Convex Optimization - Homework 6: Applications

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1 Exercise 1: Two Stage Decision Making

1.1 Demand Measured prior to Purchase

The linear program formulation for maximizing profit is:

$$\max \left\{ [25 \quad 10] \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} - [3.50 \quad 2.60 \quad 6.80] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\} \tag{1}$$

subject to

$$s_i \leq D_i, \qquad \forall i \in \{1, 2\} \tag{2}$$

$$s_i \leq y_i, \qquad \forall i \in \{1, 2\} \tag{3}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4}$$

where y_i is the quantity of drug i produced, s_i is the quantity of drug i sold, D_i is the quantity of drug i demanded, and x_i is the quantity of ingredient i purchased.

Implementing this formulation in CVX with a demand of $D_1 = 150$ and $D_2 = 200$ yields a profit of \$765 by producing and selling 150 units of Drug 1 and 0 units of Drug 2. To produce this, 450 units of Ingredient 1, 150 units of Ingredient 2, and 150 units of Ingredient 3 were acquired.

1.2 Profit Maximization for Three Demand Scenarios

The same formulation (with knowing the demand ahead of purchasing, producing, and selling) was used to maximize profit for each of the three demand scenarios. The maximum profit and associated drug production, selling, and ingredient purchases are filed in Table 1.

	Profit	s_1	s_2	y_1	y_2	x_1	x_2	x_3
Scenario 1	510	100	0	100	0	300	100	100
Scenario 2	765	150	0	150	0	450	150	150
Scenario 3	1020	200	0	200	0	600	200	200

Note that no amount of Drug 2 was produced, since we produce a \$0.60 loss with each sale.

In each scenario we have perfect information and will choose to make the maximum profit. The expectation on this maximum profit over all three scenarios is $\$790.5$.

1.3 Demand Measured prior to Selling

Both of the above analyses assume perfect information with demand being known before ingredients are acquired and any manufacturing takes place. However, a more realistic scenario would be one where we only learn demand after acquisition and assembly have already taken place. Now we need to optimize one choice of the x_i and y_i such that the expected profit is maximized.

We will use superscripts to denote versions of the variables that depend on the scenario. In this analysis, we have s_i^j denoting the amount of drug i sold in demand scenario j and D_i^j denoting the demand for drug i in the same scenario j. Therefore our expected profit optimization problem can be formulated as:

$$\max \left\{ \begin{bmatrix} 25 & 10 \end{bmatrix} \left(0.2 \begin{bmatrix} s_1^1 \\ s_2^1 \end{bmatrix} + 0.5 \begin{bmatrix} s_1^2 \\ s_2^2 \end{bmatrix} + 0.3 \begin{bmatrix} s_1^3 \\ s_2^3 \end{bmatrix} \right) - \begin{bmatrix} 3.50 & 2.60 & 6.80 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\} \tag{5}$$

subject to

$$s_i^j \leq D_i^j, \forall i \in \{1,2\}, \forall j \in \{1,2,3\} \tag{6}$$

$$s_i^j \leq y_i, \forall i \in \{1,2\}, \forall j \in \{1,2,3\} \tag{7}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{8}$$

Solving this formulation in CVX results in an expected profit of $515.00 with the profit resulting from our choice in each scenario being recorded in the next table.

	Profit	s_1	s_2	y_1	y_2	x_1	x_2	x_3
Scenario 1	-485	100	0					
Scenario 2	765	150	0	150	0	450	150	150
Scenario 3	765	150	0					

1.4 Demand Measured prior to Manufacture

A slightly better scenario would be if information on the drug demand was received prior to manufacture but after ingredient acquisition. In this case, we need to optimize one choice of the x_i such that the expected profit is maximized (we can tailor the y_i and s_i to the demand scenario). Now, our expected profit optimization problem can be formulated as:

$$\max \left\{ \begin{bmatrix} 25 & 10 \end{bmatrix} \left(0.2 \begin{bmatrix} s_1^1 \\ s_2^1 \end{bmatrix} + 0.5 \begin{bmatrix} s_1^2 \\ s_2^2 \end{bmatrix} + 0.3 \begin{bmatrix} s_1^3 \\ s_2^3 \end{bmatrix} \right) - \begin{bmatrix} 3.50 & 2.60 & 6.80 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\} \tag{9}$$

subject to

$$s_i^j \leq D_i^j, \forall i \in \{1,2\}, \forall j \in \{1,2,3\} \tag{10}$$

$$s_i^j \leq y_i^j, \forall i \in \{1,2\}, \forall j \in \{1,2,3\} \tag{11}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1^j \\ y_2^j \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \forall j \in \{1,2,3\} \tag{12}$$

Solving this formulation in CVX results in an expected profit of $615.00 with the resulting profits and units produced recorded in the next table.

	Profit	s_1	s_2	y_1	y_2	x_1	x_2	x_3
Scenario 1	15	100	50	100	50			
Scenario 2	765	150	0	150	0	450	150	150
Scenario 3	765	150	0	150	0			

2 Exercise 2: Dicsount Factor Curve Fitting

2.1 Formulation

The formulation for this problem is:

$$\begin{aligned} &\min_a \sum_{i=1}^{I} |p_0 f(t_0^{(i)}) + p_1 f(t_1^{(i)})| \\ = &\min_a \sum_{i=1}^{I} |p_0(a_{0,k_{i0}} + a_{1,k_{i0}} t_0^{(i)} + a_{2,k_{i0}} t_0^{(i)2}) + p_1(a_{0,k_{i1}} + a_{1,k_{i1}} t_1^{(i)} + a_{2,k_{i1}} t_1^{(i)2})| \end{aligned} \tag{13}$$

where k_{ij} is the interval that $t_j^{(i)}$ falls in.

Subject to the constraints:

$$\begin{aligned} a_{0,k} + a_{1,k} m_{k+1} + a_{2,k} m_{k+1}^2 &= a_{0,k+1} + a_{1,k+1} m_{k+1} + a_{2,k+1} m_{k+1}^2, & \forall k \in \{1,2,\ldots,N-1\} \\ a_{1,k} + 2a_{2,k} m_{k+1} &= a_{1,k+1} + 2a_{2,k+1} m_{k+1}, & \forall k \in \{1,2,\ldots,N-1\} \\ a_{1,k} + 2a_{2,k} m_k &\leq 0, & \forall k \in \{1,2,\ldots,N\} \\ a_{1,k} + 2a_{2,k} m_{k+1} &\leq 0, & \forall k \in \{1,2,\ldots,N\} \\ a_{0,N} + a_{1,N} T + a_{2,N} T^2 &\geq 0 \end{aligned} \tag{14}$$

where m_i is the left endpoint of the ith interval and m_{N+1} is the right endpoint of the final interval. The formula for obtaining the m_i is:

$$m_i = (i-1)\frac{T}{N}$$

2.2 CVX Solution

A MATLAB script was written up to solve this formulation using CVX. The resultant discount curve fits are shown in Figs. 1-4 below, with their error values in the caption.

3 Exercise 3: Probability Density Estimation

3.1 Reward Function

Let $F(x)$ be the true distribution from which the ten samples x_i are drawn. The optimization problem which gives the maximum likelihood estimator $f(x)$ is:

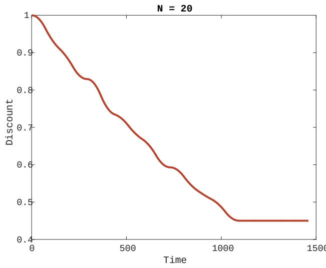

Figure 1: Discount Curve fit for N=20 (optimum error = 1.53704)

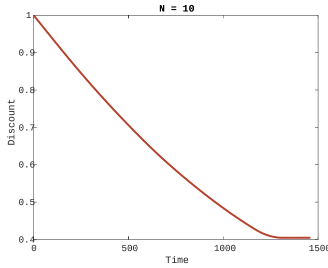

Figure 2: Discount Curve fit for N=10 (optimum error = 1.73544)

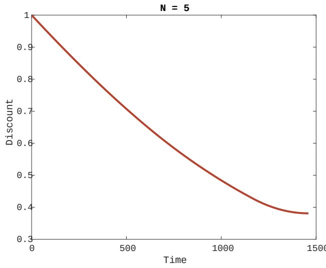

Figure 3: Discount Curve fit for N=5 (optimum error = 2.0067)

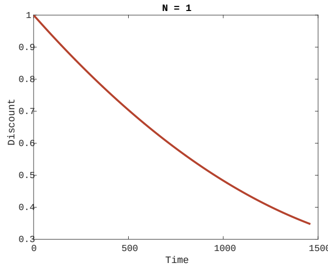

Figure 4: Discount Curve fit for N=1 (optimum error = NaN)

$$\begin{aligned} \underset{f}{\text{maximize}} \quad & \Pi_{i=1}^{10} f(x_i) \\ \text{subject to} \quad & \int_X f(x)dx = 1 \\ & f(x) \geq 0, \ \forall x \end{aligned}$$

Since the $\log$ function is monotone increasing, this is an equivalent optimization problem to:

$$\begin{aligned} \underset{f}{\text{maximize}} \quad & \log\left(\Pi_{i=1}^{10} f(x_i)\right) \\ & = \sum_{i=1}^{10} \log(f(x_i)) \\ \text{subject to} \quad & \int_X f(x)dx = 1 \\ & f(x) \geq 0, \ \forall x \end{aligned}$$

"Equivalent" in the sense that the optimization problem has the same minimizing choice of $f^*(x)$, if not necessarily the same optimal value p^*.

3.2 Optimization Formulation

We choose for the function $f(x)$ to be parametrized by N linear pieces defined over evenly subdivided intervals.

Define the interval that all the x_i fall into to be from 0 to X (in step 3 we will make this 0 to X=4). To divide this interval into N evenly spaced sub-intervals, each sub-interval must be $r = X/N$. Therefore the formulation is:

$$\begin{aligned}
\underset{c,d}{\text{maximize}} \quad & \sum_{i=1}^{10} \log\left(c_{i_j}(x - i_j r - r) + d_{i_j}\right) \\
\text{subject to} \quad & \sum_{i=1}^{N} (d_i r + \frac{1}{2} c_i r^2) = 1 \\
& d_i \geq 0, \qquad \forall i \in 1, \dots, N \\
& d_i + c_i r \geq 0, \qquad \forall i \in 1, \dots, N \\
& d_{i+1} = d_i + c_i r, \quad \forall i \in 1, \dots, N-1
\end{aligned}$$

Note here that we define our parameters c_i and d_i as a line with respect to interval-local coordinates rather than global coordinates. The interval-local coordinates start at 0 (at the interval's left endpoint) and range up to r when the interval ends. To transform from the universal coordinates x into the local coordinates use $x' = c_{i_j}(x - i_j r - r)$ like we do in the objective function. This definition in local coordinates greatly simplifies the definition of the constraints as the borders are always at 0 and r.

3.3 CVX Solution

A Matlab script for optimizing this formulation using CVX was composed and ran. The resulting best fit corresponded approximately to a sum of Kroniker deltas at each sample point each weighted to make the total sum to 1 (as constrained, since this is a PDF).The plot is shown in Fig. 5 along with the baseline and some future fits to be described in the following sections. Note that the plot is clipped, since these "deltas" actually stretch extremely high up to around $f = 25$. This clipping is to keep the other fits and the baseline visible.

3.4 Non-Increasing Requirement

We want to encode that the PDF is non-decreasing. That is to say that higher values of x are never more likely than lower values. This requirement becomes one additional constraint. Our elegant interval-local coordinate parametrization simplifies this constraint to just:

$$c_i \leq 0, \ \forall i \tag{15}$$

Adding this constraint to our CVX optimization results in the plot labeled "Non-Increasing" in Fig. 5. Note how much better this PDF estimation is then the drastically over-fit "deltas" from the previous optimization.

3.5 Bounded Slope Requirements

Since our true PDF is an exponential with mean 1, the true slope of this PDF is:

$$\frac{df}{dx} = -e^{-x}$$

To encode $\pm 20\%$ bounds on the slope at all times we use:

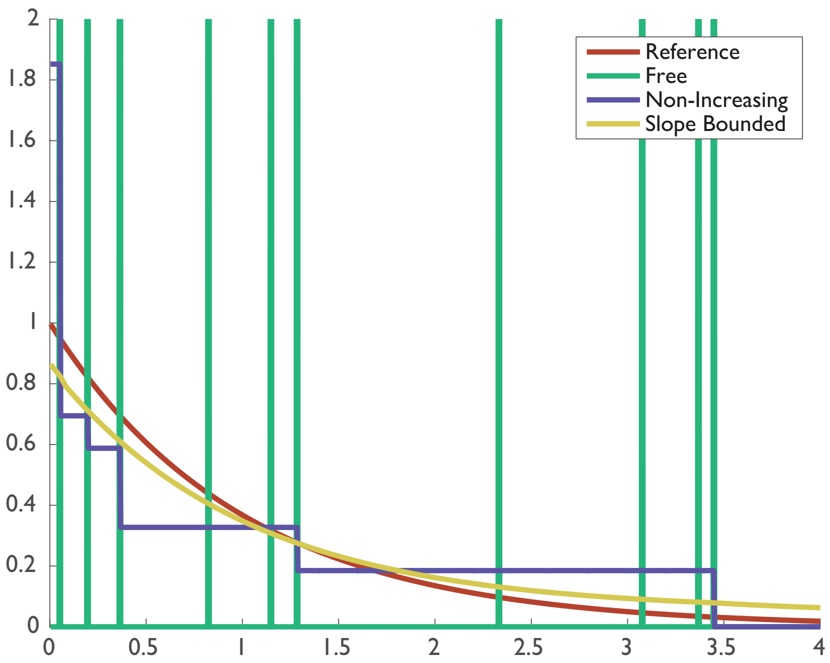

Figure 5: Probability Density Function Fits

$$
\begin{aligned}
c_i \le -0.8e^{-x} \forall x \in I_i \rightarrow \quad & c_i \le -0.8e^{-r(i-1)} \\
c_i \ge -1.2e^{-x} \forall x \in I_i \rightarrow \quad & c_i \ge -1.2e^{-ri}
\end{aligned}
\tag{16}
$$

where the implication step was made by choosing the tightest bound in the interval I_i.

Adding this constraint produces the closest approximation to the true PDF as plotted in Fig. 5. This curve is qualitatively much closer to the true PDF. Success.