Social and Economic Networks
Advanced Problems: Week 1

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1.

Consider the following variation on the betweenness measure briefly discussed in video 1.2 in the context of the Florentine Marriage data (slides 15-16). Any given shortest path between two families is weighted by inverse of the number of intermediate nodes on that path. For instance, the shortest path between the Ridolfi and Albizzi involves two links and the Medici are the only family that lies between them on that path. In contrast, between the Ridolfi and the Ginori the shortest path is three links and there are two families, the Medici and Albizzi, that lie between the Ridolfi and Ginori on that path.

More specifically, let ℓ_{ij} be the length of the shortest path between nodes i and let $W_k(ij) = P_k(ij)/(\ell_{ij} - 1)$, (setting $\ell_{ij} = \infty$ and $W_k(ij) = 0$ if i and j are not connected). Where $P_k(ij)$ is the number of the shortest paths between i and j that go through node k, and $P(ij)$ is the number of the shortest paths between i and j. Then the weighted betweenness measure for a given node k be defined by

$$WB_k = \sum_{ij:i<j,k\notin\{i,j\}} \frac{W_k(ij)/P(ij)}{(n-1)(n-2)/2}. \tag{1}$$

where we take the convention that $\frac{W_k(ij)}{P(ij)} = 0/0 = 0$ if there are no paths connecting i and j.

Show that

- $WB_k > 0$ if and only if k has more than one link in a network and some of k's neighbors are not linked to each other,
- $WB_k = 1$ for the center node in a star network that includes all nodes (with $n \geq 3$), and
- $WB_k < 1$ unless k is the center node in a star network that contains all nodes.

Calculate this measure for the the network pictured in Figure 1 for nodes 4 and 5.

Contrast this measure with the betweenness measure discussed in video 1.2.

2. Fix the probability of any given link forming in a Erdos-Renyi $G(n,p)$ random network to be p where $1 > p > 0$. Fix some arbitrary network g on k nodes.

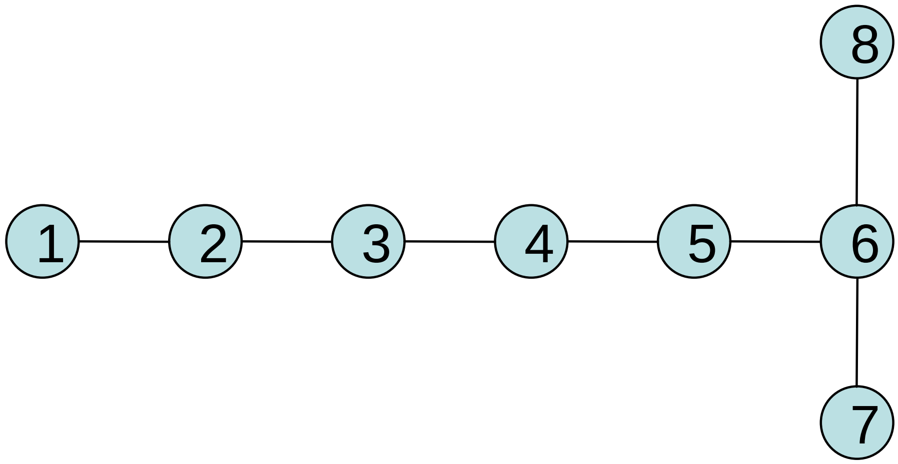

Figure 1: Differences in Betweenness measures.

Now, consider a sequence of random networks indexed by the number of nodes n, as $n \to \infty$. Show that the probability that a copy of the k node network g is a subnetwork of the random network on the n nodes goes to 1 as n goes to infinity.

[Hint: partition the n nodes into as many separate groups of k nodes as possible (with some leftover nodes) and consider the subnetworks that end up forming on each of these groups. Using the independence of link formation, show that the probability that the none of these match the desired network goes to 0 as n grows.]