Notes on Algebra

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I. Basic Properties & Facts

Arithmetic Operations

$$
\begin{aligned}
ab + ac &= a(b+c) \\
a\left(\frac{b}{c}\right) &= \frac{ab}{c} \\
\frac{\left(\frac{a}{b}\right)}{c} &= \frac{a}{bc} \\
\frac{a}{\left(\frac{b}{c}\right)} &= \frac{ac}{b} \\
\frac{a}{b} + \frac{c}{d} &= \frac{ad+bc}{bd} \\
\frac{a}{b} - \frac{c}{d} &= \frac{ad-bc}{bd} \\
\frac{a-b}{c-d} &= \frac{b-a}{d-c} \\
\frac{a+b}{c} &= \frac{a}{c} + \frac{b}{c} \\
\frac{ab+ac}{a} &= b+c,\ a \neq 0 \\
\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} &= \frac{ad}{bc}
\end{aligned}
$$

Exponent Properties

$$
\begin{aligned}
a^n a^m &= a^{n+m} \\
\frac{a^n}{a^m} &= a^{n-m} = \frac{1}{a^{m-n}} \\
(a^n)^m &= a^{nm} \\
a^0 &= 1,\ a \neq 0 \\
(ab)^n &= a^n b^n \\
\left(\frac{a}{b}\right)^n &= \frac{a^n}{b^n} \\
a^{-n} &= \frac{1}{a^n} \\
\frac{1}{a-n} &= a^n \\
\left(\frac{a}{b}\right)^{-n} &= \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n} \\
a^{\frac{n}{m}} &= \left(a^{\frac{1}{m}}\right)^n = (a^n)^{\frac{1}{m}}
\end{aligned}
$$

Properties of Radicals

$$
\begin{aligned}
\sqrt[n]{a} &= a^{\frac{1}{n}} \\
\sqrt[n]{ab} &= \sqrt[n]{a}\,\sqrt[n]{b} \\
\sqrt[n]{\sqrt[m]{a}} &= \sqrt[nm]{a} \\
\sqrt[n]{\frac{a}{b}} &= \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \\
\sqrt[n]{a^n} &= a, \text{if } n \text{ is odd} \\
\sqrt[n]{a^n} &= |a|, \text{if } n \text{ is even}
\end{aligned}
$$

Properties of Inequalities

If $a < b$ then $a + c < b + c$ and $a - c < b - c$.
If $a < b$ and $c > 0$ then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
If $a < b$ and $c < 0$ then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

Properties of Absolute Value

$$
\begin{aligned}
|a| &= \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases} \\
|a| &\geq 0 \\
|-a| &= |a| \\
|ab| &= |a||b| \\
\left|\frac{a}{b}\right| &= \frac{|a|}{|b|} \\
|a+b| &\leq |a| + |b| \text{ Triangle Inequality}
\end{aligned}
$$

Distance Formula If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are two points, the distance between them is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Complex Numbers

$$
\begin{aligned}
i &= \sqrt{-1} \\
i^2 &= -1 \\
\sqrt{-a} &= i\sqrt{a},\ a \geq 0 \\
(a+bi)+(c+di) &= a+c+(b+d)i \\
(a+bi)-(c+di) &= a-c+(b-d)i \\
(a+bi)(c+di) &= ac-bd+(ad+bc)i \\
(a+bi)(c-di) &= a^2+b^2 \\
|a+bi| &= \sqrt{a^2+b^2} \text{ Complex Modulus} \\
\overline{(a+bi)} &= a-bi \text{ Complex Conjugate} \\
(a+bi)\overline{(a+bi)} &= |a+bi|^2
\end{aligned}
$$

Logarithms and Log Properties Definition

$$y = \log_b x \quad \equiv \quad x = b^y$$

Special Logarithms

$$
\begin{aligned}
\ln x &= \log_e x \text{ natural log} \\
\log x &= \log_{10} x \text{ common log} \\
e &= 2.718281828
\end{aligned}
$$

Logarithm Properties

$$
\begin{aligned}
\log_b b &= 1 \\
\log_b 1 &= 0 \\
\log_b b^x &= x \\
b^{\log_b x} &= x \\
\log_b(x^r) &= r\log_b x \\
\log_b(xy) &= \log_b x + \log_b y \\
\log_b\left(\frac{x}{y}\right) &= \log_b x - \log_b y
\end{aligned}
$$

The domain of $\log_b x$ is $x > 0$.

II. Factoring & Solving

Factoring Formulas

$$
\begin{aligned}
x^2-a^2 &= (x+a)(x-a) \\
x^2+2ax+a^2 &= (x+a)^2 \\
x^2-2ax+a^2 &= (x-a)^2 \\
x^2+(a+b)x+ab &= (x+a)(x+b) \\
x^3+3ax^2+3a^2x+a^3 &= (x+a)^3 \\
x^3-3ax^2+3a^2x-a^3 &= (x-a)^3 \\
x^3+a^3 &= (x+a)(x^2-ax+a^2) \\
x^3-a^3 &= (x-a)(x^2-ax+a^2) \\
x^{2n}-a^{2n} &= (x^n-a^n)(x^n+a^n)
\end{aligned}
$$

If n is odd then

$$
\begin{aligned}
&x^n-a^n = (x-a)(x^{n-1}+ax^{n-2}+\cdots+a^{n-1}) \\
&x^n+a^n = (x+a)(x^{n-1}-ax^{n-2}+a^2x^{n-3}-\cdots+a^{n-1})
\end{aligned}
$$

Quadratic Formula
Solve $ax^2+bx+c=0$, $a \neq 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

If $b^2-4ac>0$ Two real unequal solutions.
If $b^2-4ac=0$ Repeated real solution.
If $b^2-4ac<0$ Two complex solutions.

Square Root Property
If $x^2=p$ then $x=\pm\sqrt{p}$.

Absolute Value Equations/Inequalities
If b is a positive number

$$
\begin{aligned}
|p| = b &\implies p=-b \text{ or } p=b \\
|p| < b &\implies -b<p<b \\
|p| > b &\implies p<-b \text{ or } p>b
\end{aligned}
$$

III. Functions & Graphs

Constant Function
Given

$$y = a \quad \text{or} \quad f(x) = a$$

The graph is a horizontal line passing through the point $(0,a)$.

Line/Linear Function
Given

$$y = mx+b \quad \text{or} \quad f(x) = mx+b$$

The graph is a line with point $(0,b)$ and slope m.

Slope
Slope of the line containing the two points (x_1,y_1) and (x_2,y_2) is

$$m = \frac{y_2-y_1}{x_2-x_1} = \frac{\text{rise}}{\text{run}}$$

Slope-intercept form
The equation of the line with slope m and y-intercept $(0,b)$ is

$$y = mx+b$$

Point-Slope form
The equation of the line with slope m and passing through the point (x_1,y_1) is

$$y = y_1 + m(x-x_1)$$

Parabola/Quadratic Function

Case 1

$$y = a(x-h)^2 + k \quad \text{or} \quad f(x) = a(x-h)^2 + k$$

The graph is a parabola that opens up if $a > 0$ or down if $a < 0$ and has a vertex at (h, k).

Case 2

$$y = ax^2 + bx + c \quad \text{or} \quad f(x) = ax^2 + bx + c$$

The graph is a parabola that opens up if $a > 0$ or down if $a < 0$ and has a vertex at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Case 3

$$x = ay^2 + by + c \quad \text{or} \quad g(y) = ay^2 + by + c$$

The graph is a parabola that opens right if $a > 0$ or left if $a < 0$ and has a vertex at $\left(g\left(-\frac{b}{2a}\right), -\frac{b}{2a}\right)$.

Circle

Equation:

$$(x-h)^2 + (y-k)^2 = r^2$$

Graph: circle with radius r and center (r, k).

Ellipse

Equation:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Graph: ellipse with center (h, k) with vertices a units right/left from the center and vertices b units up/down from the center.

Hyperbola

Case 1

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Graph: hyperbola that opens left and right, has a center at (h, k), vertices a units left/right of center and asymptotes that pass through center with slope $\pm\frac{b}{a}$.

Case 2

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Graph: hyperbola that opens up and down, has a center at (h, k), vertices b units up/down from the center and asymptotes that pass through center with slope $\pm\frac{b}{a}$.

IV. Common Algebraic Errors

Error	**Reason/Justification/Example**
$\frac{2}{0} \neq 0$ and $\frac{2}{0} \neq 2$	Division by zero is undefined!
$-3^2 \neq 9$	$-3^2 = -9, (-3)^2 = 9$. Watch parenthesis!
$(x^2)^3 \neq x^5$	$(x^2)^3 = x^2x^2x^2 = x^6$
$\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$	$\frac{1}{2} = \frac{1}{1+1} \neq \frac{1}{1} + \frac{1}{1} = 2$
$\frac{1}{x^2+x^3} \neq x^{-2} + x^{-3}$	This is a more complex version of the previous error.
$\frac{a+bx}{a} \neq 1 + bx$	$\frac{a+bx}{a} = \frac{a}{a} + \frac{bx}{a} = 1 + \frac{bx}{a}$ Beware of incorrect canceling!
$-a(x-1) \neq -ax - a$	$-a(x-1) = -ax + a$ Make sure you distribute the "-"!
$(x+a)^2 \neq x^2 + a^2$	$(x+a)^2 = (x+a)(x+a)$ $= x^2 + 2ax + a^2$
$\sqrt{x^2 + a^2} \neq x + a$	$5 = \sqrt{25} = \sqrt{3^2 + 4^2}$ $\neq \sqrt{3} + \sqrt{4} = 3 + 4 = 7$
$\sqrt{x+a} \neq \sqrt{x} + \sqrt{a}$	See previous error.
$(x+a)^n \neq x^n + a^n$ and, $\sqrt[n]{x+a} \neq \sqrt[n]{x} + \sqrt[n]{a}$	More general versions of previous three errors.
$2(x+1)^2 \neq (2x+2)^2$	$2(x+1)^2 = 2(x^2 + 2x + 1)$ $= 2x^2 + 4x + 2$, $(2x+2)^2 = 4x^2 + 8x + 4$. Square first, then distribute!
$(2x+2) \neq 2(x+1)^2$	See previous example. You cannot factor out a constant if there is a power on the parenthesis!
$\sqrt{-x^2 + a^2} \neq -\sqrt{x^2 + a^2}$	$\sqrt{-x^2 + a^2} = (-x^2 + a^2)^{\frac{1}{2}}$ Now see the previous error.
$\frac{a}{\left(\frac{b}{c}\right)} \neq \frac{ab}{c}$	$\frac{a}{\left(\frac{b}{c}\right)} = \frac{\left(\frac{a}{1}\right)}{\left(\frac{b}{c}\right)} = \left(\frac{a}{1}\right)\left(\frac{c}{b}\right) = \frac{ac}{b}$
$\frac{\left(\frac{a}{b}\right)}{c} \neq \frac{ac}{b}$	$\frac{\left(\frac{a}{b}\right)}{c} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{1}\right)} = \left(\frac{a}{b}\right)\left(\frac{1}{c}\right) = \frac{a}{bc}$