Social and Economic Networks
Advanced Problems: Week 2

Stanford University Prof. Matthew Jackson

1.

Consider a society of two groups, where the set N_1 comprises the members of group 1 and the set N_2 comprises the members of group 2, with cardinalities n_1 and n_2, respectively. Suppose that $n_1 > n_2$. For an individual i, let d_i be i's degree (total number of friends) and let s_i denote the number of friends that i has that are within own group. Let h_k denote a simple homophily index for group k, defined by $h_k = \frac{\sum_{i \in N_k} s_i}{\sum_{i \in N_k} d_i}$. Show that if h_1 and h_2 both lie above 0 and below 1, and the average degree in group 1 is at least as high as the average degree in group 2, then $h_1 > h_2$. What are h_1 and h_2 in the case where friendships are formed in percentages that correspond to the relevant populations?

2. Consider a network (g, N) such that each node has at least two neighbors ($n_i(g) \geq 2$ for each $i \in N$). Compare the average clustering measure of a network to the overall clustering measure in the following two cases:

a. $Cl_i(g) \geq Cl_j(g)$ whenever $d_i(g) \geq d_j(g)$, and

b. $Cl_i(g) \leq Cl_j(g)$ whenever $d_i(g) \geq d_j(g)$.

Hint: Write average clustering as $\sum_i Cl_i(g)\left(\frac{1}{n}\right)$ and argue that overall clustering can be written as $\sum_i Cl_i(g)\left(\frac{d_i(g)(d_i(g)-1)/2}{\sum_j d_j(g)(d_j(g)-1)/2}\right)$. Then compare these different weighted sums.

3.

Define the *Katz prestige* of a node i, denoted $P_i^K(g)$, to be a sum of the prestige of i's neighbors divided by their respective degrees. That is, the Katz prestige of a node i is

$$P_i^K(g) = \sum_{j \neq i} g_{ij} \frac{P_j^k(g)}{d_j(g)}. \tag{1}$$

Let $\widehat{g}_{ij} = g_{ij}/d_j(g)$, be the normalized adjacency matrix g so that the sum across any (non-zero) *column* is normalized to 1.[1] The relationship in (1) can then be rewritten as

$$P^K(g) = \widehat{g} P^K(g), \tag{2}$$

[1]Let 0/0=0, so that if $d_j(g) = 0$, then set $\widehat{g}_{ij} = 0$.

or

$$(\mathbf{I} - \widehat{g})P^K(g) = 0, \tag{3}$$

where P^K is written as a $n \times 1$ vector, and $\mathbf{I}$ is the identity matrix.

Katz also introduced another way of keeping track of the power or prestige of a node. The idea is based on presuming that the power or prestige of a node is simply a weighted sum of the walks that it has emanating from it. A walk of length 1 is worth a, a walk of length 2 is worth a^2, and so forth, for some parameter $0 < a < 1$.

Note that $g^k\mathbf{1}$ is the vector where each entry is the total number of walks of length k that emanate from each node. Thus, the vector of the power of nodes, or prestige of nodes, can be written as

$$P^{K2}(g, a) = ag\mathbf{1} + a^2g^2\mathbf{1} + a^3g^3\mathbf{1}\cdots \tag{4}$$

We can rewrite this as

$$P^{K2}(g, a) = \left(1 + ag + a^2g^2\cdots\right)ag\mathbf{1}. \tag{5}$$

For small enough $a > 0$, this is finite and then can be expressed as[2]

$$P^{K2}(g, a) = (\mathbf{I} - ag)^{-1}ag\mathbf{1}. \tag{6}$$

Consider a two link network among three nodes. That is let the network consist of links 12 and 23.

Calculate the Katz-prestige (based on (3)) of each node, and compare this to the degree centrality and betweenness centrality for this network.

Calculate the second measure due to Katz (based on (6)) for each node, when a=1/2, which is the Bonacich centrality of each node when $b = 1/2$ and $a = 1/2$.

How does this compare to Bonacich centrality when $b = 1/4$ and $a = 1/2$?

Which nodes are relatively favored when b increases and why?

What happens as we continue to increase b to $b = 3/4$?

[2]From (5), if $P^{K2}(g, a)$ is finite then it follows that $P^{K2}(g, a) - agP^{K2}(g, a) = ag\mathbf{1}$ or $(\mathbf{I} - ag)P^{K2}(g, a) = ag\mathbf{1}$. A sufficient condition for this to be finite is that a be smaller than 1 over the norm of the largest eigenvalue of g; and for this it is sufficient that a be smaller than 1 over the maximum degree of any agent.