DM561

Linear Algebra with Applications

Linear Programming

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Outline

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1. Leontief Input Output Models

2. Production Planning

3. Diet Problem

4. Budget Allocation

Closed Model

- Economic system consisting of a finite number of industries, $1, 2, \ldots, k$.
- Over some fixed period of time, each industry produces output of some good or service that is completely utilized in a predetermined manner by the k industries.
- Find suitable prices to be charged for these k outputs so that for each industry, total expenditures equal total income.
- Such a price structure represents an equilibrium position for the economy.

Closed Model

Three homeowners – a carpenter, an electrician, and a plumber – agree to make repairings in their three homes. They agree to work a total of 10 days each according to the following schedule:

	Work Performed by		
	Carpenter	**Electrician**	**Plumber**
Days of Work in Home of Carpenter	2	1	6
Days of Work in Home of Electrician	4	5	1
Days of Work in Home of Plumber	4	4	3

For tax purposes, they must report and pay each other a reasonable daily wage, even for the work each does on his or her own home.

Their normal daily wages are about $100, but they agree to adjust their respective daily wages so that each homeowner will come out even—that is, so that the total amount paid out by each is the same as the total amount each receives. What should be the prices of their work?

Closed Model

-

$$\begin{array}{rcrcrcl} 2p_1 & + & p_2 & + & p_3 & = & 10p_1 \\ 4p_1 & + & 5p_2 & + & p_3 & = & 10p_2 \\ 4p_1 & + & 4p_2 & + & 3p_3 & = & 10p_3 \end{array}$$

- $(I - A)\mathrm{p} = \mathrm{p}$. If $\det(I - A) \neq 0$ then non trivial solution. Moreover, it can be shown that for exchange matrices A that are stochastic matrices the solution p is such that its elements are non-negative and if A^m are positive for all m positive integer then all p entries are positive.

Open Model

- Consider a market with n industries producing n different commodities.
- The market is interdependent, meaning that each industry requires input from the other industries and possibly even its own commodity.
- In addition, there is an outside demand for each commodity that has to be satisfied.
- We wish to determine the amount of output of each industry which will satisfy all demands exactly; that is, both the demands of the other industries and the outside demand.

Open Model

- Let $n = 3$ and let a_{ij} indicate the amount of commodity i, $i = 1, 2, 3$ necessary to produce one unit of commodity j, $j = 1, 2, 3$.

- a_{ij} are given in monetary terms:
  a_{ij} cost of the commodity i necessary to produce one unit profit of commodity j.
  $\rightsquigarrow$ Hence, we will assume that $a_{ij} \geq 0$
  Example: to produce an amount of commodity j worth 100 dkk, one needs an amount of commodity i worth 30 dkk.

  For the sake of simplicity we scale all these values such that the profit of each commodity is 1 unity of currency.

  $\rightsquigarrow$ Hence, for each commodity j its production is not profitable unless

$$\sum_{i=1}^{3} a_{ij} < 1.$$

- d_i be the demand of commodity i expressed in units of currency.

For each commodity, the outside demand is covered by the production of the commodity after the subtraction of the amount of commodity that has to go in the other industries and the amount that has to go in the same industry.
Hence:

$$x_i - \sum_{j=1}^{n} a_{ij} x_j = d_i$$

For $n = 3$ we have:

$$\begin{aligned} x_1 - a_{11}x_1 - a_{12}x_2 - a_{13}x_3 &= d_1 \\ x_2 - a_{21}x_1 - a_{22}x_2 - a_{23}x_3 &= d_2 \\ x_3 - a_{31}x_1 - a_{32}x_2 - a_{33}x_3 &= d_3 \end{aligned}$$

In matrix terms:

$$I\mathrm{x} - A\mathrm{x} = \mathrm{d} \qquad \text{or} \qquad (I - A)\mathrm{x} = \mathrm{d}$$

which is a system of linear equations. To make sense the solution x must be non-negative. It can be shown that under the conditions expressd above the solution to the system is unique and non-negative.

Open Model

- Unique non-negative solution for $\mathbf{x}$ if and only if $(I - A)^{-1}$ exists and $(I - A)^{-1} \geq 0$.
- The matrix A such that $(I - A)^{-1}$ exists and $(I - A)^{-1} \geq 0$ is called productive.
- The matrix A is productive $\iff$ there exists $\mathbf{x} \geq 0$ such that $\mathbf{x} > A\mathbf{x}$
  $\iff \sum_{j=1}^{n} a_{ij} < 1$ (row sums)
  that is, there is some production plan such that each industry produces (monetarily) more than it consumes.
- A matrix is productive $\iff \sum_{i=1}^{m} a_{ij} < 1$ (column sums)
  that is, the jth industry is profitable if the total value of the outputs of all m industries needed to produce one unit of value of output of the industry j is less than one.

Decision Support Tools

- So far we considered a full economic system (country, region) and the decision making from the point of view of a Government planning
  IO Models and Linear Systems of Equations

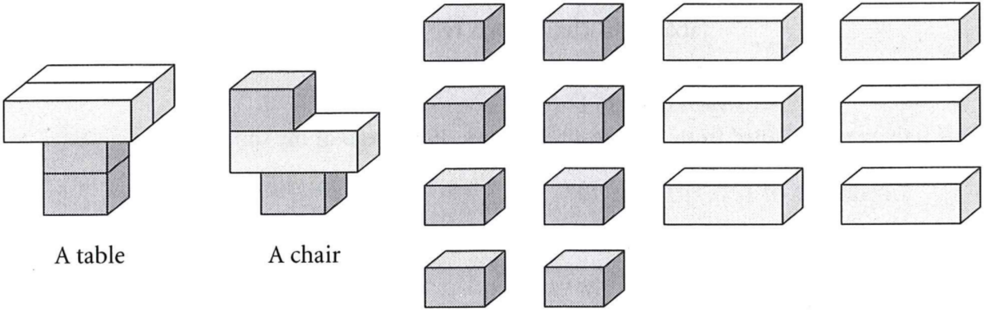
- Now, let's consider the Planning of Activities by a single Firm. Eg: Supply chain management, logistics, production scheduling
  Linear Programming

A firm have can produce their items in many different ways. The planning problem is characterized by a large number of feasible ways of providing the same output.

Outline

Production Planning

Suppose a company produces only tables and chairs.
A table is made of 2 large Lego pieces and 2 small pieces, while a chair is made of 1 large and 2 small pieces.
The resources available are 8 small and 6 large pieces.



The profit for a table is 1600 dkk and for a chair 1000 dkk. What product mix maximizes the company's profile using the available resources?

Mathematical Model

	Tables	Chairs	Capacity
Small Pieces	2	2	8
Large Pieces	2	1	6
Profit	16	10	

Decision Variables

$x_1 \geq 0$ units of small pieces
$x_2 \geq 0$ units of large pieces

Object Function

$\max\ 16x_1 + 10x_2$ maximize profit

Constraints

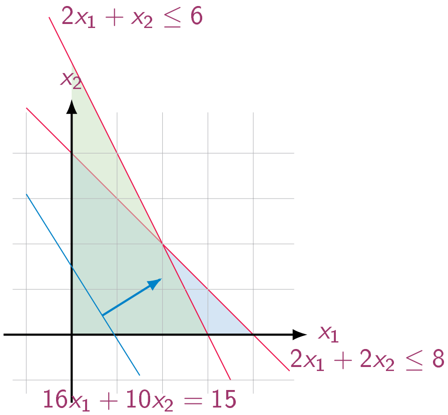
$$\begin{array}{rcrcl} 2x_1 & + & 2x_2 & \leq & 8 \quad \text{small pieces capacity} \\ 2x_1 & + & x_2 & \leq & 6 \quad \text{large pieces capacity} \end{array}$$

Mathematical Model

Materials A and B
Products 1 and 2

$$
\begin{array}{lrcrcl}
\max & 16x_1 & + & 10x_2 & & \\
 & 2x_1 & + & 2x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 6 \\
 & & & x_1 & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

Graphical Representation:



Resource Allocation - General Model

Managing a production facility

$1, 2, \ldots, n$	products
$1, 2, \ldots, m$	materials
b_i	units of raw material at disposal
a_{ij}	units of raw material i to produce one unit of product j
σ_j	market price of unit of jth product
ρ_i	prevailing market value for material i
$c_j = \sigma_j - \sum_{i=1}^{n} \rho_i a_{ij}$	profit per unit of product j
x_j	amount of product j to produce

$$\begin{array}{rcccccccccccl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \ldots & + & c_nx_n & = & z \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
& a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
& \ldots & & & & & & & & & & \\
& a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\
& & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}$$

Notation

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \dots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & \leq & b_2 \\
 & \dots & & & & & & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \dots, x_n & \geq & 0
\end{array}
$$

$$
\begin{array}{rll}
\max & \displaystyle\sum_{j=1}^{n} c_jx_j & \\
 & \displaystyle\sum_{j=1}^{n} a_{ij}x_j & \leq b_i, \quad i = 1, \dots, m \\
 & x_j & \geq 0, \quad j = 1, \dots, n
\end{array}
$$

In Matrix Form

$$
\begin{array}{lccccccccccc}
\max & c_1 x_1 & + & c_2 x_2 & + & c_3 x_3 & + & \dots & + & c_n x_n & = & z \\
\text{s.t.} & a_{11} x_1 & + & a_{12} x_2 & + & a_{13} x_3 & + & \dots & + & a_{1n} x_n & \leq & b_1 \\
 & a_{21} x_1 & + & a_{22} x_2 & + & a_{23} x_3 & + & \dots & + & a_{2n} x_n & \leq & b_2 \\
 & \dots & & & & & & & & & & \\
 & a_{m1} x_1 & + & a_{m2} x_2 & + & a_{m3} x_3 & + & \dots & + & a_{mn} x_n & \leq & b_m \\
 & & & & & & & \multicolumn{3}{r}{x_1, x_2, \dots, x_n} & & \geq & 0
\end{array}
$$

$$c^T = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}$$

$$
A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{31} & a_{32} & \dots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

$$
\begin{array}{lrcl}
\max & z & = & c^T x \\
 & Ax & \leq & b \\
 & x & \geq & 0
\end{array}
$$

Vector and Matrices in Excel

$$\sum_{j=1}^{n} c_j = c_1 + c_2 + \ldots + c_n$$

$$\mathtt{SUM(B5:B14)}$$

Scalar product

$$\begin{aligned} \mathrm{u} \cdot \mathrm{v} &= u_1 v_1 + u_2 v_2 + \ldots + u_n v_n \\ &= \sum_{j=1}^{n} u_j v_j \end{aligned}$$

$$\mathtt{SUMPRODUCT(B5:B14, C5:C:14)}$$

Our Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \dots, m \\ & x_j \ge 0, \quad j = 1, \dots, n \end{aligned}$$

$$\begin{array}{rrcrcl} \max & 16x_1 & + & 10x_2 & & \\ & 2x_1 & + & 2x_2 & \le & 8 \\ & 2x_1 & + & x_2 & \le & 6 \\ & & & x_1, x_2 & \ge & 0 \end{array}$$

$$\begin{aligned} \max \quad & c^T x \\ & Ax \le b \\ & x \ge 0 \end{aligned}$$

$$x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

$$\max \quad \begin{bmatrix} 16 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$x_1, x_2 \ge 0$$

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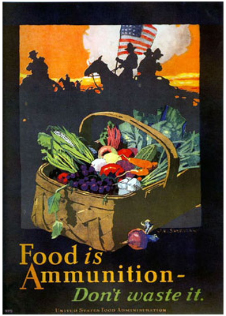
The Diet Problem (Blending Problems)

- Select a set of foods that will satisfy a set of daily nutritional requirement at minimum cost.
- Motivated in the 1930s and 1940s by US army.
- Formulated as a linear programming problem by George Stigler
- First linear programming problem
- (programming intended as planning not computer code)



min cost/weight
subject to nutrition requirements:
eat enough but not too much of Vitamin A
eat enough but not too much of Sodium
eat enough but not too much of Calories
...

The Diet Problem

Suppose there are:

- 3 foods available, corn, milk, and bread,
- there are restrictions on the number of calories (between 2000 and 2250) and the amount of Vitamin A (between 5000 and 50,000)

Food	Corn	2% Milk	Wheat bread
Vitamin A	107	500	0
Calories	72	121	65
Cost per serving	\$0.18	\$0.23	\$0.05

The Mathematical Model

Parameters (given data)

$$
\begin{aligned}
F &= \text{set of foods} \\
N &= \text{set of nutrients} \\
\\
a_{ij} &= \text{amount of nutrient } j \text{ in food } i,\ \forall i \in F,\ \forall j \in N \\
c_i &= \text{cost per serving of food } i, \forall i \in F \\
F_{mini} &= \text{minimum number of required servings of food } i, \forall i \in F \\
F_{maxi} &= \text{maximum allowable number of servings of food } i, \forall i \in F \\
N_{minj} &= \text{minimum required level of nutrient } j, \forall j \in N \\
N_{maxj} &= \text{maximum allowable level of nutrient } j, \forall j \in N
\end{aligned}
$$

Decision Variables

$$x_i = \text{number of servings of food } i \text{ to purchase/consume}, \forall i \in F$$

The Mathematical Model

Objective Function: Minimize the total cost of the food

$$\text{Minimize} \sum_{i \in F} c_i x_i$$

Constraint Set 1: For each nutrient $j \in N$, at least meet the minimum required level

$$\sum_{i \in F} a_{ij} x_i \geq N_{minj}, \qquad \forall j \in N$$

Constraint Set 2: For each nutrient $j \in N$, do not exceed the maximum allowable level.

$$\sum_{i \in F} a_{ij} x_i \leq N_{maxj}, \qquad \forall j \in N$$

Constraint Set 3: For each food $i \in F$, select at least the minimum required number of servings

$$x_i \geq F_{mini}, \qquad \forall i \in F$$

Constraint Set 4: For each food $i \in F$, do not exceed the maximum allowable number of servings.

$$x_i \leq F_{maxi}, \qquad \forall i \in F$$

The Mathematical Model

system of equalities and inequalities

$$
\begin{aligned}
\min \quad & \sum_{i \in F} c_i x_i \\
\sum_{i \in F} a_{ij} x_i \geq N_{minj}, & \qquad \forall j \in N \\
\sum_{i \in F} a_{ij} x_i \leq N_{maxj}, & \qquad \forall j \in N \\
x_i \geq F_{mini}, & \qquad \forall i \in F \\
x_i \leq F_{maxi}, & \qquad \forall i \in F
\end{aligned}
$$

Outline

Budget Allocation

- A company has six different opportunities to invest money.
- Each opportunity requires a certain investment over a period of 6 years or less.

Expected Investment Cash Flows and Net Present Value	Opp. 1	Opp. 2	Opp. 3	Opp. 4	Opp. 5	Opp. 6		Budget
Year 1	-\$5.00	-\$9.00	-\$12.00	-\$7.00	-\$20.00	-\$18.00		\$45.00
Year 2	-\$6.00	-\$6.00	-\$10.00	-\$5.00	\$6.00	-\$15.00		\$30.00
Year 3	-\$16.00	\$6.10	-\$5.00	-\$20.00	\$6.00	-\$10.00		\$20.00
Year 4	\$12.00	\$4.00	-\$5.00	-\$10.00	\$6.00	-\$10.00		\$0.00
Year 5	\$14.00	\$5.00	\$25.00	-\$15.00	\$6.00	\$35.00		\$0.00
Year 6	\$15.00	\$5.00	\$15.00	\$75.00	\$6.00	\$35.00		\$0.00
NPV	\$8.01	\$2.20	\$1.85	\$7.51	\$5.69	\$5.93		

- The company has an investment budget that needs to be met for each year.
- It also has the wish of investing in those opportunities that maximize the combined Net Present Value (NPV) after the 6th year.

Digression: What is the Net Present Value?

- P: value of the original payment presently due
- the debtor wants to delay the payment for t years,
- let r be the market rate of return that the creditor would obtain from a similar investment asset
- the future value of P is $F = P(1+r)^t$

Viceversa, consider the task of finding:

- the present value P of \$100 that will be received in five years, or equivalently,
- which amount of money today will grow to \$100 in five years when subject to a constant discount rate.

Assuming a 5% per year interest rate, it follows that

$$P = \frac{F}{(1+r)^t} = \frac{\$100}{(1+0.05)^5} = \$78.35.$$

Budget Allocation

Net Present Value calculation:
for each opportunity we calculate the NPV at time zero (the time of decision) as:

$$P_0 = \sum_{t=1}^{5} \frac{F_t}{(1+0.05)^5}$$

Expected Investment Cash Flows and Net Present Value									
	Opp. 1	Opp. 2	Opp. 3	Opp. 4	Opp. 5	Opp. 6			Budget
Year 1	-$5.00	-$9.00	-$12.00	-$7.00	-$20.00	-$18.00			$45.00
Year 2	-$6.00	-$6.00	-$10.00	-$5.00	$6.00	-$15.00			$30.00
Year 3	-$16.00	$6.10	-$5.00	-$20.00	$6.00	-$10.00			$20.00
Year 4	$12.00	$4.00	-$5.00	-$10.00	$6.00	-$10.00			$0.00
Year 5	$14.00	$5.00	$25.00	-$15.00	$6.00	$35.00			$0.00
Year 6	$15.00	$5.00	$15.00	$75.00	$6.00	$35.00			$0.00
NPV	$8.01	$2.20	$1.85	$7.51	$5.69	$5.93			

Budget Allocation - Mathematical Model

- Let B_t be the budget available for investments during the years $t = 1..5$.
- Let a_{tj} be the cash flow for opportunity j and c_j its NPV
- Task: choose a set of opportunities such that the budget is never exceeded and the expected return is maximized. Consider both the case of indivisible and divisible opportunities.

Variables $x_j = 1$ if opportunity j is selected and $x_j = 0$ otherwise, $j = 1..6$

Objective

$$\max \sum_{j=1}^{6} c_j x_j$$

Constraints

$$\textstyle\sum_{j=1}^{6} a_{tj} x_j + B_t \geq 0 \qquad \forall t = 1..5$$