Social and Economic Networks
Solutions Advanced Problems: Week 4

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Exercise 1.

Let us show that the diameter is bounded by $2K+1$. Suppose to the contrary that there are two non-isolated nodes, i and j, who are at a distance of at least $2K+2$ from each other. Consider a node h that is linked to i. Let $v_h = u_h(g) - u_h(g - hi)$ be the utility gain that h has from linking to i. This will take the form of

$$-c + \sum_{h' \in S} [b(\ell_g(h,h')) - b(\ell_{g-hi}(h,h'))]$$

for some set of nodes S that each have distance at most K from i under g. Here, S is the set of nodes that h ends up closer to as a result of adding the link hi (and note that $i \in S$) and are ones that h can actually get utility from. It follows that

$$-c + \sum_{h' \in S} [b(\ell_g(h,h')) - b(\ell_{g-hi}(h,h'))] \geq 0.$$

Note that j must be at a distance of more than K in g from every h' in S since h is at a distance of more than $2K$ from j. Note also that j is at a distance of more than K in g from h for the same reason. Thus, if j were to add a link to i the gain would be at least

$$-c + b(2) + \sum_{h' \in S} b(\ell_g(h,h'))$$

where the $b(2) > 0$ is the resulting benefit from being connected to h at a distance of 2. This is strictly higher than the previous expression and thus greater than 0.

The same argument shows that j will have a strictly positive payoff from linking to i. This contradicts the pairwise stability of g.

2.

Let there be four agents and let $\delta = .9$, $c_1 = .01$, $c_2 = c_3 = .02$, and $c_4 = 2.5$. It is easy to check that agents 1, 2, and 3 should form a clique. Given agent 4's cost, that agent should have at least one link (linking to one of the other agents brings 2.52 - 2.5 to that agent, and positive payoffs to the others) and no more than one link (adding a second or third link incurs a great cost > 2.5, and only improves benefits by .09 for each agent in the link). Given that the benefits are the same in total regardless of which agent connects to agent 4, and agent 1 has the lowest cost, it should be agent 1, and so the efficient network is unique.

For the second part, consider a case such that an efficient network has a component more than three players and show that there is a star with at least four players in it, as the other cases trivially have a star as a subnetwork.

Let us first show that one total utility maximizing way to form a component involving n players and $K \geq n-1$ links, is to form a star with a center be one of the lowest cost players, and then to add the next $K-(n-1)=m$ links among pairs of nodes that have the next m cheapest total costs per link.

First, note that this network maximizes total benefits among networks with this many links and players, as it has a total benefit of $2K\delta+(n(n-1)-2K)\delta^2$. Any network with K links can only generate direct connections benefits of $2K\delta$, and then has remaining pairs of nodes at distances of at least two from each other so can get at most $(n(n-1)-2K)\delta^2$ in terms of total indirect benefits.

This network also has minimal possible cost among all networks that involve these n players and have K links, since having at least one link for every agent involves costs of at least $(n-1)c_1+\sum_{i>1} c_i$, where c_1 is the minimum cost, and then the above network has the next m cheapest possible links by design.

If an efficient network has more than three players in some component and that component has the above form, then it clearly has a star involving at least four players in it. So, consider some other component form that achieves the same total utility. Let us show that it must also have a subnetwork that is a star and involves at least four players.

Given the minimum cost of the above-described network, to be efficient a component must have the same total benefits as the above-described network and the same total cost. Thus, an efficient component must have all nodes at distance from each other of at most 2.

If there is a node with just one link, then it must be connected to some lowest cost node, or else it would lead to a higher total cost than the network described above. Given that all nodes are at a distance of at most 2 from each other, other nodes must all be linked to the same minimum cost node, and so there is a star as a subset of the component involving all of the component's players.

So, consider a case such that all nodes have at least two links.

If the case where there are either one or two nodes at the minimum cost level, then higher cost node that has at least two links must have links to each of the lowest cost node(s), since links to these nodes are cheaper than links to other node(s). So, one of these nodes are the centers of stars involving all nodes.

So, consider the remaining case where there are three or more nodes at the minimum cost level. They must form a clique as any link between them is cheaper than any second link from a more expensive node to some other node. In that case, any one of them having a link to some other node (and at least one exists since the component has at least four nodes) is the center of a star involving at least four nodes.

[To see why the exercise does not ask for a star involving more than four links, consider a case with two cost levels, and such that $2\delta - 2\delta^2 = c_{max} + c_{min}$. Consider three nodes at each cost level, and order them by cost so that nodes 1 to 3 are low cost and 4 to 6 are high cost. Straightforward calculations show that the following is one of several efficient network structures: $\{12, 13, 23, 14, 15, 24, 26, 35, 36\}$.]

Exercise 3.

Let there be three islands of three individuals each. Let $\delta = 1 - \varepsilon$ and $c < \delta - \delta^2$, while $C = 4$. For small ε, the efficient network is to have each island completely connected internally, and to have a "star" shape among islands: that is, some unique agent i on one of the islands, who has a single link to each of the other two islands, and no other inter-island links.

The unique pairwise stable network has no links between islands. To see this, note that each island will be completely connected since $c < \delta - \delta^2$. Having a link to a another island (say the second island) only pays an agent (say on the first island) more than the cost of $C = 4$ if the second island has some of its agents having direct links to the third island, and provided the agent does not have another path to that third island that bypasses the second island. Thus, there cannot be links between all three of the islands, and at most two "inter-island" links in total. However then a link from one of the islands results in at most a value of 3 and costs 4.

Exercise 4.

Let us show that there is no example when $n = 3$ of an anonymous setting where there exists a pairwise stable network but no pairwise Nash stable network. Note that if either the empty network or a one link network is pairwise stable, then it is also pairwise Nash stable, since there is at most one link that can be deleted at a time. So, if there is such an example, the pairwise stable networks must all be either two or three link networks. Suppose a two link network is pairwise stable. It follows that $u_1(\{12, 13\}) \geq u_1(\{12\})$ (using anonymity to work with node 1). Since the empty network is not pairwise stable, it follows that $u_1(\{12\}) \geq u_1(\emptyset)$. Thus, $u_1(\{12, 13\}) \geq u_1(\emptyset)$, and so then the two-link network is pairwise Nash stable. Thus, if there is an example, then it must be that the only pairwise stable network is the complete network. Since it is pairwise stable, the only possible deviation that would make it fail to be pairwise Nash stable would be that $u_3(\{12, 13, 23\}) < u_3(\{12\})$ (using anonymity to work with node 3). However, since the one link network are not pairwise stable, it must be that $u_3(\{12, 13\}) \geq u_3(\{12\})$ (as otherwise a one link network would be pairwise stable given the empty network is not which means that one link is better for both involved than the empty network). Then, the two link network is not pairwise stable (and the deviation from a one link network is to a two-link network) it must be that the reason that the two link network is not pairwise stable is that $u_3(\{12, 13, 23\}) > u_3(\{12, 13\})$. But this, coupled with the previous inequality contradicts that fact that $u_3(\{12, 13, 23\}) < u_3(\{12\})$.