$ to predict one or more of these parameters so $\boldsymbol{\theta} = \mathbf{f}[\mathbf{x}, \phi]$ and $Pr(\mathbf{y}|\boldsymbol{\theta}) = Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}, \phi])$.

$$Pr(y|\lambda) = (1 - \lambda)^{1-y} \cdot \lambda^y$$

$$Pr(y|\mathbf{x}) = (1 - \text{sig}[\mathbf{f}[\mathbf{x}|\phi]])^{1-y} \cdot \text{sig}[\mathbf{f}[\mathbf{x}|\phi]]^y$$

Example 2: binary classification



Example 2: binary classification

3. To train the model, find the network parameters $\hat{\phi}$ that minimize the negative log-likelihood loss function over the training dataset pairs $\{\mathbf{x}_i, \mathbf{y}_i\}$:

$$\hat{\phi} = \operatorname{argmin}_{\phi} [L[\phi]] = \operatorname{argmin}_{\phi} \left[- \sum_{i=1}^I \log \left[Pr(\mathbf{y}_i | \mathbf{f}[\mathbf{x}_i, \phi]) \right] \right]. \quad (5.7)$$

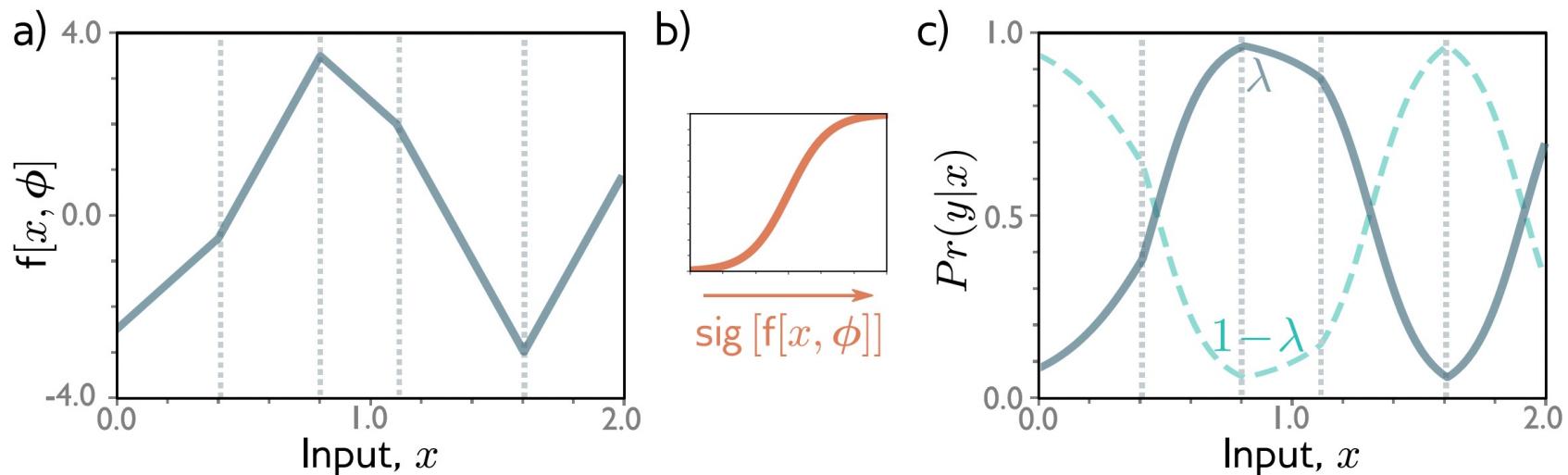
$$Pr(y|\mathbf{x}) = (1 - \operatorname{sig}[\mathbf{f}[\mathbf{x}|\phi]])^{1-y} \cdot \operatorname{sig}[\mathbf{f}[\mathbf{x}|\phi]]^y$$

$$L[\phi] = \sum_{i=1}^I -(1 - y_i) \log [1 - \operatorname{sig}[\mathbf{f}[\mathbf{x}_i|\phi]]] - y_i \log [\operatorname{sig}[\mathbf{f}[\mathbf{x}_i|\phi]]]$$

Also called binary cross-entropy loss as it is result from cross-entropy loss calculation – discussed later.

Example 2: binary classification

4. To perform inference for a new test example \mathbf{x} , return either the full distribution $Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}, \hat{\phi}])$ or the maximum of this distribution.



Choose $y=1$ where λ is greater than 0.5, otherwise 0
And we get a probability estimate!

The Slido logo consists of the word "slido" in a lowercase, sans-serif font, with a small blue square containing a white "S" character positioned above the letter "l".

Please download and install the Slido app on all computers you use



Which statement correctly differentiates least squares and cross-entropy loss?

- ① Start presenting to display the poll results on this slide.

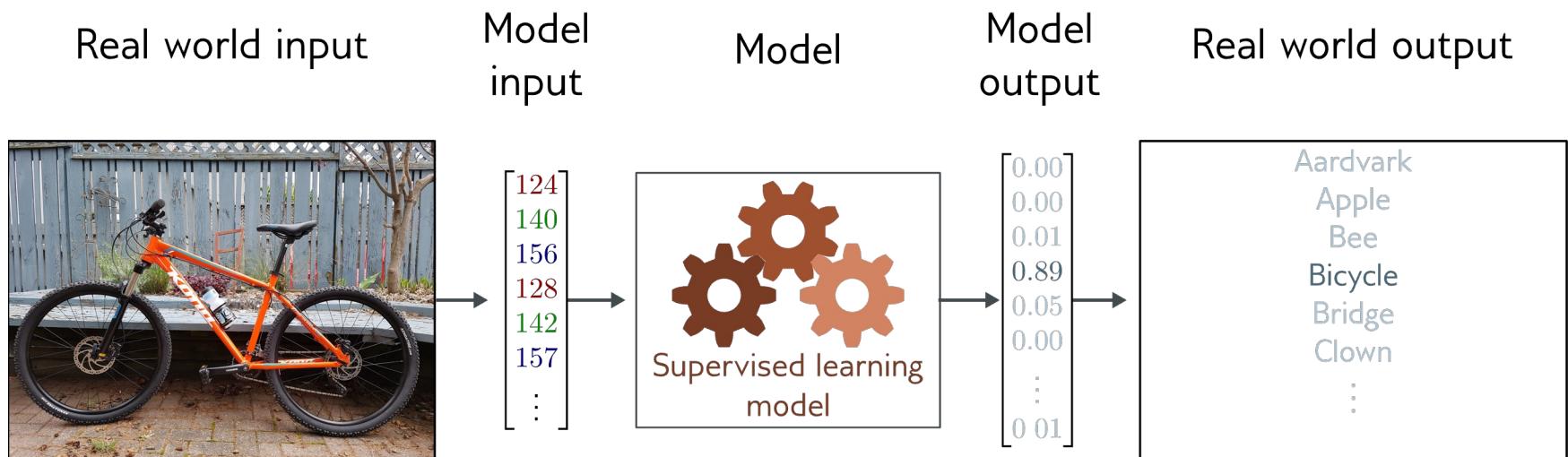
Example 2: Binary Classification Takeaways

- Binary cross entropy loss as the loss function
- Threshold to get prediction
- We also get a probability or “confidence value”

Loss functions

- Maximum likelihood
- Recipe for loss functions
- Example 1: univariate regression
- Example 2: binary classification
- Example 3: multiclass classification
- Other types of data
- Multiple outputs
- Cross entropy

Example 3: multiclass classification



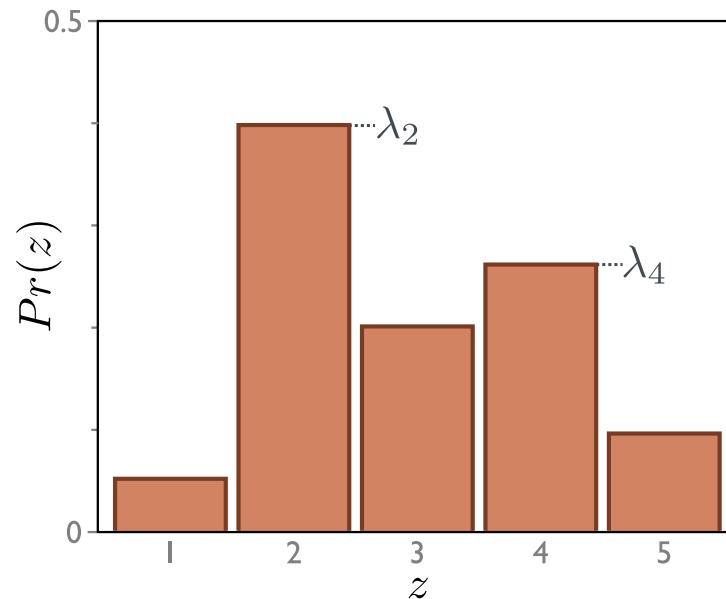
Goal: predict which of K classes $y \in \{1, 2, \dots, K\}$ the input x belongs to

Example 3: multiclass classification

1. Choose a suitable probability distribution $Pr(\mathbf{y}|\boldsymbol{\theta})$ that is defined over the domain of the predictions \mathbf{y} and has distribution parameters $\boldsymbol{\theta}$.

- Domain: $y \in \{1, 2, \dots, K\}$
- **Categorical distribution**
- K parameters $\lambda_k \in [0, 1]$
- $\sum_k \lambda_k = 1$

$$Pr(y = k) = \lambda_k$$



Example 3: multiclass classification

- Set the machine learning model $\mathbf{f}[\mathbf{x}, \phi]$ to predict one or more of these parameters so $\boldsymbol{\theta} = \mathbf{f}[\mathbf{x}, \phi]$ and $Pr(\mathbf{y}|\boldsymbol{\theta}) = Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}, \phi])$.

Problem:

- Output of neural network can be anything
- Parameters $\lambda_k \in [0,1]$, sum to one

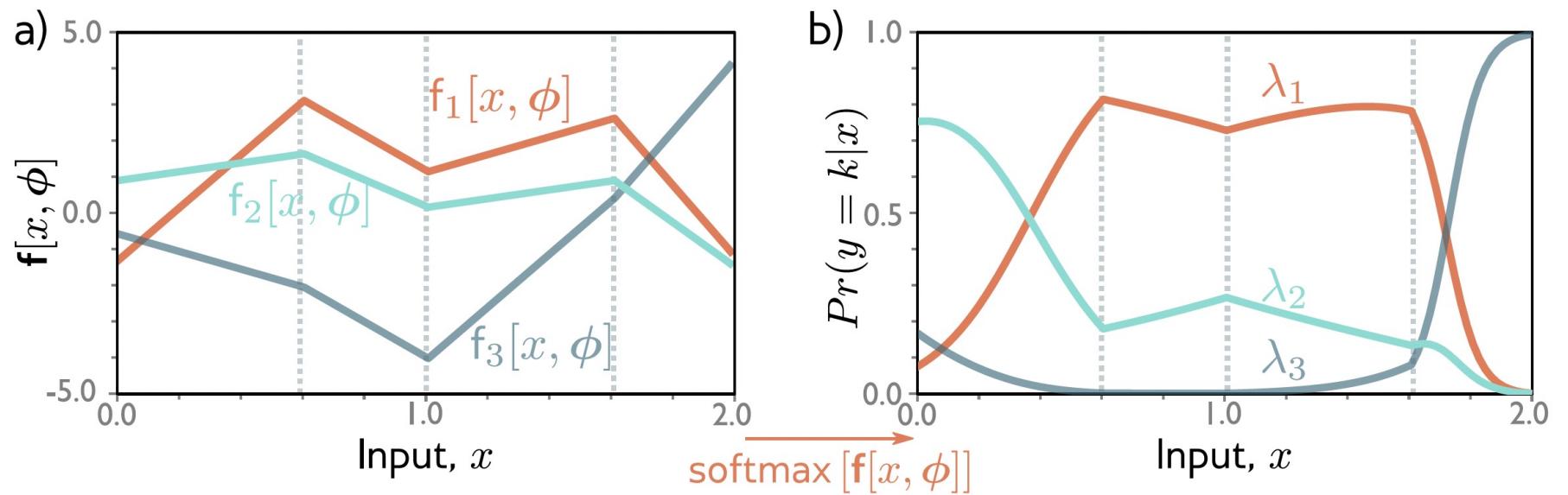
Solution:

- Pass through function that maps “anything” to $[0,1]$ and sums to one

$$\text{softmax}_k[\mathbf{z}] = \frac{\exp[z_k]}{\sum_{k'=1}^K \exp[z_{k'}]}$$

$$Pr(y = k | \mathbf{x}) = \text{softmax}_k[\mathbf{f}[\mathbf{x}, \phi]]$$

Example 3: multiclass classification



$$Pr(y = k|x) = \text{softmax}_k[\mathbf{f}[\mathbf{x}, \phi]]$$

Example 3: multiclass classification

3. To train the model, find the network parameters $\hat{\phi}$ that minimize the negative log-likelihood loss function over the training dataset pairs $\{\mathbf{x}_i, \mathbf{y}_i\}$:

$$\hat{\phi} = \operatorname{argmin}_{\phi} [L[\phi]] = \operatorname{argmin}_{\phi} \left[- \sum_{i=1}^I \log \left[Pr(\mathbf{y}_i | \mathbf{f}[\mathbf{x}_i, \phi]) \right] \right]. \quad (5.7)$$

$$L[\phi] = - \sum_{i=1}^I \log [\text{softmax}_{y_i} [\mathbf{f}[\mathbf{x}_i, \phi]]]$$

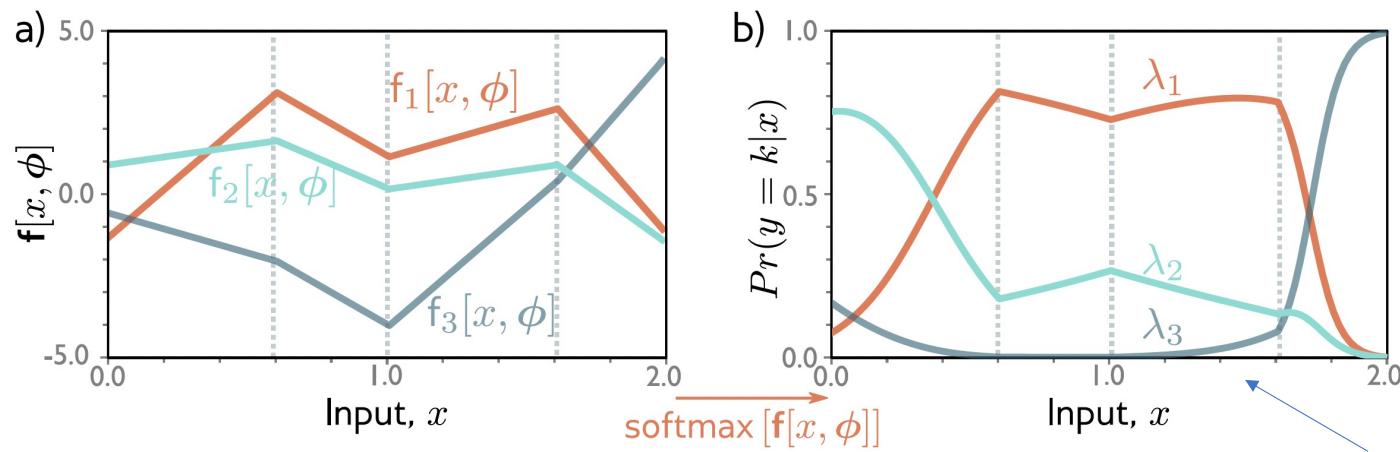
$$\text{softmax}_k[\mathbf{z}] = \frac{\exp[z_k]}{\sum_{k'=1}^K \exp[z_{k'}]}$$

$$= - \sum_{i=1}^I \mathbf{f}_{y_i} [\mathbf{x}_i, \phi] - \log \left[\sum_{k=1}^K \exp [\mathbf{f}_k [\mathbf{x}_i, \phi]] \right]$$

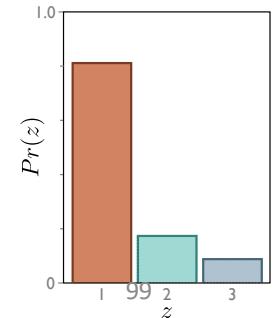
Multiclass cross-entropy loss 98

Example 3: multiclass classification

4. To perform inference for a new test example \mathbf{x} , return either the full distribution $Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}, \hat{\phi}])$ or the maximum of this distribution.



Choose the class with the largest probability
We also get probability or “confidence”



Loss functions

- Maximum likelihood
- Recipe for loss functions
- Example 1: univariate regression
- Example 2: binary classification
- Example 3: multiclass classification
- Other types of data
- Multiple outputs
- Cross entropy

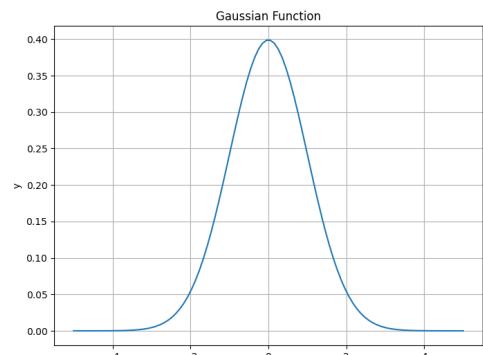
Other data types

Data Type	Domain	Distribution	Use
univariate, continuous, unbounded	$y \in \mathbb{R}$	univariate normal	regression
univariate, continuous, unbounded	$y \in \mathbb{R}$	Laplace or t-distribution	robust regression
univariate, continuous, unbounded	$y \in \mathbb{R}$	mixture of Gaussians	multimodal regression
univariate, continuous, bounded below	$y \in \mathbb{R}^+$	exponential or gamma	predicting magnitude
univariate, continuous, bounded	$y \in [0, 1]$	beta	predicting proportions
multivariate, continuous, unbounded	$\mathbf{y} \in \mathbb{R}^K$	multivariate normal	multivariate regression
univariate, continuous, circular	$y \in (-\pi, \pi]$	von Mises	predicting direction
univariate, discrete, binary	$y \in \{0, 1\}$	Bernoulli	binary classification
univariate, discrete, bounded	$y \in \{1, 2, \dots, K\}$	categorical	multiclass classification
univariate, discrete, bounded below	$y \in [0, 1, 2, 3, \dots]$	Poisson	predicting event counts
multivariate, discrete, permutation	$\mathbf{y} \in \text{Perm}[1, 2, \dots, K]$	Plackett-Luce	ranking

Figure 5.11 Distributions for loss functions for different prediction types.

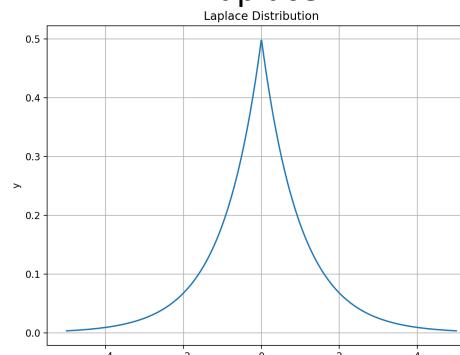
Other Distributions

Gaussian



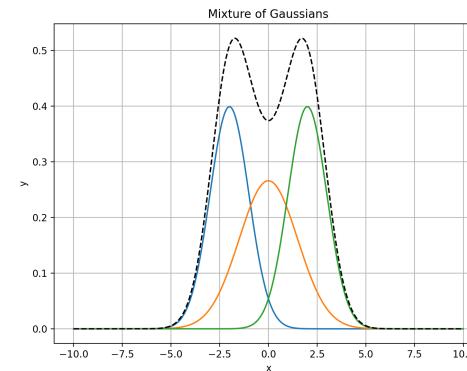
$y \in \mathbb{R}$ Regression

Laplace



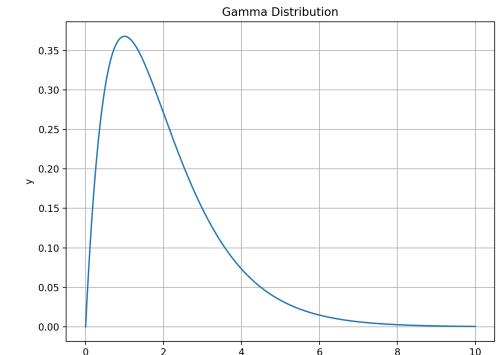
$y \in \mathbb{R}$ Robust Regression

Mixture of Gaussians



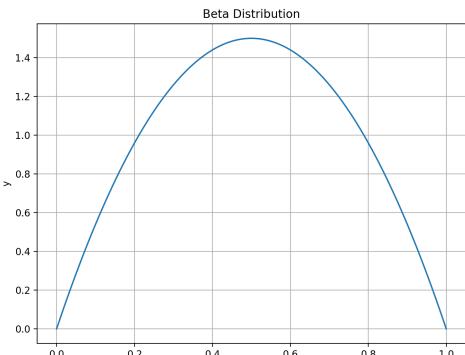
$y \in \mathbb{R}$ Multimodal Regression

Gamma



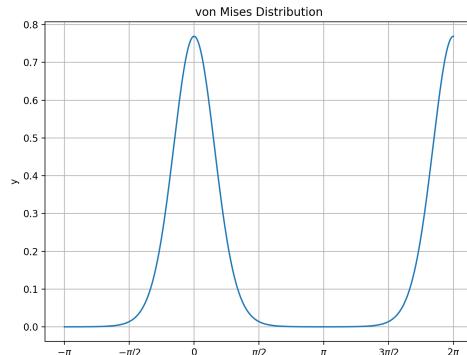
$y \in \mathbb{R}^+$ Predict Magnitude

Beta



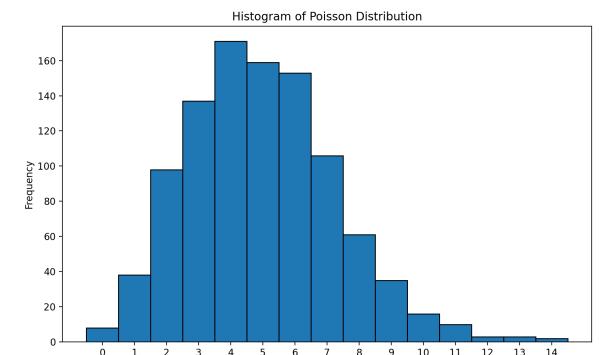
$y \in [0,1]$ Predict Proportions

Von Mises



$y \in (-\pi, \pi]$ Predict Directions

Poisson



$y \in [0,1,2, \dots]$ Predict Event Counts

Loss functions

- Maximum likelihood
- Recipe for loss functions
- Example 1: univariate regression
- Example 2: binary classification
- Example 3: multiclass classification
- Other types of data
- **Multiple outputs**
- Cross entropy

Multiple outputs

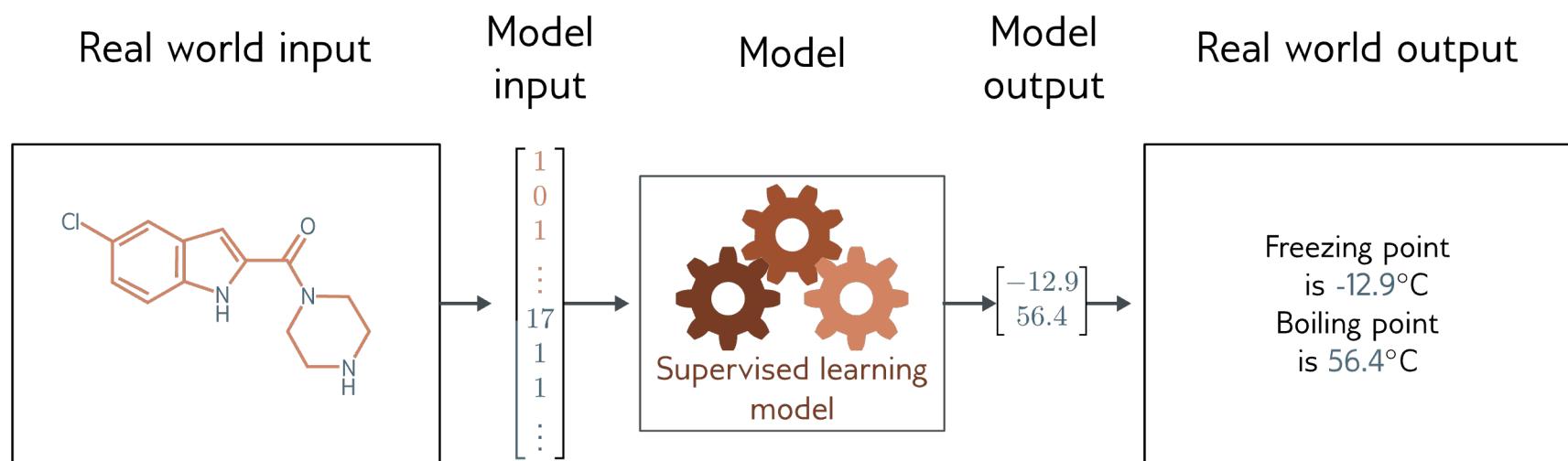
- Treat each output y_d as independent:

$$Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}_i, \boldsymbol{\phi}]) = \prod_d Pr(y_d|\mathbf{f}_d[\mathbf{x}_i, \boldsymbol{\phi}])$$

- Negative log likelihood becomes sum of terms:

$$L[\boldsymbol{\phi}] = - \sum_{i=1}^I \log \left[Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}_i, \boldsymbol{\phi}]) \right] = - \sum_{i=1}^I \sum_d \log \left[Pr(y_{id}|\mathbf{f}_d[\mathbf{x}_i, \boldsymbol{\phi}]) \right]$$

Example 4: multivariate regression



Example 4: multivariate regression

- Goal: to predict a multivariate target $\mathbf{y} \in \mathbb{R}^{D_o}$
- Solution treat each dimension independently

$$\begin{aligned} Pr(\mathbf{y}|\boldsymbol{\mu}, \sigma^2) &= \prod_{d=1}^{D_o} Pr(y_d|\mu_d, \sigma^2) \\ &= \prod_{d=1}^{D_o} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y_d - \mu_d)^2}{2\sigma^2} \right] \end{aligned}$$

- Make network with D_o outputs to predict means

$$Pr(\mathbf{y}|\mathbf{f}[\mathbf{x}, \boldsymbol{\phi}], \sigma^2) = \prod_{d=1}^{D_o} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(y_d - f_d[\mathbf{x}, \boldsymbol{\phi}])^2}{2\sigma^2} \right]$$

Example 4: multivariate regression

- What if the outputs vary in magnitude
 - E.g., predict weight in kilos and height in meters
 - One dimension has much bigger numbers than others
- Could learn a separate variance for each...
- ...or rescale before training, and then rescale output in opposite way

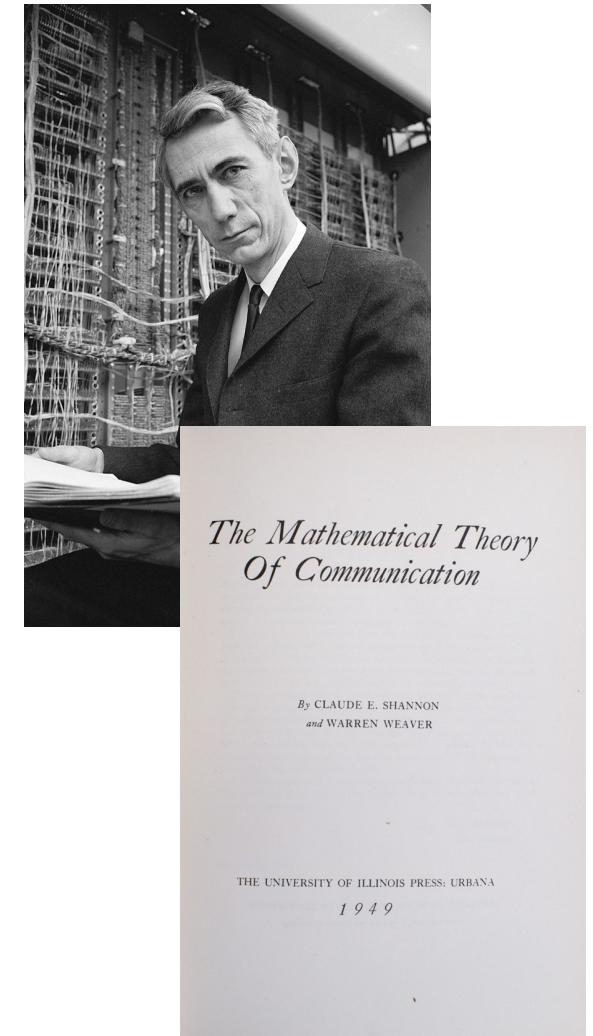
Loss functions

- Maximum likelihood
- Recipe for loss functions
- Example 1: univariate regression
- Example 2: binary classification
- Example 3: multiclass classification
- Other types of data
- Multiple outputs
- Cross entropy

Information Theory and Entropy

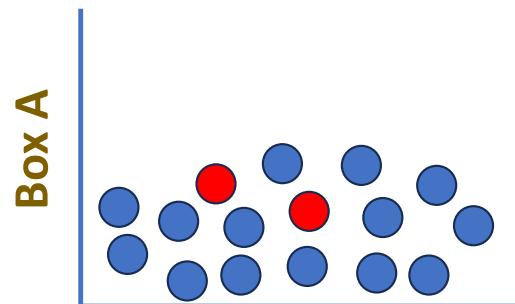
- **Claude Shannon:** the "father of information theory," was an American mathematician, electrical engineer, and cryptographer
- **Theory of Communication:** In his landmark 1948 paper, "A Mathematical Theory of Communication," Shannon introduced a formal framework for the transmission, processing, and storage of information.
- **Information Theory:** Quantified information, allowing for the measurement of information content in messages, which is crucial for data compression, error detection and correction, and more.
- **Concept of Information Entropy:** introduced entropy as a measure of the uncertainty or randomness in a set of possible messages, providing a limit on the best possible lossless compression of any communication.

$$H(x) = - \sum_x P(x) \log_2(P(x))$$

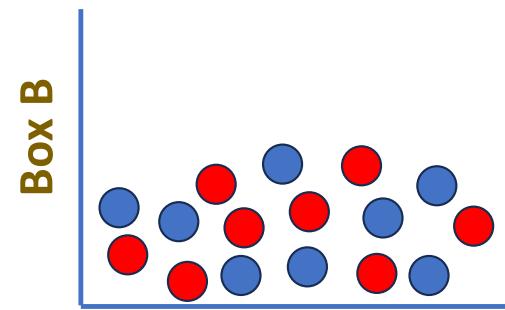


Entropy is a measure of *surprise* or *uncertainty*

Randomly pick a ball from the box



Low or High Entropy?



Low or High Entropy?

In class poll: <https://piazza.com/class/m5v834h9pcatx/post/27>

Connection to Deep Learning

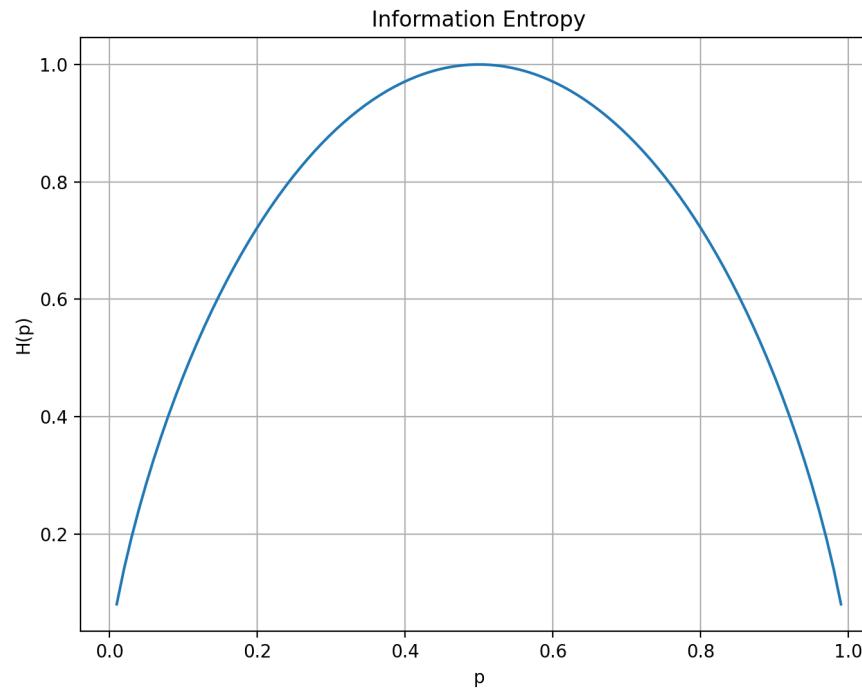
Cross-Entropy Loss

- If a neural network predicts **(0.25, 0.25, 0.25, 0.25)** for four possible classes, **high entropy** → uncertain.
- If it predicts **(0.99, 0.01, 0, 0)**, **low entropy** → confident.

Regularization & Overfitting

- A **high-entropy** model makes diverse predictions → good for exploration.
- A **low-entropy** model is overconfident → might overfit.

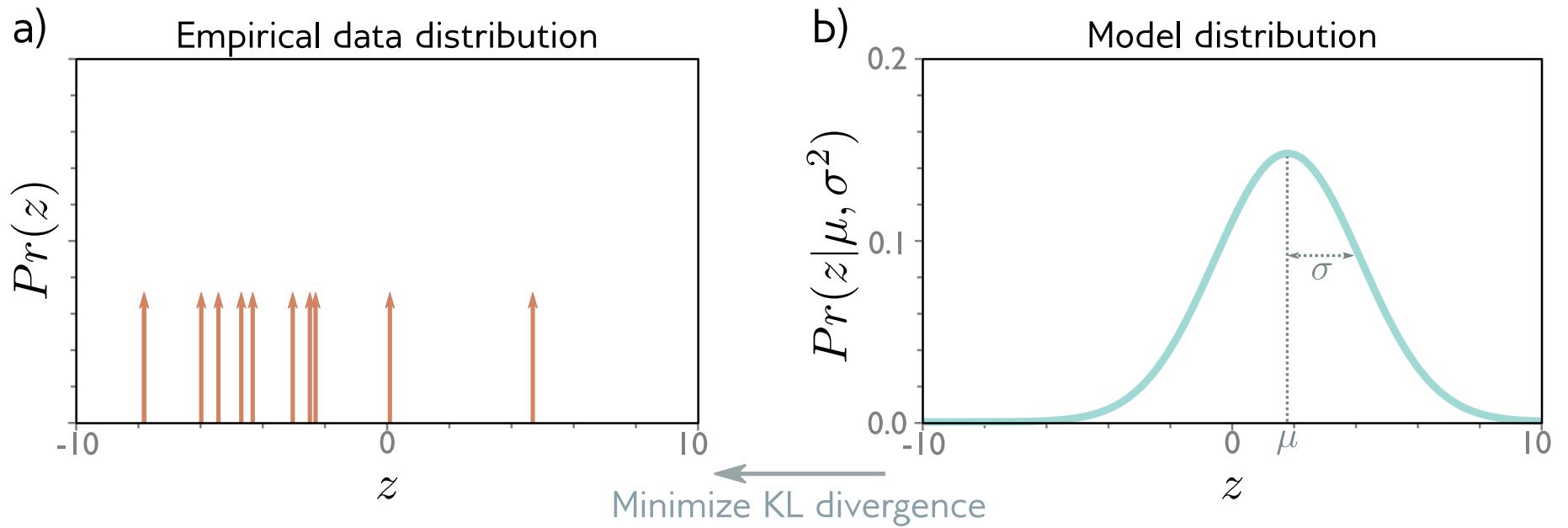
Entropy for a Binary Event $x \in \{0,1\}$



$$H(x) = - \sum_x P(x) \log_2(P(x)) = -p \log_2(p) - (1-p) \log_2(1-p)$$

Cross Entropy – Concept from Information Theory

Measures the difference between two probability distributions: the true distribution of the labels and the predicted distribution of the labels by a model.



$$\text{KL}[q||p] = \int_{-\infty}^{\infty} q(z) \log[q(z)] dz - \int_{-\infty}^{\infty} q(z) \log[p(z)] dz$$

Kullback-Leibler Divergence -- a measure between probability distributions¹¹³

Cross Entropy – Concept from Information Theory

For discrete distributions, the cross-entropy between two distributions p and q over the same underlying set of events is defined as:

$$H(p, q) = -\sum p(x) \log q(x)$$

Here, $p(x)$ is the true probability of an event x , and $q(x)$ is the estimated probability of the same event according to the model.

For instance, in binary classification:

$$H(p, q) = -[y \log(\hat{y}) + (1 - y)\log(1 - \hat{y})]$$

Here, y is the true label (0 or 1), and \hat{y} is the predicted probability of the class being 1.

Recap

- Reconsidered loss functions as fitting a parametric probability model
- Introduced Maximum Likelihood criterion for finding parameters to making the training data most probably under that model
- Introduced a 4-step recipe for (1) picking a suitable parametric probability distribution, (2) defining the model to pick one or more of the parameters, (3) training the model and (4) doing inference
- Derived loss functions for univariate regression, binary and multiclass classification
- Briefly reviewed parametric probability models for other types of data
- Discussed how this is the same as Cross Entropy from Information Theory

Minimizing Negative Log Likelihood

$$\begin{aligned}\hat{\phi} &= \operatorname{argmin}_{\phi} \left[- \sum_{i=1}^I \log[\Pr(y_i | f(x_i, \phi))] \right] \\ &= \operatorname{argmin}_{\phi} [L[\phi]]\end{aligned}$$

Recipe for loss functions

1. Choose a suitable probability distribution $\Pr(\mathbf{y}|\boldsymbol{\theta})$ that is defined over the domain of the predictions \mathbf{y} and has distribution parameters $\boldsymbol{\theta}$.
2. Set the machine learning model $\mathbf{f}[\mathbf{x}, \phi]$ to predict one or more of these parameters so $\boldsymbol{\theta} = \mathbf{f}[\mathbf{x}, \phi]$ and $\Pr(\mathbf{y} | \mathbf{f}[\mathbf{x}, \phi])$.
3. To train the model, find the network parameters $\hat{\phi}$ that minimize the negative log-likelihood loss function over the training dataset pairs $\{\mathbf{x}_i, \mathbf{y}_i\}$:

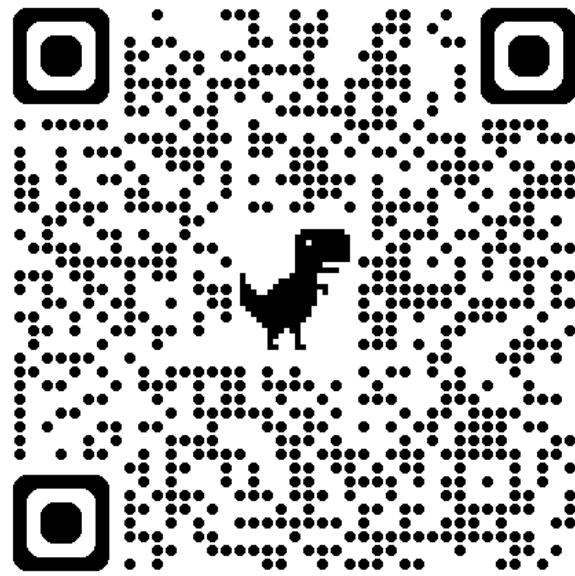
$$\hat{\phi} = \operatorname{argmin}_{\phi} [L[\phi]] = \operatorname{argmin}_{\phi} \left[- \sum_{i=1}^I \log[\Pr(y_i | f[x_i, \phi])] \right]$$

4. To perform inference for a new test example x , return either the full distribution $\Pr(\mathbf{y} | \mathbf{f}[\mathbf{x}, \phi])$ or the maximum of this distribution.

Next up

- Now let's find the parameters that give the smallest loss
 - → Training the model

Feedback?



[Link](#)