Social and Economic Networks
Solutions Advanced Problems: Week 3

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Exercise 1

The mean-field expression for the change in the degree of a node $i < t$ over time can be written as

$$\frac{dd_i(t)}{dt} = \frac{\alpha m}{t} + \frac{(1-\alpha)md_i(t)}{2mt(1-q)} - \frac{qmd_i(t)}{mt(1-q)} = \frac{\alpha m}{t} + \frac{(1-\alpha-2q)d_i(t)}{2(1-q)t}.$$

In a case where $1-\alpha > 2q$ this has solution

$$d_i(t) = \phi_t(i) = \left(d_0 + \frac{2\alpha m(1-q)}{1-\alpha-2q}\right)\left(\frac{t}{i}\right)^{(1-\alpha-2q)/(2(1-q))} - \frac{2\alpha m(1-q)}{1-\alpha-2q}, \qquad (1)$$

where d_0 is the initial number of links that a node has when it is born. From (1) we deduce that

$$\phi_t^{-1}(d) = t\left(\frac{d_0 + \frac{2\alpha m(1-q)}{1-\alpha-2q}}{d + \frac{2\alpha m(1-q)}{1-\alpha-2q}}\right)^{\frac{2(1-q)}{1-\alpha-2q}}.$$

Setting $d_0 = m$, and following the technique from the chapter we conclude that

$$F_t(d) = 1 - \left(\frac{m + \frac{2\alpha m(1-q)}{1-\alpha-2q}}{d + \frac{2\alpha m(1-q)}{1-\alpha-2q}}\right)^{2(1-q)/(1-\alpha-2q)}. \qquad (2)$$

In a case where $1-\alpha = 2q$, then

$$\frac{dd_i(t)}{dt} = \frac{\alpha m}{t}.$$

With an initial condition that $d_i(i) = m$ the solution is

$$d_i(t) = m + \alpha m \log\left(\frac{t}{i}\right).$$

Setting $F_t(d) = 1 - i(d)/t$ we conclude that

$$F_t(d) = 1 - e^{-(d-m)/(\alpha m)}.$$

Exercise 2.

Let m be the number of links that each newborn node adds each period. So, under the mean field approximation, each newborn node ends up with $m(1+f)$ links as it enters, and then $n(1-f)m$ are added via preferential attachment to older nodes. The total degree of all nodes at time t is $2tmn$, and so the probability that a node with degree $d_i(t)$ gets a new link in period t is

$$\frac{mn(1-f)d_i(t)}{2tmn} = \frac{(1-f)d_i(t)}{2t}.$$

Thus, the continuous time mean field approximation is described by

$$\frac{dd_i(t)}{dt} = \frac{(1-f)d_i(t)}{2t},$$

with initial condition $d_i(i) = m(1+f)$. This has solution

$$d_i(t) = m(1+f)\left(\frac{t}{i}\right)^{(1-f)/2}.$$

Thus, letting $i_t(d)$ be the node that has degree d at time t, it follows that

$$\frac{i_t(d)}{t} = \left(\frac{m(1+f)}{d}\right)^{2/(1-f)}.$$

Those having degree less than d are those with later birth dates, so the cdf is given by

$$F_t(d) = 1 - \left(\frac{m(1+f)}{d}\right)^{2/(1-f)} = 1 - (m(1+f))^{2/(1-f)}\,(d)^{-2/(1-f)}.$$

Exercise 3

Let the number of new nodes entering at time t be gn_t, where n_t is the number of nodes at time t and $g>0$ is a growth rate. The mean-field equation for degree evolution is then

$$\frac{dd_i(t)}{dt} = \frac{gn_t\alpha m}{n_t} + \frac{gn_t(1-\alpha)md_i(t)}{2n_tm} = g\alpha m + \frac{g(1-\alpha)d_i(t)}{2}.$$

Let us show that if $n_t = (1+g)n_{t-1}$ and the degree of a node born at time i has initial degree d_0 and evolves according to

$$\frac{dd_i(t)}{dt} = ad_i(t) + b,$$

then the cdf is

$$F_t(d) = 1 - \left(\frac{d_0 + \frac{b}{a}}{d + \frac{b}{a}} \right)^{\log(1+g)/a}.$$

To see this, first note that the solution to

$$\frac{dd_i(t)}{dt} = ad_i(t) + b.$$

with initial condition $d_i(i) = d_0$ is

$$d_i(t) = \left(d_0 + \frac{b}{a} \right) e^{a(t-i)} - \frac{b}{a}.$$

This leads to a solution of

$$t - i_t(d) = \frac{1}{a} \log \left(\frac{d + \frac{b}{a}}{d_0 + \frac{b}{a}} \right).$$

Given the exponentially growing system with a deterministic d_i, we have

$$F(d) = 1 - \frac{n_{i_t(d)}}{n_t} = 1 - (1+g)^{-(t-i_t(d))}.$$

Substituting from the expression for $t - i_t(d)$ then leads to the claimed expression.

So, since

$$\frac{dd_i(t)}{dt} = \frac{g(1-\alpha)d_i(t)}{2} + g\alpha m,$$

we set $a = g(1-\alpha)/2$ and $b = g\alpha m$, and $d_0 = m$, and then the cdf is

$$F_t(d) = 1 - \left(\frac{m + \frac{2m\alpha}{1-\alpha}}{d + \frac{2m\alpha}{1-\alpha}} \right)^{2\log(1+g)/(g(1-\alpha))}.$$