1 Derivations

1.1

We are investigating systems with control-affine deterministic dynamics with additive white Gaussian noise. The continuous dynamics are written as:

$$\dot{x} = f(x) + g(x)u + (\mu(x) + \sigma(x)w_t)$$

where w_t at time t is sampled from a standard Gaussian distribution ($\mu = 0$, $\sigma = 1$). Each random variable w_t is independent and identically distributed throughout all time t. w_t can be interpreted as a white Gaussian noise process over time. Note that our system dynamics are described by a stochastic differential equation, rather than the typical deterministic differential equation.

For the sake of derivations, we will be analyzing a discrete approximation of this continuous system and then take the limit as Δt goes to zero. For a more rigorous, strictly continuous treatment of stochastic differential equations with additive white noise seeThe mathematical treatment covered here is for intuition and understanding only.

The discrete approximation to our system is:

$$\begin{aligned} x_{k+1} - x_k &= \int_{t=k\Delta t}^{(k+1)\Delta t} (f(x) + g(x)u + (\mu(x) + \sigma(x)w_t))dt \\ &= \int_{t=k\Delta t}^{(k+1)\Delta t} (f(x) + g(x)u + \mu(x))dt + \int_{t=k\Delta t}^{(k+1)\Delta t} \sigma(x)w_t dt \\ &\simeq (f(x_k) + g(x_k)u_k + \mu(x_k))\Delta t + \sigma(x_k) \int_{t=k\Delta t}^{(k+1)\Delta t} w_t dt \end{aligned}$$

Where we've separated out the drift due to the additive noise and x_k denotes the state at time $t = k\Delta t$. We could have approximated the change due to the white noise in the same manner as we did the change due to the deterministic dynamics; namely, assume the noise is constant over the time interval Δt and approximate as $\sigma(x_k)w_{k\Delta t}\Delta t$.

However, a better approximation can be made that leverages random process theory. The integral $\int_{t=k\Delta t}^{(k+1)\Delta t} w_t dt$ is interpreted in the Ito sense. Taking the integral of i.i.d white noise over time is one method for generating the Wiener Process (which is responsible for Brownian motion, amongst other things). The value of the Wiener process W_t at time t is distributed according to another Gaussian with mean zero and variance equal to the length of time the integral is take over Δt. It is important to note that although the Wiener process is also Guassian distributed at each time point, it is not the same as the white noise it was generated from. The Wiener process is not i.i.d. That is, each value of $W_t = \int_{t_{start}}^{t} w_t dt$ is dependent on previous values of W_s (with $s < t$). This dependence makes the trajectory of W_t continuous, in contrast to the discontinuous, noisy plot of w_t.

We take the Ito integral from start time $k\Delta t$ to $(k+1)\Delta t$, and get a new random variable for each k interval that represents the drift in the state due to Brownian motion. Replacing the integral with this new random variable W_k:

$$x_{k+1} - x_k \simeq (f(x_k) + g(x_k)u_k + \mu(x_k))\Delta t + \sigma(x_k)W_k$$

Noting that this variable is Gaussian distributed with variance Δt, we can replace this variable with a new standard normal distribution scaled by the standard deviation:

$$\begin{aligned} x_{k+1} - x_k &\simeq (f(x_k) + g(x_k)u_k + \mu(x_k))\Delta t + \sigma(x_k)\sqrt{\Delta t} b_k \\ &\rightarrow \\ x_{k+1} &\simeq x_k + (f(x_k) + g(x_k)u_k + \mu(x_k))\Delta t + \sigma(x_k)\sqrt{\Delta t} b_k \end{aligned}$$

1.2

We will pick up our treatment by starting with the general stochastic reachability framework introduced in Abate et. al. Note that our system is a purely continuous state system (no discrete mode transitions) and is therefore not strictly a hybrid system. However, we can still use Abate's framework by considering our continuous system a type of hybrid system with only one mode.

$$\begin{aligned} V_{k-1} &= \mathbf{1_A}(x)\mathrm{E}_{z\sim T}(V(z)) \\ &= \mathbf{1_A}(x)\mathrm{E}_{b\sim N}(V(x + (f(x) + g(x)u + b)\Delta t)) \\ &= \mathbf{1_A}(x)\int_{-\infty}^{\infty} V(x + (f(x) + g(x)u + b)\Delta t)\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}db \end{aligned}$$

1.3 Differential Form

$$\begin{aligned} V_{k-1}(x) &= \mathbf{1_A}(x)\mathrm{E}_{z\sim T}(V_k(z)) \\ &= \mathbf{1_A}(x)\mathrm{E}_{b\sim N}(V_k(x + (f(x) + g(x)u + b)\Delta t)) \\ &= \mathbf{1_A}(x)\mathrm{E}_{b\sim N}(V_k(x) + \nabla V_k^T(f(x) + g(x)u + b)\Delta t) \\ &= \mathbf{1_A}(x)V_k(x) + \mathrm{E}_{b\sim N}(\nabla V_k^T(f(x) + g(x)u + b)\Delta t) \\ &= \mathbf{1_A}(x)V_k(x) + \nabla V_k^T\mathrm{E}_{b\sim N}(f(x) + g(x)u + b)\Delta t \\ &\rightarrow \\ \frac{V_{k-1} - V_k(x)}{\Delta t} &= \nabla V_k^T\mathrm{E}_{b\sim N}(f(x) + g(x)u + b) \end{aligned}$$

$$\begin{aligned} &\Delta t \rightarrow 0 \\ &-\frac{\delta V}{\delta t} = \nabla V_k^T\mathrm{E}_{b\sim N}(f(x) + g(x)u + b) \\ &-\frac{\delta V}{\delta t} = \nabla V_k^T(f(x) + g(x)u + \mu) \end{aligned}$$

1.4 Mitchell Magic

$$dx = F(x, u)dt + \sigma(x)dB$$

$$-\frac{\delta V}{\delta t} = \nabla V_k^T \mathrm{E}_{b \sim N}(f(x) + g(x)u + b) + 0.5\texttt{trace}[\sigma(x)\sigma(x)^T \nabla_x^2 V_k]$$

$$dx = (f(x) + g(x)u + \mu(x))dt + \sigma(x)dB$$