CHAPTER - 4

PRINCIPLE OF MATHEMATICAL INDUCTION

Mathematical induction

Mathematical induction is a method of proving results involving natural numbers

Eg:- $n(n+1)$ is even for any natural number

Principle of mathematical induction

Let $P(n)$ denotes the statement involving natural numbers.

Then (i) $P(1)$ is true

(ii) $P(k+1)$ is true whenever $P(k)$ is true. Then $P(n)$ is true for all natural number 'n'.

Prove by Principle of mathematical Induction that $1+3+3^2+\dots+3^{n-1} = \frac{3^n-1}{2}$ for all $n \in N$

Let $P(n)$ denotes the given statement

$P(n)$: $1+3+3^2+\dots+3^{n-1} = \frac{3^n-1}{2}$

~~* Using mathematical Induction P.~~

Then $P(1) = \frac{3^1-1}{2} = \frac{2}{2} = 1$

$P(n)$ is true for $n=1$

Assume $P(n)$ for $n=k$ i.e. $1+3+3^2 \dots 3^{k-1}$

$= \frac{3^k-1}{2}$

(adding 3^k on both sides

$1+3+3^2+\dots 3^{k-1}+3^k = \frac{3^k-1}{2}+3^k$

$\Rightarrow \frac{3^k-1}{2}+3^k \Rightarrow \frac{3^k-1+2\times 3^k}{2}$ $\boxed{a^n \cdot a^m = a^{m+n}}$

$\Rightarrow \frac{3\times 3^k-1}{2} \Rightarrow \frac{3^{k+1}-1}{2}$

$\therefore P(k+1)$ is true

Hence by mathematical induction $P_{(n)}$ is true for all $n \in N$.

* Using mathematical induction prove that

$$1.2 + 2.3 + 3.4 + \dots n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for $n \in N$

Sol. Let $P_{(n)}$ denotes the given statement.

$$P_{(n)} : 1.2 + 2.3 + 3.4 + n(n+1)$$

$$= \frac{n(n+1)(n+2)}{3}$$

$P_{(1)} = 1.2 = 2.$ | $P_{(1)} = \frac{(1+1)(1+2)}{3}$

$= \frac{2 \times 3}{3} = 2$

$P_{(1)}$ is true.

Assume that P(k) is true ie,

$$1\cdot2 + 2\cdot3 + 3\cdot4 \ldots k(k+1) = \frac{k(k+1)(k+2)}{3}$$

On adding (k+1)(k+2) on both sides

$$1\cdot2 + 2\cdot3 + 3\cdot4 \ldots k(k+1) + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} \quad (k+1)(k+2)$$

$$\Rightarrow \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$

$$(k+1)(k+2)\left[\frac{k}{3} + \frac{3}{3}\right] \quad \left[\frac{k}{3}+1 = \frac{k+3}{3}\right.$$

$$\Rightarrow \frac{(k+1)(k+2)(k+3)}{3}$$

∴ P(k+1) is true Hence by mathematical induction P(n) is true for all $n \in N$

Q) Using mathematical induction P.T

$a + ar + ar^2 + \dots ar^{n-1} = \frac{a(r^n - 1)}{r-1}$

for all $n \in N$.

Sol Let $P(n) = a + ar + ar^2 + \dots + ar^{n-1}$

$= \frac{a(r^n - 1)}{r-1}$

$P(1) = \frac{a(r^1 - 1)}{r-1} = a$

which is true

assume $P(k)$ is true

ie, $a + ar + ar^2 \dots ar^{k-1} = \frac{a(r^k - 1)}{(r-1)}$

On adding ar^k on both sides

$a + ar + ar^2 \dots ar^{k-1} + ar^k = \frac{a(r^k - 1)}{(r-1)} + ar^k$

$\Rightarrow \frac{a(r^k - 1) + ar^k(r-1)}{(r-1)}$

$$\Rightarrow \frac{ar^k - a + ar^{k+1} - ar^k}{(r-1)}$$

$$\frac{ar^{k+1} - a}{r-1} = \frac{a(r^{k+1} - 1)}{r-1}$$

$P(k+1)$ is true. Hence by PMI

$P(n)$ is true for all $n \in N$

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* Using Mathematical Induction P.T

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

for all $n \in N$

$$P(1) = 1 - \frac{1}{2^1} = 1 - \frac{1}{2} = \frac{1}{2}$$

which is true

Assume that $P(k)$ is true

$$P(k): \frac{1}{2} + \frac{1}{4} + \cdots \frac{1}{2^k} = 1 - \frac{1}{2^k}$$

on adding $\frac{1}{2^{k+1}}$ on both sides

$$\Rightarrow \frac{1}{2} + \frac{1}{4} \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

$$\Rightarrow \frac{2^k-1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^k - 1 + 2^k}{2^k \cdot 2^{k+1}}$$

$$\cancel{\Rightarrow \frac{2^{k+1}-1}{2^{2k+1}}} \quad \cancel{2^{k+1}-1}$$

$$\cancel{\frac{(2^k-1)\,2^{k+1} + 2^k}{2^k \cdot 2^{k+1}}} \Rightarrow \cancel{\frac{2^{2k+1} - 2^{k+1} + 2^k}{2^k \cdot 2^{2k+1}}}$$

$$\Rightarrow \cancel{\frac{2^k\,(2^{k+1}-1) + 2^k}{2^k \cdot 2^{k+1}}}$$

$$\Rightarrow \frac{(2^k-1)}{2^k}\cdot\frac{2}{2} + \frac{1}{2^{k+1}}$$

$$\Rightarrow \frac{2^{k+1}-2}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{2^{k+1}-1}{2^{k+1}}$$

$$= 1 - \frac{1}{2^{k+1}}$$

$\therefore$ $P(k+1)$ is true, Hence by PMI $P(n)$ is true for all $n \in N$ //

Q Using PMI, P.T

$$\left(1-\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right) = n+1 \quad \forall n \in N$$

Let $P(n) = \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right) = n+1$

$P(1) \Rightarrow \left(1+\frac{1}{1}\right) = 1+1 = 2$ — True

assume $P(k) \rightarrow$ True.

ie, $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{k}\right) = k+1$

We have to Prove $P(k+1)$ is also True

$P(k+1) = (k+1)+1 = k+2$ (RHS)

LHS $\rightarrow$

$$\underbrace{\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{k}\right)}\left(1+\frac{1}{k+1}\right)$$

$\cancel{k+1 + 1+\frac{1}{k+1} \Rightarrow k+2+\frac{1}{k+1}}$

$(k+1)\left(1+\frac{1}{k+1}\right) \Rightarrow \cancel{k+1}$

$$\cancel{\frac{(k+2)(k+1)+1}{(k+1)}} \Rightarrow$$

$$(k+1)\left(1+\frac{1}{k+1}\right) \Rightarrow (k+1)\left(\frac{(k+1)+1}{k+1}\right)$$

$$\Rightarrow \frac{\cancel{(k+1)}(k+2)}{\cancel{(k+1)}} = k+2 \Rightarrow LHS$$

which is equal to RHS Hence proved using PMI, True $\forall\, n \in N$

Q) Using PMI, P.T

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\cdots\left(1+\frac{(2n+1)}{n^2}\right)$$

$$=(n+1)^2 \quad \forall\, n \in N$$

$$P(n) = \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\cdots\left(1+\frac{(2n+1)}{n^2}\right)$$

$$=(n+1)^2$$

$P(1) = 1 + \frac{3}{1} = 4$ LHS

RHS $\Rightarrow (1+1)^2 = 2^2 = 4$ (True)

$P(k) \xrightarrow{\text{assume}}$ True

i.e. $\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right) \cdots \left(1 + \frac{(2k+1)}{k^2}\right)$

$= (k+1)^2$

$P(k+1)$ LHS $\rightarrow ((k+1)+1)^2 = (k+2)^2$

RHS $\rightarrow \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\cdots\left(1+\frac{(2k+1)}{k^2}\right)$

$\boxed{\begin{aligned} &2(k+1)+1 \\ &= 2k+2+1 \\ &= 2k+3 \end{aligned}}$

$\left(1 + \frac{2k+3}{(k+1)^2}\right) =$

$\Rightarrow (k+1)^2 \left(1 + \frac{2k+3}{(k+1)^2}\right)$

$\cancel{\Rightarrow (k+1)^2 \times \frac{(k+3)}{(k+1)}}$

$\Rightarrow \cancel{k^2 + 4k + 3}$

$1 + \frac{2k+3}{(k+1)^2}$

$\frac{(k+1)^2 + 2k+3}{(k+1)^2}$

$k^2 + 2k + 1 + 2k + 3$

$\Rightarrow \frac{k^2 + 4k + 4}{(k+1)^2}$

$\cancel{\frac{(k+1)(k+3)}{(k+1)^2}}$

$$\Rightarrow \cancel{(k+1)^2} \times \frac{(k+2)^2}{\cancel{(k+1)^2}} = (k+2)^2 \Rightarrow RHS$$

Here $RHS = LHS$ so, $P(k+1)$ holds True

Hence True for $\forall n \in N$ (Proved)

Q) Using MI P.T.

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} \cdots + \frac{1}{(3n-1)(3n+2)}$$

$$= \frac{n}{6n+4} \quad \forall\, n \in N$$

as $P(n) = \frac{1}{2.5} + \frac{1}{5.8} \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$

$P(1) = \frac{1}{2.5} = \frac{1}{(3-1)(3+2)} = \frac{1}{10}$ LHS

$RHS \rightarrow \frac{1}{6(1)+4} = \frac{1}{10} \rightarrow$ True.

$P(k) \xrightarrow{\text{assumed to be}}$ True i.e, $\frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} \cdots \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4}$

we have to prove $P(k+1) \rightarrow$ True.

$P(k+1)$ LHS $- \frac{k+1}{6(k+1)+4} \Rightarrow \frac{(k+1)}{\cancel{(k+5)}\ 6k+10}$

RHS $\rightarrow \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \cdots + \frac{1}{(3k-1)(3k+2)} +$

$$+ \frac{1}{(3(k+1)-1)(3(k+1)+2}$$

$$\Rightarrow \frac{k}{(6k+4)} + \frac{1}{(3k+2)(3k+5)}$$

$$\Rightarrow \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

$$\Rightarrow \frac{1}{3k+2}\left[\frac{k}{2} + \frac{1}{(3k+5)}\right]$$

$$\frac{1}{3k+2}\left[\frac{k(3k+5)+2}{2(3k+5)}\right]$$

$$\left(\frac{3k^2 + 5k + 2}{2(3k+5)}\right)\frac{1}{(3k+2)}$$

$$\frac{1}{\cancel{3k+2}}\left[\frac{(k+1)\cancel{(3k+2)}}{2(3k+5)}\right]$$

$3k^2 + 5k + 2$

$(k+1)(3k+2)$

$$= \frac{k+1}{2(3k+5)} =$$

$$\Rightarrow \frac{k+1}{6k+10} \Rightarrow \text{LHS}$$ which is equal to RHS

So, $P(k+1)$ is True

So, it is True $\forall\ n \in N$

Q) Using MI, PT $n(n+1)(n+5)$ is multiple of 3 for all $n \in N$

A) Let $P(n)$ denotes the statement

$P(n) : n(n+1)(n+5)$

$P(1) = 1(1+1)(1+5) = 2 \times 6 = 12$

12 m is a multiple of 3. so, True.

Say $P(k) - k(k+1)(k+5) = 3l$

$P(k+1)$:- ~~l some~~ ~~$\in N$~~ for som $l \in N$

$(k+1)((k+1)+1)((k+1)+5)$

$\cancel{= (k+1)(k+2) \cdot k}$

$\cancel{\Rightarrow (k+1)(k+2)(k+6)}$

$(k+1)(k+2)((k+5)+1)$

$\Rightarrow (k+1)(k+2)(k+5) + (k+1)(k+2)$

$\Rightarrow [(k+1)(k+5)(k+2)] + (k+1)(k+2)$

$\Rightarrow (k+1)(k+5)k + (k+1)(k+5)2$

$+ (k+1)k + 2(k+1)$

$\Rightarrow k(k+1)(k+5) + (k+1)[2(k+5) + (k+2)]$

$\Rightarrow 3l + (k+1)[2k+10+k+2]$ $[k(k+1)(k...$ $= 3l$

$\Rightarrow 3l + (k+1)[3k+12]$

$\Rightarrow 3l + (k+1)\, 3[k+4]$

$\Rightarrow 3l + 3(k+1)(k+4)$

$\Rightarrow \underline{3\,[l+(k+1)(k+4)]}$ which is a multiple of 3. So, $P(k+1)$ is divisible by 3. Hence by PMI $P(n)$ is true $\forall\, n \in N$

Q) Using MI Pro. T. $3^{2n+2} - 8n - 9$ is divisible by 8 for all $n \in N$.

a) $P(n) = 3^{2n+2} - 8n - 9$

Then $P(1) = 3^{2(1)+2} - 8(1) - 9$

$\Rightarrow 3^4 - 8 - 9 \rightarrow 81 - 17 = 64$

64 is divisible by 8 so, It is true

Assuming P(k) is true

Then, $3^{2k+2} - 8k - 9$ is divisible by 8.

Then $- \quad 3^{2(k+1)+2} - 8(k+1) - 9$

$\Rightarrow 3^{2k+2} \cdot 3^2 - 8k - 8 - 9$

$\Rightarrow (9)(3^{2k+2}) - 8k \times 9 + 8k \times 9 - 9 \times 9$
$+ 9 \times 9 - 8k - 17$

$\Rightarrow 9(3^{2k+2} - 8k - 9) + 9 \times 8k + 9 \times 9$
$- 8k - 17$

$\Rightarrow 9(3^{2k+2} - 8k - 9) + \underbrace{72k - 8k}_{64k} + \underbrace{81 - 17}_{64}$

$\Rightarrow 9(3^{2k+2} - 8k - 9) + 64(k+1)$

which is divisible by 8.

$\therefore$ P(k+1) is true.

Q) Using MI, P.T

$41^n - 14^n$ is a multiple of 27 $\forall n \in N$

a) $P(n) \rightarrow 41^n - 14^n$

$P(1) = 41^1 - 14^1 = \underline{27}$

which is a multiple of 27 so,

$P(1)$ is true.

Assuming $P(k)$ is true

ie, $41^k - 14^k$ is a multiple of 27

Say $41^k - 14^k = 27l$ for some $l \in N$

Then $41^{k+1} - 14^{k+1}$

$\Rightarrow 41^k * 41 - 14^k \cdot 14$

$\Rightarrow 41^k * 41 - \underbrace{41 * 14^k} + \underbrace{41 * 14^k} - 14^k * 14$

$41[41^k - 14^k] + 41 * 14^k - 14^k * 14$

$\Rightarrow \underbrace{41[41^k - 14^k]}_{} + 14^k[41-14]$

This Term is divisible by 27 ($P(k)$ assumed to be True)

This is also divisible by 27

~~So~~ Means, $P(k+1)$ is True

Hence, $P(n)$ is True $\forall n \in N$
Proved,

Q) Using MI, PT $10^{2n-1}+1$ is divisible by 11 for all $n \in N$

A) Let $P(n)$: $10^{2n-1}+1$

$P(1) = 10^{2(1)-1}+1 = 10^1+1 = 11$

11 is divisible by 11 so, $P(1)$ is True

Assuming $P(k)$ is true

ie, $10^{2k-1}+1$ is divisible by 11

Say $10^{2k-1}+1 = 11\lambda$ for some $\lambda \in N$

Then $10^{2(k+1)-1}+1 = 10^{2k+2-1}+1$

$= 10^{2k+1}+1$

$\Rightarrow 10^{2k} * 10 + 1$

$\Rightarrow 10 * 10^{2k} * \frac{10}{10} + 1$

$\Rightarrow 10 * 10 * \frac{10^{2k}}{10^{1}} + 1$

$= 100 * 10^{2k} * 10^{-1} + 1$

$\Rightarrow 100 * \{10^{2k-1} + 1\}$

$\Rightarrow 100 * 10^{2k-1} + 100 - 100 + 1$

$100(\underbrace{10^{2k-1}+1}) - 100 + 1\}$

$\Rightarrow 100 * 11l - 99$

$\Rightarrow 11[100l - 9]$ which is divisible by 11. $\therefore$ $P(k+1)$ is true. Hence by PMI $P(n)$ is true for all $n \in N$