Social and Economic Networks
Solutions Advanced Problems: Week 2

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Exercise 1

Let $\overline{d}^1 \geq \overline{d}^2$ be the average degrees in the respective groups, and $D_k = n_k\overline{d}^k$ be the total degree of group k. Let $A_k = \sum_{i\in N_k} d_i - s_i$ be the total number of links from members of group k to the other group. It follows that $A_1 = A_2 = A$ since each link between groups involves a member of each group. Then note that

$$h_k = 1 - \frac{A}{D_k}.$$

Since $D_1 > D_2$ (given that $n_1 > n_2$ and $\overline{d}^1 \geq \overline{d}^2$), and $D_1 > A > 0$ (given that h_1 is between 0 and 1), it follows that $h_1 > h_2$.

If π_k is the share of population of group k, and friendships are formed in percentages that correspond to the total population, then

$$h_k = \frac{\sum_{i\in N_k} s_i}{\sum_{i\in N_k} d_i} = \frac{\sum_{i\in N_k} \pi_k d_i}{\sum_{i\in N_k} d_i} = \pi_k.$$

Exercise 2

Average clustering is $\sum_i Cl_i(g)\left(\frac{1}{n}\right)$.

Overall clustering is

$$Cl(g) = \frac{\sum_{i;j\neq i;k\neq j;k\neq i} g_{ij}g_{ik}g_{jk}}{\sum_{i;j\neq i;k\neq j;k\neq i} g_{ij}g_{ik}}.$$

The denominator is $\sum_i d_i(g)(d_i(g)-1)$ and the numerator is $\sum_i Cl_i(g)d_i(g)(d_i(g)-1)$. It then follows that overall clustering is

$$\sum_i Cl_i(g)\left(\frac{d_i(g)(d_i(g)-1)/2}{\sum_j d_j(g)(d_j(g)-1)/2}\right).$$

Thus, both are weighted averages of the individual clustering measures. Average clustering has an equal weighting, while overall clustering has weights that place relatively more weight on higher degree nodes. Thus, in case (a) overall clustering puts higher weight on nodes with higher clustering, and so overall clustering will be at least as high as average clustering. In case (b), this is reversed.

Exercise 3

The degrees of the nodes are $(1,2,1)$ and so the degree centralities are $(\frac{1}{2},1,\frac{1}{2})$.

The betweenness centralities are $(0,1,0)$ as only node 2 lies on a path between the other two nodes.

In an undirected network the Katz Prestige is simply any rescaling of the degrees of the nodes, and so will be the same as the degree centralities $(\frac{1}{2},1,\frac{1}{2})$. You can check this by noting that

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{1}$$

and then

$$I - \widehat{g} = \begin{pmatrix} 1 & -1/2 & 0 \\ -1 & 1 & -1 \\ 0 & -1/2 & 1 \end{pmatrix}. \tag{2}$$

The product of this matrix and the degrees $(1,2,1)$ is the 0 vector.

To calculate the Bonacich centrality measures, note that

$$I - bg = \begin{pmatrix} 1 & -b & 0 \\ -b & 1 & -b \\ 0 & -b & 1 \end{pmatrix}. \tag{3}$$

From this we see that

$$(I - \frac{1}{2}g)^{-1} = \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix}. \tag{4}$$

$$(I - \frac{1}{4}g)^{-1} = \begin{pmatrix} 15/14 & 4/14 & 1/14 \\ 4/14 & 16/14 & 4/14 \\ 1/14 & 4/14 & 15/14 \end{pmatrix}. \tag{5}$$

$$(I - \frac{3}{4}g)^{-1} = \begin{pmatrix} -7/2 & -6 & -9/2 \\ -6 & -8 & -6 \\ -9/2 & -6 & -7/2 \end{pmatrix}. \tag{6}$$

From this, we find that the $Ce^B(g,1/2,1/2) = (2,3,2)$; $Ce^B(g,1/2,1/4) = (6/7,10/7,6/7)$; $Ce^B(g,1/2,3/4) = (-10,-14,-10)$. As we increase b we increase the relative weight put on longer walks in the Bonacich centrality measure. In this network, that benefits the "peripheral" nodes 1 and 3, as their relative centrality is higher at $b = 1/2$ than at $b = 1/4$. As we continue to increase b to be too high, the sum no longer converges and the expression of $I - bg$ is not the proper expression for the sum and so the centrality measure is no longer well defined and the resulting term no longer has the correct interpretation.