**Social and Economic Networks**
**Advanced Problems: Week 4**

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1. Consider a variation on the symmetric connections model, such that instead of a value of $\delta^k$ for being path connected to another node at distance $k$, a node gets a benefit $b(k)$, with $b(k)$ decreasing in $k$ and being positive as long as the node is within a distance $K > 1$, and otherwise the value is 0. Suppose that the cost of a link is $c > b(1) > b(2) > b(3) > \cdots > b(k) > b(k+1) > \cdots b(K) > 0$. Show that as $n$ grows, the diameter of any nontrivial pairwise stable network is bounded above.

Define diameter relative to the nonempty part of the network (so isolated nodes are ignored, but count the diameter as infinite if there are two or more nonempty components to the network).

2. Consider an asymmetric version of the connections model, where players all have a common $\delta$ parameter where $0 < \delta < 1$ and the only asymmetry is that individuals might have different costs per link. In particular, suppose that each individual's cost for a link is the same for all links, so that individual $i$ has a cost $c_i$ for each link that $i$ is involved with; but where it is possible that these costs differ across players.

Provide an example where the unique efficient network is not a star network, nor a complete network, nor the empty network.

Show that every nonempty efficient network has a subnetwork that is a star network (possibly only involving a subset of the players, but at least four if there are at least four players in the network) and such that the center player in that star has a minimal cost (that is $c_i \leq c_j$ for all $j$).

3. Consider the islands model where the cost of linking within an island is $c < \delta - \delta^2$ and show an example for some value of $\delta$ and $C$ (the cost of linking across islands) with three islands and three players per island, such that all pairwise stable networks are nonempty and distinct from all efficient networks.

4. Show that if payoffs are anonymous, [1] $n = 3$, and isolated players get a payoff of 0, then whenever there exists a pairwise stable network, there also exists exist a pairwise Nash stable network. (You can play with some settings with $n > 3$ to see why that restriction is stated in this problem.)

[1] A profile of utility functions is anonymous if for every $\pi$ that is a permutation on $N$ (a one-to-one function mapping the set of agents $N$ to $N$), it follows that $u_{\pi(i)}(g^{\pi}) = u_i(g)$, where $g^{\pi} = \{\{\pi(i), \pi(j)\} | ij \in g\}$ is the network obtained from $g$ by permuting the positions of agents according to $\pi$.