



Complete global stability for an SIR epidemic model with delay – Distributed or discrete

C. Connell McCluskey

Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, Canada

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ABSTRACT

SIR models with distributed delay and with discrete delay are studied. The global dynamics are fully determined for $\mathcal{R}_0 > 1$ by using a Lyapunov functional. For each model it is shown that the endemic equilibrium is globally asymptotically stable whenever it exists.

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1. Introduction

In modelling the transmission of an infectious disease, a common model structure involves dividing the population into susceptible, infectious and recovered individuals. If the immunity that is obtained upon recovery is permanent, then one gets an SIR model. In this paper we consider SIR models with mass action incidence and constant recruitment. We now review some works on related models.

In [6], an SIR model using ordinary differential equations was given. That model assumed that the infectivity of individuals was constant over the duration of infection. The model exhibits threshold behaviour where the disease dies out if a key parameter $\mathcal{R}_0 \leq 1$ and the disease limits to an endemic equilibrium if $\mathcal{R}_0 > 1$. In [4], a complete analysis of the global dynamics of an ordinary differential equation model with multiple infectious stages was presented, showing the same threshold behaviour.

In [7], an SIR model with a discrete delay was analyzed. The delay was used to model the fact that an individual may not be infectious until some time after becoming infected. This system also exhibits threshold behaviour. The disease dies out for $\mathcal{R}_0 \leq 1$, whereas it is permanent (survives) for $\mathcal{R}_0 > 1$. Additionally, it was shown that if $\mathcal{R}_0 > 1$ and the delay is small enough, then the endemic equilibrium is globally asymptotically stable. The global stability for larger delay remained an open problem. A solution to that problem is given in Section 5 of this paper, where it is shown that the endemic equilibrium is globally asymptotically stable whenever $\mathcal{R}_0 > 1$.

An SIR model with distributed delay was studied in [1,8,10]. The distributed delay allows infectivity to be a function of the duration since infection, up to some maximum duration. Once again, the model displays threshold dynamics. For $\mathcal{R}_0 \leq 1$, the disease dies out. It was shown in [8] that the disease is permanent for $\mathcal{R}_0 > 1$. In [1,10], conditions are found on the model parameters that guarantee that the endemic equilibrium is globally asymptotically stable for $\mathcal{R}_0 > 1$. These conditions would typically hold for small delays but not larger delays. The global stability when these conditions fail remained an open problem. A solution to that problem is given in Section 4 of this paper, where it is shown that the endemic equilibrium is globally asymptotically stable whenever $\mathcal{R}_0 > 1$.

E-mail address: ccmcc8@gmail.com.

The global attractivity proofs in this paper use Lyapunov functionals that are similar in nature to the one used in [9] in which the global dynamics are resolved for an SEIR model with infinite delay. In each case, the Lyapunov functional includes an integral over the prior history.

In this paper, an SIR model with distributed delay is given in Section 2. In Section 3, some results from the literature relating to earlier work on this model are given. Section 4 contains a proof of the global asymptotic stability of the endemic equilibrium for $\mathcal{R}_0 > 1$. In Section 5, an SIR model with discrete delay is presented and the endemic equilibrium is shown to be globally asymptotically stable for $\mathcal{R}_0 > 1$.

2. SIR model with distributed delay

A population is divided into three classes: susceptible, infectious and recovered, denoted by s , i and r , respectively. All recruitment is into the susceptible class at a constant rate b . The death rates for the classes are μ_1 , μ_2 and μ_3 , respectively. The average duration of infectiousness is $1/\lambda$. Infectiousness is assumed to vary over time from the initial time of infection until a duration h has passed; this variation is described by the function f which accounts for the relative infectivity, as well as the survival of individuals infected at earlier times until time t . The mass action coefficient is β and is chosen so that $\int_{\tau=0}^h f(\tau) d\tau = 1$. (It is assumed that f takes on non-zero values arbitrarily close to h , so that the interval of integration is not artificially extended by concluding with an interval for which the integrand is automatically zero.)

The model is given in [10] by the equations

$$\begin{aligned}\frac{ds(t)}{dt} &= b - \beta s(t) \int_{\tau=0}^h f(\tau) i(t-\tau) d\tau - \mu_1 s(t) \\ \frac{di(t)}{dt} &= \beta s(t) \int_{\tau=0}^h f(\tau) i(t-\tau) d\tau - (\mu_2 + \lambda) i(t)\end{aligned}\quad (2.1)$$

and

$$\frac{dr(t)}{dt} = \lambda i(t) - \mu_3 r(t).$$

Since the variable r does not appear in the equations for $\frac{ds}{dt}$ and $\frac{di}{dt}$, it is sufficient to analyze the behaviour of solutions to (2.1).

The initial condition for (2.1) is

$$s(0) \in \mathbb{R}_{\geq 0} \quad \text{and} \quad i_0 \in \mathcal{C}([-h, 0], \mathbb{R}_{\geq 0}).$$

Thus, the number of infectives is determined from time $-h$, allowing (2.1) to describe how the variables change for all $t \geq 0$.

3. Previous results

The basic reproduction number [3] for the model is

$$\mathcal{R}_0 = \frac{\beta b}{\mu_1(\mu_2 + \lambda)}.$$

For all parameter values, the disease-free equilibrium is given by

$$E_0 = \left(\frac{b}{\mu_1}, 0 \right).$$

If $\mathcal{R}_0 \leq 1$, then E_0 is the only equilibrium. If $\mathcal{R}_0 > 1$, then there is also a unique endemic equilibrium

$$E_* = (s^*, i^*) = \left(\frac{\mu_2 + \lambda}{\beta}, \frac{b - \mu_1 s^*}{\beta s^*} \right).$$

By noting that $s' > 0$ for $s = 0$ and $i \geq 0$, that $i' \geq -(\mu_2 + \lambda)i$ for $s, i \geq 0$ and that $(s + i)' = b - \mu_1 s - (\mu_2 + \lambda)i$, one may obtain the following result [8].

Proposition 3.1. *The region*

$$\Omega = \left\{ (s, i) \in \mathbb{R}_{\geq 0}^2 : s + i \leq \frac{b}{\bar{\mu}} \right\},$$

where $\bar{\mu} = \min\{\mu_1, \mu_2 + \lambda\}$, is positively invariant and is attracting.

We now state without proof some key results from [2,8,10], which provide the setting in which the main results of this paper (Sections 4 and 5) are to be enjoyed.

Theorem 3.2. Suppose $\mathcal{R}_0 \leq 1$. Then E_0 is globally attractive, and is locally asymptotically stable if the inequality is strict.

Theorem 3.3. Suppose $\mathcal{R}_0 > 1$. Then the E_0 is unstable and E_* is locally asymptotically stable.

Theorem 3.4. Suppose $\mathcal{R}_0 > 1$. Then the system is permanent; that is, there exists $\eta > 0$ such that for any solution for which the disease is initially present, there exists $T \geq 0$ such that for any $t \geq T$, we have

$$s(t) \geq \eta \quad \text{and} \quad i(t) \geq \eta.$$

4. Distributed delay—global asymptotic stability for $\mathcal{R}_0 > 1$.

The only issue to resolve, with regard to the long-term behaviour of Eq. (2.1), is the global dynamics for $\mathcal{R}_0 > 1$. We recall that the endemic equilibrium E^* exists if and only if $\mathcal{R}_0 > 1$.

Theorem 4.1. Suppose $\mathcal{R}_0 > 1$. Then E_* is globally asymptotically stable.

Proof. Evaluating both sides of (2.1) at E_* and recalling that $\int_{\tau=0}^h f(\tau) d\tau = 1$, gives

$$b = \beta s^* i^* + \mu_1 s^* \tag{4.1}$$

and

$$(\mu_2 + \lambda) i^* = \beta s^* i^*, \tag{4.2}$$

which will be used as substitutions in the calculation which follows. Let

$$\begin{aligned} g(y) &= y - 1 - \ln y \quad \text{and} \quad U_s(t) = g\left(\frac{s(t)}{s^*}\right) \\ U_i(t) &= g\left(\frac{i(t)}{i^*}\right) \\ U_+(t) &= \int_{\tau=0}^h \alpha(\tau) g\left(\frac{i(t-\tau)}{i^*}\right) d\tau, \end{aligned} \tag{4.3}$$

where

$$\alpha(\tau) = \int_{\sigma=\tau}^h f(\sigma) d\sigma.$$

We will study the behaviour of the Lyapunov functional

$$U(t) = \frac{1}{\beta i^*} U_s + \frac{1}{\beta s^*} U_i + U_+. \tag{4.4}$$

We note that $\alpha(\tau) > 0$ for each $\tau \in [0, h]$. Also, $g: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$ has the global minimum $g(1) = 0$. Thus, $U(t) \geq 0$ with equality if and only if $s(t) = i(t) = 1$ and $i(t - \tau) = 1$ for almost all $\tau \in [0, h]$.

By Proposition 3.1 and Theorem 3.4, solutions are bounded above and are bounded away from zero for large time. Without loss of generality, we may assume that the solution in question satisfies these bounds for all $t \geq 0$. Thus, $U(t)$ is defined (and finite) for all $t \geq 0$.

The derivatives of U_s , U_i and U_+ will be calculated separately and then combined to get the desired quantity $\frac{dU}{dt}$.

$$\begin{aligned} \frac{dU_s}{dt} &= \frac{1}{s^*} \left(\frac{s - s^*}{s} \right) \frac{ds}{dt} \\ &= \frac{1}{s^*} \left(\frac{s - s^*}{s} \right) \left(b - \beta s \int_{\tau=0}^h f(\tau) i(t - \tau) d\tau - \mu_1 s \right). \end{aligned}$$

Using (4.1) to replace b gives

$$\begin{aligned} \frac{dU_s}{dt} &= \frac{1}{s^*} \left(\frac{s - s^*}{s} \right) \left(\beta \int_{\tau=0}^h f(\tau) (s^* i^* - s i(t - \tau)) d\tau + \mu_1 (s^* - s) \right) \\ &= -\frac{\mu_1}{s^*} \frac{(s - s^*)^2}{s} + \beta i^* \int_{\tau=0}^h f(\tau) \left(1 - \frac{s}{s^*} \right) \left(1 - \frac{s}{s^*} \frac{i(t - \tau)}{i^*} \right) d\tau. \end{aligned}$$

Defining $x = s(t)/s^*$, $y = i(t)/i^*$ and $z = i(t - \tau)/i^*$, we obtain

$$\frac{dU_s}{dt} = -\frac{\mu_1}{s^*} \frac{(s - s^*)^2}{s} + \beta i^* \int_{\tau=0}^h f(\tau) \left(1 - \frac{1}{x} - xz + z \right) d\tau. \quad (4.5)$$

Next we calculate $\frac{dU_i}{dt}$.

$$\begin{aligned} \frac{dU_i}{dt} &= \frac{1}{i^*} \left(1 - \frac{i^*}{i} \right) \left(\beta s \int_{\tau=0}^h f(\tau) i(t - \tau) d\tau - (\mu_2 + \lambda) i \right) \\ &= \frac{1}{i^*} \left(1 - \frac{i^*}{i} \right) \left(\beta s^* i^* \int_{\tau=0}^h f(\tau) \frac{s}{s^*} \frac{i(t - \tau)}{i^*} d\tau - (\mu_2 + \lambda) i^* \frac{i}{i^*} \right). \end{aligned}$$

Using (4.2) to replace $(\mu_2 + \lambda) i^*$ gives

$$\begin{aligned} \frac{dU_i}{dt} &= \frac{1}{i^*} \left(1 - \frac{i^*}{i} \right) \beta s^* i^* \int_{\tau=0}^h f(\tau) \left(\frac{s}{s^*} \frac{i(t - \tau)}{i^*} - \frac{i}{i^*} \right) d\tau \\ &= \beta s^* \int_{\tau=0}^h f(\tau) \left(xz - y - \frac{xz}{y} + 1 \right) d\tau. \end{aligned} \quad (4.6)$$

We now calculate the derivative of $U_+(t)$.

$$\begin{aligned} \frac{dU_+}{dt} &= \frac{d}{dt} \int_{\tau=0}^h \alpha(\tau) g \left(\frac{i(t - \tau)}{i^*} \right) d\tau \\ &= \int_{\tau=0}^h \alpha(\tau) \frac{d}{dt} g \left(\frac{i(t - \tau)}{i^*} \right) d\tau \\ &= - \int_{\tau=0}^h \alpha(\tau) \frac{d}{d\tau} g \left(\frac{i(t - \tau)}{i^*} \right) d\tau. \end{aligned}$$

Using integration by parts, we get

$$\frac{dU_+}{dt} = -\alpha(\tau) g \left(\frac{i(t - \tau)}{i^*} \right) \Big|_{\tau=0}^h + \int_{\tau=0}^h \frac{d}{d\tau} \alpha(\tau) g \left(\frac{i(t - \tau)}{i^*} \right) d\tau. \quad (4.7)$$

Observe that $\alpha(h) = 0$ and $\alpha(0) = \int_{\sigma=0}^h f(\sigma) d\sigma$. Also, note that $\frac{d}{d\tau} \alpha(\tau) = -f(\tau)$. Filling into (4.7) and rearranging yields

$$\begin{aligned} \frac{dU_+}{dt} &= \int_{\sigma=0}^h f(\sigma) d\sigma g \left(\frac{i}{i^*} \right) - \int_{\tau=0}^h f(\tau) g \left(\frac{i(t - \tau)}{i^*} \right) d\tau \\ &= \int_{\tau=0}^h f(\tau) \left(g \left(\frac{i}{i^*} \right) - g \left(\frac{i(t - \tau)}{i^*} \right) \right) d\tau. \end{aligned} \quad (4.8)$$

Combining Eqs. (4.5), (4.6) and (4.8), multiplying appropriately by coefficients determined by (4.4), we obtain

$$\frac{dU}{dt} = -\frac{\mu_1}{\beta s^* i^*} \frac{(s - s^*)^2}{s} - \int_{\tau=0}^h f(\tau) C(\tau) d\tau,$$

where

$$\begin{aligned} C(\tau) &= -2 + \frac{1}{x} + \frac{xz}{y} + \ln y - \ln z \\ &= \left(\frac{1}{x} - 1 - \ln \frac{1}{x} \right) + \left(\frac{xz}{y} - 1 - \ln \frac{xz}{y} \right) \\ &= g \left(\frac{1}{x} \right) + g \left(\frac{xz}{y} \right) \\ &\geq 0. \end{aligned}$$

Thus, $\frac{dU}{dt} \leq 0$. By Theorem 5.3.1 of [5], solutions limit to M , the largest invariant subset of $\left\{ \frac{dU}{dt} = 0 \right\}$. We now show that M consists of only the endemic equilibrium.

Note that $C(\tau) = 0$ if and only if $x = 1$ and $xz/y = 1$, or equivalently, if and only if $s(t) = s^*$ and $i(t) = i(t - \tau)$. Thus, $\frac{dU}{dt} = 0$ if and only if

$$s(t) = s^* \quad \text{and} \quad f(\tau) i(t) = f(\tau) i(t - \tau) \quad (4.9)$$

for almost all $\tau \in [0, h]$. For each element of M , we have $s(t) = s^*$ and, since M is invariant, $\frac{ds}{dt} = 0$. Using (4.9) we obtain

$$\begin{aligned} 0 &= \frac{ds}{dt} \\ &= b - \beta s^* \int_{\tau=0}^h f(\tau) i(t - \tau) d\tau - \mu_1 s^* \\ &= b - \beta s^* \int_{\tau=0}^h f(\tau) i(t) d\tau - \mu_1 s^*. \end{aligned}$$

Rearranging gives

$$\begin{aligned} i(t) &= \frac{b - \mu_1 s^*}{\beta s^*} \\ &= i^*. \end{aligned}$$

Thus, each element of M satisfies $s(t) = s^*$ and $i(t) = i^*$ for all t . This shows that $\lim_{t \rightarrow \infty} (s(t), i(t)) = (s^*, i^*)$. Since, by Theorem 3.3, $E_* = (s^*, i^*)$ is locally asymptotically stable, it now follows that E_* is globally asymptotically stable. ■

5. Discrete delay—global asymptotic stability for $\mathcal{R}_0 > 1$.

The same approach that worked for the distributed delay model (2.1) also works for a discrete delay model found in [7]. The model equations are

$$\begin{aligned} \frac{ds(t)}{dt} &= b - \beta s(t) i(t - h) - \mu_1 s(t) \\ \frac{di(t)}{dt} &= \beta s(t) i(t - h) - (\mu_2 + \lambda) i(t) \\ \frac{dr(t)}{dt} &= \lambda i(t) - \mu_3 r(t). \end{aligned} \tag{5.1}$$

All of the results given in Section 3 for Eq. (2.1) have analogous results for Eq. (5.1) which can be found in [7]. We note that the endemic equilibrium exists if and only if $\mathcal{R}_0 > 1$. The global stability result in Theorem 4.1 has the following analog.

Theorem 5.1. Suppose $\mathcal{R}_0 > 1$. Then E_* is globally asymptotically stable.

The proof is similar to the proof of Theorem 4.1, but with the Lyapunov functional

$$V = \frac{1}{\beta i^*} U_s + \frac{1}{\beta s^*} U_i + U_-,$$

where U_s and U_i are given by Eq. (4.3), and

$$U_- = \int_{\tau=0}^h g \left(\frac{i(t - \tau)}{i^*} \right) d\tau.$$

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