Social and Economic Networks
Solutions Advanced Problems: Week 7

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Exercise 1

Consider an example of a directed triad: agent 1 has a directed link to agent 2, agent 2 has a directed link to agent 3 and agent 3 has a directed link to agent 1. The claim is there is no pure strategy Nash equilibrium. Suppose there is a Nash Equilibrium where $x_1 = 1$, it follows that agent 3's best response is $x_3 = 0$ and agent 2's best response is $x_2 = 1$. Then, agent 1's best response is $x_1 = 0$, which is a contradiction. Similarly, if there is a Nash Equilibrium where $x_1 = 0$, it follows that agent 3's best response is $x_3 = 1$ and agent 2's best response is $x_2 = 0$. Then, agent 1's best response is $x_1 = 1$, which is again a contradiction. Thus, there is no pure strategy Nash equilibrium.

Let $p = 1 - c$, where c is the common cost of taking the action 1. We claim that playing 1 with probability p and 0 otherwise is a mixed strategy equilibrium. For any agent i, if his/her uses the mixed strategy, by choosing $x_i = 1$, the payoff is $1-c$, and by choosing $x_i = 0$, the expected payoff is $p \cdot 1 + (1-p) \cdot 0 = p = 1-c$, and so agents are indifferent.

Exercise 2.

(a) First, note that in an equilibrium, it cannot be that all players choose 0. This follows since there is a positive level at which $f'(x^*) = c'(x^*)$, which then given the strict concavity and convexity implies that $f'(0) > c'(0)$, and so 0 is not an equilibrium since any agent can then increase payoffs by raising his or her x_i. Thus, at least one player has $x_i > 0$.

In a pure strategy equilibrium, since players are best responding it must be that

$$f'(\sum_k x_k) \leq c'(x_i)$$

for each i (or else raising x_i would increase a player's payoff), and this must hold with equality whenever $x_i \geq 0$ (or an agent could gain by lowering the action). Thus,

$$f'(\sum_k x_k) = c'(\max_k x_k).$$

Given the strict concavity and convexity of f and c, this implies that

$$f'(\sum_k x_k) > c'(x_i),$$

when $x_i < \max_k x_k$. Thus, $x_i < \max_k x_k$ cannot be part of an equilibrium, and so any pure strategy equilibrium must have all agents choose the same positive action x which then must be the solution to

$$f'(nx) = c'(x).$$

There is a solution to this since we already argued that $f'(0) > c'(0)$, and we also know that $f'(x^*) = c'(x^*)$ and that implies that $f'(nx^*) < c'(x^*)$, and so x lies between 0 and x^*. It is unique given the strict concavity of f and strict convexity of c.

(b) On the circle, order the players so that even numbered players are linked only to odd numbered players and vice versa. There is an equilibrium where even numbered players choose x^* and odd numbered players choose 0 (and there is another with the roles reversed). Note that this is a best reply for the even-numbered players since $f'(x^*) = c'(x)$. For the odd-numbered players, their neighbors produce $2x^*$ in total, and $f'(2x^*) < c'(0)$, and thus given the concavity of f and convexity of c, the derivative of $f(2x^* + x_i) - c(x_i)$ is negative for all $x_i \geq 0$ and so the best reply is 0.

Exercise 3.

Suppose that there is a contagion from m nodes. Let us show that there is a labeling ℓ such that $\alpha_\ell(i) \geq q$ for all $i \geq m + 1$. Let M^0 be the original set of m nodes that take action 1, and let M^t for $t \geq 1$ be the set of nodes that are infected after t iterations of the best response function. Given that there is contagion, $\cup_{t \leq T} M^t = N$ for some $T \leq n$. Set $\ell(i)$ such that $i \in M^t$ and $j \in M^{t'}$ with $t' < t$ implies that $\ell(j) < \ell(i)$. Given that the best response of $i \in M^\tau$ for a $\tau \geq 1$ is 1 when the set of agents choosing 1 is $\cup_{t \leq \tau-1} M^t$, it follows that at least a fraction q of i's neighbors lie in $\cup_{t \leq \tau-1} M^t$. Thus by the construction of ℓ, at least a fraction $q$of i's neighbors have $\ell(j) < \ell(i)$, which is the desired claim.

Next, suppose that there is a labeling ℓ such that $\alpha_\ell(i) \geq q$ for all $i \geq m + 1$. Let us show that there is a contagion starting from the nodes such that $\ell(i) \leq m$. Let M^0 be the the nodes such that $\ell(i) \leq m$ be the starting set of nodes that take action 1, and let M^t for $t \geq 1$ be the set of nodes that are infected after t iterations of the best response function. Suppose to the contrary that $M^n = M$ where $M \neq N$. Let $i = argmin_{j \notin M} \ell(j)$. It follows that $j \in M$ for all j such that $\ell(j) < \ell(i)$. But then i's best response to having M take action 1 is to take action 1 since by the definition of ℓ it follows that i has a fraction of at least q neighbors in M. This contradicts the presumption that M is the limit of the best response iteration.

Finally, let us show there exists a set S that is uniformly less than $1-q$-cohesive if and only if there is a labeling ℓ such that $\alpha_\ell(i) \geq q$ for all $i \geq m + 1$ where m is the cardinality of the complement of S.

Suppose that there exists a set S that is uniformly less than $1 - q$-cohesive. Then there is a contagion from the complement of S (when q is the threshold), which is the seeding m nodes. The above then implies that there is a labeling ℓ such that $\alpha_\ell(i) \geq 1 - r$ for all $i \geq m + 1$ where m is the cardinality of the complement of S.

Suppose there is a labeling ℓ such that $\alpha_\ell(i) \geq q$ for all $i \geq m + 1$ where m is the cardinality of the complement of S. Then from above we know that there is a contagion from m nodes (when q is the threshold). The result then implies that there exists a set S of m nodes that is uniformly less than $1 - q$-cohesive.