aircond: An Example for Optimization Under Uncertainty

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Abstract

This paper describe a scalable model used to create test instances for optimization under uncertainty. See `https://github.com/DLWoodruff/aircond`.

We present a multi-period production planning model with overtime production and inventory carry-forward based on a model called "Air Conditioning" in lecture notes by Jeff Linderoth. As in [1], we present notation for a multi-stage optimization model with uncertainty.

We are considering a T-stage stochastic program. We use $t \in \{1, \ldots, T\}$ to index stages, and use $\{\boldsymbol{\xi}_t\}_{t=2}^T$ to denote the associated stochastic process. The bold letter $\boldsymbol{\xi}_t$ is used to denote random variables, which can be vector valued, and ξ_t is used to denote their realizations. The values of ξ_t for $t = 2, \ldots, T$ are revealed at the end of stage $t-1$, therefore we refer to $\boldsymbol{\xi}_t$ and ξ_t only for $t = 2, \ldots, T$. We use the symbol $\vec{\xi}^t$ to refer to the realized values of all random variables up to and including stage t, i.e.,

$$\vec{\xi}^t = (\xi_2, \xi_3, \ldots, \xi_t) .$$

When $t = T$, $\vec{\xi}^T$ is known as a *scenario.*

We use x_t to represent the decision that is made at stage t, and use the notation $\vec{x}^t$ for $1 \leq t \leq T$ to represent the decisions for all stages up to, and including, stage t, i.e.

$$\vec{x}^t = (x_1, x_2, \ldots, x_t)$$

For a T-stage stochastic program, the decision x_t for $t = 2, \ldots, T$ is made such that given the realized $\vec{\xi}^t$, a problem-specific objection function is minimized. To form a recursive expression, we rely on a functional ϕ to capture the objectives for the problem at hand given scenario data. Note that at each stage t, the decision x_t is made with only the knowledge of $\vec{x}^{t-1}$ and $\vec{\xi}^t$ and is independent of the future decisions and realizations of $\boldsymbol{\xi}_{t+1}, \ldots, \boldsymbol{\xi}_T$. Such independence is known as non-anticipativity. The first stage solution x_1 is made so that it minimizes the expected value of $\phi_2(x_1; \boldsymbol{\xi}_2)$.

The above notation allows us to write the multi-stage stochastic problem (MSSP) recursively as

$$\begin{aligned} z^* = \min_{x_1} \quad & \mathbb{E}_{\boldsymbol{\xi}_2} \phi_2 \left(x_1; \boldsymbol{\xi}_2 \right), \\ s.t. \quad & x_1 \in X_1 \end{aligned} \tag{1}$$

where for each $t = 2, \ldots, T-1$, we have

$$\begin{aligned} \phi_t(\vec{x}^{t-1}; \vec{\xi}^t) \quad &= \quad \min_{x_t} \mathbb{E}_{\vec{\boldsymbol{\xi}}^{t+1}} [\phi_{t+1}((\vec{x}^{t-1}, x_t); \vec{\boldsymbol{\xi}}^{t+1}) | \vec{\boldsymbol{\xi}}^t = \vec{\xi}^t] \\ s.t. \qquad & \qquad x_t \in X_t \left(\vec{x}^{t-1}, \vec{\xi}^t \right), \end{aligned} \tag{2}$$

and

$$\begin{aligned} \phi_T(\vec{x}^{T-1}; \vec{\xi}^T) \quad &= \quad \min_{x_T} \phi'_T(\vec{x}^T; \vec{\xi}^T) \\ s.t. \qquad & \qquad x_T \in X_T \left(\vec{x}^{T-1}, \vec{\xi}^T \right), \end{aligned}$$

where $\phi'_T(\vec{x}^T; \vec{\xi}^T)$ is the problem-specific objective function for the final stage given the full scenario data and solutions to the previous stage(s). At each stage we impose a constraint for the feasibility of x_t, such that x_t belongs to a feasible region $x_t \in X_t \left(\vec{x}^{t-1}, \vec{\xi}^t \right)$. We incorporate feasible region constraints into the objective function by introducing the extended representation

$$f_{t+1}(\vec{x}^t; \vec{\xi}^{t+1}) = \begin{cases} \phi_{t+1}(\vec{x}^t; \vec{\xi}^{t+1}), & x_t \in X_t(\vec{x}^{t-1}, \vec{\xi}^t) \\ \infty, & \text{otherwise.} \end{cases}$$

Then, the MSSP problem (1) can be simplified to

$$z^* = \min_{x_1} \mathbb{E}_{\boldsymbol{\xi}_2} f_2 \left(x_1; \boldsymbol{\xi}_2 \right), \tag{3}$$

where for each $t = 2, \ldots, T-1$, we have

$$f_t(\vec{x}^{t-1}; \vec{\xi}^t) = \min_{x_t} \mathbb{E}_{\vec{\boldsymbol{\xi}}^{t+1}}[f_{t+1}((\vec{x}^{t-1}, x_t); \vec{\boldsymbol{\xi}}^{t+1}) | \vec{\boldsymbol{\xi}}^t = \vec{\xi}^t], \qquad (4)$$

and

$$f_T(\vec{x}^{T-1}; \vec{\xi}^T) = \min_{x_T} f'_T(\vec{x}^T; \vec{\xi}^T)$$

where

$$f'_T(\vec{x}^T; \vec{\xi}^T) = \begin{cases} \phi'_T(\vec{x}^T; \vec{\xi}^T), & x_T \in X_T(\vec{x}^{T-1}, \vec{\xi}^T) \\ \infty, & \text{otherwise.} \end{cases}$$

For theoretical purposes, we will assume that we are dealing with stochastic programs with relatively complete recourse, which means that for solution values that are feasible in earlier stages there exist feasible solution values in subsequent stages.

A special case is the T-stage stochastic linear program, which is defined as

$$\min_{x_1 \in X_1} c_1 x_1 + \mathbb{E}\left[\min_{x_2 \in X_2(x_1, \xi_2)} c_2 x_2 + \mathbb{E}\left[\cdots + \mathbb{E}[\min_{x_T \in X_T(x_{T-1}, \xi_T)} c_T x_T]\right]\right] \qquad (5)$$

where ξ_t represents the random data at stage t, $\xi_t = \{c_t, A_t, B_t, b_t\}$, and feasible region $X_1 = \{x_1 : A_1 x_1 = b_1, x_1 \geq 0\}$, and for $t = 2, \ldots, T$, we have

$$X_t(x_{t-1}; \xi_t) = \{x_t : B_t x_{t-1} + A_t x_t = b_t, x_t \geq 0\}$$

To draw a connection between the multi-stage linear program and the general MSSP problem defined above, notice that by taking

$$\phi'_T(\vec{x}^T; \vec{\xi}^T) = c_1 x_1 + \ldots + c_T x_T,$$

then by recursive form (2), we have

$$\phi_T(\vec{x}^{T-1}; \vec{\xi}^T) = c_1 x_1 + \ldots + c_{T-1} x_{T-1} + \min_{x_T} c_T x_T,$$

$$\begin{aligned}
\phi_{T-1}(\vec{x}^{T-2}; \vec{\xi}^{T-1}) &= \min_{x_{T-1}} \mathbb{E}_{\vec{\boldsymbol{\xi}}^T}[\phi_T(\vec{x}^{T-1}; \vec{\boldsymbol{\xi}}^T) | \vec{\boldsymbol{\xi}}^{T-1} = \vec{\xi}^{T-1}] \\
&= \min_{x_{T-1}} \mathbb{E}_{\vec{\boldsymbol{\xi}}^T}[c_1 x_1 + \ldots + \min_{x_T} c_T x_T | \vec{\boldsymbol{\xi}}^{T-1} = \vec{\xi}^{T-1}] \\
&= c_1 x_1 + \ldots + \min_{x_{T-1}}\left[c_{T-1} x_{T-1} + \mathbb{E}_{\boldsymbol{\xi}_T} \min_{x_T} c_T x_T\right]
\end{aligned}$$

and similarly we can recursively define the object function for $t = T - 2, \ldots, 2$, and in the end we would have

$$\begin{aligned}
\phi_2(x_1; \xi_2) &= \min_{x_2} \mathbb{E}_{\vec{\boldsymbol{\xi}}^3}[\phi_3(\vec{x}^2; \vec{\boldsymbol{\xi}}^3)|\boldsymbol{\xi}_2 = \xi_2] \\
&= \min_{x_2} \mathbb{E}_{\vec{\boldsymbol{\xi}}^3}[c_1x_1 + c_2x_2 + \min_{x_3}[c_3x_3 + \mathbb{E}[\ldots + \mathbb{E}\min_{x_T} c_Tx_T]]|\boldsymbol{\xi}_2 = \xi_2] \\
&= c_1x_1 + \min_{x_2}[c_2x_2 + \mathbb{E}[\min_{x_3} c_3x_3 + \mathbb{E}[\ldots + \mathbb{E}\min_{x_T} c_Tx_T]]],
\end{aligned}$$

and finally $\min_{x_1} \mathbb{E}_{\boldsymbol{\xi}_2} \phi_2(x_1, \boldsymbol{\xi}_2)$ would be in the form of (5).

1 Aircond Model

An earlier version of this model appeared in [1].

1.1 Variables

Given a set of stages (periods) $\mathcal{T} = \{1, 2, \ldots, T\}$, and a single product, we define each variable over each stage $t \in \mathcal{T}$. For each stage we have the following:

> $x_t \in \mathbb{R}_{\geq 0}$ number of each product to make in regular time (decided right before period t)
>
> $w_t \geq \delta_t \in \mathbb{R}_{\geq 0}$ number of each product to make in overtime (decided right before period t)
>
> $\delta_t \in \{0, 1\}$ start-up variable indicating $x_t > 0$ (and/or $w_t > 0$)
>
> $y_t \in \mathbb{R}$ number of each product to carry forward (determined by x, w and the demand for period t)
>
> $y_t^+ \in \mathbb{R}$ y_t if $y_t \geq 0$
>
> $y_t^- \in \mathbb{R}$ $|y_t|$ if $y_t < 0$

1.2 Parameters

The following quantities are all taken as parameters in our construction of the problem. For our purposes, only the demand, D_t, is stochastic, though this problem may be extended to have stochastic inventory capacity and costs. Each parameter is real-valued and all but N_t are non-negative. All are known the beginning of period 1 except for the demand.

D_t demand in period t (known at the end of period t)

I_t single period inventory cost

N_t single period negative inventory cost (failure to meet demand)

Q_t quadratic coefficient for single period negative inventory cost (failure to meet demand)

K_t regular time capacity

C_t regular time production cost

O_t overtime production cost

S_t start-up cost

y_0 beginning inventory

1.3 Objective Function

The objective is to minimize the expected cost of production subject to inventory and demand constraints. At each stage we can only produce so much of each product and we must meet the given demand. Using the notation introduced in (1), with $\boldsymbol{\xi} = \mathbf{D}$, and, subsequently,

$$\vec{\xi}^t = (\xi_2, \xi_3, \dots, \xi_t) = (D_2, D_3, \dots, D_t),$$

the objective is defined recursively by:

$$\begin{aligned}
\min_{x_1} \quad & \Big(\delta_1 \cdot S_1 + C_1 \cdot x_1 + O_1 \cdot w_1 + \max\{I_1 y_1^+, N_1 y_1^- + Q_1 \left(y^-\right)_1^2 + \mathbb{E}_{\boldsymbol{\xi}_2} \phi_2(x_1; \boldsymbol{\xi}_2)\Big) \\
s.t. \quad & x_1 \leq K_1, \\
& y_0 + x_1 + w_1 - y_1 = D_1 \\
& y_1^+ - y_1^- = y_1 \\
& M\delta_1 \geq x_1 + w_1
\end{aligned}$$

where

$$\begin{aligned}
\phi_t(x_{t-1}; \vec{\xi}^t) = \min_{x_t} \quad & \Big(\delta_t \cdot S_t + C_t \cdot x_t + O_t \cdot w_t + \max\{I_t y_t^+, N_t y_t^- + Q_t \left(y_t^-\right)^2\} + \mathbb{E}_{\vec{\boldsymbol{\xi}}^{t+1}} \Big[\phi_{t+1}(x_t; \vec{x}^{t-1}, \\
s.t. \quad & x_t \leq K_t, \\
& y_{t-1} + x_t + w_t - y_t = D_t \\
& y_t^+ - y_t^- = y_t \\
& M\delta_t \geq x_t + w_t
\end{aligned}$$

and

$$
\begin{aligned}
\phi_T(x_{T-1};\vec{\xi}^T) = \min_{x_T} \quad & \left(\delta_T \cdot S_T + C_T \cdot x_T + O_T \cdot w_T + \max\{I_T y_T^+, N_T y_T^- + Q_T \left(y_T^-\right)^2\}\right) \\
s.t. \quad & x_T \leq K_T, \\
& y_{T-1} + x_T + w_T - y_T = D_T \\
& y_T^+ - y_T^- = y_T \\
& M\delta_T \geq x_T + w_T.
\end{aligned}
$$

Here M is a large number (for most data, $\sum_{t=1}^{T} D_t$ would be large enough).

References

[1] Xiotie Chen, Sylvain Cazaux, Brian C. Knight, and David L. Woodruff. Confidence interval software for multi-stage stochastic programs, 2021.