A Half-Blood Half-Pipe, A Perfect Performance

Enhao Gong
Xiaoyun Wang
Rongsha Li

Tsinghua University
Beijing, China

Advisor: Jimin Zhang

Abstract

Our basic model has two parts: to find a half-pipe shape that can maximize vertical air, and to adapt the shape to maximize the possible total angle of rotation. In an extended model, we analyze the snowboarder's effect on vertical air and on rotation. Finally, we discuss the feasibility and the tradeoffs of building a practical course.

The major assumption is that resistance includes the friction of snow plus air drag, with the former proportional to the normal force. We find air drag negligible.

We first obtain and solve a differential equation for energy lost to friction and drag based on force analysis and energy conservation. We calculate vertical air by analyzing projectile motion. We then calculate the angular momentum before the flight and discuss factors influencing it. In an extended model, we take the snowboarder's influence into account.

We compare analytical and numerical results with reality, using default parameters; we validate that our method is correct and robust. We analyze the effects on vertical air of width, height, and gradient angle of the half-pipe. We find that a wider, steeper course with proper depth and the path of a skilled snowboarder are best for vertical air. Using a genetic algorithm, we globally optimize the course shape to provide either the greatest vertical air or maximal potential rotation; there is a tradeoff. Implementing a hybrid scoring system as the objective function, we optimize the course shape to a "half-blood" shape that would provide the eclectically best snowboard performance.

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Background

A half-pipe is the venue for extreme sports such as snowboarding and skateboarding. It usually consists of two concave ramps (including a transition and a vert), topped by copings and decks, facing each other across a transition as shown in **Figure 1**. Half-pipe snowboarding has been a part of the Winter Olympics since 2002; the riders take two runs, performing tricks such as straight airs, grabs, spins, flips, and inverted rotations.

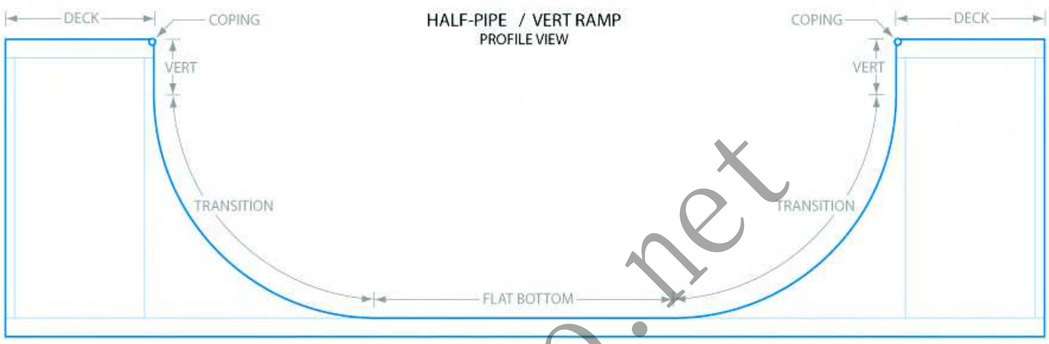


Figure 1. End-on schematic view of a half-pipe. (Source: Wikimedia Commons; created by Dennis Dowling.)

We find no analysis of the "best" shape for a half-pipe. However, usually it is 100–150 m long, 17–19.5 m wide, and 5.4–6.5 m from floor to crown, with slope angle 16–18.5° [Postins n.d.]. In addition, the Fédération Internationale de Ski (FIS) recommends that the width, height, transition, and the bottom flat be 15 m, 3.5 m, 5 m, and 5 m, respectively [2003, 36].

Half-pipe snowboarding is currently judged using subjective measures. Still, there is strong community perception that air time and degree of rotation play a major role in competition success [Harding et al. 2008a; Harding et al. 2008b]. According to Harding et al. [2008b] and Harding and James [2010], who have attempted to introduce objective analysis into the scoring, air time and total rotation are the two most critical evaluation criteria.

Terminology and Definitions

Cycle: The start of a cycle is when the snowboarder reaches the edge of the half-pipe after a flight, and the end of a cycle is the next start.

Flight: the part of the movement when the snowboarder is airborne.

Flight distance (S_f): displacement along the z direction during the flight.

Flight time (t_f): duration of the flight.

Cycle distance (S_c): displacement along z direction during a cycle.

Assumptions

- The cross section of the half-pipe is a convex curve that is smooth (has second-order derivative) everywhere except the endpoints.
- The snowboarder crouches during the performance until standing up to gain speed right at the edge of the half-pipe before the flight.
- We neglect the rotational kinetic energy of the snowboarder before considering the twist performance.
- The friction of the snow is proportional to the normal force of the snow exerted on the snowboarder but has nothing to do with velocity (that is, the angle between the direction of the snowboard and the snowboarder's velocity is constant).
- Air drag is proportional to the square of speed.
- The snowboarder's body is perpendicular to the tangential surface of the half-pipe during movement on the half-pipe.
- The force exerted on the board can be considered as acting at its center.
- We neglect the influence of natural factors such as uneven sunshine (which may result from an east or west orientation), altitude, etc.

Basic Model

Model Overview

A cycle can be divided into two parts: movement on the half-pipe, and the airborne performance.

For the first, we focus on the conversion and conservation of energy. The loss of mechanical energy E_{lost} due to the resistance of snow and air is the key. We derive a differential equation for it. We cannot neglect the snowboarder's increasing the mechanical energy by stretching the body (standing up) and doing work against the centrifugal force.

To derive an expression for vertical air, we apply Newton's Second Law. If we neglect air drag during the flight (we later show that it is indeed negligible), we can calculate vertical air, duration of the flight, flight distance, gravitational potential decrease, etc.

Next, we discuss the airborne rotation of the snowboarder. Since the shape of the half-pipe directly influences the initial angular momentum of the snowboarder, and the angular momentum cannot change during the flight, the relationship between the half-pipe shape and the initial angular momentum is the key to our discussion. After deriving an expression for the initial angular momentum, we can find the optimal shape of the half-pipe.

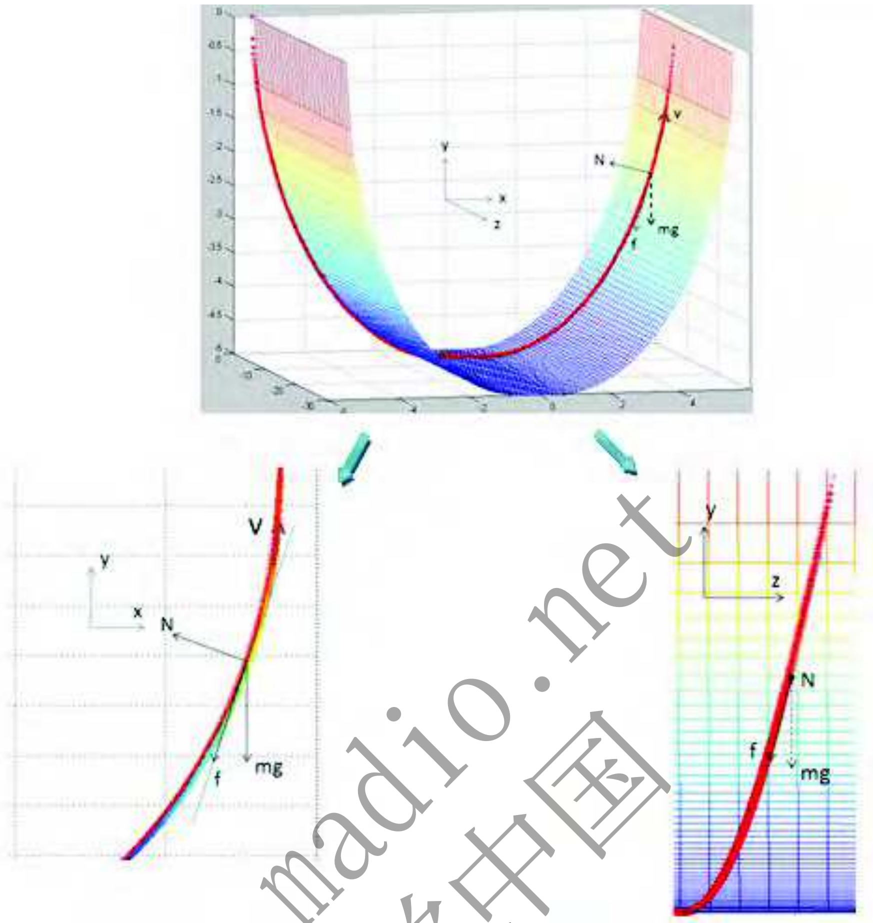


Figure 2. Force analysis.

The Model

Vertical Air

Step 1. Force Analysis: The top part of **Figure 2** shows the definition of the coordinate variables: x is the free variable, while y and z are functions of x. The relationship between y and x depends on the shape of the half-pipe, while the relationship between z and x depends on the path chosen by the snowboarder.

Three forces act on the snowboarder: gravity (mg), normal force (N), and resistance (f).

Resistance can be represented as

$$f = \alpha N + \beta\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \alpha N + \beta\left(1 + y'^2 + z'^2\right)\dot{x}^2.$$

For the normal force, only the part of centripetal acceleration that is parallel

Table 1.
Model parameters.

Parameter	Meaning
x, y, z	Coordinate variables
$\dot{x}$, $\dot{y}$, $\dot{z}$	Velocities
y', y''	$\partial y/\partial x$, $\partial^2 y/\partial x^2$
z'	$\partial z/\partial x$
s	Length of the path
E_0	Initial mechanical energy at the beginning of a cycle
E_{leave}	Kinetic energy right before the flight
E_{reach}	Kinetic energy at the end of the flight
E_{lost}	Mechanical energy lost due to friction of the snow and air drag
N	Normal force of the snow exerted on the snowboarder
W_{human}	Work done by the snowboarder at the edge of the half-pipe when (s)he stands up
W_G	Decrease in gravitational potential during the flight
f	Friction of the snow plus air drag
m	Mass of the snowboarder
α	Friction coefficient between the snow and snowboard
β	Drag coefficient of air
θ	Angle between z-axis and the horizontal plane
Δh	Rise of the mass point of the snowboarder when (s)he stands up from a crouching position
ρ	Radius of curvature at a point on the cross section of the half-pipe
x_t, y_t, z_t, y'_t, z'_t, $\dot{y}_t$, $\dot{z}_t$, y''_t	Values right before the flight
H_f	Vertical air

to the direction of N needs to be considered:

$$\rho = \frac{\left(1+y'^2\right)^{3/2}}{y''};$$

$$N = \frac{\dot{x}^2+\dot{y}^2}{\rho}\,m + \frac{mg\cos\theta}{\sqrt{1+y'^2}} = \left(y''\dot{x}^2 + g\cos\theta\right)\frac{m}{\sqrt{1+y'^2}}.$$

Path length unit can be represented as

$$ds = \sqrt{1+\dot{y}^2+\dot{z}^2}\,dx.$$

Step 2. Energy Conservation: According to the Energy Conservation Principle, we have

$$\tfrac{1}{2}m\left(1+y'^2+z'^2\right)\dot{x}^2 = E_0 - E_{\text{lost}} - mg(y\cos\theta - z\sin\theta).$$

Then we have

$$\dot{x}^2 = \frac{2}{m\left(1+y'^2+z'^2\right)}\left[E_0 - E_{\text{lost}} - mg(y\cos\theta - z\sin\theta)\right].$$

Step 3. E_{lost}:

$$
\begin{aligned}
E_{\text{lost}} &= \int_{-x_0}^{x} f \cdot ds \\
&= \int_{-x_0}^{x} \left[\alpha N \sqrt{1 + y'^2 + z'^2} + \beta \left(1 + y'^2 + z'^2\right)^{3/2} \cdot \dot{\tau}^2 \right] d\tau \\
&= \int_{-x_0}^{x} \left\{ \left(\alpha \frac{m y''}{\sqrt{(1+y'^2)(1+y'^2+z'^2)}} + \beta \right) \times \right. \\
&\qquad \frac{2}{m} \left[E_0 - E_{\text{lost}} - mg(y\cos\theta - z\sin\theta) \right] \\
&\qquad\qquad \left. + \alpha m g \cos\theta \frac{\sqrt{1+y'^2+z'^2}}{\sqrt{1+y'^2}} \right\} d\tau.
\end{aligned}
$$

The integral has a variable upper limit. Differentiating both sides and solving the resulting first-order linear ordinary differential equation, we get an expression for E_{lost}, which we would like to minimize. However, since the relationship between y and x is unknown, as is that between z and x, the expression is a functional. The expression and the calculation are too complicated, so we use a numerical method to solve the problem.

Step 4. W_{human}: When the snowboarder stands up, (s)he does work overcoming the centrifugal force. At high speed, the centrifugal force is huge, so W_{human} is considerable and cannot be neglected. The work done by the snowboarder is

$$
W_{\text{human}} = \frac{\dot{t}_t^2}{\rho} m \cdot \Delta h = \frac{y_t'^2 y_t''}{\left(1 + y_t'\right)^{3/2}} m \cdot \Delta h.
$$

Step 5. Vertical Air: At the edge of the half-pipe before the flight, we have $\dot{x} = 0$. From the Energy Conservation Principle, we get

$$
\tfrac{1}{2} m \left(\dot{y}_t^2 + \dot{z}_t^2\right) = E_0 - E_{\text{lost}}(x_t) = mg \cdot z_t \sin\theta + \frac{y_t'^2 y_t''}{\left(1 + y_t'\right)^{3/2}} m \cdot \Delta h.
$$

Since $\dfrac{\dot{y}_t}{\dot{z}_t} = \dfrac{y_t'}{z_t'}$, we have $\dot{z}_t = \dfrac{z_t'}{y_t'} \dot{y}_t$ and

$$
\dot{y}_t^2 = \frac{2}{m\left[1 + \left(\frac{z_t'}{y_t'}\right)^2\right]} \left(E_0 - E_{\text{lost}}(x_t) + mg \cdot z_t \sin\theta + \frac{y_t'^2 y_t''}{\left(1 + y_t'\right)^{3/2}} m \cdot \Delta h \right).
$$

If we neglect air drag during the flight, **Figure 3** shows that vertical air is

$$H_f = \frac{1}{\cos\theta}\frac{\dot{y}_t^2}{2g\cos\theta} = \frac{E_0 - E_{\text{lost}}(x_t) + mg\cdot z_t\sin\theta + \dfrac{y_t'^2 y_t''}{(1+y_t')^{3/2}}\, m\cdot\Delta h}{mg\cdot\cos^2\theta\left[1+\left(\frac{z_t'}{y_t'}\right)^2\right]}.$$

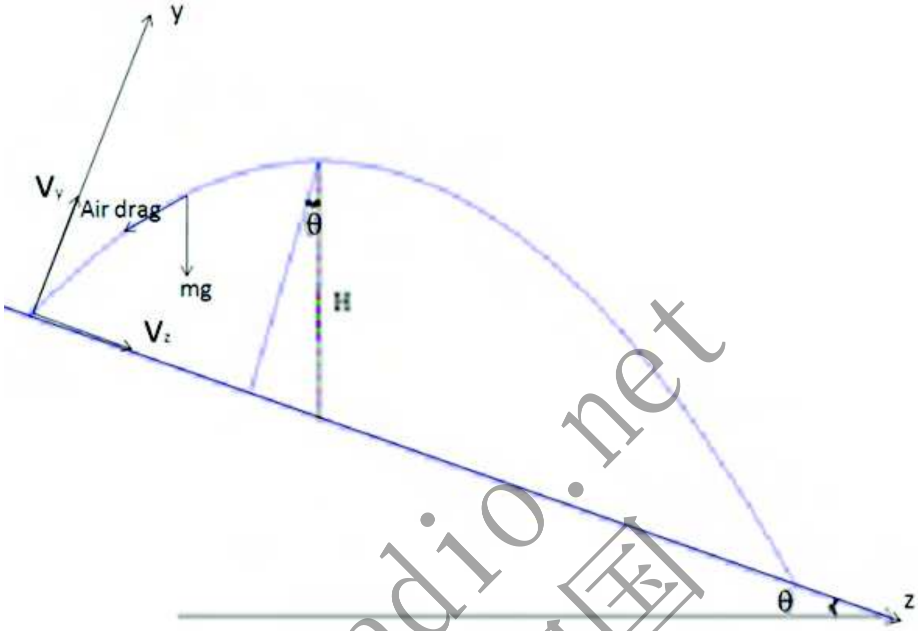


Figure 3. The path and force analysis of the flight.

Step 6. Flight Distance: To compare the initial kinetic energy of two adjacent cycles, we must calculate the decrease of the gravitational potential at the beginning and at the end of the flight. The duration of the flight is

$$t = \frac{2\dot{y}_t}{g\cos\theta}.$$

The flight distance is $S_f = \dot{z}_t t + \frac{1}{2}\, g\sin\theta\cdot t^2$.
The decrease of gravitational potential is $W_G = mg\cdot S_f\sin\theta$.

Finally, we give an equation to describe the energy conversion and conservation relationship at the beginning and the end of a cycle:

$$E_0 = E_{\text{lost}} + mg\cdot z_t\sin\theta + W_G + W_{\text{human}} + E_{\text{reach}}.$$

From this, we get

$$E_{\text{reach}} = E_0 - E_{\text{lost}} - mg\cdot z_t\sin\theta - W_G - W_{\text{human}}.$$

Rotation

[EDITOR'S NOTE: The authors regard the snowboarder as a stick and use considerations of conservation of angular momentum (in the absence of air drag) to explain various tricks. We omit the details.]

Numerical Computation

We determine values for some parameters.

The coefficient α of kinetic friction between snow and snowboard is generally 0.03–0.2 [Yan et al. 2009], while Chen et al. [1992] determined it to be 0.0312. Thus, we generally assume $\alpha = 0.03$

The air drag coefficient β is about 0.15 [Yan 2006].

Since the mass of Olympic Champion Shaun White is 63 kg, we assume that the mass of a snowboarder is typically 60 kg. Moreover, according to ZAUGG AG EGGIWIL [2008], the drop-in ramp height should be at least 5.5 m and the distance from ramp to pipe should be at least 9 m. Since the slope angle is about 18°, the potential energy is $mg(5.5 + 9\cos 18^\circ \approx 8267$ J.

Since $\alpha < 0.2$ (for which E_0 would be 6613 J), we conservatively assume the initial mechanical energy at the beginning of a cycle (E_0) to be 7000 J.

We need to define $y(x)$ and $z(x)$, the shape of the ramp cross-section and the path that the snowboarder chooses. As we assumed before, $y(x)$ is a smooth convex curve, which is symmetric in reality. There is a horizontal flat connecting two parts of the ramp and each part consists of a smooth transition part and possibly vertical part.

As **Figure 4** shows, with constant friction the energy loss is minimized if z is proportional to $\tau(x)$, the length of the projection curvature of the three-dimensional curve $\vec{p}(x) = \big(x, y(x), z(x)\big)$ on the xy-plane. Thus, we define the coordinate along the z-axis as $z = z\big(\tau(x)\big) \approx k\tau(x)$. Practically, we control $z(x)$ using the location of the point right before the flight.

Given the relationships between $y(x)$ and x and between $z(x)$ and x, we use numerical approximation to solve the differential equation for E_{lost}. **Figure 4** shows the output figure from numerical simulation.

To validate our results, we compared analytical results and numerical results for the dimensions

width = 15 m, flat = 5 m, depth = 3.5 m, z_t = 10 m,

with conditions

$\theta = 18^\circ$, m = 60 kg, g = 9.8 m/s^2, β = 0.15 kg/m,

and transition part standard elliptic.

Analytical and numerical results differed negligibly, yielding the results of **Table 2**. The resulting maximum vertical air of about 7 m matches Shaun White's best performance and is consistent with common performances (12–20 ft for men, 6–15 ft for women).

Sensitivity

We modified parameter values by 10%, and the results (**Table 3**) support that the numerical simulation is robust.

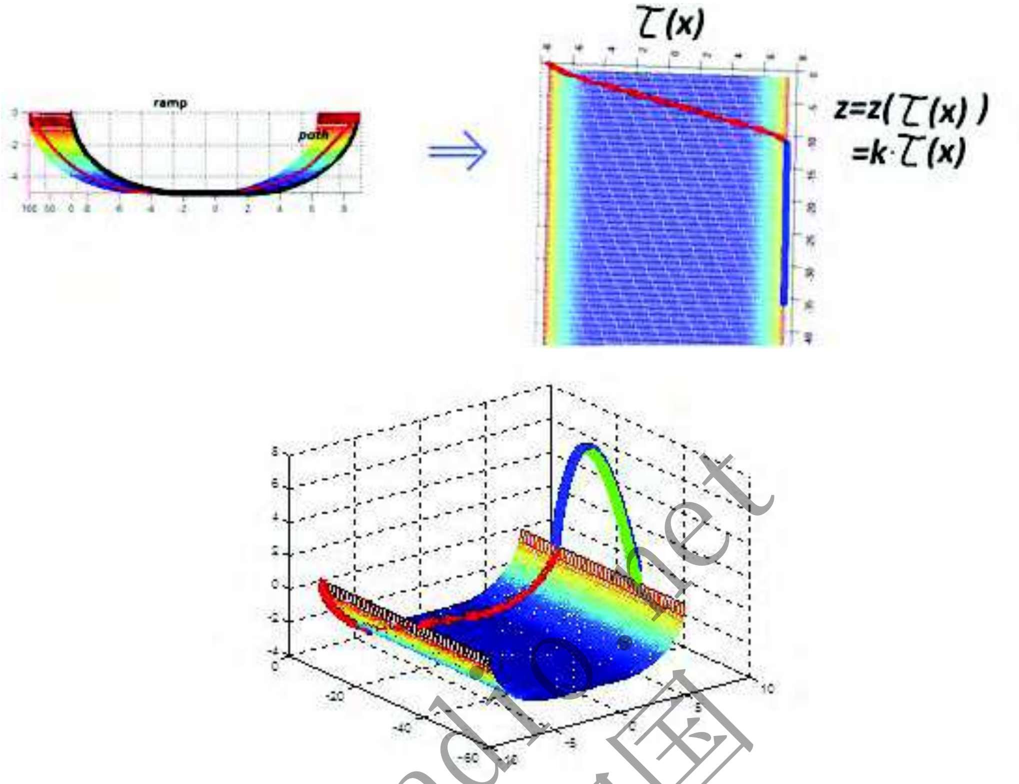


Figure 4. Path of the snowboarder.

Table 2.
Results.

Condition	$\dot{y}$ (m/s)	Vertical air (m)	Flight distance (m)	Duration of flight (s)
No air resistance	11.2	7.12	25.5	2.41
Air resistance	11.2	7.00	24.5	2.38

Table 3.
Sensitivity of vertical air to ±10% change in parameter values.

Parameter	Percentage change in vertical air for +10% in parameter	Percentage change in vertical air for −10% change in parameter
Width	+5.5%	−6.1%
Depth	+2.5%	−2.6%
Flat	+0.9%	−0.9%
Z_t (Flight point)	+8.2%	−8.1%

Extended Model

[EDITOR'S NOTE: We omit the authors' extended model, which takes into account the snowboarder's actions' effect on $\dot{z}_t$ and total degree of rotation.]

Solutions to the Requirements

Question 1: Snowboard Course for Maximal Vertical Air

Maximum vertical air is determined by the parameters width W, depth H, flat B, flight point z_t, and transition shape $(y(x))$. Using an elliptical transition shape and standard values for the other parameters, we find:

- The wider the ramp, the faster the snowboarder can speed up before the flight and consequently the greater vertical air is.
- A higher ramp can result in higher speed and greater vertical air. But when the ramp is higher than 15 m, the speed and the vertical air decrease with height. Commonly, the height of ramp is around 3–6 m.
- Longer flat provides greater vertical air. But the accompanying decreasing value of E_{leave} means that the potential maximum vertical air is decreased; the actual vertical air is affected by the direction in which the snowboarder flies.
- The steeper the venue is (described by θ), the faster snowboarders can slide and the higher they can fly.
- The path of the snowboarder (z_t) plays a significant part in vertical air. A steeper path provides higher speed, and a greater chance to fly farther down the pipe, but at the expense of fewer tricks. A shallower path ($z_t < 5$ m) means cutting straight across and straight up the wall, but it provides less momentum and lower speed. The snowboarder's skill in choosing a path can help optimize vertical air in some tricks.

Global Optimal

We examine a multidimensional space of parameter values for an elliptical transition:

- width from 13 m to 18 m,
- height from 2 m to 6 m,
- flat length from 4 m to 6 m,
- z_t from 0 m to 15 m, and
- θ from 15° to 20°.

Using the kinetic energy before the flight (E_{leave}) and vertical air as criteria, we find that the best values of the parameters for vertical air (when the snowboarder does not change direction right before the flight) are:

$$\text{width} = 18 \text{ m}, \qquad \text{height} = 3 \text{ m}, \qquad \text{flat} = 4 \text{ m}, \qquad z_t = 6 \text{ m}, \qquad \theta = 20°.$$

The shape of the transition plays a significant role, since the curve directly determines the energy loss caused by friction along the path. To find the optimal transition shape, we applied a genetic algorithm, generating random convex curves and using third-order controlling splines. Vertical air was the fitness function, and the genes were the position of the control points of the B-splines. **Figure 5** shows the result, which matches previous work claiming that the transition should be an ellipse [Fédération de Ski Internationale 2003].

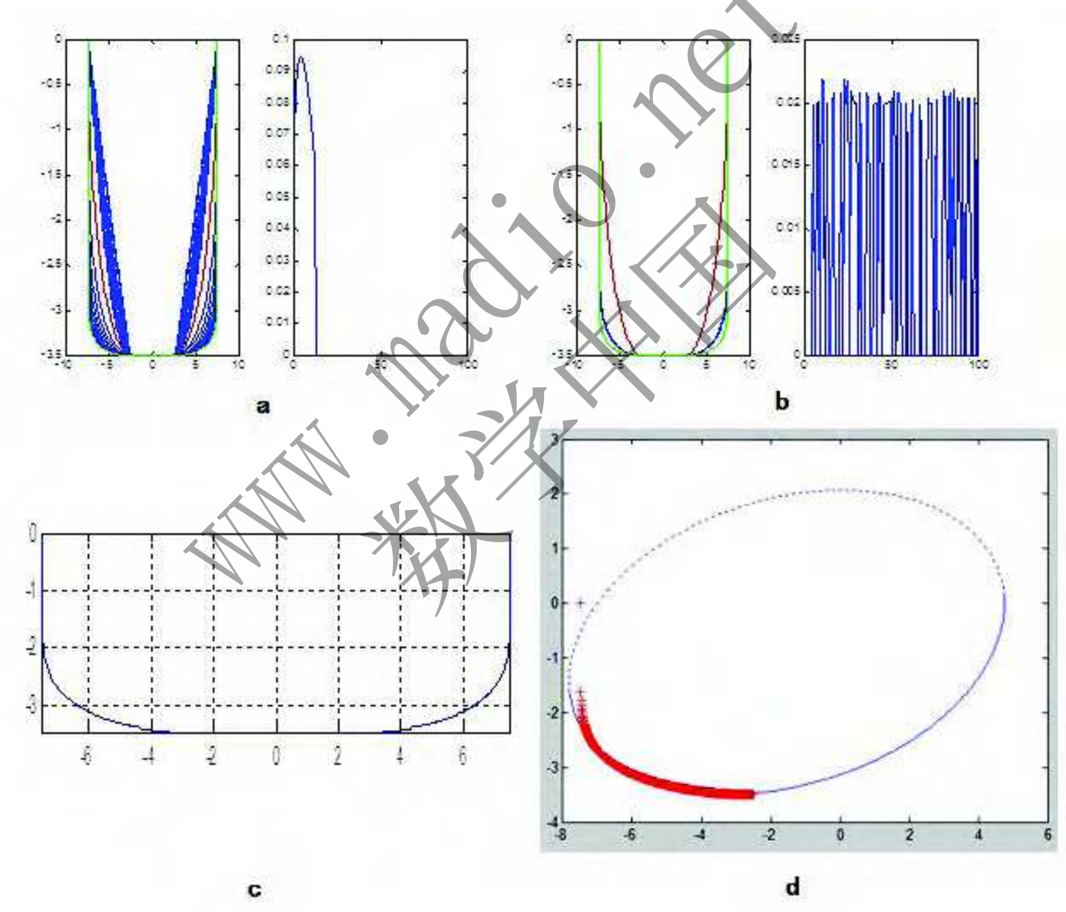


Figure 5. Result of optimizing transition shape using a genetic algorithm.
(a) Initial generation of curves (blue) and their fitness values.
(b) Final generation of curves (blue) and their fitness values.
(c) Shape of the optimized course.
(d) Fitting the optimized course with a second-order curve, which turns out to be a segment of an ellipse.

Question 2: Tailoring the Shape to Other Requirements

[EDITOR'S NOTE: Here the authors apply their genetic algorithm technique to tailor the shape of the course to optimize separately total duration of flight, initial angular momentum in the z-direction, and total degree of rotation.]

We also modified the objective function and implemented a genetic algorithm for the tradeoff between vertical air and total degree of rotation. As Harding et al. [2008c] have shown, the overall performance of half-pipe snowboard can be treated as a complex combination of vertical air, average air time (AAT), and average degree of rotation (ADR). (As we have derived, air time (flight time) is almost proportion to the square root of vertical air in the motion of a projectile, so they can be regarded as a single criterion). They put forward an equation to predict the score of a snowboarder:

$$\text{Predicted score} = 11.424(\text{AAT}) + 0.013(\text{ADR}) - 2.223.$$

They justify this function empirically.

We set the objective function for the genetic algorithm to be the predicted score. The resulting optimal shape of the course can be fitted by a combination of the two ellipses that fit optimal curves for vertical air and total degree of rotation. Judging by the predicted score, a snowboarder could get a score greater than 46 on this course.

Question 3: Tradeoffs for a Practical Course

Radius of curvature: The radius of curvature of the half-pipe cross-section cannot be too small, or the snow may fall off. Since the snowboard is more than 1 m long, too small a radius of curvature is dangerous.

Flat bottom: Since the 1980s, half-pipes have had extended flat ground (the flat bottom) added between the quarter-pipes; the original-style half-pipes have become deprecated. The flat ground gives the athlete time to regain balance after landing and more time to prepare for the next trick.

Moreover, according to our numerical computation, when a half-pipe has a wider flat bottom, then E_{reach} and E_{leave} decrease, whereas the duration of flight and vertical air increase. The change in number of cycles is negligible.

E_{reach}**:** E_{reach} is the final kinetic energy of a cycle, as well as the E_0 of the next cycle. Considering that the snowboarder experiences more than one cycle, it is not wise to chase a higher jump at the cost of less E_{reach}.

According to our results, the wider the course, the larger the vertical air. However, a wider course leads to lower final kinetic energy (E_{reach}), which is unwelcome. The situation is similar with the height of the half-pipe: As long as the height is less than the width, a higher course results in larger vertical air but less E_{reach}. Flat also has the same effect.

Therefore, the half-pipe should not be too wide, too high, or too flat.

Number of cycles: In the Olympics, a snowboarder must perform 5 to 8 acrobatic tricks (hence 5 to 8 cycles) along the half-pipe's 110 m extent.

From our computation, both vertical air and E_{reach} become larger for larger θ. Nonetheless, since S_f (the displacement along the z direction during the flight) increases as θ increases, a snowboarder would find it difficult to perform enough acrobatic tricks on a very steep half-pipe.

Hence, θ should not be too large.

Strengths and Weaknesses

- The model describes the motion in detail, with coordination among the many physical quantities.
- The numerical computations are precise.
- The results generated by numerical computation agree with empirical data, lending support to the model.
- The model takes the subjective influence of snowboarders into account.
- We establish an objective function to compare different course shapes.
- We optimize the course locally (to learn the individual impact of the parameters) as well as globally (to shed light on the design of a half-pipe), and obtain numerical solutions.
- The model does not provide an analytic solution for the optimal course.
- The model does not take into account detailed mechanical characteristics and on-snow performance of snowboards.

References

Chen, J., R. Li, and Z. Guan. 1992. Study of the coefficient of friction between snow-boards and snow in skiing sports. *China Winter Sports* 1992 (4).

Fédération de Ski Internationale. 2003. FIS Snowboard Judges Manual. Oberhofen Switzerland: International Ski Federation. `http://lactat.at/news/news_files/FIS%20Judgesbook%200304.pdf`.

Harding, J.W., K. Toohey, D.T. Martin, C. Mackintosh, A.M. Lindh, and D.A. James. 2008a. Automated inertial feedback for half-pipe snowboard competition and the community perception. In *The Impact of Technology on Sport II*, edited by F.K. Fuss, A. Subic, and S. Ujihashi, 845–850. New York: Taylor & Francis London. `http://www.anarchistathlete.com/wp-content/uploads/2009/05/harding-et-al-2007a-automated-inertial-feedback-for-half-pipe-snowboard-competition-and-the-community-perception.pdf`.

Harding, J.W., K. Toohey, D.T. Martin, A.G. Hahn, and D.A. James. 2008b. Technology and half-pipe snowboard competition—Insight from elite-level judges. In *The Engineering of Sport 7*, Vol. 2, edited by Margaret Estivalet and Pierre Brisson, 467–476. Paris: Springer-Verlag France.

Harding, J.W., C.G. Mackintosh, D.T. Martin, A.G. Hahn, and D.A. James. 2008c. Automated scoring for elite half-pipe snowboard competition: Important sporting development or techno distraction? *Sports Technology* 1 (6) 277–290.

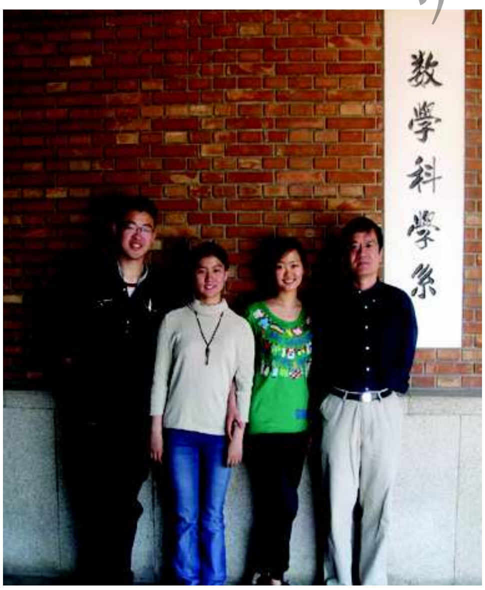
Harding, J.W., and D.A. James. 2010. Performance assessment innovations for elite snowboarding. In *The Engineering of Sport 8*, edited by A. Sabo, P. Kafka, S. Litzenberger, and C. Sabo, 2919–2924. Amsterdam: Elsevier.

Postins, M.C. n.d. Olympic Half Pipe Snowboarding Rules. `http://www.ehow.com/way_5150384_olympic-half-pipe-snowboarding-rules.html`.

Yan, Hongguang, Ping Liu, and Feng Guo. 2009. Factors influencing velocity away from decks in snowboard half-pipe. *Journal of Shenyang Sport University* 2009 (3). `http://en.cnki.com.cn/Article_en/CJFDTotal-SYTB200903006.htm`. `doi:CNKI:SUN:SYTB.0.2009-03-006`.

Yan, Nai-peng. 2006. Primary determinant influencing the skiing resistance against the alpine skier. *China Winter Sports* 2006 (5). `http://en.cnki.com.cn/Article_en/CJFDTotal-BXYD200605007.htm`. `doi:CNKI:ISSN:1002-3488.0.2006-05-007`.

ZAUGG AG EGGIWIL. 2008. LG Snowboard FIS World Cup Snowboard Resort Information Sheet: Halfpipe. `http://www.fis-ski.com/data/document/snowboard-resort-information-sheet_new-version-12.11.2008.pdf`.



Enhao Gong, Rongsha Li, and Xiaoyun Wang, with advisor Jimin Zhang.

(The sign says "Department of Mathematics.")