Social and Economic Networks
Solutions Advanced Problems: Week 1

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Exercise 1.

(a) $WB_k > 0$ if and only if k has more than one link in a network and some of k's neighbors are not linked to one another:

Let us first show that if $WB_k > 0$ then k has more than one link in a network and some of k's neighbors are not linked to one another. If $WB_k > 0$ for some k, then it follows from the definition of WB_k that there exists $ij : i \neq j, k \notin \{i, j\}$ such that $W_k(ij) > 0$. From the definition of $W_k(ij)$ it follows that $P_k(ij) > 0$. Thus, at least one geodesic path between i and j passes through k. Consider such a path, denoted $i_0 i_1, ..., i_{m-1} i_m$ where the path consists of $m \geq 2$ links, with $i_0 = i$ and $i_m = j$, and $i_h = k$ where $0 < h < m$. It follows that i_{h-1} and i_{h+1} are neighbors of k. These nodes cannot be connected or else the path as above, but which eliminates links $i_{h-1}i_h$ and $i_h i_{h+1}$ and replaces them by $i_h i_{h+1}$ would lead to a shorter path connecting i to j which would contradict the fact that original path is a shortest path.

Next, let us show that if k has more than one link in a network and some of k's neighbors are not linked to one another then $WB_k > 0$. Let k be linked to i and j and these nodes not be linked to each other. Then $\ell_{ij} = 2$ and $P_k(ij) = 1$. Thus, $W_k(ij) > 0$ and then since $P(ij) > 0$ it follows that $WB_k > 0$.

(b) Let k be the center of a star that includes all nodes when $n \geq 3$. Then for each pair $ij : i \neq j, k \notin \{i, j\}$, there is a unique path from i to j and it is of length 2 and passes through k. Then $\ell_{ij} = 2$ and $P_k(ij) = 1$ and so $W_k(ij) = 1$ and then since $P(ij) = 1$ it follows that $WB_k = \sum_{ij:i\neq j,k\notin\{i,j\}} \frac{1}{(n-1)(n-2)/2}$. Since there are $(n-1)(n-2)/2$ pairs of nodes ij such that $i \neq j, k \notin \{i, j\}$, it follows that $WB_k = 1$.

(c) It is enough to show that if k is not the center of a star network, then there is some pair of nodes ij such that $i \neq j, k \notin \{i, j\}$ and for which $W_k(ij) < P(ij)$. This happens if either $P_k(ij) < P(ij)$ or if $\ell_{ij} \geq 2$. In the first case there is a shortest path between i and j that does not contain k which implies that k is not the center of a star, and in the second case there must be a node that lies on the shortest path between k and either i or j in which case k is not the center of a star.

Both nodes 4 and 5 have the same betweenness measure as measured in the standard manner: They each have three nodes on one side of them and four on the other, so they lie on all of the paths between 12 pairs of nodes, so they each have a Freeman betweenness measure of 12/21. If we adjust these to account for the lengths of these various paths, direct calculation of the measures leads to $WB_4 = 269/(60 \cdot 21) = .2135$ and $WB_5 = 279/(60 \cdot 21) = .2214$. While 4 and 5 are equivalent according to the betweenness measure of Freeman, the weighted measure favors node 5 since both nodes lie on the same number of shortest paths, but 5 lies on some shorter shortest paths than 4 does.

Exercise 2

Given that $1 > p > 0$, the probability of a particular network g on a given set of k nodes is $q = p^m(1-p)^{[k(k-1)/2]-m} > 0$, where m is the number of links in g. (Note that this considers a labeled version of the network, but you could also work with any permutation of it, which will increase the probability; so this is a lower bound.) Partitioning n into $K(n)$, the largest integer smaller than n/k, different sets of k nodes with possibly some remainder, the probability that none of these sets exhibit a copy of the network g is at most $(1-q)^{K(n)}$. As n grows, this converges to 0, and so the probability that at least one of these sets of k nodes contains a copy of this network goes to 1.