Social and Economic Networks
Solutions Advanced Problems: Week 6

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Exercise 1.

If the third agent sees a B, then the agent knows there are at least two B signals and at most one A signal, and given the symmetry of the setting it follows that the conditional probability that the state is good is more than 1/2 and so choosing B is an optimal decision.

So, consider the case where the third agent sees an A. The only two possibilities in terms of signals are B, B, A and B, A, A. Let us assess the relative probability of these two events. Let s_i be an agent's signal and a_i be an agent's action. We know that $s_1 = a_1 = B = a_2$.

By Bayes' rule $\Pr(s_2 = B|s_1 = B, a_2 = B, s_3 = A) = \frac{\Pr(a_2=B,s_2=B|s_1=B,s_3=A)}{\Pr(a_2=B|s_1=B,s_3=A)}$. Noting that agent 2 takes action a_2 with certainty if $s_2 = B$ the numerator simplifies to $\Pr(s_2 = B|s_1 = B, s_3 = A)$, and expanding the denominator this is equal to $\frac{\Pr(s_2=B|s_1=B,s_3=A)}{\Pr(a_2=B,s_2=A|s_1=B,s_3=A)+\Pr(s_2=B|s_1=B,s_3=A)}$. Then noting that by the symmetry of the setting $\Pr(s_2 = B|s_1 = B, s_3 = A) = 1/2$, and noting that agent 2 takes action B with a probability of 1/2 conditional on seeing $s_1 = B$ and $S_2 = A$, so the expression becomes $\frac{1/2}{\frac{1}{2}\Pr(s_2=A|s_1=B,s_3=A)+1/2}$. Again by symmetry, this simplifies to $\frac{1/2}{\frac{1}{4}+1/2} = \frac{2}{3}$. Thus, conditional on $s_1 = B, a_2 = B, s_3 = A$, there is 2/3 probability that the three signals are B, B, A and a 1/3 probability that they are B, A, A. The conditional probability of the state being good is thus greater than 1/2. Therefore, conditional on seeing A, the third agent will still choose the action B. Thus, the third agent chooses action B regardless of s_3.

It follows that any subsequent agent does not learn anything about the third agent's signal, so any subsequent agent's decision problem is the same as the third agent's, and so all subsequent agents choose action B.

Exercise 2.

Given the strong connection and aperiodicity there is a unique solution to $sT = s$ up to a rescaling. Therefore, the influence vector s is the unique (up to a rescaling) vector that satisfies

$$s_i = \sum_j T_{ji} s_j.$$

If $\sum_j T_{ji} = 1$ for every i, then

$$1/n = \sum_j T_{ji}/n,$$

and so $1/n$ is a solution, and thus the unique influence vector has all agents having equal influence whenever $\sum_j T_{ji} = 1$ for every i. Given that $\sum_j T_{ij} = 1$ since T is row-stochastic, if $T_{ij} = T_{ji}$, then $\sum_j T_{ji} = 1$, and so this must also have equal influences.

Exercise 3.

The first eigenvector is $(1/3, 2/3)$ and the second eigenvector is $(1, -1)$. Since $(1, -1)T$ is $(1/4, -1/4)$, it follows that the second eigenvalue is $1/4$.

Using the hint it follows that

$$T^t = \begin{pmatrix} \frac{1}{3}\left(1 + \frac{2}{4^t}\right) & \frac{2}{3}\left(1 - \frac{1}{4^t}\right) \\ \frac{1}{3}\left(1 - \frac{1}{4^t}\right) & \frac{1}{3}\left(2 + \frac{1}{4^t}\right) \end{pmatrix}. \tag{1}$$

Thus, $b_1(t) = \frac{1}{3}\left(1 + \frac{2}{4^t}\right)$.