Social and Economic Networks
Solutions Advanced Problems: Week 5

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Exercise 1.

The requirement for a non-zero steady-state is that for each d

$$0 = \frac{d\rho(d)}{dt} = (1 - \rho(d))\nu\theta - \rho(d)\delta. \tag{1}$$

Letting $\lambda = \nu/\delta$, it follows that

$$\rho(d) = \rho = \frac{\lambda\theta}{\lambda\theta + 1}. \tag{2}$$

Equation (2) leads to the following requirement for a non-zero steady state, noting that θ is an average of $\rho(d)$'s and so is simply ρ:

$$\theta = \rho = \frac{\lambda\theta}{\lambda\theta + 1}. \tag{3}$$

This simplifies to

$$\theta = \frac{\lambda - 1}{\lambda}, \tag{4}$$

which has a positive solution only when $\lambda > 1$, and otherwise the only solution to (3) is $\theta = 0$.

Exercise 2

A mean-field approximation to steady state requires that $0 = (1 - \rho(d))v\theta d - \rho(d)\delta$. Recalling that $\lambda = v/\delta$, it follows that

$$\rho(d) = \frac{\lambda\theta d + \epsilon/\delta}{\lambda\theta d + 1 + \epsilon/\delta},$$

and so following the same techniques as in lecture it follows that the steady-state satisfies

$$\theta = \sum_d \frac{P(d)d(\lambda\theta d + \epsilon/\delta)}{E[d](\lambda\theta d + 1 + \epsilon/\delta)}.$$

If all individuals have the same degree d, the equation above becomes

$$\theta = \frac{\lambda\theta d + \epsilon/\delta}{\lambda\theta d + 1 + \epsilon/\delta}.$$

Solving this leads to

$$\theta = \frac{\lambda d - 1 - \epsilon/\delta + \sqrt{(1 + \epsilon/\delta - \lambda d)^2 + 4\lambda d\epsilon/\delta}}{2\lambda d}.$$

where only the positive solution makes sense, since $\theta \geq 0$.

3.

Suppose that $B \bigcup A$ is the eventual set of nodes adopting the technology. For any $i \in C$, it must be that more than $1 - q$ of his/her neighbors do not adopt the technology in all periods, and so these neighbors are all in C. So for any $i \in C$, a fraction of more than $1 - q$ of his/her neighbors are in C, and thus C is more than $1 - q$-cohesive. If there is a nonempty subset D of B such that $D \cup C$ has a cohesiveness of more than $1 - q$, for any node in D every time there is a fraction of more than $1 - q$ of his/her neighbors play the action 0, and then all nodes in $D \cup C$ should stay at 0, which contradicts that fact that $D \subset B$. Thus, for every nonempty subset D of B, $D \cup C$ has a cohesiveness of no more than $1 - q$.

To see the converse, suppose that C is more than $1 - q$-cohesive and for every nonempty subset D of B, $D \cup C$ has a cohesiveness of no more than $1 - q$. It follows from an argument similar to that above that all nodes in C do not adopt. Suppose there is a nonempty subset D of B such that all nodes in $D \cup C$ never adopt, we know that $D \cup C$ has a cohesiveness of no more than $1 - q$ so there exists at least one $i \in D$, with a fraction of no more than $1 - q$ of his/her neighbors in $D \cup C$. This implies for this $i \in D$, there is a fraction of at least q of his/her neighbors who eventually adopt, then i also padopts eventually. (And note that it only takes finite steps to get to an end of the process.) This contradiction shows that D is empty and all nodes in $B \cup A$ will adopt eventually.