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Global asymptotic properties of a delay *SIR* epidemic model with finite incubation times

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1. Introduction to *SIR* model with incubation times

In this section we present the equations for epidemics (e.g. malaria) spread by vectors (mosquitoes) which have an incubation time to become infectious. The model will give rise to a system of distributed delay differential equations.

A model for the spread of an infectious disease (involving only susceptibles and infective individuals) transmitted by a vector (e.g. mosquitoes) after an incubation time, was proposed by Cooke [6]. Extensions of his model were considered by Busenberg and Cooke [5], Marcati and Pozio [11] and Volz [13]. Thieme [12] has also considered the model by Cooke, extending the model by continuously stratifying the population according to the average frequency of the bites by a mosquito (the vector). The present model still can be considered an extension of Cooke's model. We assume that:

(A) Human population is described by an *SIR* model with vital dynamics where the total size N of human population is constant (see [8]). Hence the death constant rate

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μ is the same for the classes S (susceptible), I (infectious), R (removed) individuals and equal to the birth rate b. All newborns are susceptibles.

(B) The vector (e.g. mosquitoes) is divided into susceptibles V_s and infectious vectors V_i. When a susceptible vector (V_s) is infected by a person (I), there is a time "τ", $\tau > 0$, during which the infectious agents develop in the vector, and only after that time the infected vector becomes itself infectious (V_i).

(C) Susceptible individuals (S) receive the infection from infectious vector (V_i). We assume a spatially homogeneous mixing among vector and human populations.

(D) The turnover of vector population is very high and the total size of vector population "V_T" is assumed to be constant and largely exceeding the human total size (i.e. $\varepsilon = N/V_T \ll 1$). Hence, death and birth constant rates must be equal and, once denoted by "δ", this must be "large" in the sense we are specifying in the following.

According to (A)–(D) assumptions the human population follows the *SIR* model with vital dynamics:

$$\begin{aligned}
\frac{dS(t)}{dt} &= -\tilde{\beta}S(t)V_i(t) - \tilde{\mu}S(t) + \tilde{\mu}N,\\
\frac{dI(t)}{dt} &= \tilde{\beta}S(t)V_i(t) - \tilde{\mu}I(t) - \tilde{\lambda}I(t),\\
\frac{dR(t)}{dt} &= \tilde{\lambda}I(t) - \tilde{\mu}R(t),
\end{aligned} \tag{1.1}$$

where $\tilde{\beta}$ is the constant rate constant between susceptible humans and infectious vectors, $\tilde{\mu}$ is birth and death constant rate and $\tilde{\lambda}$ is the recovery rate. Of course $N(t) = S(t) + I(t) + R(t) = N(t_0)$ for any $t \geq t_0 \in \mathscr{R} = (-\infty, +\infty)$.

As far as the vector population is considered we assume two population classes, i.e. the susceptible vectors V_s and infectious vectors V_i. The "δ" is the birth and death constant rate for vectors, and we assume that all newborns are susceptibles. K is the constant rate constant between infected people and susceptible vectors. Hence

$$\begin{aligned}
\frac{dV_s(t)}{dt} &= -Ke^{-\delta\tau}I(t-\tau)V_s(t-\tau) - \delta V_s(t) + \delta(V_s(t) + V_i(t)),\\
\frac{dV_i(t)}{dt} &= Ke^{-\delta\tau}I(t-\tau)V_s(t-\tau) - \delta V_i(t).
\end{aligned} \tag{1.2}$$

Owing to (1.2) we see that $V_T(t) = V_s(t) + V_i(t) = V_T(t_0)$ for all $t \geq t_0$ ($t_0 \in \mathscr{R}$), i.e., the total vector population is constant in time. Hence, according to (D) we have

$$\varepsilon = \frac{N}{V_T} \ll 1. \tag{1.3}$$

Let us set in dimensionless form the equations. We have two dimensionless time scales, say,

$$\begin{aligned}
&\sigma = KNt \text{ (slow scale) such that } \sigma = 1 \text{ at } t_\sigma = 1/KN,\\
&\rho = KV_T t \text{ (fast scale) such that } \rho = 1 \text{ at } t_\rho = 1/KV_T = \varepsilon t_\sigma,
\end{aligned} \tag{1.4}$$

which naturally arise, the first to rescale the human variables on N (constant) and the second to rescale the vector variables on V_T (constant). The nomenclature in (1.4) is justified from the fact that if we assume, say, $\varepsilon = 10^{-4} \div 10^{-3}$, then one time unit of σ scale t_σ is about $10^{-3} \div 10^{-4}$ the time unit of ρ scale t_ρ:

$$t_\sigma = (1/\varepsilon)t_\rho = (10^{-3} \div 10^{-4})t_\rho. \tag{1.5}$$

According to above, denote by

$$s(t) = S(t)/N, \quad i(t) = I(t)/N, \quad r(t) = R(t)/N \tag{1.6}$$

the dimensionless human variables, and by

$$v_i(t) = V_i(t)/V_T, \quad v_s(t) = V_s(t)/V_T \tag{1.7}$$

the dimensionless vector variables.

Accordingly, the dimensionless vector equations become

$$\begin{aligned} \frac{dv_i}{d\rho} &= \varepsilon\left(e^{-\delta\tau} i(t-\tau)v_s(t-\tau) - \frac{\delta}{KN} v_i(t)\right), \\ \frac{dv_s}{d\rho} &= -\frac{dv_i(t)}{d\rho}, \end{aligned} \tag{1.8}$$

where it is suitable to remind that non-negativity implies $v_i, v_s \geq 0$ and $s, i, r \geq 0$ for all $t \geq t_0$. Furthermore, owing to (1.6), (1.7) and constant size both total human and vector populations, it follows that $v_i + v_s = 1$, $s + i + r = 1$ for all $t \geq t_0$. Then from (1.8) it is easy to check that

$$-\frac{\varepsilon\delta}{KN} \leq \frac{dv_i}{d\rho} \leq \varepsilon e^{-\delta\tau}, \quad \varepsilon = \frac{N}{V_T}. \tag{1.9}$$

On the fast time scale "ρ", if $\varepsilon = N/V_T \to 0$, from (1.9) and the second of (1.8), we can assume that

$$\frac{dv_i}{d\rho} = -\frac{dv_s}{d\rho} = 0, \tag{1.10}$$

i.e. v_i, v_s attain their equilibrium values:

$$\begin{aligned} v_i(t) &= \frac{e^{-\delta\tau}}{\delta} KNi(t-\tau)v_s(t-\tau), \\ v_i(t) &+ v_s(t) = 1. \end{aligned} \tag{1.11}$$

Hence, from (1.11) we get

$$v_s(t) = \frac{1}{1 + (KN/\delta)e^{-\delta\tau} i(t-\tau)} \simeq 1, \tag{1.12}$$

i.e. $V_s(t) \simeq V_T$, if the death–birth constant rate "δ" for vectors is sufficiently large to ensure $KNe^{-\delta\tau}/\delta \ll 1$, i.e.

$$\delta e^{\delta\tau} \gg KN. \tag{1.13}$$

Under this assumption it becomes reasonable to assume $v_s(t) \simeq 1$ in (1.11) and therefore we get

$$v_i(t) \simeq \frac{KN}{\delta} e^{-\delta\tau} i(t-\tau)$$

or equivalently

$$V_i(t) \simeq \frac{KV_T}{\delta} e^{-\delta\tau} I(t-\tau), \tag{1.14}$$

where $I(t-\tau)$, evolving on the "slow" time scale "σ", can be assumed "constant" on the fast scale "ρ". According to (1.6) and (1.14), if we put in dimensionless variables Eq. (1.1) on the slow time scale "σ", we get

$$\begin{aligned}
\frac{ds(\sigma)}{d\sigma} &= -\beta s(\sigma) i(\sigma-\tau_\sigma) - \mu s(\sigma) + \mu,\\
\frac{di(\sigma)}{d\sigma} &= \beta s(\sigma) i(\sigma-\tau_\sigma) - \mu i(\sigma) - \lambda i(\sigma),\\
\frac{dr(\sigma)}{d\sigma} &= \lambda i(\sigma) - \mu r(\sigma),
\end{aligned} \tag{1.15}$$

where, of course, $s(\sigma)+i(\sigma)+r(\sigma)=1$ for all σ. Furthermore, the parameters appearing in (1.15) are defined as

$$\beta = \frac{\tilde{\beta} V_T e^{-\delta\tau}}{\delta}, \qquad \mu = \frac{\tilde{\mu}}{KN}, \qquad \lambda = \frac{\tilde{\lambda}}{KN}. \tag{1.16}$$

From now on we consider system (1.15) evolving on the slow time scale "σ". However, to avoid heavy notation, σ will be substituted by "t" and τ_σ by τ, keeping in mind that "t" stands for slow time scale "σ". Accordingly we rewrite (1.15) as

$$\begin{aligned}
\frac{ds(t)}{dt} &= -\beta s(t) i(t-\tau) - \mu s(t) + \mu,\\
\frac{di(t)}{dt} &= \beta s(t) i(t-\tau) - \mu i(t) - \lambda i(t),\\
\frac{dr(t)}{dt} &= \lambda i(t) - \mu r(t),
\end{aligned} \tag{1.17}$$

which represents an *SIR* model with epidemics spread via a vector with an incubation time "τ".

Still model (1.17) could be improved more realistically, assuming that the incubation time "τ" is not the same for whole vector population, but it is a distributed parameter over the interval $[0,h]$, where "h" ($h \in \mathscr{R}_+$) is the superior limit of incubation times "τ" in the vector population. Hence, in the model equations (1.17) we substitute the force of infection

$$\beta s(t) i(t-\tau) \tag{1.18}$$

with

$$\beta s(t) \int_0^h f(\tau) i(t-\tau)\, d\tau, \tag{1.19}$$

where $\int_0^h f(\tau)\,d\tau = 1$. The "$f$", which is usually non-negative and continuous, is the distribution function of incubation times "τ" among the vectors. With this further assumption, the model equations becomes (see [2,3]):

$$\begin{aligned}
\frac{ds(t)}{dt} &= -\beta s(t)\int_0^h f(\tau)i(t-\tau)\,d\tau - \mu s(t) + \mu,\\
\frac{di(t)}{dt} &= \beta s(t)\int_0^h f(\tau)i(t-\tau)\,d\tau - \mu i(t) - \lambda i(t),\\
\frac{dr(t)}{dt} &= \lambda i(t) - \mu r(t).
\end{aligned} \tag{1.20}$$

System (1.20) needs new suitable initial conditions. Denote by $x_t=(s(t+\theta), i(t+\theta), r(t+\theta)), \theta\in[-h,0]$, and by $\mathscr{C}([-h,0],\Omega)$, the set of continuous functions $\phi : [-h,0]\to\Omega$ with supremum norm, where

$$\Omega = \{x\in\mathscr{R}^3_{+0}\,|\,x_1+x_2+x_3\le 1\}. \tag{1.21}$$

Furthermore, by $\mathscr{C}_\delta$ $(\delta>0)$, we mean the subset of $\mathscr{C}$ such that

$$\mathscr{C}_\delta = \{\phi\in\mathscr{C}\,|\,\|\phi\|<\delta\}. \tag{1.22}$$

At any $t_0\in\mathscr{R}$ the initial conditions for (1.20) are

$$x_{t_0} = \phi(\theta),\quad \theta\in[-h,0],\quad \phi\in\mathscr{C}([-h,0]). \tag{1.23}$$

We remark that in deviation of (1.20) we have neglected the possibility of a latency class for the humans, i.e. for the sake of simplicity, we do not account for any latency time, say "T", from the instant "$t-T$" at which a susceptible receives the infection from an infectious vector and instant "t" at which becomes itself infectious. This could be a further improvement of the model.

2. Summary of the results

Eq. (1.20) with initial conditions (1.23) have been studied by Beretta and Takeuchi [3]. In the following we recall the main results. The equilibria of (1.20) are the "disease-free" equilibrium

$$E_0 = (s^*=1,\ i^*=0,\ r^*=0), \tag{2.1}$$

which is feasible for all parameter values, and the "endemic" or positive equilibrium

$$E_+ = \left(s^*=\frac{\mu+\lambda}{\beta},\ i^*=\frac{\mu(1-s^*)}{\beta s^*},\ r^*=\frac{\lambda}{\mu}i^*\right) \tag{2.2}$$

which exists provided that $s^*=(\mu+\lambda)/\beta<1$. At $s^*=1$, i.e. $\mu+\lambda=\beta$, E_+ coincides with the disease free equilibrium E_0.

The local stability results have been obtained by coupling the technique of Liapunov functional with characteristic equation, are as follows:

Theorem 2.1. *Whenever E_+ of system* (1.20) *exists, it is locally asymptotically stable.*

Theorem 2.2. *A necessary and sufficient condition for E_0 of system* (1.20) *to be asymptotically stable is*

$$\beta < \mu + \lambda, \tag{2.3}$$

i.e. E_+ is not feasible.

To study the global properties of (1.20), was considered a reduced space, say only the variables (s,i), since $s(t)+i(t)+r(t)=1$ for all $t\in\mathscr{R}$. Accordingly we define

$$\Omega = \{(s,i)\in\mathscr{R}^2_{+0} \,|\, s+i\le 1\},$$

which is the projection of (1.21) on the plane of the first two components. We obtained (see [3]):

Theorem 2.3. *E_+ of system* (1.20) *is globally asymptotically stable with respect to the set $\tilde{\Omega} = \{(s,i)\in\Omega \,|\, s < s^* + i^*\}$ if $(s(t_0), i(t_0))\in\tilde{\Omega}$ for all $t > t_0$.*

As far as E_0 is concerned we get:

Theorem 2.4. *Whenever $\beta < \mu + \lambda$, E_0 of system* (1.20) *is globally asymptotically stable with respect to Ω.*

Theorems 2.1 and 2.2 show that the incubation time τ distributed on $[0,h]$ does not affect the local behaviour of an *SIR* model without delays: in both cases E_0 is locally asymptotically stable whenever E_+ is not feasible, whereas if $\beta > \mu+\lambda$ ($s^* < 1$ and E_+ is feasible) the endemic equilibrium E_+ becomes locally asymptotically stable. As shown by Theorem 2.3, the effect of distributed delays seems to affect the global stability properties. Recently, Beretta and Takeuchi [3] further considered a simple "mathematical" generalization of *SIR* model (1.20) in which we have different death and birth rates:

$$\begin{aligned}
\frac{\mathrm{d}s(t)}{\mathrm{d}t} &= -\beta s(t)\int_0^h f(\tau)i(t-\tau)\,\mathrm{d}\tau - \mu_1 s(t) + b,\\
\frac{\mathrm{d}i(t)}{\mathrm{d}t} &= \beta s(t)\int_0^h f(\tau)i(t-\tau)\,\mathrm{d}\tau - \mu_2 i(t) - \lambda i(t),\\
\frac{\mathrm{d}r(t)}{\mathrm{d}t} &= \lambda i(t) - \mu_3 r(t).
\end{aligned} \tag{2.4}$$

Of course the original *SIR* model with incubation times corresponds to the particular case $\mu_j = \mu = b$, $j = 1,2,3$ in (2.4). By a biological meaning, the initial conditions of (2.4) are given as

$$s(t_0+\tau)=\varphi_1,\quad i(t_0+\tau)=\varphi_2,\quad r(t_0+\tau)=\varphi_3,\quad -h\le\tau\le 0, \tag{2.5}$$

where $t_0\in\mathscr{R}$, $\varphi=(\varphi_1,\varphi_2,\varphi_3)^{\mathrm{T}}\in C$ such that $\varphi_{\mathrm{i}}\ge 0$ and $\varphi_j(0)>0$ for $j=1,2,3$, and C denotes the Banach space $C([-h,0],\mathscr{R}^3)$ of continuous functions mapping the interval $[-h,0]$ into $\mathscr{R}^3$.

It is easy to check that the solution $(s(t),\ i(t), r(t))$ of (2.4) with the initial conditions (2.5) exists and is unique for all $t \geq t_0$ (see [7] or [9]). Also it is trivial that $s(t) > 0,\ i(t) > 0$ and $r(t) > 0$ for all $t \geq t_0$.

Clearly, for any parameters $h,\ \beta,\ b,\ \lambda,\ \mu_j,\ j = 1, 2, 3$, (2.4) always has a disease-free equilibrium

$$E_0 = \left(\frac{b}{\mu_1}, 0, 0\right). \tag{2.6}$$

If

$$\frac{b}{\mu_1} > s^* \equiv \frac{\mu_2 + \lambda}{\beta}, \tag{2.7}$$

then (2.4) also has an endemic equilibrium

$$E_+ = \left(s^* = \frac{\mu_2 + \lambda}{\beta},\ i^* = \frac{b - \mu_1 s^*}{\beta s^*},\ r^* = \frac{\lambda(b - \mu_1 s^*)}{\mu_3 \beta s^*}\right). \tag{2.8}$$

If we use $\beta s(t) i(t)$ as the force of infection at time t instead of $\beta s(t) \int_0^h f(\tau) i(t - \tau)\,\mathrm{d}\tau$, and assume that a population has a constant size and equal birth and death rates (that is, $s(t) + i(t) + r(t) = 1$ and $\mu_1 = \mu_2 = \mu_3 = b$), then, (2.4) is reduced to the plane system in $s(t)$ and $i(t)$, which was proposed and considered by Hethcote [8] in detail. In this case, it is shown that the following threshold theorem is true [8]. Define $\delta \equiv \beta/(\mu_2 + \lambda)$.

Theorem 2.5. (i) *If* $\delta \leq 1$, *then* $\lim_{t \to +\infty}(s(t), i(t)) = (1, 0)$, *i.e. the infective and the removed fractions approach zero, so that the entire population is eventually susceptible*;

(ii) *if* $\delta > 1$, *then* $\lim_{t \to +\infty}(s(t), i(t)) = (s^*, i^*)$, *i.e. the susceptible and infective fractions eventually approach a constant positive endemic value.*

The following results [3] are obtained for (2.4):

Theorem 2.6. (i) *The disease free equilibrium* E_0 *of system* (2.4) *is globally asymptotically stable whenever* $b/\mu_1 < s^* = (\mu_2 + \lambda)/\beta$ (*thus, the endemic equilibrium* E_+ *does not exist*);

(ii) *whenever* $b/\mu_1 > s^* = (\mu_2 + \lambda)/\beta$ (*thus, the endemic equilibrium* E_+ *exists*), E_+ *of system* (2.4) *is locally asymptotically stable. An attractive region of* E_+ *which is explicitly given by the parameters was also obtained*;

(iii) *if the average incubation time* $T \equiv \int_0^h \tau f(\tau)\,\mathrm{d}\tau$ *is small enough* (*more exactly, if* $T < (\beta \bar{s})^{-1}$), *then there exist some solution* $(s(t), i(t), r(t))$ *of* (2.4) *and some time* $\bar{t} \geq t_0$ *such that* $s(\bar{t}) \leq \bar{s}$, *where* $\bar{s} = [(\lambda + \mu_2)^2 + b\beta]/[\beta(\mu_1 + \mu_2 + \lambda)]$.

After the introduction of Section 1 in which are derived the model equations, the purpose of the remaining of this paper is to further consider the global attractivity of the disease-free equilibrium E_0 in the case of $b/\mu_1 = (\mu_2 + \lambda)/\beta$, and the global asymptotic stability of the endemic equilibrium E_+ of (2.4). The remaining of this paper is organized as follows.

In Section 3, it is shown that E_0 is globally attractive even for $b/\mu_1 = (\mu_2 + \lambda)/\beta$, based on the classical Liapunov–LaSalle invariance principle. It should be pointed out here that though the case of equality may not in general be realized by the parameters, the analysis on the global attractivity of E_0 is still important and interesting from the mathematical and biological points of view.

In Section 4, we consider the global asymptotic stability of the endemic equilibrium E_+ based on some difference inequality and the construction of Liapunov functionals. This problem actually was proposed as an open problem in [3,4]. Our results show that, while E_+ exists, it is still globally asymptotically stable as long as the delay h is short enough and βb is relatively large (or μ_1, μ_2 and λ are small enough). A brief discussion is included in the last section.

Since epidemics will increase the death rates of infectives and removed, it is natural biologically to assume that

$$\mu_1 \leq \min\{\mu_2, \mu_3\}.$$

We further assume that the function $f(\tau)$ is also positive on $[0, h]$.

3. Disease free equilibrium

It is known that the disease free equilibrium E_0 is globally asymptotically stable whenever $b/\mu_1 < s^* = (\mu_2 + \lambda)/\beta$ [4]. In this section, we shall further show that the following result is true.

Theorem 3.1. *The disease free equilibrium E_0 of system* (2.4) *is globally attractive whenever $b/\mu_1 \leq s^* = (\mu_2 + \lambda)/\beta$.*

Proof. See the proof in [4] for the strict inequality.

Let us consider the equality case. For any solution $(s(t), i(t), r(t))$ of (2.4), let us first consider the case (a): $s(t) > s^*$ for all $t \geq t_0$. In this case, from (2.4), we see that for all $t \geq t_0$,

$$\begin{aligned}\dot{s}(t) - \dot{s}^* + \dot{i}(t) + \dot{r}(t) &= -\mu_1(s(t) - s^*) - \mu_2 i(t) - \mu_3 r(t) \\ &\leq -\mu_1(s(t) - s^* + i(t) + r(t)).\end{aligned}$$

Thus,

$$\lim_{t\to+\infty} s(t) = s^*, \quad \lim_{t\to+\infty} i(t) = \lim_{t\to+\infty} r(t) = 0.$$

Now let us consider the case (b): $\varphi_1 < s^*$ and $s(t) < s^*$ for all $t \geq t_0$. Set

$$G = \{\varphi = (\varphi_1, \varphi_2, \varphi_3) \in C \mid 0 \leq \varphi_1 \leq s^*,\ \varphi_2 \geq 0,\ \varphi_3 \geq 0\}.$$

We define

$$V(\varphi) = \varphi_2(0) + \beta s^* \int_0^h f(\tau) \int_{-\tau}^0 \varphi_2(u)\,\mathrm{d}u\,\mathrm{d}\tau.$$

Then,

$$\dot{V}(\varphi)|_{(2.4)} = -\beta(s^* - \varphi_1(0)) \int_0^h f(\tau)\varphi_2(-\tau)\,\mathrm{d}\tau \le 0 \tag{3.1}$$

for $\varphi \in G$. Thus, $V(\varphi)$ is a Liapunov function on the subset G in C. Let

$$Q = \{\varphi \in G \mid \dot{V}(\varphi)|_{(2.4)} = 0\}$$

and M be the largest set in Q which is invariant with respect to (2.4). Clearly, M is not empty since $(s^*, 0, 0) \in M$. From (3.1) we see that $\dot{V}(\varphi)|_{(2.4)} = 0$ only if $s^* - \varphi_1(0) = 0$ or $\varphi_2 = 0$. Note that $s^* - \varphi_1(0) = s^* - s(t) = 0$ and (2.4) imply that $\varphi_2 = 0$. Thus, we always have $\varphi_2 = 0$ if $\dot{V}(\varphi)|_{(2.4)} = 0$. Observe that G is invariant with respect to (2.4) and that any solution of (2.4) is bounded by the following inequality:

$$\dot{s}(t) + \dot{i}(t) + \dot{r}(t) \le -\mu_1(s(t) + i(t) + r(t)) + b. \tag{3.2}$$

Thus, it follows from the Liapunov–LaSalle invariance principle that $\lim_{t\to+\infty} i(t) = 0$ (see [7,9]). Hence, $\lim_{t\to+\infty} r(t) = 0$ by $\lim_{t\to+\infty} i(t) = 0$ and the last equation of (2.4). Furthermore, note that boundedness of $s(t)$ and $\int_0^h f(\tau)i(t-\tau)\,\mathrm{d}\tau \to 0$ as $t \to +\infty$ by $\lim_{t\to+\infty} i(t) = 0$, we can also easily have that $\lim_{t\to+\infty} s(t) = s^*$ by the first equation of (2.4).

In the rest, let us consider the case (c): there is some $\hat{\tau}$ with $0 \le \hat{\tau} < h$ such that $\varphi_1(-\hat{\tau}) \ge s^*$ and $s(t) < s^*$ for all $t \ge t_0$, or there is some $\hat{t}_0 \ge t_0$ such that $s(\hat{t}_0) = s^*$.

If $s(t) < s^*$ for all $t \ge t_0$, observe that system (2.4) is autonomous and the solution of (2.4) with any initial function $\varphi \in C$ is unique, by the same argument as used in case (b) with $t_0 = t_0 + 2h$, we can show that $\lim_{t\to+\infty} s(t) = s^*$ and $\lim_{t\to+\infty} i(t) = \lim_{t\to+\infty} r(t) = 0$.

If there is some $\hat{t}_0 \ge t_0$ such that $s(\hat{t}_0) = s^*$, by (2.4) we see that

$$\dot{s}(\hat{t}_0) - \dot{s}^* = -\beta s(\hat{t}_0) \int_0^h f(\tau)i(\hat{t}_0 - \tau)\,\mathrm{d}\tau - \mu_1 s(\hat{t}_0) + b$$

$$= -\beta s^* \int_0^h f(\tau)i(\hat{t}_0 - \tau)\,\mathrm{d}\tau < 0.$$

Thus, for all $t > \hat{t}_0$, $s(t) - s^* < 0$, i.e. $s(t) < s^*$. Again by the same argument as used in case (b) with $t_0 = \hat{t}_0 + 2h$, we can show that $\lim_{t\to+\infty} s(t) = s^*$ and $\lim_{t\to+\infty} i(t) = \lim_{t\to+\infty} r(t) = 0$.

This completes the proof of Theorem 3.1. □

4. Endemic equilibrium

Throughout this section, we always assume that the endemic equilibrium E_+ exists for (2.4), that is, we assume that (2.7) is true. Let us define

$$T \equiv \int_0^h \tau f(\tau)\,\mathrm{d}\tau.$$

The following Theorem 4.1 is actually the main result of this paper.

Theorem 4.1. *If there is some $\tilde{s}$ satisfying $s^* < \tilde{s} < b/(\mu_2+\lambda)$ such that the following conditions hold true*:

(i) $h < \min\{(2\beta\tilde{s})^{-1}, (\tilde{s}-s^*)/(b-\mu_1 s^*)\}$;

(ii) $b < \tilde{s}[\beta(b/(\mu_2+\lambda)-\tilde{s})+\mu_1]$,

then the endemic equilibrium E_+ of system (2.4) *is globally asymptotically stable.*

Proof. From (3.2), we see that, for any sufficiently small $\varepsilon > 0$, there is a $t_0^* \geq t_0$ such that for $t \geq t_0^*$,

$$s(t)+i(t)+r(t) \leq \frac{b}{\mu_1}+\varepsilon \equiv K_\varepsilon.$$

For any positive constant $\tilde{s}$ satisfying $s^* < \tilde{s} < b/(\mu_2+\lambda)$, define

$$\Omega_\varepsilon \equiv \{(s,i,r) \in \mathscr{R}^3 \mid s+i+r \leq K_\varepsilon,\ s>0,\ i>0,\ r>0\},$$

$$\Omega_{\varepsilon,\tilde{s}} \equiv \{(s,i,r) \in \Omega_\varepsilon \mid s \leq \tilde{s}\}.$$

Let us first show that the following *Assertion* A is true.

Assertion A: *For any positive constant $\tilde{s}$ satisfying $s^* < \tilde{s} < b/(\mu_2+\lambda)$, if $h < (2\beta\tilde{s})^{-1}$, then any solution of* (2.4) *will not ultimately stay in $\Omega_\varepsilon \backslash \Omega_{\varepsilon,\tilde{s}}$.*

In fact, if not, there are some $\tilde{s}$ satisfying $s^* < \tilde{s} < b/(\mu_2+\lambda)$ and some solution $(s(t), i(t), r(t))$ of (2.4) such that $s(t) > \tilde{s}$ for all $t \geq t_0^* \geq t_0$ and $h < (2\beta\tilde{s})^{-1}$. Define a function

$$V(t) = i(t) + \beta\tilde{s}\int_0^h f(\tau)\int_{t-\tau}^t i(u)\,\mathrm{d}u\,\mathrm{d}\tau. \tag{4.1}$$

Thus,

$$\begin{aligned} i(t) &\leq V(t) + \beta\tilde{s}\int_0^h f(\tau)\int_{t-\tau}^t i(u)\,\mathrm{d}u\,\mathrm{d}\tau \\ &\leq V(t) + \beta\tilde{s}T \max_{-h\leq\theta\leq 0} i(t+\theta) \end{aligned} \tag{4.2}$$

for $t \geq t_0$. The time derivative of $V(t)$ along solution $(s(t), i(t), r(t))$ satisfies

$$\begin{aligned} \dot{V}(t) &= \dot{i}(t) + \beta\tilde{s}\left(i(t) - \int_0^h f(\tau)i(t-\tau)\,\mathrm{d}\tau\right) \\ &= \beta s(t)\int_0^h f(\tau)i(t-\tau)\,\mathrm{d}\tau - \beta s^* i(t) + \beta\tilde{s}\left(i(t) - \int_0^h f(\tau)i(t-\tau)\,\mathrm{d}\tau\right) \\ &\geq \beta(\tilde{s}-s^*)i(t) > 0 \end{aligned} \tag{4.3}$$

for $t \geq t_0^*$. Set

$$\omega(t) = \begin{cases} i(t) - V(t)/(1-\beta\tilde{s}T) & \text{if } i(t) \geq V(t)/(1-\beta\tilde{s}T), \\ 0 & \text{if } i(t) < V(t)/(1-\beta\tilde{s}T). \end{cases} \tag{4.4}$$

Here $1-\beta\tilde{s}T > 1-2\beta\tilde{s}T > 1-2\beta\tilde{s}h > 0$ by $T < h$. Clearly, $\omega(t)$ is non-negative and continuous for $t \geq t_0^*$, and for $i(t) < V(t)/(1-\beta\tilde{s}T)$ and $t \geq t_0^* + h$,

$$\omega(t) = 0 \leq \beta\tilde{s}T \max_{-h\leq\theta\leq 0} \omega(t+\theta).$$

For $i(t) \geq V(t)/(1-\beta\tilde{s}T)$ and $t \geq t_0^* + h$, by (4.2)–(4.4), we have

$$\begin{aligned} \omega(t) &= i(t) - \frac{1}{1-\beta\tilde{s}T} V(t) \\ &\leq V(t) + \beta\tilde{s}T \max_{-h\leq\theta\leq 0} i(t+\theta) - \frac{1}{1-\beta\tilde{s}T} V(t) \\ &= V(t) + \beta\tilde{s}T \max_{-h\leq\theta\leq 0} \max\left\{ \omega(t+\theta) + \frac{1}{1-\beta\tilde{s}T} V(t+\theta), \frac{1}{1-\beta\tilde{s}T} V(t+\theta) \right\} \\ &\quad - \frac{1}{1-\beta\tilde{s}T} V(t) \\ &= \left(1 - \frac{1}{1-\beta\tilde{s}T}\right) V(t) + \beta\tilde{s}T \max_{-h\leq\theta\leq 0} \left\{ \omega(t+\theta) + \frac{1}{1-\beta\tilde{s}T} V(t+\theta) \right\} \\ &\leq \left(1 - \frac{1}{1-\beta\tilde{s}T} + \frac{\beta\tilde{s}T}{1-\beta\tilde{s}T}\right) V(t) + \beta\tilde{s}T \max_{-h\leq\theta\leq 0} \omega(t+\theta) \\ &= \beta\tilde{s}T \max_{-h\leq\theta\leq 0} \omega(t+\theta). \end{aligned}$$

Thus, for all $t \geq t_0^* + h$, we have

$$\omega(t) \leq \beta\tilde{s}T \max_{-h\leq\theta\leq 0} \omega(t+\theta). \tag{4.5}$$

By $h < (2\beta\tilde{s})^{-1}$, we can choose a positive constant α which is only dependent on $\beta, \tilde{s}$ and h such that

$$\beta\tilde{s}h\mathrm{e}^{\alpha h} < 1.$$

We next show that, for any constant $k > 0$ and all $t \geq t_0^* + h$, the following inequality holds (also see [10]):

$$\omega(t) < \left(k + \max_{-h\leq\theta\leq 0} \omega(t_0^* + h + \theta)\right) \mathrm{e}^{-\alpha(t-t_0^*-h)} \equiv g_k(t). \tag{4.6}$$

Clearly, for $t_0^* \le t \le t_0^* + h$, $\omega(t) < g_k(t)$. If (4.6) is not true, by the continuity of $\omega(t)$ and $g_k(t)$, there are some constant $k_0 > 0$ and $\bar{t}_0 > t_0^* + h$ such that

$$\omega(t) < g_{k_0}(t), \quad t_0^* \le t < \bar{t}_0, \tag{4.7}$$

$$\omega(\bar{t}_0) = g_{k_0}(\bar{t}_0). \tag{4.8}$$

On the other hand, from (4.5)–(4.7), we have

$$\begin{aligned}
\omega(\bar{t}_0) &\le \beta\tilde{s}T \max_{-h\le\theta\le 0} \omega(\bar{t}_0 + \theta) \\
&\le \beta\tilde{s}T \max_{-h\le\theta\le 0} g_{k_0}(\bar{t}_0 + \theta) \\
&= \beta\tilde{s}T \max_{-h\le\theta\le 0} \left\{ \left(k_0 + \max_{-h\le\theta\le 0} \omega(t_0^* + h + \theta) \right) \mathrm{e}^{-\alpha(\bar{t}_0+\theta-t_0^*-h)} \right\} \\
&= \beta\tilde{s}T\mathrm{e}^{\alpha h} \left(k_0 + \max_{-h\le\theta\le 0} \omega(t_0^* + h + \theta) \right) \mathrm{e}^{-\alpha(\bar{t}_0-t_0^*-h)} \\
&= \beta\tilde{s}T\mathrm{e}^{\alpha h} g_{k_0}(\bar{t}_0) \\
&< g_{k_0}(\bar{t}_0),
\end{aligned}$$

which contradicts (4.8). This proves (4.6).

In (4.6), letting $k \to 0^+$, we have that for $t \ge t_0^* + h$,

$$\omega(t) \le \max_{-h\le\theta\le 0} \omega(t_0^* + h + \theta)\mathrm{e}^{-\alpha(t-t_0^*-h)} \equiv M\mathrm{e}^{-\alpha(t-t_0^*-h)}, \tag{4.9}$$

where $M \equiv \max_{-h\le\theta\le 0} \omega(t_0^* + h + \theta)$. Therefore, it follows from (4.4) and (4.9) that for $t \ge t_0^* + h$,

$$i(t) \le M\mathrm{e}^{-\alpha(t-t_0^*-h)} + \frac{1}{1-\beta\tilde{s}T} V(t). \tag{4.10}$$

Thus, it follows from (4.1), (4.3) and (4.10) that for $t \ge t_0^* + 2h$,

$$\begin{aligned}
\dot{V}(t) &\ge \beta(\tilde{s} - s^*) i(t) \\
&= \beta(\tilde{s} - s^*) \left[V(t) - \beta\tilde{s} \int_0^h f(\tau) \int_{t-\tau}^t i(u)\,\mathrm{d}u\,\mathrm{d}\tau \right] \\
&\ge \beta(\tilde{s} - s^*) \left[V(t) - \beta\tilde{s} \int_0^h f(\tau) \int_{t-\tau}^t \left(M\mathrm{e}^{-\alpha(u-t_0^*-h)} + \frac{1}{1-\beta\tilde{s}T} V(u) \right) \mathrm{d}u\,\mathrm{d}\tau \right] \\
&\ge \beta(\tilde{s} - s^*) \left[\frac{1-2\beta\tilde{s}T}{1-\beta\tilde{s}T} V(t) - \beta\tilde{s}M \int_0^h f(\tau) \int_{t-\tau}^t \mathrm{e}^{-\alpha(u-t_0^*-h)}\,\mathrm{d}u\,\mathrm{d}\tau \right].
\end{aligned} \tag{4.11}$$

We have used that $V(t)$ is non-decreasing and $\tilde{s} > s^*$ in (4.11).

Note that $\int_0^h f(\tau)\int_{t-\tau}^{t} e^{-\alpha(u-t_0^*-h)}\,du\,d\tau \to 0$ as $t \to +\infty$, $\tilde{s} > s^*$ and $1 - 2\beta\tilde{s}T > 0$, we easily have that $V(t) \to +\infty$ as $t \to +\infty$ by (4.11), which contradicts to that $i(t)$ is bounded for $t \geq t_0$. This proves *Assertion* A.

Next, let us further show that the following *Assertion* B is also true.

Assertion B: *If conditions* (i) *and* (ii) *hold, then any solution of* (2.4) *will eventually stay in* $\Omega_{\varepsilon,\tilde{s}}$.

In fact, if not, by *Assertion* A, there is some solution $(s(t), i(t), r(t))$ of (2.4) such that, for any positive constant $\tilde{s}_1$ satisfying $s^* < \tilde{s}_1 < \tilde{s} < b/(\mu_2+\lambda)$, there are two time sequences $\{t_n\}$ and $\{t_n'\}$ with $t_n < t_n' < t_{n+1} < t_{n+1}'$, $t_n \to +\infty$ and $t_n' \to +\infty$, such that

$$s(t_n) = \tilde{s}_1, \quad s(t_n') = \tilde{s}, \qquad \tilde{s}_1 \leq s(t) \leq \tilde{s} \quad \text{for } t_n \leq t \leq t_n', \tag{4.12}$$

and $\dot{s}(t_n') \geq 0$.

From (2.4), we have

$$\begin{aligned}\tilde{s} - \tilde{s}_1 &= s(t_n') - s(t_n)\\ &= -\beta\int_{t_n}^{t_n'} s(v)\int_0^h f(\tau)i(v-\tau)\,dv\,d\tau - \mu_1\int_{t_n}^{t_n'} s(v)\,dv + b(t_n' - t_n),\end{aligned}$$

which, together with (4.12), yields

$$\begin{aligned}b(t_n' - t_n) &= \tilde{s} - \tilde{s}_1 + \beta\int_{t_n}^{t_n'} s(v)\int_0^h f(\tau)i(v-\tau)\,dv\,d\tau + \mu_1\int_{t_n}^{t_n'} s(v)\,dv\\ &\geq \tilde{s} - \tilde{s}_1 + \mu_1\tilde{s}_1(t_n' - t_n).\end{aligned}$$

Thus,

$$t_n' - t_n \geq \frac{\tilde{s} - \tilde{s}_1}{b - \mu_1\tilde{s}_1} \tag{4.13}$$

and

$$\frac{\tilde{s} - \tilde{s}_1}{b - \mu_1\tilde{s}_1} \to \frac{\tilde{s} - s^*}{b - \mu_1 s^*} > h \quad \text{as } \tilde{s}_1 \to s^* \tag{4.14}$$

by condition (i). From (2.4), we also have that for $t \geq t_0$,

$$\begin{aligned}\dot{s}(t) + \dot{i}(t) &= -\mu_1 s(t) - (\mu_2+\lambda)i(t) + b\\ &\geq -(\mu_2+\lambda)(s(t) + i(t)) + b,\end{aligned}$$

which, together with $\tilde{s} < b/(\mu_2+\lambda)$, implies that, for any sufficiently small positive constant η, there is a large $t_1^* \geq t_0^*$ such that for $t \geq t_1^*$,

$$s(t) + i(t) \geq \frac{b}{\mu_2+\lambda} - \eta \equiv N(\eta) > \tilde{s} \tag{4.15}$$

and

$$s^* + i^* = \frac{(\mu_2+\lambda)(\mu_2+\lambda-\mu_1) + b\beta}{\beta(\mu_2+\lambda)} > N(\eta) > i^* = \frac{b\beta - \mu_1(\mu_2+\lambda)}{\beta(\mu_2+\lambda)}. \tag{4.16}$$

Eqs. (4.15) and (4.16) show that the points $(\tilde{s},0,0)$ and $(0,i^*,0)$ are in the lower left-hand side of the plane $s+i=N(\eta)$, and the positive equilibrium (s^*,i^*,r^*) is in the upper right-hand side of plane $s+i=N(\eta)$. We see that the planes $s=\tilde{s}$ and $s+i=N(\eta)$ intersect at $(\tilde{s},N(\eta)-\tilde{s},r)$ for any $r>0$.

Thus, it follows from (4.12)–(4.15) that, for large $t_n' \geq t_1^*$ and $\tilde{s}_1$ which is sufficiently close to s^*,

$$i(t_n'-\tau) \geq N(\eta)-\tilde{s}>0, \quad 0\leq\tau\leq h. \tag{4.17}$$

Eq. (4.17) and condition (ii) enable us to show that $\dot{s}(t_n')<0$ which is a contradiction to $\dot{s}(t_n')\geq 0$. In fact, from (2.4) and (4.17), we have that

$$\begin{aligned}\dot{s}(t_n') &= -\beta s(t_n')\int_0^h f(\tau)i(t_n'-\tau)\,\mathrm{d}\tau-\mu_1 s(t_n')+b\\ &= -\beta\tilde{s}\int_0^h f(\tau)i(t_n'-\tau)\,\mathrm{d}\tau-\mu_1\tilde{s}+b\\ &\leq -\beta\tilde{s}(N(\eta)-\tilde{s})-\mu_1\tilde{s}+b\\ &= -\tilde{s}[\beta(N(\eta)-\tilde{s})+\mu_1]+b\\ &\equiv G(\tilde{s},\eta).\end{aligned} \tag{4.18}$$

By condition (ii), we see that

$$G(\tilde{s},0)=-\tilde{s}[\beta(N(0)-\tilde{s})+\mu_1]+b<0. \tag{4.19}$$

Thus, it follows from (4.18), (4.19) and the continuity of $G(\tilde{s},\eta)$ with respect to η that $\dot{s}(t_n')\leq G(\tilde{s},\eta)<0$ for sufficiently small $\eta>0$. This proves our second assertion.

Now, by *Assertions* A and B, we can complete the proof of Theorem 4.1 by using the following Liapunov functional:

$$\begin{aligned}V(t,s,i_t) &= s-s^*\ln\frac{s}{s^*}+\frac{\omega_1}{2}(s-s^*+i-i^*)^2\\ &\quad+\omega_2\int_0^h f(\tau)\int_{t-\tau}^t (i(u)-i^*)^2\,\mathrm{d}u\,\mathrm{d}\tau,\end{aligned}$$

where ω_1 and ω_2 are some positive constants chosen later and $(s(t),i(t),r(t))$ is any solution of (2.4).

By *Assertion* B, for $\tilde{s}$ satisfying $s^*<\tilde{s}<b/(\mu_2+\lambda)$, there is a sufficiently large time $\hat{t}>t_0$ such that for $t\geq\hat{t}$,

$$s(t)\leq\tilde{s}. \tag{4.20}$$

The derivative $\dot{V}(t,s,i_t)$ of $V(t,s,i_t)$ along the solution of (2.4) satisfies

$$\begin{aligned}\dot{V}(t,s,i_t) &= -\delta[(s-s^*)^2+(i-i^*)^2]\\ &\quad-\frac{1}{2}\int_0^h f(\tau)[W(t,\tau)B(s(t))W^{\mathrm{T}}(t,\tau)]\,\mathrm{d}\tau\end{aligned} \tag{4.21}$$

for all $t \geq \hat{t}$, where δ is some positive constant chosen later,

$$B(s(t)) = \begin{bmatrix} 2(\omega_1\mu_1 - \delta + (\mu_1 + \beta i^*)/s(t)) & \omega_1(\mu_1 + \beta s^*) & \beta \\ \omega_1(\mu_1 + \beta s^*) & 2(\omega_1\beta s^* - \omega_2 - \delta) & 0 \\ \beta & 0 & 2\omega_2 \end{bmatrix},$$

$$W(t,\tau) = (s(t) - s^*, i(t) - i^*, i(t-\tau) - i^*).$$

We can easily see that the symmetric matrix $B(s(t))$ is positive dominant diagonal for every $t \geq \hat{t}$, if

$$\frac{2(\mu_1 + \beta i^*)}{s(t)} - 4\delta - \beta > \omega_1(\beta s^* - \mu_1) - 2\delta > 2\omega_2 > \beta. \tag{4.22}$$

Let us choose δ small enough such that

$$0 < \delta < \frac{\beta}{2\tilde{s}}\left(\frac{b}{\mu_2 + \lambda} - \tilde{s}\right).$$

Then, for all $t \geq \hat{t}$,

$$\frac{2(\mu_1 + \beta i^*)}{s(t)} - 4\delta - \beta > \beta.$$

Thus, note that $\beta s^* - \mu_1 = \mu_2 + \lambda - \mu_1 > 0$, we can easily choose the positive constants ω_1, ω_2 and δ satisfying (4.22). Hence, it follows from (4.21) that for all $t \geq \hat{t}$,

$$\dot{V}(t,s,i_t) \leq -\delta[(s - s^*)^2 + (i - i^*)^2],$$

from which we have that for all $t \geq \hat{t}$,

$$V(t,s,i_t) \leq V(\hat{t}, s(\hat{t}), i_{\hat{t}}) - \delta \int_{\hat{t}}^{t} [(s(u) - s^*)^2 + (i(u) - i^*)^2]\,\mathrm{d}u.$$

Thus,

$$\int_{t_0}^{+\infty} (s(u) - s^*)^2\,\mathrm{d}u < +\infty, \quad \int_{t_0}^{+\infty} (i(u) - i^*)^2\,\mathrm{d}u < +\infty.$$

By (2.4), we see that $\mathrm{d}/\mathrm{d}t\ (s(t) - s^*)^2$ and $\mathrm{d}/\mathrm{d}t\ (i(t) - i^*)^2$ are also uniformly bounded for $t \geq t_0$. Thus, the well-known Barbălat's lemma (see [1]) shows that

$$(s(t) - s^*)^2 + (i(t) - i^*)^2 \to 0 \quad \text{as } t \to +\infty. \tag{4.23}$$

That $r(t) - r^* \to 0$ as $t \to +\infty$ is an immediate result of (4.23) and the third equation of (2.4) (see Lemma 4 in [4]).

The proof of Theorem 4.1 is completed.

In the following, let us give a simpler and more practical criterion than one given by Theorem 4.1.

By (4.18),

$$G(\tilde{s},0) \equiv b - \tilde{s}\left[\beta\left(\frac{b}{\mu_2+\lambda} - \tilde{s}\right) + \mu_1\right] = \beta\tilde{s}^2 - \left(\frac{b\beta}{\mu_2+\lambda} + \mu_1\right)\tilde{s} + b.$$

It is easy to see that equation $G(\tilde{s},0)=0$ has two different positive real roots g_1 and g_2 $(g_1 < g_2)$:

$$g_{1,2} = \frac{1}{2\beta}\left[\frac{b\beta}{\mu_2+\lambda} + \mu_1 \mp \sqrt{\left(\frac{b\beta}{\mu_2+\lambda} + \mu_1\right)^2 - 4\beta b}\right],$$

if

(iii) $b\beta > (\lambda+\mu_2)^2(2 - \frac{\mu_1}{\lambda+\mu_2} + 2\sqrt{1 - \frac{\mu_1}{\lambda+\mu_2}})$.

It is not difficult to see that condition (iii) is more restrictive than the necessary condition (2.7) for the existence of the endemic equilibrium E_+. Also note that (iii) ensures that $s^* < b/(\mu_2+\lambda)$.

Theorem 4.2. *Assume that condition* (iii) *and*
(iv) $h < \min\{(2\beta g_2)^{-1},\ (g_2 - s^*)/(b - \mu_1 s^*)\}$
are satisfied, then the endemic equilibrium E_+ *of system* (2.4) *is globally asymptotically stable.*

Proof. By $G(s^*,0) = (\mu_2+\lambda)(\mu_2+\lambda-\mu_1)/\beta > 0$ and (iii), we see that $s^* < g_1$. We can also easily check that $g_2 < b/(\mu_2+\lambda)$. Thus, $s^* < g_1 < g_2 < b/(\mu_2+\lambda)$. Choose $\tilde{s}$ such that $s^* < g_1 < \tilde{s} < g_2$. Then, $G(\tilde{s},0) < G(g_2,0) = 0$, which together with condition (iv) of Theorem 4.2 shows that, while $\tilde{s}$ is sufficiently close to g_2, conditions (i) and (ii) of Theorem 4.1 can also be satisfied. This proves Theorem 4.2. □

5. Conclusion

The *SIR* model with distributed delays has been used for epidemics spread by a vector and has been studied at least in three papers (see [2,4,6]). In this paper we devote the first section to state clearly the assumptions of the model and hence to derive the mathematical structure of the (distributed) delay differential equations (1.20). In Section 2, we consider (2.4), a mathematical generalization of (1.20), and for both systems (1.20) and (2.4) we present a summary of the known results. Finally, in the remaining two sections, we have considered the global asymptotic properties of the disease free equilibrium and the endemic equilibrium of the delay *SIR* epidemic model (2.4) in which population size may vary with respect to time and the death and birth rates of population may not be all the same. In Section 3, by the Liapunov–LaSalle invariance principle, we show that the disease free equilibrium is still globally attractive whenever $b/\mu_1 = s^* = (\mu_2+\lambda)/\beta$. Clearly, Theorem 3.1 shows that the conclusion (i) of Theorem 2.5 is also true for the delay *SIR* epidemic model (2.4) for any delay $h \geq 0$. That is to say that the disease will eventually disappear independent of time delay h whenever the endemic equilibrium does not exist. In Section 4, we give sufficient conditions to

ensure the global asymptotic stability of the endemic equilibrium E_+ whenever it exists, i.e. conditions that the disease always remains endemic. These sufficient conditions mean that, while E_+ exists, it is still globally asymptotically stable as long as the delay h is short enough and the product βb of the contact constant β and the birth rate b is relatively large (or the death rates μ_1 and μ_2, and the recovery rate λ are small enough).

Based on Hethcote's result for the *SIR* epidemic model without delay (i.e. Theorem 2.5) and general properties for delay differential equations, it is natural to conjecture that *for sufficiently small delay h*, *condition* (2.7) *implies the global asymptotic stability of the endemic equilibrium* E_+, *i.e. condition* (2.7) *should be the threshold of* (2.4) *for an epidemic to occur*.

Unfortunately, we need more restrictive conditions (ii) and (iii) in Theorems 4.1 and 4.2 in order to ensure the global asymptotic stability of the endemic state. Our proofs suggest that, to complete the analysis on the above problem, we need to construct new Liapunov functionals and to give better estimate on the lower bound of $i(t)$ than one given by (4.17).

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