EE 127 / EE 227AT

Homework Assignment #6

Due date: 4/26/16 11AM. Please LaTeX or scan your homework solution and submit the pdf on bCourses.

Exercise 1 (Linear Program Uncertainty) Real-world optimization problems always face uncertainty. In particular, production problems usually face uncertainty in the demand for products. Before completing this exercise, read the paper by Higle and Wallace posted on bCourses.

We wish to maximize profit in the following production problem faced by a pharmaceutical company. We can manufacture two drugs using three ingredients, according to the following table:

Resource	Price	Drug 1	Drug 2
Ingredient 1	\$3.50	3	1
Ingredient 2	\$2.60	1	0
Ingredient 3	\$6.80	1	1

A unit of Drug 1 requires 3 units of Ingredient 1, which cost \$3.50 each, and so on. Drug 1 sells for \$25 per unit, and Drug 2 sells for \$10 per unit.

1. First, we will assume that we know the demand ahead of time. Say there is demand for 150 units of Drug 1 and 200 units of Drug 2. Formulate the appropriate linear program to maximize profit and solve it using CVX. Report the profit achieved, the amount of each ingredient purchased, and the amount of each drug produced and sold.

2. For the rest of this problem, we will assume that there are three demand scenarios that occur with different probability:

	Probability	Drug 1 Demand	Drug 2 Demand
Scenario 1	0.2	100	100
Scenario 2	0.5	150	200
Scenario 3	0.3	200	250

   If we still know the demand ahead of time, what is the maximum expected profit we can achieve? How much of each ingredient do we purchase and how much of each drug do we produce in each scenario?

3. Production happens in three stages: first we acquire ingredients, then we produce the drugs, and finally we sell them. What if the demand remains unknown throughout resource acquisition and production, but is revealed just prior to selling? Formulate and solve a new LP that considers this uncertainty and maximizes the expected profit. Report the expected profit, the amount of each ingredient purchased, and the amount of each drug produced and sold in each scenario.

4. What if we still do not know the demand until after we acquire the ingredients, but it is revealed before we begin production? Formulate and solve the LP which maximizes expected profit in this scenario. Report the expected profit, the amount of each ingredient purchased, the amount of each drug produced, and the amount of each drug sold in each scenario. What is different about this solution compared to the other models? Why?

Exercise 2 (Discount Factor Curve Fitting) In this exercise we will estimate a discount factor curve by fitting second-order epi-splines to data. We have I instruments, each associated with a payment at a certain time and a return at a later time. Let $p_0^{(i)}$ and $p_1^{(i)}$ denote the payment and return on instrument i, occuring at $t_0^{(i)}$ and $t_1^{(i)}$ respectively. Then we wish to fit a function of time f so that

$$p_0^{(i)} f(t_0^{(i)}) + p_1^{(i)} f(t_1^{(i)}) \approx 0, \ \forall i \in \{1, \ldots, I\}$$

The curve represents the value today of one dollar recieved in the future. The function f should be nonnegative, nonincreasing, and continuously differentiable, with $f(0) = 1$. We will also restrict f to be piecewise quadratic.

1. Assume that the timeline runs from 0 to T. We will divide the timeline into N equally sized segments, and find the optimal quadratic function for each segment. That is, when t falls in segment k, $f(t) = a_{0,k} + a_{1,k}t + a_{2,k}t^2$ for some set of $3N$ coeefficients a_{ij}. We seek to minimize the quantity

$$\sum_{i=1}^{I} \left| p_0^{(i)} f(t_0^{(i)}) + p_1^{(i)} f(t_1^{(i)}) \right|$$

   Formulate the convex optimization problem that you will solve to estimate the discount factor curve. Make sure that $f(t)$ has all the desired properties for $t \in \{0, \ldots, T\}$.

2. The file `data.mat` contains a 21×4 matrix. Each row corresponds to one instrument. The elements of row i are, in order, $p_0^{(i)}, t_0^{(i)}, p_1^{(i)}$, and $t_1^{(i)}$. In this dataset, $T = 1459$. Use CVX to solve the problem given these instruments for $N = 1, 5, 10, 20$, and plot each resulting discount factor curve estimate over time.

Exercise 3 (Probability Density Estimation) In this exercise we will estimate a probability density function based on a few observations. We will construct the estimate using first-order epi-splines. Data is provided in `samples.mat`.

1. For a probability density function f and samples $x_1, \ldots, x_{10}$, write down the optimization problem whose solution is the maximum likelihood estimator of f.

2. We will restrict f to be piecewise linear and continuous, with N segments as in Exercise 2. Formulate the appropriate convex optimization problem for maximum likelihood estimation, and explain why it is convex in this case. Make sure to enforce constraints to ensure that f is a valid PDF, and has the desired properties.

3. Use CVX to solve the problem you formulated above, fitting f to the data provided with $N = 1000$ on the interval $[0, 4]$. The samples were drawn from an exponential distribution with mean 1. Plot the function you recovered along with the true PDF and comment qualitatively on the accuracy.

4. What if we have more information about the nature of the PDF we are trying to estimate? Add the constraint that f is nonincreasing. Solve the new problem and plot the result and true PDF again. Comment on the accuracy and any improvement you see.

5. What if we also knew upper and lower bounds on the slope of the PDF? Add constraints that restrict the slope of f to be within $\pm 20\%$ of the true value. For this exercise, you can compute these bounds from the exponential distribution PDF with mean 1. Plot once more and comment.