Tehnici de Optimizare

Facultatea de Matematica si Informatica

Universitatea Bucuresti

- Department Informatica-

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Probleme de programare neliniara

$$\min_{x} f(x)$$
$$s.l. \; g_i(x) = 0, \qquad i = 1, \cdots, m$$
$$h_i(x) \leq 0, \qquad i = 1, \cdots, p$$

unde f, g, h sunt functii diferentiabile.

Problema convexa: $\boldsymbol{f}$ convexa + $\boldsymbol{h_i}$ convexe + $\boldsymbol{g_i}$ liniare!

Exemplu convex:

$$\min_{x} \frac{1}{2} \left|\left|Ax - b\right|\right|^2$$
$$s.l. \qquad Cx \leq d$$

Probleme de programare neliniara

$$\min_x f(x)$$
$$s.l.\ g_i(x) = 0, \qquad i = 1, \cdots, m$$
$$h_i(x) \leq 0, \qquad i = 1, \cdots, p$$

Notam $I(x) = \{i: h_i(x) = 0(< 0)\}$, indicii inegalitatilor **active (inactive)**.

Functia Lagrangian: $L: R^n \times R^m \times R_+^p$

$$L(x, \lambda, \mu) = f(x) + \lambda^T g(x) + \mu^T h(x); L(x^*, \lambda^*, \mu^*) = \cdots + \sum \mu_i^* h_i(x^*)$$

Egalitatile sunt penalizate cand $||\lambda|| \to \infty$

Inegalitatile sunt penalizate cand $\mu \geq 0\ (\mu^T h(x) \geq 0)\ si\ ||\mu|| \to \infty$

$$h(x) = [h_1(x) \quad h_2(x)] = [(< 0) \quad (> 0)] \Rightarrow \mu^T h(x) = \mu_1 h_1(x) + \mu_2 h_2(x)$$

Probleme de programare convexa

$$\min_x f(x)$$
$$s.l.\ Ax = b,$$
$$h(x) \leq 0,$$

unde f, h_i sunt functii convexe.

Teorema Kuhn-Tucker. Fie f, h_i convexe, daca conditia Slater are loc, i.e.:

$$\exists\, x: \quad Ax = b, h_i(x) < 0, \qquad i = 1, \cdots, m, \qquad (relaxare\ la\ h_i\ neliniare)$$

Atunci x^* este o solutie a problemei daca si numai daca: gasim $\mu^* \geq 0$

$\mu_i^* h_i(x^*) = 0, \quad i = 1, \cdots, m$ si $L(x, \lambda^*, \mu^*) \geq L(x^*, \lambda^*, \mu^*)$ sau $\nabla_x L(x^*, \lambda^*, \mu^*) = 0$

Probleme de programare convexa

$$\min_x f(x)$$
$$s.l.\ Ax = b,$$
$$h(x) \leq 0,$$

Regularitate: (x^*, μ^*) minim regulat (sau problema regulata) daca conditia Slater are loc si $\mu^* \geq 0, \mu_i^* h_i(x^*) = 0$.

Concluzia teoremei: **Daca problema are minim regulat, atunci se poate reduce la o problema fara constrangeri (obiectivul este functia Lagrangian)!**

Probleme de programare convexa

$$\min_x f(x)$$
$$s.l.\ Ax = b,$$
$$h(x) \leq 0,$$

Regularitate: (x^*, μ^*) minim regulat daca $\mu^* \geq 0, \mu_i^* h_i(x^*) = 0$ si conditia Slater:

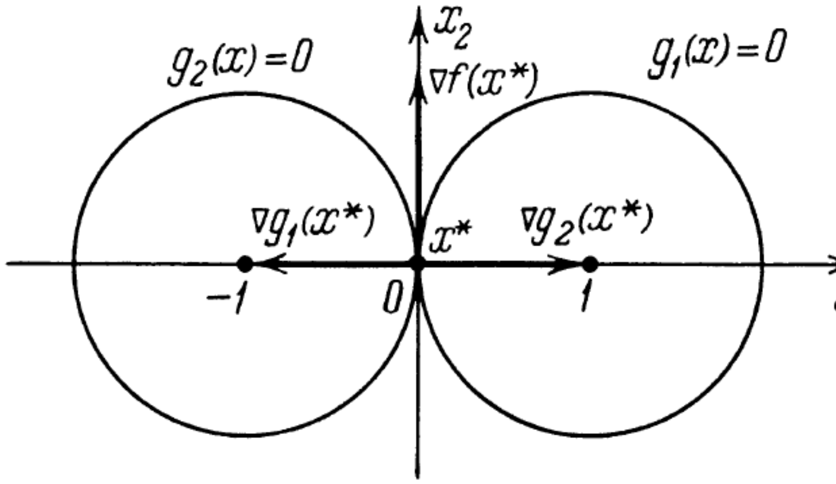
$$h_i(x^*) < 0, \qquad i = 1, \cdots, m$$

Concluzia teoremei: **Daca problema are minim regulat,**

$$\boldsymbol{x^*} \Rightarrow \boldsymbol{\nabla_{\mathbf{x}} \mathbf{L}(\mathbf{x}^*, \lambda^*, \mu^*) = 0}$$

Probleme de programare convexa

$$\min_{x} x_2$$
$$s.l.\ (x_1 - 1)^2 + x_2^2 \leq 1,$$
$$(x_1 + 1)^2 + x_2^2 \leq 1,$$



Conditia Slater nu are loc:

$[\ 0\ 0\ ]^T$ singurul punct fezabil!

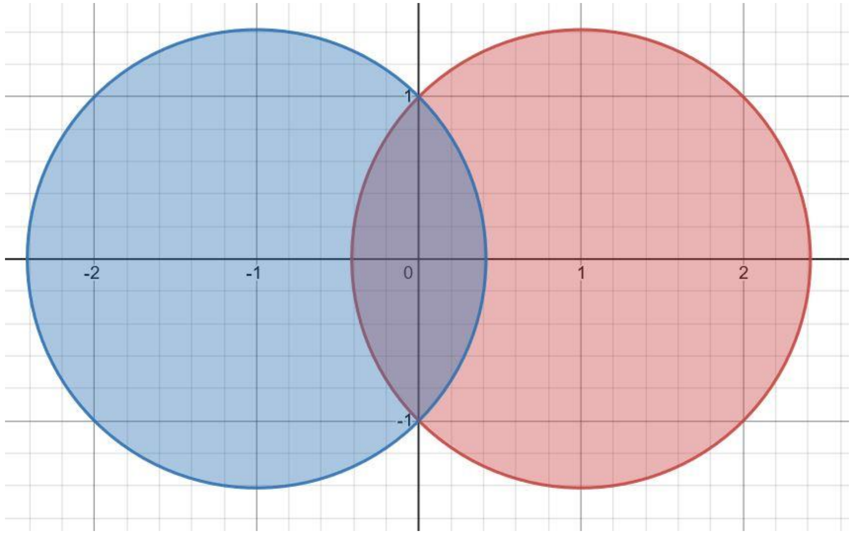
Se arata usor ca nu exista $\mu_1^*, \mu_2^* > 0$

Incat $\nabla L(0, \mu^*) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\mu_1^* \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 2\mu_2^* \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0.$

Probleme de programare convexa

$$\min_x x_2$$
$$s.l.\ (x_1 - 1)^2 + x_2^2 \leq 2,$$
$$(x_1 + 1)^2 + x_2^2 \leq 2$$

Conditia Slater are loc!



C.N.: $\nabla_x L(x, \mu) = 0$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \mu_1 2 \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} + \mu_2 2 \begin{bmatrix} x_1 + 1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mu_1[(x_1 - 1)^2 + x_2^2 - 2] = 0$$

$$\mu_2[(x_1 + 1)^2 + x_2^2 - 2] = 0$$

Probleme de programare convexa

$$\min_{x} x_2$$
$$s.l.\ (x_1 - 1)^2 + x_2^2 \leq 2,$$
$$(x_1 + 1)^2 + x_2^2 \leq 2$$

Cazul $\boldsymbol{\mu_1 = \mu_2 = 0}$

C.N.: $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (Exclus)

Probleme de programare convexa

$$\min_x x_2$$
$$s.l.\ (x_1-1)^2+x_2^2\le 2,$$
$$(x_1+1)^2+x_2^2\le 2$$

Cazul $\boldsymbol{\mu_1=0,\mu_2>0}$

C.N.: $\begin{bmatrix}0\\1\end{bmatrix}+2\mu_2\begin{bmatrix}x_1+1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\Rightarrow x_1=-1\ (violeaza\ prima\ ineg.)$

$$(x_1-1)^2+x_2^2<2$$
$$(x_1+1)^2+x_2^2=2$$

Probleme de programare convexa

$$\min_x x_2$$
$$s.l.\ (x_1-1)^2+x_2^2 \le 2,$$
$$(x_1+1)^2+x_2^2 \le 2$$

Cazul $\boldsymbol{\mu_1 > 0, \mu_2 = 0}$

C.N.: $\begin{bmatrix}0\\1\end{bmatrix} + 2\mu_1\begin{bmatrix}x_1-1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow x_1 = 1\ (violeaza\ a\ doua\ ineg.)$

$$(x_1-1)^2+x_2^2 = 2$$
$$(x_1+1)^2+x_2^2 < 2$$

Probleme de programare convexa

$$\min_x x_2$$
$$s.l.\ (x_1-1)^2+x_2^2 \le 2,$$
$$(x_1+1)^2+x_2^2 \le 2$$

Cazul $\boldsymbol{\mu_1 > 0, \mu_2 > 0}$

C.N.: $\begin{bmatrix}0\\1\end{bmatrix} + 2\mu_1\begin{bmatrix}x_1-1\\x_2\end{bmatrix} + 2\mu_2\begin{bmatrix}x_1+1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow x_1 = \frac{\mu_2-\mu_1}{\mu_1+\mu_2}, x_2 = -\frac{1}{\mu_1+\mu_2}$

$$(x_1-1)^2+x_2^2 = 2$$
$$(x_1+1)^2+x_2^2 = 2$$

Probleme de programare convexa

$$\min_{x} x_2$$
$$s.l.\ (x_1 - 1)^2 + x_2^2 \leq 2,$$
$$(x_1 + 1)^2 + x_2^2 \leq 2$$

Cazul $\boldsymbol{\mu_1 > 0, \mu_2 > 0}$

C.N.: $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\mu_1 \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} + 2\mu_2 \begin{bmatrix} x_1 + 1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2}, x_2 = -\frac{1}{\mu_1 + \mu_2}$

$$(x_1 - 1)^2 + x_2^2 = 2 \quad \Rightarrow \mu_1 = \mu_2 = \frac{1}{2}$$
$$(x_1 + 1)^2 + x_2^2 = 2$$

Probleme de programare convexa

$$\min_{x} x_2$$
$$s.l.\ (x_1-1)^2+x_2^2 \le 2,$$
$$(x_1+1)^2+x_2^2 \le 2$$

Cazul $\boldsymbol{\mu_1 > 0, \mu_2 > 0}$

C.N.: $\begin{bmatrix}0\\1\end{bmatrix} + 2\mu_1\begin{bmatrix}x_1-1\\x_2\end{bmatrix} + 2\mu_2\begin{bmatrix}x_1+1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow x_1 = \frac{\mu_2-\mu_1}{\mu_1+\mu_2}, x_2 = -\frac{1}{\mu_1+\mu_2}$

$$(x_1-1)^2+x_2^2 = 2 \quad \Rightarrow \mu_1 = \mu_2 = \frac{1}{2} \Rightarrow x_1 = 0, x_2 = -1$$
$$(x_1+1)^2+x_2^2 = 2$$

Dualitate

Functia Lagrangian: $L: R^n \times R^m \times R_+^p$

$$L(x, \lambda, \mu) = f(x) + \lambda^T(Ax - b) + \mu^T h(x)$$

Dualitate

Functia Lagrangian: $L: R^n \times R^m \times R_+^p$

$$L(x, \lambda, \mu) = f(x) + \lambda^T(Ax - b) + \mu^T h(x)$$

Functia duala: $\phi(\lambda, \mu) = \ min_x \, L(x, \lambda, \mu)$ [$L(x, \lambda, \mu)$ convexa in x!]

Dualitate

Functia Lagrangian: $L: R^n \times R^m \times R_+^p$
$$L(x, \lambda, \mu) = f(x) + \lambda^T(Ax - b) + \mu^T h(x)$$

Functia duala: $\phi(\lambda, \mu) = \ min_x\, L(x, \lambda, \mu)$ [$L(x, \lambda, \mu)$ convexa in x!]

Fie $\tilde{\mathrm{x}}, \mu$ fezabil, observam $\phi(\lambda, \mu) \leq L(\tilde{x}, \lambda, \mu) \leq f(\tilde{x})$ pentru orice $\lambda \in R^m, \mu \geq 0$

In particular, $\max \phi(\lambda, \mu) = \phi^* \leq f(x^*) = f^*$

Dualitate

Functia Lagrangian: $L: R^n \times R^m \times R_+^p$

$$L(x, \lambda, \mu) = f(x) + \lambda^T(Ax - b) + \mu^T h(x)$$

$$L(x, (1-\alpha)\lambda_1 + \alpha\lambda_2, (1-\alpha)\mu_1 + \alpha\mu_2) = f(x) + \left((1-\alpha)\lambda_1 + \alpha\lambda_2\right)^T (Ax - b) + \left((1-\alpha)\mu_1 + \alpha\mu_2\right)^T h(x)$$
$$= f(x) + (1-\alpha)\lambda_1^T(Ax - b) + \alpha{\lambda_2}^T(Ax - b) + (1-\alpha)\mu_1^T h(x) + \alpha{\mu_2}^T h(x)$$
$$= (1-\alpha)f(x) + \alpha f(x) + (1-\alpha)\lambda_1^T(Ax - b) + \alpha{\lambda_2}^T(Ax - b) + (1-\alpha)\mu_1^T h(x) + \alpha{\mu_2}^T h(x)$$
$$= (1-\alpha)[f(x) + \lambda_1^T(Ax - b) + \mu_1^T h(x)] + \alpha\left[f(x) + {\lambda_2}^T(Ax - b) + {\mu_2}^T h(x)\right]$$
$$= (1-\alpha)L(x, \lambda_1, \mu_1) + \alpha L(x, \lambda_2, \mu_2)$$

$$\boldsymbol{min\ f_1(x) + f_2(x) = f_1(x^*) + f_2(x^*) \geq f_1^* + f_2^*}$$

Functia duala: $\phi(\lambda, \mu) = \ min_x\, L(x, \lambda, \mu)$ [$L(x, \lambda, \mu)$ convexa in x!]

$$\phi\left((1-\alpha)\lambda_1 + \alpha\lambda_2, (1-\alpha)\mu_1 + \alpha\mu_2\right) = \min_x L(x, (1-\alpha)\lambda_1 + \alpha\lambda_2, (1-\alpha)\mu_1 + \alpha\mu_2)$$
$$= \min_x (1-\alpha)L(x, \lambda_1, \mu_1) + \alpha L(x, \lambda_2, \mu_2)$$
$$\geq (1-\alpha)\min_x L(x, \lambda_1, \mu_1) + \alpha \min_x L(x, \lambda_2, \mu_2)$$
$$= (1-\alpha)\phi(\lambda_1, \mu_1) + \alpha\, \phi(\lambda_2, \mu_2)$$

Dualitate

Functia Lagrangian: $L: R^n \times R^m \times R_+^p$
$$L(x, \lambda, \mu) = f(x) + \lambda^T(Ax - b) + \mu^T h(x)$$

Functia duala: $\phi(\lambda, \mu) = \ min_x \, L(x, \lambda, \mu)$ [$L(x, \lambda, \mu)$ convexa in x!]

- Fie x̃ fezabil, observam $\phi(\lambda, \mu) \leq L(\tilde{x}, \lambda, \mu) \leq f(\tilde{x})$ pentru orice $\lambda \in R^m, \mu \geq 0$
- Functia duala ϕ este **functie concava**!

Dualitate

Functia Lagrangian: $L: R^n \times R^m \times R_+^p$

$$L(x, \lambda, \mu) = f(x) + \lambda^T g(x) + \mu^T h(x)$$

Functia duala: $\phi(\lambda, \mu) = \ min_x \, L(x, \lambda, \mu)$

Problema duala: $$\max_{\mu \geq 0, \lambda} \phi(\lambda, \mu) = -\min_{\mu \geq 0, \lambda} -\phi(\lambda, \mu)$$

- Solutiile problemei duale: μ^*, λ^* reprezinta multiplicatorii Lagrange!
- Soluta problemei primale: x^* se numeste solutie primala!

Dualitate

Teorema dualitate. Pentru oricare $\boldsymbol{x}, \boldsymbol{\mu}$ fezabili avem:

$$f(x) \geq \phi(\lambda, \mu).$$

Daca x^* minim regulat si (λ^*, μ^*) multiplicatori Lagrange, atunci

$$f^* = \phi^* \text{ (}\textbf{Dualitate tare}\text{)}$$

- **Sub regularitate, $\boldsymbol{f}$ si $\boldsymbol{\phi}$ au minime, iar valorile optime coincid.**
- **Mai mult, rezolvarea problemei duale poate mai simpla decat primala!**

Dualitate-exemple

$$\min_x c^T x$$
$$s.l. \left||x|\right|^2 \leq 1$$

Dualitate-exemple

$$\min_x c^T x$$
$$s.l. \left||x|\right|^2 \leq 1$$

Functia Lagrangian: $L(x, \mu) = c^T x + \mu \left(\left||x|\right|^2 - 1\right) \Rightarrow \nabla_{xx}^2 L(x, \mu) = 2\mu I_n \geq 0$

Functia duala: $\phi(\mu) = \min_x L(x, \mu)$

Dualitate-exemple

$$\min_x c^T x$$
$$s.l. \left|\left|x\right|\right|^2 \leq 1$$

Functia Lagrangian: $L(x, \mu) = c^T x + \mu \left(\left|\left|x\right|\right|^2 - 1\right)$

Functia duala: $\phi(\mu) = \min_x L(x, \mu)$

$$x(\mu) = arg\min_x c^T x + \mu \left(\left|\left|x\right|\right|^2 - 1\right) \Rightarrow c + 2\mu x^*(\mu) = 0$$

Dualitate-exemple

$$\min_x c^T x$$
$$s.l. \left||x|\right|^2 \leq 1$$

Functia Lagrangian: $L(x, \mu) = c^T x + \mu \left(\left||x|\right|^2 - 1\right)$

Functia duala: $\phi(\mu) = \min_x L(x, \mu)$

$$x(\mu) = arg\min_x c^T x + \mu \left(\left||x|\right|^2 - 1\right) \Rightarrow c + 2\mu x(\mu) = 0$$

Deci: $x^*(\mu) = -\frac{1}{2\mu} c \Rightarrow \phi(\mu) = L(x^*(\mu), \mu) = c^T x(\mu) + \mu \left(\left||x(\mu)|\right|^2 - 1\right)$

Dualitate-exemple

$$\min_x c^T x$$
$$s.l. \left||x|\right|^2 \leq 1$$

Functia Lagrangian: $L(x, \mu) = c^T x + \mu \left(\left||x|\right|^2 - 1 \right)$

Functia duala: $\phi(\mu) = \min_x L(x, \mu)$

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Deci: $x(\mu) = -\frac{1}{2\mu} c \Rightarrow \phi(\mu) = L(x(\mu), \mu) = c^T x(\mu) + \mu \left(\left||x(\mu)|\right|^2 - 1 \right)$

$$\phi(\mu) = -\frac{1}{4\mu} \left||c|\right|^2 - \mu \Rightarrow \mu^* = \frac{\left||c|\right|}{2}$$

Dualitate - exemple

$$\min_{x} c^T x$$
$$s.l. \left|\left|x\right|\right|^2 \leq 1$$

Problema duala:

$$\max_{\mu \geq 0} -\frac{1}{4\mu}\left|\left|c\right|\right|^2 - \mu$$

Dualitate-exemple

$$\min_{x} \frac{1}{2}||Ax - b||^2$$
$$s.l. ||x||^2 \leq 1$$

Dualitate-exemple

$$\min_x \frac{1}{2}\left||Ax - b|\right|^2$$
$$s.l. \left||x|\right|^2 \leq 1$$

F.L.: $L(x,\mu) = \frac{1}{2}\left||Ax - b|\right|^2 + \mu\left(\left||x|\right|^2 - 1\right) \Rightarrow \nabla^2 L(x,\mu) = A^T A + 2\mu I_n > 0$

Functia duala: $\phi(\mu) = \min_x L(x,\mu)$

Dualitate-exemple

$$\min_x \frac{1}{2}\left|\left|Ax - b\right|\right|^2$$
$$s.l. \left|\left|x\right|\right|^2 \leq 1$$

Functia Lagrangian: $L(x, \mu) = c^T x + \mu\left(\left|\left|x\right|\right|^2 - 1\right)$

Functia duala: $\phi(\mu) = \min_x L(x, \mu)$

$$x(\mu) = arg\min_x \frac{1}{2}\left|\left|Ax - b\right|\right|^2 + \mu\left(\left|\left|x\right|\right|^2 - 1\right) \Rightarrow A^T(Ax(\mu) - b) + 2\mu x(\mu) = 0$$

Dualitate-exemple

$$\min_x \frac{1}{2}\left||Ax-b|\right|^2$$
$$s.l.\left||x|\right|^2 \leq 1$$

Functia Lagrangian: $L(x,\mu) = \frac{1}{2}\left||Ax-b|\right|^2 + \mu\left(\left||x|\right|^2 - 1\right)$

Functia duala: $\phi(\mu) = \min_x L(x,\mu)$

$$x(\mu) = arg\min_x \frac{1}{2}\left||Ax-b|\right|^2 + \mu\left(\left||x|\right|^2 - 1\right) \Rightarrow A^T(Ax(\mu)-b) + 2\mu x(\mu) = 0$$
$$(A^TA + 2\mu I_n)x(\mu) = A^Tb$$

Deci: $x(\mu) = (A^TA + 2\mu I_n)^{-1}A^Tb \Rightarrow \phi(\mu) = L(x(\mu),\mu) = \frac{1}{2}\left||Ax(\mu)-b|\right|^2 + \mu\left(\left||x(\mu)|\right|^2 - 1\right)$

Dualitate-exemple

$$\min_x \frac{1}{2}\left\|Ax - b\right\|^2$$
$$s.l. \left\|x\right\|^2 \leq 1$$

Functia Lagrangian: $L(x,\mu) = \frac{1}{2}\left\|Ax - b\right\|^2 + \mu\left(\left\|x\right\|^2 - 1\right)$

Functia duala: $\phi(\mu) = \min_x L(x,\mu)$

$$\phi(\mu) = -\frac{1}{2}b^T A(A^T A + 2\mu I_n)^{-1}A^T b - \mu c + \frac{1}{2}\left\|b\right\|^2$$

Problema duala: $\max_{\mu \geq 0} -\frac{1}{2}b^T A(A^T A + 2\mu I_n)^{-1}A^T b - \mu c + \frac{1}{2}\left\|b\right\|^2$

Probleme de programare neliniara

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Notam $I(x) = \{i: h_i(x) = 0\}$, indicii inegalitatilor **active**.

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Egalitatile sunt penalizate cand $||\lambda|| \to \infty$

Inegalitatile sunt penalizate cand $\mu \geq 0\ (\mu^T h(x) \geq 0)\ si\ ||\mu|| \to \infty$
$$h(x) = [h_1(x) \quad h_2(x)] = [(< 0) \quad (> 0)] \Rightarrow \mu^T h(x) = \mu_1 h_1(x) + \mu_2 h_2(x)$$

PN – Conditii necesare

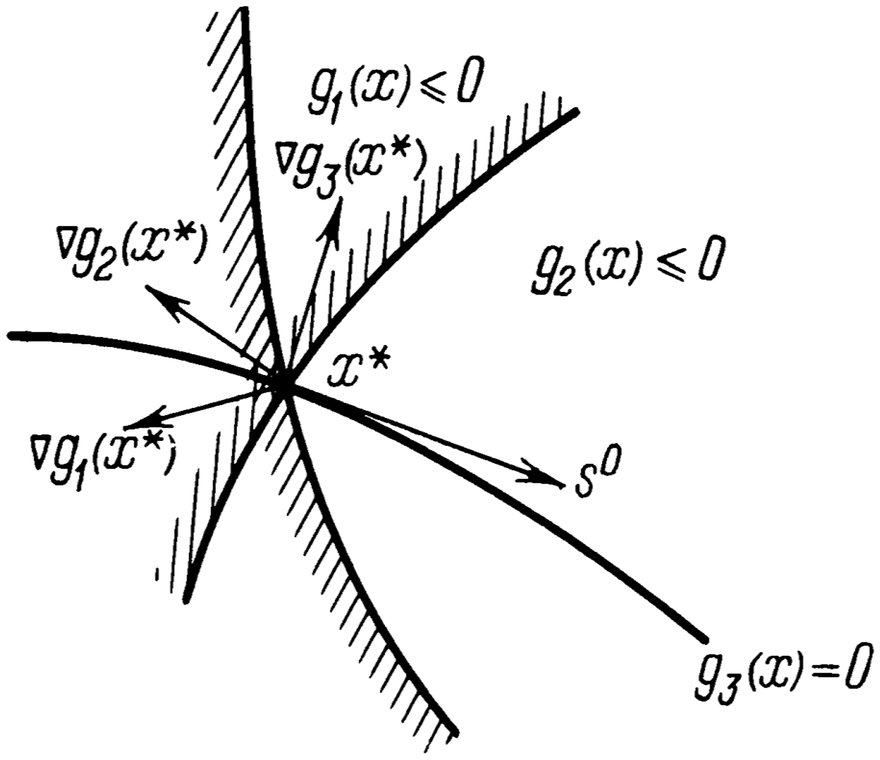
$$\min_x f(x)$$
$$s.l. \quad g_i(x) = 0, \quad i = 1, \cdots, m$$
$$h_i(x) \leq 0, \quad i = 1, \cdots, p$$

In cazul convex, conditia Slater este necesara pt C.O.

Conditie de regularitate: daca $\nabla g_i(x^*)$ linear ind. si
$\exists d$: $\nabla g_i(x^*)^T d = 0, \nabla h_i(x^*)^T d < 0, pt\ i \in I(x^*)$

atunci x^* este minim regulat.

(in figura se foloseste uniform g pt. toate constrangerile)



PN – Conditii necesare

Conditie de regularitate: daca $\nabla g_i(x^*)$ linear ind. si

$$\exists d\colon\ \nabla g_i(x^*)^T d = 0, \nabla h_i(x^*)^T d < 0, pt\ i \in I(x^*)$$

atunci x^* este minim regulat.

Exemplu: $h_1(x) = (x_1 + 1)^2 + x_2^2 - 1, h_2(x) = (x_1 - 1)^2 + x_2^2 - 1$

Fie $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Observam $\nabla \mathrm{h}_1(\mathrm{x}^*) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \nabla \mathrm{h}_2(\mathrm{x}^*) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, h(x^*) = 0$

$$\nabla h_1(x^*)^T d = d_1 < 0$$
$$\nabla h_2(x^*)^T d = -d_1 < 0$$

Constrangeri neregulate!

Probleme de programare neliniara

$$\min_{x} f(x)$$
$$s.l.\ g_i(x) = 0, \qquad i = 1, \cdots, m$$
$$h_i(x) \leq 0, \qquad i = 1, \cdots, p$$

Teorema [Conditii necesare KT]. Fie x^* minim regulat fezabil, atunci exista (λ^*, μ^*) a.i.

$$\nabla_{\mathrm{x}} \mathrm{L}(\mathrm{x}^*, \lambda^*, \mu^*) = 0, \qquad \mu^* \geq 0, \qquad \mu_i h_i(x^*) = 0$$

Probleme de programare neliniara

$$(PN): \qquad \min_x f(x) \quad s.l.\ g(x) = 0, h(x) \leq 0$$

Teorema [Conditii suficiente]. Fie x^* KT regulat fezabil si conditiile necesare in x^* au loc. Fie f, g, h dublu diferentiabile in x^*. Daca pentru orice d:

$$\nabla g_i(x^*)^T d = 0, \qquad i = 1, \dots, m$$
$$\nabla h_i(x^*)^T d = 0, \qquad i \in I(x^*), \mu_i^* > 0$$
$$\nabla h_i(x^*)^T d \geq 0, \qquad i \in I(x^*), \mu_i^* = 0$$

avem

$$\mathrm{d}^{\mathrm{T}} \nabla_{xx}^2 L(x^*, \lambda^*, \mu^*) d > 0$$

Atunci x^* minim local al problemei PN.

C.S. - Exemplu

$$\min_{x} x_2 \ \ s.l. \ \left||x|\right|^2 \leq 1$$

C.S. - Exemplu

$$\min_{x} x_2 \;\; s.l. \; ||x||^2 \leq 1$$

$$L(x, \mu) = x_2 + \mu\left(||x||^2 - 1\right)$$

C.S. - Exemplu

$$\min_{x} x_2 \;\; s.l. \; \left| |x| \right|^2 \leq 1$$

$$L(x,\mu) = x_2 + \mu \left(\left| |x| \right|^2 - 1 \right)$$

C.N.-KT : $\mu^* x_1^* = 0, \;\; 1 + 2\mu^* x_2^* = 0 \Rightarrow \mu^* > 0 \; (constrangere \; activa)$

Probleme de programare neliniara

$$\min_x x_2 \ \ s.l. \ \left||x|\right|^2 \leq 1$$

$$L(x,\mu) = x_2 + \mu\left(\left||x|\right|^2 - 1\right)$$

C.N.-KT : $\mu^* x_1^* = 0, \ 1 + 2\mu^* x_2^* = 0 \Rightarrow \mu^* > 0 \ (constrangere\ activa)$

Adica $(x_1^*)^2 + (x_2^*)^\wedge 2 = 1$. Solutie unica: $x^* = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mu^* = \frac{1}{2}$.

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Plan tangent: $\mathrm{T_Q}(\mathrm{x}^*) = \left\{d\!:\;\; [0\;\; -1]\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0\right\} = \{d\!:\;\; d_2 = 0\}$

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$$\nabla^2 L(x^*,\mu^*) = \begin{bmatrix} \mu^* & 0 \\ 0 & 2\mu^* \end{bmatrix} \Rightarrow d^T \nabla^2 L(x^*,\mu^*) d = \mu^* d_1^2 > 0 \; (d \neq 0)$$

Probleme de programare neliniara

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Concluzie: x^* minim local (global!).

Algoritmi

- Modelul convex: $\min_{x} f(x) \quad s.l.\ h(x) \le 0$
- $h_i(x) \ge h_i\left(x^k\right) + \nabla h_i\left(x^k\right)^T\left(x - x^k\right)$

Metoda Gradient Proiectat cu Liniarizare:

$$x^{k+1} = argmin_x\, \nabla f\left(x^k\right)^T\left(x - x^k\right) + \frac{1}{2\alpha_k}\left|\left|x - x^k\right|\right|^2$$

$$s.l.\ h_i\left(x^k\right) + \nabla h_i\left(x^k\right)^T\left(x - x^k\right) \le 0, \qquad \forall i$$

Algoritmi

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$$x^{k+1} = argmin_x \, \nabla f(x^k)^T (x - x^k) + \frac{1}{2\alpha_k} \left\| x - x^k \right\|^2$$

$$s.l. \;\; h_i(x^k) + \nabla h_i(x^k)^T (x - x^k) \le 0, \qquad \forall i$$

THEOREM 1. Let the set X^* of solutions of problem (1) be nonempty, let the functions $f(x)$, $g_i(x)$ be differentiable, let their gradients satisfy a Lipschitz condition, and let Slater's condition hold. Then we can find a $\bar{\gamma} > 0$ such that for $0 < \gamma < \bar{\gamma}$ method (7) converges to a point $x^* \in X^*$. If, also, $f(x)$ is strongly convex, then $\|x - x^*\| \le cq^k$, $0 \le q \le 1$. □

Algoritmi

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$$s.l. \;\; h_i\left(x^k\right) + \nabla h_i\left(x^k\right)^T\left(x - x^k\right) \leq 0, \qquad \forall i$$

La fiecare iteratie se rezolva:

$$\min_x \frac{1}{2}\left|\left|x \, - \left(x^k - \alpha_k \nabla f\left(x^k\right)\right)\right|\right|^2 = \frac{1}{2}\left|\left|x \, - y^k\right|\right|^2$$
$$s.l. \; Ax \leq b$$

unde $A = \begin{bmatrix} \nabla h_1\left(x^k\right)^T \\ \ldots \\ \nabla h_p\left(x^k\right)^T \end{bmatrix}, b = Ax^k - h(x^k)$

Conditia Slater are loc!

MPGL

$$\min_{x} \frac{1}{2}\left\|x - y^k\right\|^2$$
$$s.l.\ Ax \leq b$$
$$L(x,\mu) = \frac{1}{2}\left\|x - y^k\right\|^2 + \mu^T(Ax - b)$$
$$\nabla_x L(x,\mu) = 0 \Rightarrow x(\mu) = y^k - A^T\mu$$
$$\max_{\mu \geq 0} \phi(\mu) = \max_{\mu \geq 0} -\frac{1}{2}\left||A^T\mu|\right|^2 + \mu^T\left(Ay^k - b\right)\ (QP)$$

- MGP, MGC…