PERGAMON

Automatica 37 (2001) 1407–1416

automatica

www.elsevier.com/locate/automatica

Brief Paper

Multivariable anti-windup controller synthesis using linear matrix inequalities☆

Eric F. Mulder[a], Mayuresh V. Kothare[a,*], Manfred Morari[b]

[a]Department of Chemical Engineering, Lehigh University, 111 Research Drive, Bethlehem, PA 18015, USA

[b]Automatic Control Laboratory, Swiss Federal Institute of Technology (ETH), Physikstrasse 3, 8092 Zürich, Switzerland

Received 14 July 1999; revised 20 December 2000; received in final form 22 January 2001

Abstract

We present a general formulation of the problem of multi-variable anti-windup bumpless transfer (AWBT) controller synthesis which provides sufficient conditions for stability covering a large class of typical input nonlinearities such as saturation, deadzone, and switching/override nonlinearities. The resulting synthesis method demonstrates graceful performance degradation whenever input nonlinearities are active through minimization of a weighted $\mathscr{L}_2$ gain. The main contribution of the paper is that the AWBT controller synthesis, using static compensation, is cast as a convex optimization over linear matrix inequalities. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Input nonlinearity; Constrained system; Anti-windup; Bumpless transfer; Passivity theorem; Linear matrix inequality; Multiplier theory; Absolute stability

1. Introduction

Linear time-invariant (LTI) anti-windup bumpless transfer (AWBT) refers to the class of control schemes which account for control input nonlinearities in the controller design of otherwise linear systems using the following two-step design paradigm:

Design first the linear controller ignoring control input nonlinearities and then add AWBT compensation to minimize the adverse effects of any control input nonlinearities on closed-loop performance.

Such AWBT compensation schemes provide a computationally efficient technique for "retro-fitting" existing unconstrained controllers to account for input nonlinearities thereby eliminating "controller windup" problems. The AWBT design paradigm has been the basis for a number of diverse control techniques proposed in the literature over the past several decades (Fertik & Ross, 1967; Hanus, Kinnaert, & Henrotte, 1987; Åström & Rundqwist, 1989; Walgama & Sternby, 1990; Campo & Morari, 1989; Doyle, Smith, & Enns, 1987; Zheng, Kothare, & Morari, 1994). Although each technique was a contribution in its own right, significant weaknesses were pointed out by Kothare, Campo, Morari, and Nett (1994) for several of these techniques, such as lack of rigorous stability analysis and clear exposition of performance objectives. These deficiencies motivated the development of a general AWBT framework (see Kothare et al., 1994) which was shown to unify *all* previously known linear time-invariant AWBT schemes in terms of two matrix parameters. In addition, Kothare and Morari developed a rigorous stability analysis (Kothare & Morari, 1999) and proposed guidelines for AWBT controller synthesis in Kothare and Morari (1997). This framework formalized the AWBT problem and established a firm theoretical basis for the eventual development of AWBT controller synthesis techniques.

Unfortunately, the literature is still lacking a satisfactory solution to the AWBT synthesis problem using the general framework of Kothare et al. (1994). The main difficulty with the general framework is the utilization of static AWBT compensation. As with the general static output feedback control problem (see Cao, Lam, & Sun, 1998), the static AWBT controller *synthesis* problem has

☆This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor T. Sugie under the direction of Editor Roberto Tempo.

* Corresponding author. Tel.: + 1-610-758-6654; fax: + 1-610-758-5057.

E-mail address: mayuresh.kothare@lehigh.edu (M. V. Kothare).

PII: S0005-1098(01)00075-9

not been shown to be cast as a convex optimization problem. For example, both Marcopoli and Phillips (1996) and Mulder, Kothare, and Morari (1999, 2000) have presented synthesis solutions to the general framework in the form of bilinear matrix inequalities (BMIs) while only partial convex solutions to the static AWBT synthesis problem have been reported recently in Wada and Saeki (1999). Other techniques avoid the difficulties of static compensation by incorporating dynamic AWBT compensation such as the methods reported in Teel and Kapoor (1997), Park and Choi (1995), Edwards and Postlethwaite (1998) and Miyamoto and Vinnicombe (1996).

In this paper, we present a method, utilizing tools from absolute stability theory and based on the general framework, for the synthesis of a quadratically stabilizing AWBT compensator which provides optimal performance based on minimization of an induced $\mathscr{L}_2$ norm objective. The novelty of this method is that, for the first time, the static AWBT controller synthesis problem has been reduced to a *convex* optimization problem over linear matrix inequalities (LMIs).

Notation. $\Re$ is the set of real numbers. For a matrix A, A^{T} denotes its transpose. The matrix inequality $A > B$ ($A \geqslant B$) means that A and B are square Hermitian matrices and $A - B$ is positive (semi-)definite. $\mathscr{L}_2$ is the Hilbert space of m-vector valued signals defined on $(-\infty, \infty)$, with scalar product $\langle x|y\rangle = \int_{-\infty}^{\infty} x(t)^* y(t)\,\mathrm{d}t$ and such that $\|x\|_2 \triangleq \langle x|x\rangle^{1/2} < \infty \ \forall x \in \mathscr{L}_2$. A transfer function matrix in terms of state-space data is denoted

$$G(s) = C(sI - A)^{-1}B + D \triangleq \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

2. A general AWBT framework

Fig. 1 shows a standard error-feedback control system. The LTI controller $K(s)$ is designed using existing linear control techniques (PID, $\mathscr{H}_2$, $\mathscr{H}_\infty$) to stabilize the LTI plant $P(s)$ and meet certain performance criteria. The presence of control input nonlinearities such as actuator saturation or overrides/mode selection schemes introduce the block $N \neq I$ as in Fig. 1. As a result, the plant input, $\hat{u}$, is not always equal to the controller output, u. This mismatch can cause serious performance degradation and instability in an otherwise stable system (Campo & Morari, 1990).

Using the two-step design paradigm, Kothare et al. (1994) proposed the following axiomatic compensation framework shown in Fig. 2. Within this framework, the system components in Figs. 1 and 2 take the following state-space realizations:

$$P(s) = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right], \qquad K(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right]$$

$$\hat{K}(s) = \left[\begin{array}{c|cc} A_k & B_k & [I \ \ 0] \\ \hline C_k & D_k & [0 \ \ I] \end{array}\right]$$

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = -\Lambda v = -\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} v \tag{1}$$

where Λ_1, Λ_2 are static gain matrices for AWBT compensation. Recall from Kothare et al. (1994) that Λ is chosen to be static such that Λ acts independently on the states of $\hat{K}(s)$ via ξ_1 and on the output of $\hat{K}(s)$ via ξ_2 *if and only if* $N \neq I$. Thus, the AWBT problem has been reduced to that of selecting Λ_1 and Λ_2 to stabilize the closed-loop and provide graceful performance degradation when $N \neq I$.

For the purpose of analysis, we substitute the nonlinearity N in Fig. 2 with the nonlinearity $\Delta = (I - N)$, giving the completely equivalent system in Fig. 3. Our particular choice for modeling the saturation nonlinearity is motivated in part by the work in Mulder et al.

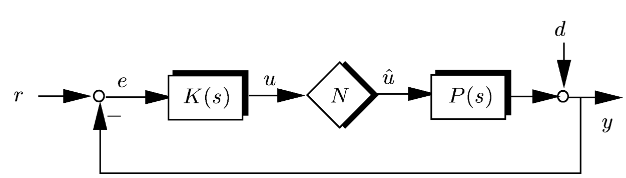

Fig. 1. Linear control design error-feedback case.

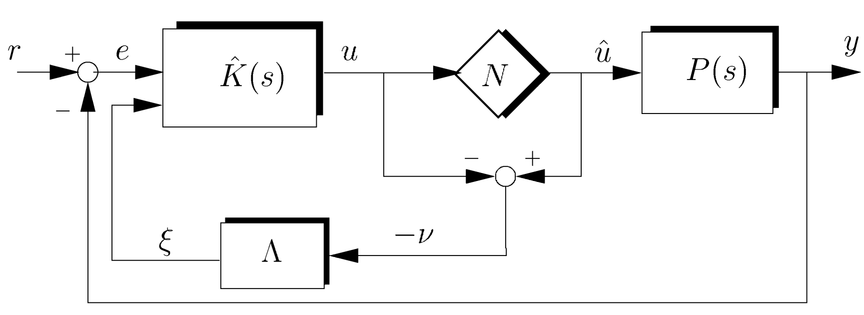

Fig. 2. The AWBT problem-error-feedback case.

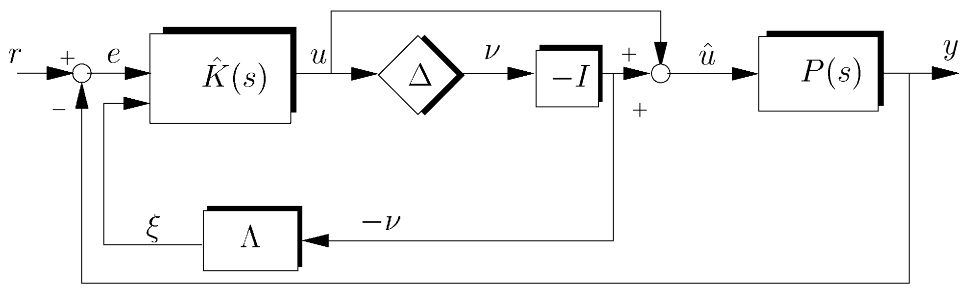

Fig. 3. General AWBT problem with shifted nonlinearity.

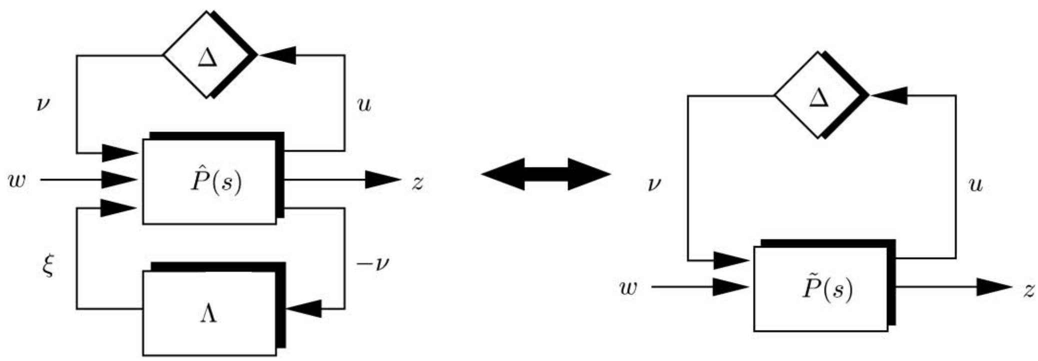

Fig. 4. Standard interconnection for the AWBT problem.

(1999) which demonstrated an apparent convexity to the AWBT synthesis problem despite the form of the solution (BMIs). This apparent convexity was also observed in Marcopoli and Phillips (1996). Such observations motivated the search for an alternate representation which would allow the problem to be cast as an LMI and thus prove convexity. As we will see, choosing the form in Fig. 3 allows us to recast the synthesis problem in the form of LMIs. In order not to restrict ourselves to the error feedback case, consider the linear fractional transformation (LFT) shown in Fig. 4. Any feedback/feedforward interconnection of linear elements, such as the one in Fig. 3, can be brought into this LFT form. The exogenous input w includes all signals which enter the system from its environment such as commands, disturbances, and sensor noise. The input signal v represents the output of Δ where $v = u - \hat{u}$. The output signal z represents all controlled outputs which the controller is designed to keep small (e.g., tracking error). The signal ξ represents the AWBT compensation to the system. Notice that when $v = 0$, we recover the original closed-loop dynamics of the linear design.

We can equivalently write the system in Fig. 4 in state-space form as:

$$\begin{aligned} \dot{x} &= Ax + (B_v - B_\xi \Lambda)v + B_w w, \\ u &= C_u x + (D_{uv} - D_{u\xi}\Lambda)v + D_{uw} w, \\ z &= C_z x + (D_{zv} - D_{z\xi}\Lambda)v + D_{zw} w, \end{aligned} \tag{2}$$

where A, B_v, B_w, B_ξ, C_u, D_{uv}, D_{uw}, $D_{u\xi}$, C_z, D_{zv}, D_{zw}, and $D_{z\xi}$ are functions of the original parameters in the feedback/feedforward interconnection in Fig. 3. Here, we have implictly used the relationship $\xi = -\Lambda v$, to absorb Λ into the LFT.

Thus, we have reduced the AWBT synthesis problem to a simple LFT interconnection where the objective is to intelligently design Λ. Notice that the particular modeling of N as $I - \Delta$ ensures that Λ appears only in the "B" and "D" matrices relating v to $\dot{x}$, u and z and this facilitates the convex formulation of the synthesis of Λ. Next, we provide conditions for asymptotic stability of the system in Fig. 4.

3. AWBT stability

Development of nonconservative conditions for stability of the system in Fig. 3 or 4 for the specific nonlinearity Δ is, in general, a difficult problem. On the other hand, we can apply results from absolute stability theory (Lur'e, 1957; Megretski & Rantzer, 1997; Zames, 1966a, b) to develop sufficient conditions which guarantee stability for an entire class of nonlinearities, $\tilde{\Delta}$, of which the specific nonlinearity, Δ, is a member. This has been the prevalent approach in Campo and Morari (1990), Campo, Morari, and Nett (1989), Glattfelder and Schaufelberger (1983) and Kapasouris and Athans (1985), as well as in Kothare and Morari (1999), and simplifies the nonlinear AWBT analysis at the unavoidable price of conservatism. We begin our approach by defining the set of nonlinearities, $\tilde{\Delta}$, which we will consider:

Definition 1. Let $f: \Re^n \times \Re \to \Re^n$ with $f(0,t) = 0\ \forall t \geqslant 0$ be a memoryless (possibly time-varying) diagonal nonlinearity $f = \mathrm{diag}\{f_1, \dots, f_n\}$. We say that $f \in$ sector$[K_1, K_2]$, with $K_1 = \mathrm{diag}(K_{11}, \dots, K_{1n})$, $K_2 = \mathrm{diag}(K_{21}, \dots, K_{2n})$, $K_2 - K_1 > 0$ if

$$K_{1i}x_i^2 \leqslant x_i f_i(x_i, t) \leqslant K_{2i}x_i^2 \quad \text{for all } x_i \in \Re,\ t \geqslant 0,\ i = 1,2,\dots,n. \tag{3}$$

We define the set $\tilde{\Delta}$ of all allowable structured memoryless time-varying nonlinearities as follows:

$$\tilde{\Delta} = \{\Delta : \Re^{n_u} \times \Re \to \Re^{n_u} | \Delta(0,t) = 0\ \forall t \geqslant 0,\ \Delta = \mathrm{diag}\{\Delta_1, \Delta_2, \dots, \Delta_{n_u}\}, \Delta_i \in \text{ sector}[0,1]\}. \tag{4}$$

$\tilde{\Delta}$ includes typical input nonlinearities such as input saturation, deadzone, and relay as well as typical overrides and mode selectors. Sufficient conditions for stability of the system in (2) for all nonlinearities satisfying (4) were derived in Kothare and Morari (1999) using the passivity theorem and multi-loop circle multiplier. From those analysis results, we state the following explicit AWBT synthesis result:

Theorem 1 (Synthesis with multiloop circle criterion). *Let $\tilde{P}_{-11}(s)$ denote the transfer function relating* $-v$ to u *in Fig.* 4. *Then, the closed-loop in Fig.* 4 *is $\mathscr{L}_2$ stabilizable for all $\Delta \in \tilde{\Delta}$ if*

1. *$\tilde{P}(s)$ in Fig.* 4 *is asymptotically stable*; *and*
2. *$\exists W = \mathrm{diag}(W_1, W_2, \dots, W_{n_u}) \in \Re^{n_u \times n_u}$ with $W > 0$ and $\delta > 0$ such that*

$$W\tilde{P}_{-11}(j\omega) + \tilde{P}^*_{-11}(j\omega)W + 2W \geqslant \delta I \quad \forall \omega \in \Re. \tag{5}$$

Moreover, *condition* (5) *is equivalent to the existence of a diagonal matrix $M = \mathrm{diag}(M_1, M_2, \dots, M_{n_u}) = W^{-1}$ with $M > 0$, $Q = Q^{\mathrm{T}} > 0$, and $\delta_1 = \delta^{-1} > 0$ such that the following LMI in Q, M, X, δ_1 is satisfied*

$$\begin{bmatrix} QA^{\mathrm{T}} + AQ & B_v M - B_\xi X + QC_u^{\mathrm{T}} & 0 \\ MB_v^{\mathrm{T}} - X^{\mathrm{T}}B_\xi^{\mathrm{T}} + C_u Q & \mathscr{X} & M \\ 0 & M & -\delta_1 I \end{bmatrix} < 0 \tag{6}$$

with $\mathscr{X} = -2M + D_{uv}M + MD_{uv}^{\mathrm{T}} - D_{u\xi}X - X^{\mathrm{T}}D_{u\xi}^{\mathrm{T}}$.

If the LMI is feasible, *then $\Lambda = XM^{-1}$* (*where X and M are feasible solutions to* (6)) *is the AWBT compensator which stabilizes the nonlinear system in Fig.* 4. *Furthermore*, *the stability multiplier*, $W = M^{-1}$ *is explicitly computed from the solution of* (6).

Proof. See the appendix

The main result is that the AWBT compensator Λ can be readily computed from a convex solution of the LMI in (6). We note that since we ensure stability for all $\Delta \in \tilde{\Delta}$, and since $\Delta = I \in \tilde{\Delta}$ corresponds to an open connection between u and $\hat{u}$, we are restricting ourself to open-loop stable plants $P(s)$ in our formulation of Fig. 1. However, by replacing the sector [0,1] by the sector [β,1], with $0 < \beta < 1$ and applying a loop transformation, we can derive locally stabilizing anti-windup designs for unstable plants. Lack of space precludes a detailed discussion of these issues.

In addition to stability, we wish to impose certain performance criteria such that the closed-loop system exhibits graceful performance degradation when $N \neq I$ ($\Delta \neq 0$). For this purpose, we choose to define an alternate but equivalent stability criterion using Lyapunov functions, which facilitates the inclusion of performance criteria.

Theorem 2 (Synthesis with Lyapunov stability criterion). *Define a Lyapunov function of the form*

$$V(x) = x^{\mathrm{T}}Px + \int_0^t v^{\mathrm{T}}(\tau)\delta v(\tau)\,\mathrm{d}\tau + 2\sum_{i=1}^{n_u} W_i \int_0^t (v_i(\tau)u_i(\tau) - v_i(\tau)v_i(\tau))\,\mathrm{d}\tau. \tag{7}$$

Then, *the closed-loop in Fig.* 4 *is $\mathscr{L}_2$ stabilizable for all $\Delta \in \tilde{\Delta}$ if $\exists P = P^{\mathrm{T}} > 0$, $\delta > 0$, $W = \mathrm{diag}(W_1, W_2, \dots, W_{n_u}) \in \Re^{n_u \times n_u}$ with $W > 0$ such that*

1. *$V(x) > 0$ for $x \neq 0$ and $V(0) = 0$*; *and*
2. *$V(x)$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x) < 0 \quad \text{for } x \neq 0. \tag{8}$$

As with condition (5), *satisfying the Lyapunov stability condition* (8) *reduces to determining the feasibility of the LMI in* (6) *with $M = \mathrm{diag}(M_1, M_2, \dots, M_{n_u}) = W^{-1} > 0$, $Q = P^{-1} > 0$ and $\delta_1 = \delta^{-1} > 0$. If the LMI is feasible, then $\Lambda = XM^{-1}$* (*where X and M are feasible solutions to* (6)) *is the AWBT compensator which stabilizes the nonlinear system in Fig.* 4. *Furthermore*, *the stability multiplier*, $W = M^{-1}$ *is explicitly computed from the solution of* (6).

Proof. See the appendix

The form of the Lyapunov function in (6) was chosen specifically to be equivalent to Theorem 1. In the next section, we show how we can combine a performance criterion with the Lyapunov based stability result into a single controller synthesis problem.

4. AWBT performance

In addition to stability of the system, we require that the AWBT compensation provide graceful performance degradation in the presence of actuator saturation. We choose to define our performance objective for the AWBT compensator as the minimization of the weighted induced $\mathscr{L}_2$ norm from the exogenous inputs, w, to the desired outputs, z

$$\sup_{\|w\|_2 \neq 0} \frac{\|\gamma^{1/2} z\|_2}{\|w\|_2}. \tag{9}$$

We now state the main result.

Theorem 3 ($\mathscr{L}_2$ gain criterion). *The AWBT system in Fig.* 4 *is $\mathscr{L}_2$ stabilizable for all $\Delta \in \tilde{\Delta}$ and has a weighted induced $\mathscr{L}_2$ gain less than $\sqrt{\Gamma}$ with weight $\gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_{n_z}) \in \Re^{n_z \times n_z}$, $\gamma > 0$ if there exists a matrix $P = P^{\mathrm{T}} > 0$, $W = \mathrm{diag}(W_1, W_2, \ldots, W_{n_u}) \in \Re^{n_u \times n_u}$ with $W > 0$, and scalars $\delta > 0$, $\Gamma > 0$, such that*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{ x^{\mathrm{T}} P x + \int_0^t v^{\mathrm{T}}(\tau)\delta v(\tau)\,\mathrm{d}\tau \right. \\ &\left. + 2\sum_{i=1}^{n_u} W_i \int_0^t (v_i(\tau)u_i(\tau) - v_i(\tau)v_i(\tau))\,\mathrm{d}\tau \right\} + z^{\mathrm{T}}\gamma z \\ &- \Gamma w^{\mathrm{T}} w < 0. \end{aligned} \tag{10}$$

Moreover, condition (10) *is equivalent to the existence of a matrix $M = \mathrm{diag}(M_1, M_2, \ldots, M_{n_u}) = W^{-1}$ with $M > 0$, $\delta_1 = \delta^{-1} > 0$, $Q = P^{-1} > 0$, and $\Gamma > 0$ such that the following LMI in Q, M, X, δ_1, and Γ is satisfied*:

$$\begin{bmatrix} QA^{\mathrm{T}} + AQ & B_w & B_v M - B_\xi X + QC_u^{\mathrm{T}} & QC_z^{\mathrm{T}} & 0 \\ B_w^{\mathrm{T}} & -\Gamma & D_{uw}^{\mathrm{T}} & D_{zw}^{\mathrm{T}} & 0 \\ MB_v^{\mathrm{T}} - X^{\mathrm{T}} B_\xi^{\mathrm{T}} + C_u Q & D_{uw} & \mathscr{X} & MD_{zv}^{\mathrm{T}} - X^{\mathrm{T}} D_{z\xi}^{\mathrm{T}} & M \\ C_z Q & D_{zw} & D_{zv} M - D_{z\xi} X & -\gamma^{-1} & 0 \\ 0 & 0 & M & 0 & -\delta_1 I \end{bmatrix} < 0 \tag{11}$$

with $\mathscr{X} = -2M + D_{uv} M + MD_{uv}^{\mathrm{T}} - D_{u\xi} X - X^{\mathrm{T}} D_{u\xi}^{\mathrm{T}}$.

A well-defined upper bound on the weighted $\mathscr{L}_2$ gain is obtained by finding the minimum feasible Γ subject to (11), *which is a standard LMI eigenvalue optimization problem* (*Boyd, Ghaoui, Feron, & Balakrishnan*, 1994). *Moreover, $\Lambda = XM^{-1}$ (where X and M are feasible solutions to* (11) *for the minimum feasible Γ) is the proposed solution to the AWBT synthesis problem.*

Proof. See the appendix

The variable, γ, is a user defined positive, diagonal matrix which can be used to provide higher weighting to more critical output variables in the performance criterion.

5. AWBT synthesis example

This example was taken from Zheng et al. (1994) where the anti-windup IMC method was applied. Consider the following plant:

$$\begin{aligned} P(s) &= \frac{10}{100s+1}\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} -0.01 & 0.00 & 1 & 0 \\ 0.00 & -0.01 & 0 & 1 \\ \hline 0.4 & -0.5 & 0 & 0 \\ -0.3 & 0.4 & 0 & 0 \end{array}\right]. \end{aligned} \tag{12}$$

Within the error-feedback context of Fig. 1, the controller for the linear system is

$$\begin{aligned} K(s) &= \left(1 + \frac{1}{100s}\right)\begin{bmatrix} 2.0 & 2.5 \\ 1.5 & 2.0 \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0.020 & 0.025 & 2.0 & 2.5 \\ 0.015 & 0.020 & 1.5 & 2.0 \end{array}\right]. \end{aligned} \tag{13}$$

A setpoint change of $[0.63\ \ 0.79]^{\mathrm{T}}$ applied to the linear system at $t = 0$ results in a smooth closed-loop response. However, if we apply the saturation limits of ± 1 on the controller output, u, the same set-point change results in large, wild oscillations in the output. Both responses are shown in Fig. 5.

In order to compare our method to other existing *static* AWBT methods, we take advantage of the fact that all other known methods can be shown to be special cases of the general framework of Kothare et al. (1994) where $\Lambda_2 = 0$ such as those in Marcopoli and Phillips (1996) and Wada and Saeki (1999). Since it is a trivial matter to apply our synthesis technique for $\Lambda_2 = 0$, we can determine the best possible one parameter static compensator, Λ_1, for our particular performance objective. Thus, setting $w = r, z = e$ and $\gamma = I$, we determine that the best one parameter compensator is

$$\Lambda_1 = \begin{bmatrix} 89.9 & 100.6 \\ 104.3 & 144.8 \end{bmatrix}$$

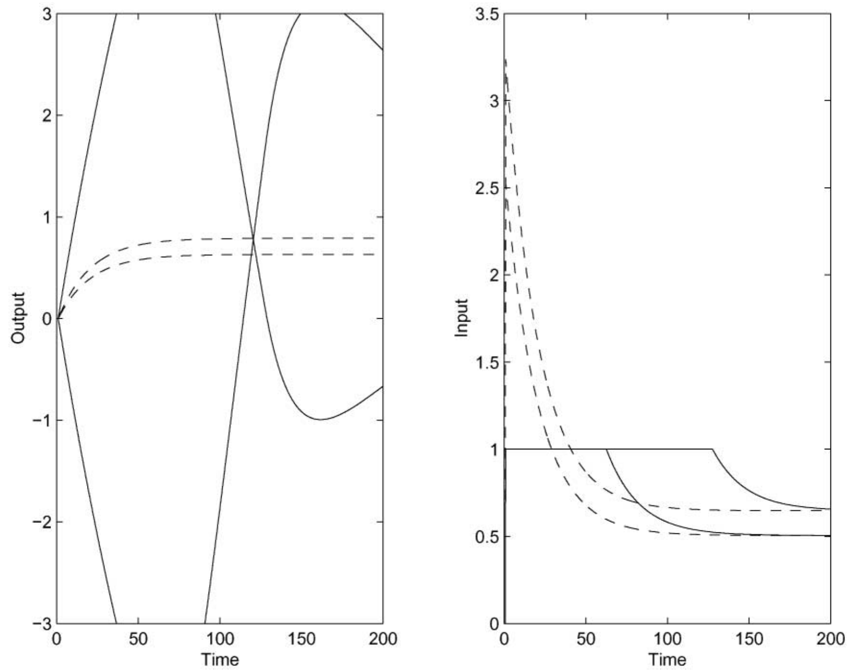

Fig. 5. Example process response: —, constrained without compensation; - -, unconstrained.

with an upper bound of 65.3 on the induced $\mathscr{L}_2$ norm. The closed-loop responses with this compensator are shown in Fig. 6. These can be compared with responses using the best two parameter compensator (with $\gamma = I$) which was determined to be

$$\Lambda_1 = \begin{bmatrix} 0.87 & 0.82 \\ 1.09 & 1.09 \end{bmatrix} \quad \text{and} \quad \Lambda_2 = \begin{bmatrix} 3.45 & 4.37 \\ 3.48 & 2.42 \end{bmatrix}$$

with an upper bound of 1.55 on the $\mathscr{L}_2$ gain (the *linear*, *unconstrained* closed-loop induced $\mathscr{L}_2$ gain or the $\mathscr{H}_\infty$ norm is 1). The corresponding closed-loop step responses are shown in Fig. 7, clearly showing superiority of the two parameter framework ($\Lambda_1 \neq 0$, $\Lambda_2 \neq 0$) over the one parameter framework ($\Lambda_2 = 0$, $\Lambda_1 \neq 0$).

As an additional design criterion, suppose we wish to recover as much of the linear performance as is possible for output 1 at the expense of the performance in output 2. For this problem, we apply our method with the output weight,

$$\gamma = \begin{bmatrix} 1.00 & 0 \\ 0 & 0.01 \end{bmatrix}.$$

The optimization problem results in

$$\Lambda_1 = \begin{bmatrix} 0.91 & 0.85 \\ 64.3 & 64.4 \end{bmatrix} \quad \text{and} \quad \Lambda_2 = \begin{bmatrix} 210.4 & 211.7 \\ 170.0 & 168.3 \end{bmatrix}$$

with an upper bound of 1.55 on the weighted $\mathscr{L}_2$ gain. From the simulation results in Fig. 8, we can see complete recovery of the linear response for output 1 at the expense of output 2.

As a final remark, we note that the values of the AWBT parameters Λ_1 and Λ_2 obtained are specific to the state-space realizations of the transfer functions $P(s)$ and $K(s)$. For different realizations of the same transfer functions $K(s)$ and $P(s)$, the AWBT parameters will have correspondingly different values or "realizations" but will give the same closed-loop response.

6. Conclusions

We have presented a static AWBT synthesis technique which has several important features. Firstly, it provides an explicit guarantee of asymptotic stability through the use of the passivity theorem with the multi-loop circle multiplier. This sufficient stability condition is applicable to any multi-variable linear AWBT system subject to input nonlinearities such as saturation, deadzone, switching/override nonlinearities. Secondly, the technique provides graceful performance degradation under saturation by minimizing a weighted $\mathscr{L}_2$ gain where the performance weight, γ, is specified by the user. Finally, the technique leads to AWBT synthesis of Λ_1 and Λ_2 in the form of a convex optimization problem over LMIs. Static AWBT synthesis has not been previously shown to be a convex problem. Thus, we provide for the first time a numerically efficient tractable solution to the static AWBT synthesis problem in LMI form.

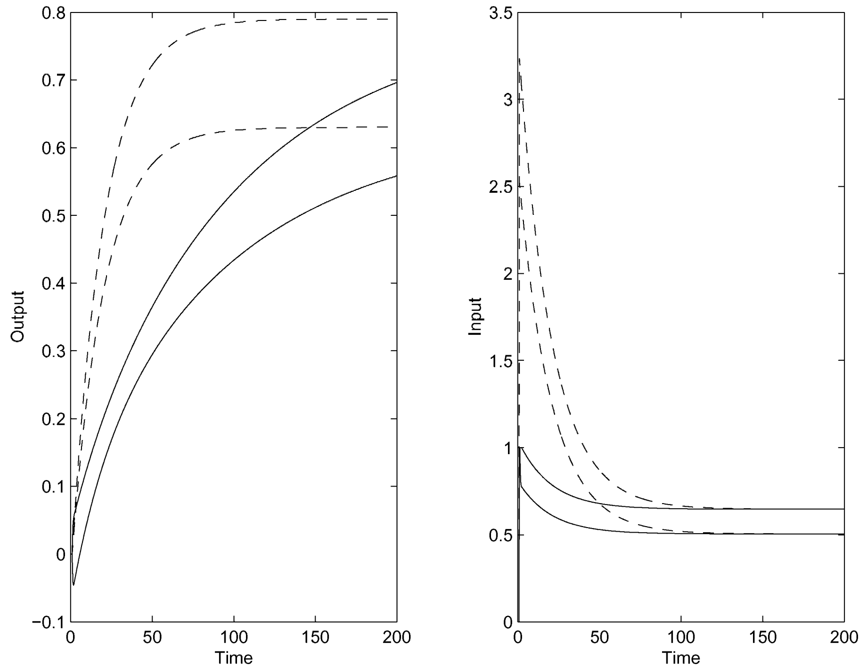

Fig. 6. Example process response: —, Theorem 3 with $\gamma = I$, $\Lambda_2 = 0$; - -, unconstrained.

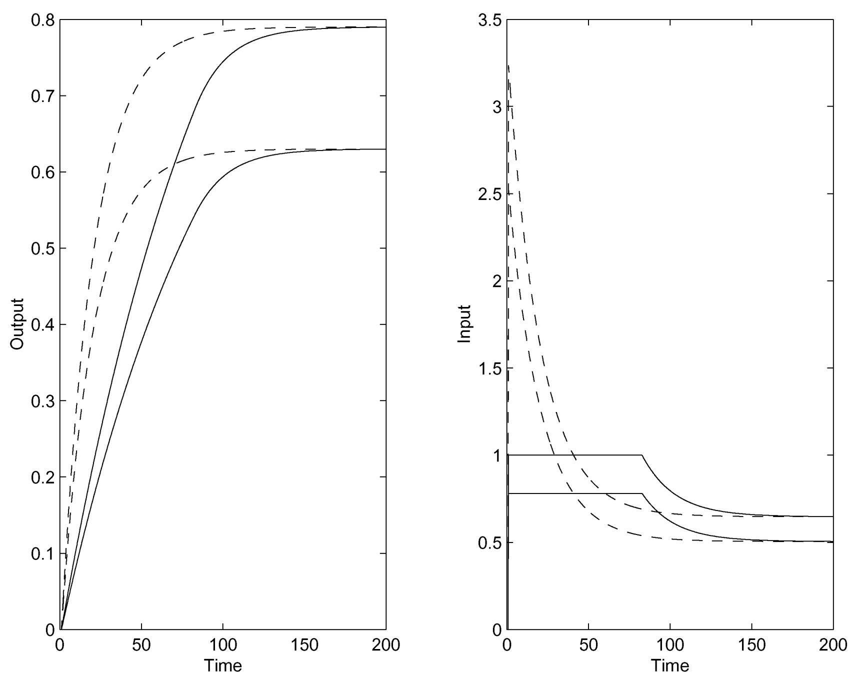

Fig. 7. Example process response: —, Theorem 3 with $\gamma = I$, $\Lambda_2 \neq 0$; - -, unconstrained.

Appendix

Proof of Theorem 1. Using the passivity theorem, it was clearly explained in Kothare and Morari (1999) that asymptotic stability of the system in Fig. 4 is guaranteed if $\tilde{P}_{-11}(s)$ is strictly passive and Δ is passive. Determining that $\tilde{P}_{-11}(s)$ is strictly passive is equivalent to determining that $\tilde{P}_{-11}(s)$ is positive real, or

$$\tilde{P}_{-11}(j\omega) + \tilde{P}^{*}_{-11}(j\omega) \geqslant \delta I \quad \forall \omega \in \Re,\ \delta > 0. \tag{A.1}$$

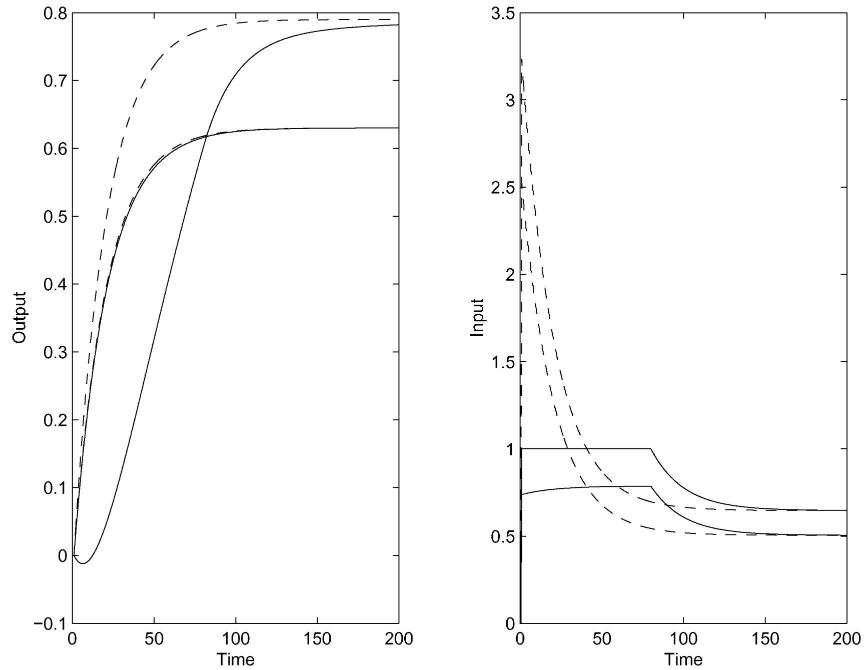

Fig. 8. Example process response; —, Theorem 3 with $\gamma = \mathrm{diag}(1.00, 0.01)$; - -, unconstrained.

Depending on the structure of Δ, this result may be highly conservative. Thus, using a common loop transformation along with multiplier theory, it was shown in Kothare and Morari (1999) that asymptotic stability is guaranteed for all $\Delta \in \tilde{\Delta}$, if

$$W\tilde{P}_{-11}(j\omega) + \tilde{P}^{*}_{-11}(j\omega)W + 2W \geqslant \delta I \quad \forall \omega \in \Re, \tag{A.2}$$

where W is a positive definite diagonal matrix, or equivalently,

$$\begin{bmatrix} A^{\mathrm{T}}P + PA & -PB_v + PB_\xi\Lambda - C_u^{\mathrm{T}}W \\ -B_v^{\mathrm{T}}P + \Lambda^{\mathrm{T}}B_\xi^{\mathrm{T}}P - WC_u & \delta I - 2W + WD_{uv} + D_{uv}^{\mathrm{T}}W - WD_{u\xi}\Lambda - \Lambda^{\mathrm{T}}D_{u\xi}^{\mathrm{T}}W \end{bmatrix} < 0. \tag{A.3}$$

We can apply a simple congruence transformation $\mathrm{diag}(-P^{-1}, \ W^{-1})$ to (A.3). Defining $Q = P^{-1}$, $M = W^{-1}$, $X = \Lambda M$, we obtain the equivalent condition

$$\begin{bmatrix} QA^{\mathrm{T}} + AQ & B_vM - B_\xi X + QC_u^{\mathrm{T}} \\ MB_v^{\mathrm{T}} - X^{\mathrm{T}}B_\xi^{\mathrm{T}} + C_uQ & M\delta M - 2M + D_{uv}M + MD_{uv}^{\mathrm{T}} - D_{u\xi}X - X^{\mathrm{T}}D_{u\xi}^{\mathrm{T}} \end{bmatrix} < 0. \tag{A.4}$$

Finally, apply a simple Schur complement to remove the nonlinear term in M and δ. Then define $\delta_1 = \delta^{-1}$ to obtain the main result in (6). □

Proof of Theorem 2. By inspection of (7), it can easily be seen that $P = P^{\mathrm{T}} > 0$ implies the first term is positive for nonzero x; $\delta > 0$ implies the second term is positive for nonzero x; and $W > 0$ implies that the third term is non-negative (since we require $\Delta \in$ sector[0,1]). Thus, $V(x) > 0$ for all nonzero x. Also, $V(0) = 0$. Therefore, if (8) is satisfied, then the closed-loop system is globally asymptotically stable. Condition (8) is guaranteed for all $(x,v) \neq 0$ if (A.3) is satisfied which then leads to (A.4) and finally the main result (6). □

Proof of Theorem 3. Following (Boyd et al., 1994), integrate (10) from 0 to t with the initial condition $x_0 = 0$, to get

$$V(x) + \int_0^t (z^{\mathrm{T}}\gamma z - \Gamma w^{\mathrm{T}}w)\,\mathrm{d}\tau \leqslant 0, \tag{A.5}$$

where $V(x)$ is as defined in (7). Since $V > 0$, (A.5) implies

$$\frac{\|\gamma^{1/2}z\|_2}{\|w\|_2} \leqslant \sqrt{\Gamma}. \tag{A.6}$$

Condition (10) is guaranteed for all $(x, w, v) \neq 0$ if

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + C_z^{\mathrm{T}}\gamma C_z & PB_w + C_z^{\mathrm{T}}\gamma D_{zw} & PB_v - PB_\xi \Lambda + C_u^{\mathrm{T}}W + C_z^{\mathrm{T}}\gamma(D_{zv} - D_{z\xi}\Lambda) \\ B_w^{\mathrm{T}}P + D_{zw}^{\mathrm{T}}\gamma C_z & -\Gamma + D_{zw}^{\mathrm{T}}\gamma D_{zw} & D_{uw}^{\mathrm{T}}W + D_{zw}^{\mathrm{T}}\gamma(D_{zv} - D_{z\xi}\Lambda) \\ B_v^{\mathrm{T}}P - \Lambda^{\mathrm{T}}B_\xi^{\mathrm{T}}P + WC_u + (D_{zv}^{\mathrm{T}} - \Lambda^{\mathrm{T}}D_{z\xi}^{\mathrm{T}})\gamma C_z & WD_{uw} + (D_{zv}^{\mathrm{T}} - \Lambda^{\mathrm{T}}D_{z\xi}^{\mathrm{T}})\gamma D_{zw} & \delta\mathrm{I} - 2W + W(D_{uv} - D_{u\xi}\Lambda) + (D_{uv}^{\mathrm{T}} - \Lambda^{\mathrm{T}}D_{u\xi}^{\mathrm{T}})W + (D_{zv}^{\mathrm{T}} - \Lambda^{\mathrm{T}}D_{z\xi}^{\mathrm{T}})\gamma(D_{zv} - D_{z\xi}\Lambda) \end{bmatrix} < 0. \quad \text{(A.7)}$$

After applying a Schur complement to remove the quadratic terms in Λ, (A.7) is equivalent to

$$\begin{bmatrix} A^{\mathrm{T}}P + PA & PB_w & PB_v - PB_\xi\Lambda + C_u^{\mathrm{T}}M^{-1} & C_z^{\mathrm{T}} \\ B_w^{\mathrm{T}}P & -\Gamma & D_{uw}^{\mathrm{T}}M^{-1} & D_{zw}^{\mathrm{T}} \\ B_v^{\mathrm{T}}P - \Lambda^{\mathrm{T}}B_\xi^{\mathrm{T}}P + M^{-1}C_u & M^{-1}D_{uw} & \delta\mathrm{I} + M^{-1}\mathscr{X}M^{-1} & D_{zv}^{\mathrm{T}} - \Lambda^{\mathrm{T}}D_{z\xi}^{\mathrm{T}} \\ C_z & D_{zw} & D_{zv} - D_{z\xi}\Lambda & -\gamma^{-1} \end{bmatrix} < 0 \quad \text{(A.8)}$$

with $\mathscr{X} = -2M + D_{uv}M + MD_{uv}^{\mathrm{T}} - D_{u\xi}X - X^{\mathrm{T}}D_{u\xi}^{\mathrm{T}}$, $M = W^{-1}$, $X = \Lambda M$.

We can apply a simple congruence transformation $\mathrm{diag}(P^{-1}, I, M, I)$ to (A.8). Defining $Q = P^{-1}$, we obtain the equivalent condition

$$\begin{bmatrix} QA^{\mathrm{T}} + AQ & B_w & B_vM - B_\xi X + QC_u^{\mathrm{T}} & QC_z^{\mathrm{T}} \\ B_w^{\mathrm{T}} & -\Gamma & D_{uw}^{\mathrm{T}} & D_{zw}^{\mathrm{T}} \\ MB_v^{\mathrm{T}} - X^{\mathrm{T}}B_\xi^{\mathrm{T}} + C_uQ & D_{uw} & M\delta M + \mathscr{X} & MD_{zv}^{\mathrm{T}} - X^{\mathrm{T}}D_{z\xi}^{\mathrm{T}} \\ C_zQ & D_{zw} & D_{zv}M - D_{z\xi}X & -\gamma^{-1} \end{bmatrix} < 0. \quad \text{(A.9)}$$

Finally, we apply a simple Schur complement to remove the nonlinear term in M and δ. Defining $\delta_1 = \delta^{-1}$, we obtain the main result in (11). □

References

Åström, K. J., & Rundqwist, L. (1989). Integrator windup and how to avoid it. *Proceedings of the 1989 American Control Conference*, 1693–1698.

Boyd, S., El Ghaoui, L., Feron, E., & Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory, vol. 15. Studies in Applied Mathematics*. Philadelphia, PA: SIAM.

Campo, P. J., & Morari, M. (1990). Robust control of processes subject to saturation nonlinearities. *Computers & Chemical Engineering*, *14*(4/5), 343–358.

Campo, P. J., Morari, M., & Nett, C. N. (1989). Multivariable anti-windup and bumpless transfer: A general theory. *Proceedings of the 1989 American Control Conference*, pp. 1706–1711.

Cao, Y. -Y., Lam, J., & Sun, Y. -X. (1998). Static output feedback stabilization: An ILMI approach. *Automatica*, *34*(12), 1641–1645.

Doyle, J. C., Smith, R. S., & Enns, D. F. (1987). Control of plants with input saturation nonlinearities. *Proceedings of the 1987 American Control Conference*, Minneapolis, MN, pp. 1034–1039.

Edwards, C., & Postlethwaite, I. (1998). Anti-windup and bumpless transfer schemes. *Automatica*, *34*(2), 199–210.

Fertik, H. A., & Ross, C. W. (1967). Direct digital control algorithm with anti-windup feature. *ISA Transactions*, *6*(4), 317–328.

Glattfelder, A. H., & Schaufelberger, W. (1983). Stability analysis of single-loop control systems with saturation and antireset-windup circuits. *IEEE Transactions on Automatic Control*, *28*(12), 1074–1081.

Hanus, R., Kinnaert, M., & Henrotte, J. L. (1987). Conditioning technique, a general anti-windup and bumpless transfer method. *Automatica*, *23*(6), 729–739.

Kapasouris, P., & Athans, M. (1985). Multivariable control systems with saturating actuators, antireset windup strategies. *Proceedings of the 1985 American Control Conference*, pp. 1579–1584.

Kothare, M. V., Campo, P. J., Morari, M., & Nett, C. N. (1994). A unified framework for the study of anti-windup designs. *Automatica*, *30*(12), 1869–1883.

Kothare, M. V., & Morari, M. (1997). Multivariable anti-windup controller synthesis using multi-objective optimization. *Proceedings of the 1997 American Control Conference*, Albuquerque, New Mexico, pp. 3093–3097.

Kothare, M. V., & Morari, M. (1999). Multiplier theory for stability analysis of anti-windup control systems. *Automatica*, *35*(5), 917–928.

Lur'e, A. I. (1957). *Some nonlinear problems in the theory of automatic control*. London: Her Majesty's Stationery Office, in Russian, 1951.

Marcopoli, V. R., & Phillips, S. M. (1996). Analysis and synthesis tools for a class of actuator-limited multivariable control systems: A Linear Matrix Inequality approach. *International Journal of Robust and Nonlinear Control*, 6, 1045–1063. Special Issue on Linear Matrix Inequalities and their applications.

Megretski, A., & Rantzer, A. (1997). System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control*, *42*(6), 819–829.

Miyamoto, S., & Vinnicombe, G. (1996). Robust control of plants with saturation nonlinearity based on coprime factor representations. *Proceedings of the 35th IEEE Conference on Decision and Control*, Kobe, Japan, pp. 2838–2840.

Mulder, E. F., Kothare, M. V., & Morari, M. (1999). Multivariable anti-windup controller synthesis using iterative linear matrix inequalities. *Proceedings of the 1999 European Control Conference*, Karlsruhe, Germany.

Mulder, E. F., Kothare, M. V., & Morari, M. (2000). Multivariable anti-windup controller synthesis using Bilinear Matrix Inequalities. *European Journal of Control*, *6*(5), 455–464.

Park, J.-K., & Choi, C.-H. (1995). Dynamic compensation method for multivariable control systems with saturating actuators. *IEEE Transactions on Automatic Control*, *40*(9), 1635–1640.

Teel, A. R., & Kapoor, N. (1997). The $\mathscr{L}_2$ anti-windup problem: Its definition and solution. *Proceedings of the 1997 European Control Conference (ECC)*, Brussels, Belgium, May.

Wada, N., & Saeki, M. (1999). Design of a static anti-windup compensator which guarantees robust stability (in Japanese). *Transactions of the Institute of Systems, Control and Information Engineers*, *12*(11), 664–670.

Walgama, K. S., & Sternby, J. (1990). Inherent observer property in a class of anti-windup compensators. *International Journal of Control*, *52*(3), 705–724.

Zames, G. (1966a). On the input-output stability of time varying nonlinear feedback systems—Part I: Conditions derived using concepts of loop gain, conicity, and positivity. *IEEE Transactions on Automatic Control*, *11*(2), 228–238.

Zames, G. (1966b). On the input-output stability of time varying nonlinear feedback systems—Part II: Conditions involving circles in the frequency plane and sector nonlinearities. *IEEE Transactions on Automatic Control*, *11*(3), 465–476.

Zheng, A., Kothare, M. V., & Morari, M. (1994). Anti-windup design for internal model control. *International Journal of Control*, 60(5), 1015–1024. Also presented at the 1993 *A.I.Ch.E. Annual Meeting*, MO.

Eric Mulder was born in Martinsville, Indiana, USA in 1974. He received his B.S. degree in Chemical Engineering from Purdue University in 1997 along with certification in the cooperative education program for service to Union Carbide Corporation from 1994 to 1997. He also completed an internship at ExxonMobil Corporation in 2000. He is currently completing his Ph.D. thesis at Lehigh University. His main research interests are in constrained control of linear and nonlinear processes, and in the application of linear matrix inequalities to process control.

Mayuresh V. Kothare received the degree of Bachelor of Technology (B.Tech.) in Chemical Engineering from the Indian Institute of Technology, Bombay, in 1991. He was recipient of the Institute Silver Medal for ranking first in Chemical Engineering at the Indian Institute of Technology, Bombay, the J. N. Tata Endowment award and the Sumant Mulgaonkar Memorial award for academic excellence in chemical engineering. He then joined California Institute of Technology where he received an M.S. degree in Chemical Engineering in June 1995 and a Ph.D. degree in Chemical Engineering in June 1997 with a minor in Control and Dynamical Systems. From September 1994 to December 1996, he was a research assistant in the Automatic Control Laboratory of the Swiss Federal Institute of Technology in Zürich. From January to June 1997, he was a visiting scholar in the Department of Chemical Engineering at City College New York. From July 1997 to June 1998, he held a postdoctoral appointment in the Modeling and Advanced Control group of Mobil Technology Company, a subsidiary of Mobil Oil Corporation. In July 1998, he joined Lehigh University where he is currently an Assistant Professor of Chemical Engineering.

Dr. Kothare's interests are in the areas of control of chemical processes with input and output constraints; and modeling, dynamics and control of microchemical systems. He has made contributions in the analysis and synthesis of anti-windup control systems, model predictive control and application of linear matrix inequalities to solving receding horizon control problems. He is recipient of the year 2000 Ted Peterson Student Paper Award from the Computing & Systems Technology Division of the American Institute of Chemical Engineers.

Manfred Morari obtained the diploma from ETH Zürich and the Ph.D. from the University of Minnesota, both in chemical engineering. In 1994 Manfred Morari was appointed head of the Automatic Control Laboratory at the Swiss Federal Insitute of Technology (ETH) in Zürich. Prior to that he was the McCollum-Corcoran Professor of Chemical Engineering and Executive Officer for Control and Dynamical Systems at the California Institute of Technology. His interests are in the areas of hybrid systems and control of biomedical systems. In recognition of his research contributions, he received numerous awards among them, the Donald P. Eckman Award of the Automatic Control Council, the Allan P. Colburn Award and the Professional Progress Award of the AIChE, the Curtis W. McGraw Research Award of the ASEE and was elected to the National Academy of Engineering (US). Prof. Morari has held appointments with Exxon R & E and ICI and has consulted internationally for a number of major corporations.