Formula de convoluție

Fie X_1 și X_2 două variabile aleatoare continue independente și $Y = X_1 + X_2$.

Atunci:
$$\boxed{f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y-x_2)\cdot f_{X_2}(x_2)\,dx_2 = \int_{-\infty}^{\infty} f_{X_2}(y-x_1)\cdot f_{X_1}(x_1)\,dx_1}$$

Exemple

$\boxed{1}$ $X_1 \sim Unif(0, a_1)$, $X_2 \sim Unif(0, a_2)$, X_1, X_2 independente, iar $0 < a_1 < a_2$ și $Y = X_1 + X_2$

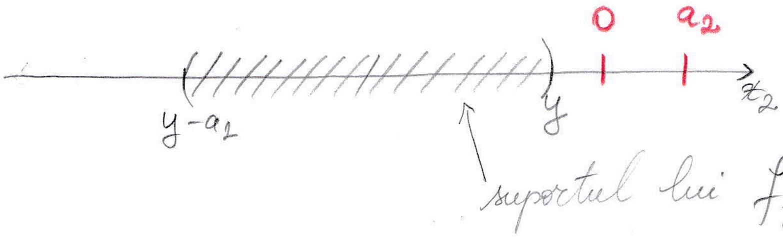
$$f_{X_i}(x) = \begin{cases} \dfrac{1}{a_i}, & x \in (0, a_i) \\ 0, & x \notin (0, a_i) \end{cases} \qquad i=\overline{1,2}$$

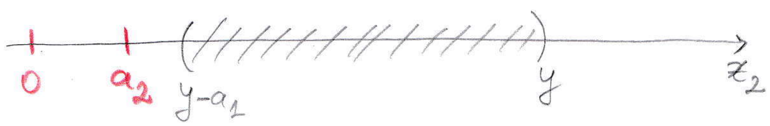
$$f_{X_1}(y-x_2) = \begin{cases} \dfrac{1}{a_1}, & y-x_2 \in (0,a_1) \\ 0, & \text{în rest} \end{cases}, \quad f_{X_1}(y-x_2) = \begin{cases} \dfrac{1}{a_1}, & x_2 \in (y-a_1, y) \\ 0, & \text{în rest} \end{cases}$$

$$y-x_2 \in (0,a_1) \Leftrightarrow 0 < y-x_2 < a_1 \Leftrightarrow -a_1 < x_2-y < 0 \Leftrightarrow y-a_1 < x_2 < y$$

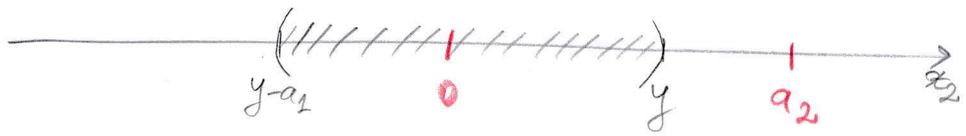
$\Rightarrow x_2 \in (y-a_1, y)$

Atunci $f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y-x_2)\cdot f_{X_2}(x_2)\,dx_2 = \frac{1}{a_2}\cdot \underbrace{\int_0^{a_2} f_{X_1}(y-x_2)\,dx_2}_{I}$

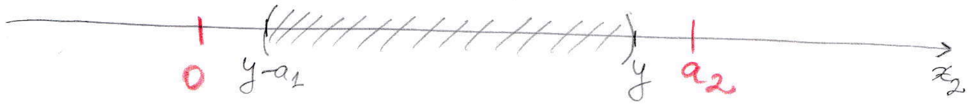




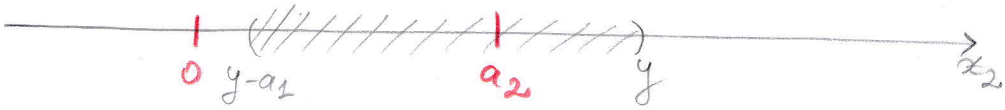
I Dacă $y<0$ sau $y \geq a_1+a_2$ atunci $I=0$



II Dacă $y \in [0, a_1]$ atunci $I = \int_0^y \frac{1}{a_1} dx_2 = \frac{1}{a_1} \cdot y$



III Dacă $y \in (a_1, a_2]$ atunci $I = \int_{y-a_1}^{y} \frac{1}{a_1} dx_2 = \frac{1}{a_1} \cdot (y-(y-a_1)) = 1$



IV Dacă $y \in (a_2, a_1+a_2)$ atunci $I = \int_{y-a_1}^{a_2} \frac{1}{a_1} dx_2 = \frac{1}{a_1} \cdot (a_2-(y-a_1))$

Așadar $f_Y(y) = \begin{cases} 0, & y<0 \text{ sau } y \geq a_1+a_2 \\ \frac{y}{a_1 \cdot a_2}, & y \in [0, a_1] \\ \frac{1}{a_2}, & y \in (a_1, a_2] \\ \frac{a_1+a_2-y}{a_1 \cdot a_2}, & y \in (a_2, a_1+a_2) \end{cases}$

$\boxed{2}$ Fie $X_1, X_2 \sim Norm(0,1)$ independente și $Y = X_1 + X_2$.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y - x_2) \cdot f_{X_2}(x_2)\, dx_2$$

$$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x_i^2}{2}}, \quad x_i \in \mathbb{R}, \; i = \overline{1,2}$$

$$f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(y-x_2)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x_2^2}{2}} dx_2$$

$$f_Y(y) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} e^{-\frac{y^2 - 2x_2 y + 2x_2^2}{2}} dx_2 = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} e^{-\frac{\cancel{2}(x_2^2 - x_2 y + \frac{1}{2}y^2)}{\cancel{2}}} dx_2$$

$$f_Y(y) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} e^{-\left(x_2 - \frac{1}{2}y\right)^2} \cdot e^{-\frac{1}{4}y^2} dx_2 = \frac{e^{-\frac{1}{4}y^2}}{2\pi} \cdot \int_{-\infty}^{\infty} e^{-\left(x_2 - \frac{1}{2}y\right)^2} dx_2$$

$$\boxed{\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}}$$ Integrala Euler - Poisson

S.V. $x_2 - \frac{1}{2}y = t \Rightarrow dx_2 = dt$

$x_2 \to -\infty \Rightarrow t \to -\infty$

$x_2 \to \infty \Rightarrow t \to \infty$

$$f_Y(y) = \frac{e^{-\frac{1}{4}\cdot y^2}}{2\pi} \cdot \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{e^{-\frac{1}{4}\cdot y^2}}{2\pi} \cdot \sqrt{\pi} = \frac{e^{-\frac{1}{2}\cdot\left(\frac{y - \overset{m}{0}}{\sqrt{2}}\right)^2}}{\sqrt{2\pi} \cdot \underbrace{\sqrt{2}}_{\sigma}}$$

$\Rightarrow$ $Y \sim N(0, 2)$