Social and Economic Networks
Advanced Problems: Week 6

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1. Consider the following variation of the observational learning model, which is due to Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992). Agents choose an action only once. The action A pays 1, while the action B pays either 0 or 2, with equal probability. Agents choose sequentially and cannot communicate other than to observe each others' actions (but not payoffs). Agents see all of the previous actions before making a choice themselves. So, agent 1 makes a choice and gets a payoff. Then agent 2 sees agent 1's choice (but not payoff) and chooses an action and gets a payoff. Then agent 3, having see the actions of 1 and 2, but not their payoffs, chooses an action and gets a payoff, and so forth. In terms of uncertainty: action B is either "good" and pays 2 for all of the agents who choose it, or "bad" and pays 0 for all of the agents who choose it. Agents know that the chances that B is bad or good are equal. In addition, they each get to observe a private signal about the state of nature ("good" or "bad"). The signal of agent i, denoted s_i takes on a value "good" or "bad". If the true state is "good", then the signal is "good" with a probability p, where $1 > p > 1/2$, and similarly if the state is "bad" then the signal is "bad" with probability p. The signals are independent across the agents, conditional on the state. All of the agents have had a course in basic probability, and choose the action which they think leads to the highest expected payoff conditional on all the information that they have. If they think there is an equal chance of good or bad, then they flip a coin.

So, let us think about how this process will evolve. The first agent has only her own signal. Thus, if she sees "good" then she thinks there is a p probability that the state is "good" and thus chooses action B. If she sees "bad" then she thinks that there is a p probability that the state is "bad", and so chooses action A. Therefore, based on the action of the first agent, all of the subsequent agents can deduce her signal. So, consider the case where the first agent chose B (the other case is analogous). If the second agent sees "good" then he has effectively seen two good signals, and so chooses A. If the second agent sees "bad", then he has effectively seen one good and one bad signal, and so chooses with equal probability. Note that the third agent cannot always be sure of the second agent's signal. If the third agent sees an action sequence B, A, then she can be sure that the signals were "good", "bad". But if the third agent sees B, B she cannot be sure of what the second agent's signal was.

Show that the third agent, conditional on seeing B, B will ignore her own signal and choose B regardless. Show that this will be true for all of the subsequent agents.

(You can deduce that a herd eventually forms by inducting on your argument.)

2. Show that, under the DeGroot model, if $T_{ji} = T_{ij}$ for all i in a strongly connected and aperiodic society, then every agent has the same influence. Show this is a consequence of the stronger claim that in a strongly connected and aperiodic society if $\sum_j T_{ji} = 1$ for all i, then every agent has the same influence.

3.

Consider the DeGroot model with two agents and

$$T = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}. \tag{1}$$

Find the exact expression for $b_1(t)$ when $b_1(0) = 1$ and $b_2(0) = 0$.

Hint: To see how T^t behaves as we increase t, it is useful to rewrite T using what is known as its "diagonal decomposition." In particular, let u be the matrix of left hand eigenvectors of T. Then u has the following form

$$u = \begin{pmatrix} s_1 & s_2 \\ 1 & -1 \end{pmatrix}, \tag{2}$$

where (s_1, s_2) is the unit eigenvector corresponding to the social influence weights, and (1,-1) is the other eigenvector. It is easy to check that here $s_1 = \frac{T_{21}}{1+T_{21}-T_{11}}$ and $s_2 = \frac{1-T_{11}}{1+T_{21}-T_{11}}$. Also, in the case of a row stochastic T with $n = 2$, (1,-1) is always the second eigenvector, since

$$(1, -1)T = (T_{11} - T_{21}, T_{12} - T_{22}) = (T_{11} - T_{21}, -(T_{11} - T_{21})) = (T_{11} - T_{21})(1, -1).$$

Moreover, not only is (1,-1) the second eigenvector, but its associated eigenvalue is $T_{11} - T_{21}$.

Since u is the matrix with rows being the eigenvectors, we know that

$$uT = \Lambda u, \tag{3}$$

where Λ is the matrix with the first and second eigenvalues λ_1 and λ_2 (ranked in terms of absolute values) on its diagonal:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & T_{11} - T_{21} \end{pmatrix}.$$

From (2) and given that $s_1 > 0$ and $s_2 > 0$ we know that u has an inverse, u^{-1} which is easily seen to be

$$u^{-1} = \begin{pmatrix} 1 & s_2 \\ 1 & -s_1 \end{pmatrix}. \tag{4}$$

From (3) it follows that

$$T = u^{-1}\Lambda u, \tag{5}$$

This is the *diagonal decomposition* of T.[1] From (5) it follows that

$$T^2 = u^{-1}\Lambda u u^{-1}\Lambda u = u^{-1}\Lambda^2 u,$$

and more generally that

$$T^t = u^{-1}\Lambda^t u. \tag{6}$$

So, the convergence of T^t is directly related to the convergence of Λ^t.

[1] It is sometimes useful to note that u^{-1} is the matrix of right-hand eigenvectors of T, and that they have the same matrix of eigenvalues as u. To see this, note that from (3) it follows that $uTu^{-1} = \Lambda u u^{-1} = \Lambda$, Thus $u^{-1}uTu^{-1} = u^{-1}\Lambda$ and so $Tu^{-1} = u^{-1}\Lambda$, and u^{-1} is the vector of right hand eigenvectors. The left-hand and right-hand eigenvectors also go by the names of the row and column eigenvectors, respectively. The diagonal decomposition can then be stated in terms of u and its inverse, or equivalently in terms of the two matrices of column and row eigenvectors.