DM561

Linear Algebra with Applications

Linear Programming

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Outline

Outline

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Closed Model

- Economic system consisting of a finite number of industries, $1, 2, \ldots, k$.
- Over some fixed period of time, each industry produces output of some good or service that is completely utilized in a predetermined manner by the k industries.
- Find suitable prices to be charged for these k outputs so that for each industry, total expenditures equal total income.
- Such a price structure represents an equilibrium position for the economy.

Closed Model

Three homeowners – a carpenter, an electrician, and a plumber – agree to make repairs in their three homes. They agree to work a total of 10 days each according to the following schedule:

	Work Performed by		
	Carpenter	**Electrician**	**Plumber**
Days of Work in Home of Carpenter	2	1	6
Days of Work in Home of Electrician	4	5	1
Days of Work in Home of Plumber	4	4	3

For tax purposes, they must report and pay each other a reasonable daily wage, even for the work each does on his or her own home.

Their normal daily wages are about \$100, but they agree to adjust their respective daily wages so that each homeowner will come out even—that is, so that the total amount paid out by each is the same as the total amount each receives. What should be the prices of their work?

Closed Model

- $$\begin{array}{rcrcrcl} 2p_1 & + & p_2 & + & p_3 & = & 10p_1 \\ 4p_1 & + & 5p_2 & + & p_3 & = & 10p_2 \\ 4p_1 & + & 4p_2 & + & 3p_3 & = & 10p_3 \end{array}$$
- $(I-A)\mathbf{p} = \mathbf{p}$. If $\det(I-A) \neq 0$ then non trivial solution. Moreover, it can be shown that for exchange matrices A that are stochastic matrices the solution $\mathbf{p}$ is such that its elements are non-negative and if A^m are positive for all m positive integer then all $\mathbf{p}$ entries are positive.

Open Model

- Consider a market with n industries producing n different commodities.
- The market is interdependent, meaning that each industry requires input from the other industries and possibly even its own commodity.
- In addition, there is an outside demand for each commodity that has to be satisfied.
- We wish to determine the amount of output of each industry which will satisfy all demands exactly; that is, both the demands of the other industries and the outside demand.

Open Model

- Let $n = 3$ and let a_{ij} indicate the amount of commodity i, $i = 1, 2, 3$ necessary to produce one unit of commodity j, $j = 1, 2, 3$.

- a_{ij} are given in monetary terms:
  a_{ij} cost of the commodity i necessary to produce one unit profit of commodity j.
  $\rightsquigarrow$ Hence, we will assume that $a_{ij} \geq 0$
  Example: to produce an amount of commodity j worth 100 dkk, one needs an amount of commodity i worth 30 dkk.

  For the sake of simplicity we scale all these values such that the profit of each commodity is 1 unity of currency.

- d_i be the demand of commodity i expressed in units of currency.

For each commodity, the outside demand is covered by the production of the commodity after the subtraction of the amount of commodity that has to go in the other industries and the amount that has to go in the same industry.
Hence:

$$x_i - \sum_{j=1}^{n} a_{ij} x_j = d_i$$

For $n = 3$ we have:

$$\begin{aligned} x_1 - a_{11}x_1 - a_{12}x_2 - a_{13}x_3 &= d_1 \\ x_2 - a_{21}x_1 - a_{22}x_2 - a_{23}x_3 &= d_2 \\ x_3 - a_{31}x_1 - a_{32}x_2 - a_{33}x_3 &= d_3 \end{aligned}$$

In matrix terms:

$$I\mathbf{x} - A\mathbf{x} = \mathbf{d} \qquad \text{or} \qquad (I - A)\mathbf{x} = \mathbf{d}$$

which is a system of linear equations. To make sense the solution $\mathbf{x}$ must be non-negative. It can be shown that under the conditions expressd above the solution to the system is unique and non-negative.

Open Model

- Unique non-negative solution for $\mathbf{x}$ if and only if $(I - A)^{-1}$ exists and $(I - A)^{-1} \geq 0$.
- The matrix A such that $(I - A)^{-1}$ exists and $(I - A)^{-1} \geq 0$ is called productive.
- The matrix A is productive $\iff$ there exists $\mathbf{x} \geq 0$ such that $\mathbf{x} > A\mathbf{x}$
  $\iff \sum_{j=1}^{n} a_{ij} < 1$ (row sums)
  that is, there is some production plan such that each industry produces (monetarily) more than it consumes.
- A matrix is productive $\iff \sum_{i=1}^{m} a_{ij} < 1$ (column sums)
  that is, the jth industry is profitable if the total value of the outputs of all m industries needed to produce one unit of value of output of the industry j is less than one.

A More General I/O Model

- $(I - A)^{-1}\mathbf{x} = \mathbf{d}$
- Let $\tilde{A} = (I - A)^{-1} \implies \tilde{A}\mathbf{x} = \mathbf{d}$
- We can relax this assumption to say that net production of commodities must be greater or equal to demand:

$$\tilde{A}\mathbf{x} \geq \mathbf{d}$$

- The total cost of the economy is $\sum_j x_j$, hence:

$$\begin{aligned} \min \quad & \mathbf{1}^T\mathbf{x} \\ \tilde{A}\mathbf{x} \quad & \geq \quad \mathbf{d} \\ x \quad & \geq \quad 0 \end{aligned}$$

Decision Support Tools

- So far we considered a full economic system (country, region) and the decision making from the point of view of a planning Government

  I/O Models and Linear Systems of Equations

- Now, let's consider the Planning of Activities by a single Firm.

  A firm have can produce multifunctional outputs in many different ways. The planning problem is characterized by a large number of feasible ways of providing the same output.

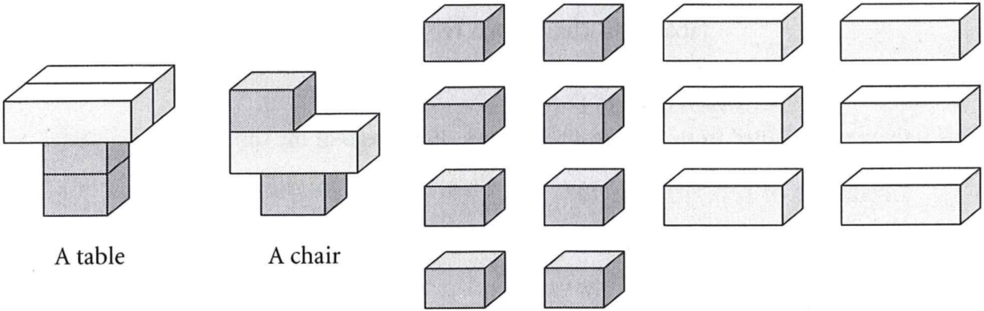
  Eg: Supply chain management, logistics, production scheduling

  Linear Programming

Outline

Production Planning

Suppose a company produces only tables and chairs.
A table is made of 2 large Lego pieces and 2 small pieces, while a chair is made of 1 large and 2 small pieces.
The resources available are 8 small and 6 large pieces.



The profit for a table is 1600 dkk and for a chair 1000 dkk. What product mix maximizes the company's profile using the available resources?

Mathematical Model

	Tables	Chairs	Capacity
Small Pieces	2	2	8
Large Pieces	2	1	6
Profit	16	10	

Decision Variables

$x_1 \geq 0$ units of tables
$x_2 \geq 0$ units of chairs

Object Function

$$\max\ 16x_1 + 10x_2 \quad \text{maximize profit}$$

Constraints

$$\begin{array}{rcrcll} 2x_1 & + & 2x_2 & \leq & 8 & \text{small pieces capacity} \\ 2x_1 & + & x_2 & \leq & 6 & \text{large pieces capacity} \end{array}$$

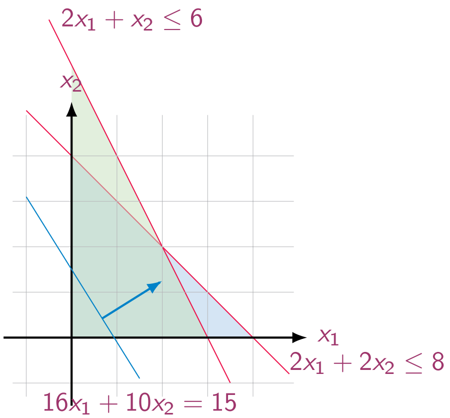
Mathematical Model

Materials A and B
Products 1 and 2

$$
\begin{array}{lrcrcl}
\max & 16x_1 & + & 10x_2 & & \\
 & 2x_1 & + & 2x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 6 \\
 & & & x_1 & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

a_{ij}	1	2	b_i
A	2	2	8
B	2	1	6
c_j	16	10	

Graphical Representation:



Resource Allocation - General Model

Managing a production facility

$1, 2, \ldots, n$	products
$1, 2, \ldots, m$	materials
b_i	units of raw material at disposal
a_{ij}	units of raw material i to produce one unit of product j
σ_j	market price of unit of jth product
ρ_i	prevailing market value for material i
$c_j = \sigma_j - \sum_{i=1}^{n} \rho_i a_{ij}$	profit per unit of product j
x_j	amount of product j to produce

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \ldots & + & c_nx_n & = & z \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
& a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
& \ldots & & & & & & & & & & \\
& a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\
& & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

Notation

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \ldots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
& a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
& \ldots & & & & & & & & & & \\
& a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\
& & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

$$
\begin{array}{rrl}
\max & \sum_{j=1}^{n} c_jx_j & \\
& \sum_{j=1}^{n} a_{ij}x_j & \leq b_i, \quad i = 1, \ldots, m \\
& x_j & \geq 0, \quad j = 1, \ldots, n
\end{array}
$$

In Matrix Form

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \dots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & \leq & b_2 \\
 & \dots & & & & & & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \dots, x_n & \geq & 0
\end{array}
$$

$$
c^T = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}
$$

$$
A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{31} & a_{32} & \dots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

$$
\begin{array}{lrcl}
\max & z & = & c^T x \\
 & Ax & \leq & b \\
 & x & \geq & 0
\end{array}
$$

Our Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\ & x_j \geq 0, \quad j = 1, \ldots, n \end{aligned}$$

$$\begin{array}{lrcrcl} \max & 16x_1 & + & 10x_2 & & \\ & 2x_1 & + & 2x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 6 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{aligned} \max \quad & c^T x \\ Ax & \leq b \\ x & \geq 0 \end{aligned}$$

$$x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

$$\max \quad \begin{bmatrix} 16 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

Outline

Duality

Resource Valuation problem: Determine the value of the raw materials on hand such that:
(i) it would be convenient selling and (ii) an outside company would be willing to buy them.

z_i	value of a unit of raw material i
$\sum_{i=1}^m b_i z_i$	total expenses for buying (or opportunity cost, cost of having instead of selling)
ρ_i	prevailing unit market value of material i
σ_j	prevailing unit product price

Goal: for the outside company to minimize the total expenses;
(for the owing company the minimum amount of opportunity cost to accept for selling)

$$\min \sum_{i=1}^{m} b_i z_i \tag{1}$$

$$z_i \geq \rho_i, \quad i = 1 \ldots m \tag{2}$$

$$\sum_{i=1}^{m} z_i a_{ij} \geq \sigma_j, \quad j = 1 \ldots n \tag{3}$$

(2) otherwise selling to someone else and (3) otherwise not selling

Let

$$y_i = z_i - \rho_i$$

markup that the company would make by reselling the raw material instead of producing.

$$\min \sum_{i=1}^{m} y_i b_i + \cancel{\sum_i \rho_i b_i}$$

$$\sum_{i=1}^{m} y_i a_{ij} \geq c_j, \quad j = 1 \dots n$$

$$y_i \geq 0, \quad i = 1 \dots m$$

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \dots, m$$

$$x_j \geq 0, \quad j = 1, \dots, n$$

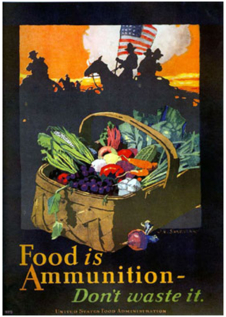
Dual Problem

Primal Problem

Outline

The Diet Problem (Blending Problems)

- Select a set of foods that will satisfy a set of daily nutritional requirement at minimum cost.
- Motivated in the 1930s and 1940s by US army.
- Formulated as a linear programming problem by George Stigler
- First linear programming problem solved
- (programming intended as planning not computer code)



min cost/weight
subject to nutrition requirements:
eat enough but not too much of Vitamin A
eat enough but not too much of Sodium
eat enough but not too much of Calories

The Diet Problem

Suppose there are:

- 3 foods available: corn, milk, and bread,
- there are restrictions on the number of calories (between 2000 and 2250) and the amount of Vitamin A (between 5000 and 50 000)

Food	Corn	2% Milk	Wheat bread
Vitamin A	107	500	0
Calories	72	121	65
Cost per serving	\$0.18	\$0.23	\$0.05

The Mathematical Model

Parameters (given data)

F = set of foods
N = set of nutrients

a_{ij} = amount of nutrient j in food i, $\forall i \in F$, $\forall j \in N$
c_i = cost per serving of food $i, \forall i \in F$
F_{mini} = minimum number of required servings of food $i, \forall i \in F$
F_{maxi} = maximum allowable number of servings of food $i, \forall i \in F$
N_{minj} = minimum required level of nutrient $j, \forall j \in N$
N_{maxj} = maximum allowable level of nutrient $j, \forall j \in N$

Decision Variables

x_i = number of servings of food i to purchase/consume, $\forall i \in F$

The Mathematical Model

Objective Function: Minimize the total cost of the food

$$\text{Minimize} \sum_{i \in F} c_i x_i$$

Constraint Set 1: For each nutrient $j \in N$, at least meet the minimum required level

$$\sum_{i \in F} a_{ij} x_i \geq N_{minj}, \qquad \forall j \in N$$

Constraint Set 2: For each nutrient $j \in N$, do not exceed the maximum allowable level.

$$\sum_{i \in F} a_{ij} x_i \leq N_{maxj}, \qquad \forall j \in N$$

Constraint Set 3: For each food $i \in F$, select at least the minimum required number of servings

$$x_i \geq F_{mini}, \qquad \forall i \in F$$

Constraint Set 4: For each food $i \in F$, do not exceed the maximum allowable number of servings.

$$x_i \leq F_{maxi}, \qquad \forall i \in F$$

The Mathematical Model

system of equalities and inequalities

$$
\begin{aligned}
\min \quad & \sum_{i \in F} c_i x_i \\
\sum_{i \in F} a_{ij} x_i \geq N_{minj}, & \qquad \forall j \in N \\
\sum_{i \in F} a_{ij} x_i \leq N_{maxj}, & \qquad \forall j \in N \\
x_i \geq F_{mini}, & \qquad \forall i \in F \\
x_i \leq F_{maxi}, & \qquad \forall i \in F
\end{aligned}
$$

Budget Allocation

- A company has six different opportunities to invest money.
- Each opportunity requires a certain investment over a period of 6 years or less.

Expected Investment Cash Flows and Net Present Value	Opp. 1	Opp. 2	Opp. 3	Opp. 4	Opp. 5	Opp. 6			Budget
Year 1	-$5.00	-$9.00	-$12.00	-$7.00	-$20.00	-$18.00			$45.00
Year 2	-$6.00	-$6.00	-$10.00	-$5.00	$6.00	-$15.00			$30.00
Year 3	-$16.00	$6.10	-$5.00	-$20.00	$6.00	-$10.00			$20.00
Year 4	$12.00	$4.00	-$5.00	-$10.00	$6.00	-$10.00			$0.00
Year 5	$14.00	$5.00	$25.00	-$15.00	$6.00	$35.00			$0.00
Year 6	$15.00	$5.00	$15.00	$75.00	$6.00	$35.00			$0.00
NPV	$8.01	$2.20	$1.85	$7.51	$5.69	$5.93			

- The company has an investment budget that needs to be met for each year.
- It also has the wish of investing in those opportunities that maximize the combined Net Present Value (NPV) after the 6th year.

Digression: What is the Net Present Value?

- P: value of the original payment presently due
- the debtor wants to delay the payment for t years,
- let r be the market rate of return that the creditor would obtain from a similar investment asset
- the future value of P is $F = P(1+r)^t$

Viceversa, consider the task of finding:

- the present value P of \$100 that will be received in five years, or equivalently,
- which amount of money today will grow to \$100 in five years when subject to a constant discount rate.

Assuming a 5% per year interest rate, it follows that

$$P = \frac{F}{(1+r)^t} = \frac{\$100}{(1+0.05)^5} = \$78.35.$$

Budget Allocation

Net Present Value calculation:
for each opportunity we calculate the NPV at time zero (the time of decision) as:

$$P_0 = \sum_{t=1}^{5} \frac{F_t}{(1+0.05)^t}$$

Expected Investment Cash Flows and Net Present Value									
	Opp. 1	**Opp. 2**	**Opp. 3**	**Opp. 4**	**Opp. 5**	**Opp. 6**			**Budget**
Year 1	-$5.00	-$9.00	-$12.00	-$7.00	-$20.00	-$18.00			$45.00
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Year 3	-$16.00	$6.10	-$5.00	-$20.00	$6.00	-$10.00			$20.00
Year 4	$12.00	$4.00	-$5.00	-$10.00	$6.00	-$10.00			$0.00
Year 5	$14.00	$5.00	$25.00	-$15.00	$6.00	$35.00			$0.00
Year 6	$15.00	$5.00	$15.00	$75.00	$6.00	$35.00			$0.00
NPV	$8.01	$2.20	$1.85	$7.51	$5.69	$5.93			

Budget Allocation - Mathematical Model

- Let B_t be the budget available for investments during the years $t = 1..5$.
- Let a_{tj} be the cash flow for opportunity j and c_j its NPV
- Task: choose a set of opportunities such that the budget is never exceeded and the expected return is maximized. Consider both the case of indivisible and divisible opportunities.

Variables $x_j = 1$ if opportunity j is selected and $x_j = 0$ otherwise, $j = 1..6$

Objective

$$\max \sum_{j=1}^{6} c_j x_j$$

Constraints

$$\sum_{j=1}^{6} a_{tj} x_j + B_t \geq 0 \qquad \forall t = 1..5$$

Linear Programming

The Syntax of a Linear Programming Problem

objective func.	$\max / \min \ \mathbf{c}^T\mathbf{x}$	$\mathbf{c} \in \mathbb{R}^n$
constraints	s.t. $A\mathbf{x} \ \substack{\geq \\ = \\ \leq} \ \mathbf{b}$	$A \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m$
	$\mathbf{x} \geq \mathbf{0}$	$\mathbf{x} \in \mathbb{R}^n, \mathbf{0} \in \mathbb{R}^n$

Essential features: continuity, linearity (proportionality and additivity), certainty of parameters

- Any vector $\mathbf{x} \in \mathbb{R}^n$ satisfying all constraints is a feasible solution.
- Each $\mathbf{x}^* \in \mathbb{R}^n$ that gives the best possible value for $\mathbf{c}^T\mathbf{x}$ among all feasible $\mathbf{x}$ is an optimal solution or optimum
- The value $\mathbf{c}^T\mathbf{x}^*$ is the optimum value

Outline

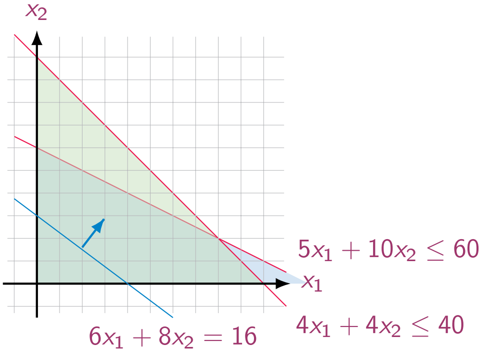
Mathematical Model

Machines/Materials A and B
Products 1 and 2

$$\begin{array}{rrcrcr}
\max & 6x_1 & + & 8x_2 & & \\
 & 5x_1 & + & 10x_2 & \leq & 60 \\
 & 4x_1 & + & 4x_2 & \leq & 40 \\
 & & & x_1 & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}$$

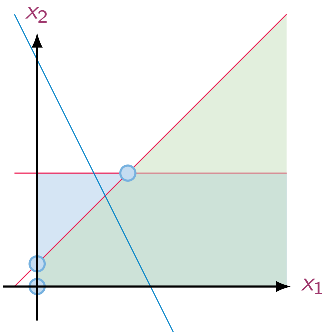
a_{ij}	1	2	b_i
A	5	10	60
B	4	4	40
c_j	6	8	

Graphical Representation:



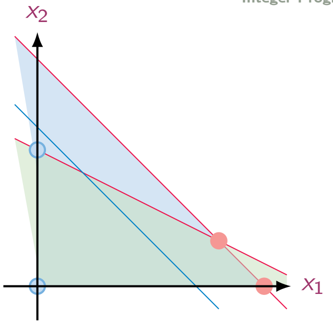
Unboundedness

$$\begin{array}{rrcrcl} \max & 2x_1 & + & x_2 & & \\ & & & x_2 & \leq & 5 \\ & -x_1 & + & x_2 & \leq & 1 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$



Infinite Solutions

$$\begin{array}{lrcrcl} \max & x_1 & + & x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$



$$\mathbf{x} = \sum_i \alpha_i \mathbf{x}_i$$

$$\alpha_i \geq 0$$

$$\sum_i \alpha_i = 1$$

$$\mathbf{x}_1^T = [8, 2, 0, 0]$$

$$\mathbf{x}_2^T = [10, 0, 10, 0]$$

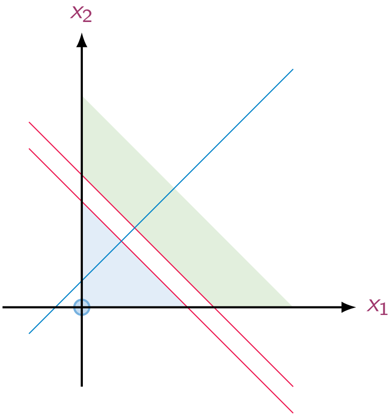
$$\alpha_1 = \alpha$$

$$\alpha_2 = 1 - \alpha$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} 8 \\ 2 \\ 0 \\ 0 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} 10 \\ 0 \\ 10 \\ 0 \end{bmatrix}$$

Infeasibility

$$
\begin{array}{rcrcl}
\max & x_1 & - & x_2 & \\
& x_1 & + & x_2 & \leq 2 \\
& 2x_1 & + & 2x_2 & \geq 5 \\
& & & x_1, x_2 & \geq 0
\end{array}
$$



Outline

Definitions

- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ closed interval
  $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ open interval

- linear combination

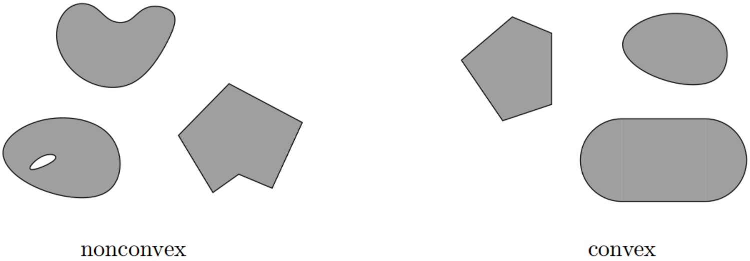
$$\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_k \in \mathbb{R}^n \qquad \boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_k]^T \in \mathbb{R}^k \qquad \mathbf{x} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k = \sum_{i=1}^{k} \lambda_i \mathbf{v}_i$$

moreover:

$\boldsymbol{\lambda} \geq \mathbf{0}$	conic combination	
$\boldsymbol{\lambda}^T \mathbf{1} = 1$	affine combination	$\left(\sum_{i=1}^{k} \lambda_i = 1 \right)$
$\boldsymbol{\lambda} \geq \mathbf{0}$ and $\boldsymbol{\lambda}^T \mathbf{1} = 1$	convex combination	

Definitions

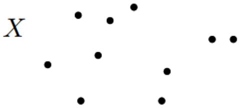
- convex set: if $\mathbf{x}, \mathbf{y} \in S$ and $0 \leq \lambda \leq 1$ then $\lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in S$

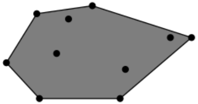


Definitions

- For a set of points $S \subseteq \mathbb{R}^n$

$\text{lin}(S)$	linear hull (span)
$\text{cone}(S)$	conic hull
$\text{aff}(S)$	affine hull
$\text{conv}(S)$	convex hull





the convex hull of X

$$\boxed{\textbf{conv}(X) = \{\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \ldots + \lambda_n x_n \mid \mathbf{x}_i \in X,\ \lambda_1, \ldots, \lambda_n \geq 0 \text{ and } \textstyle\sum_i \lambda_i = 1\}}$$

Definitions

- $G \subseteq \mathbb{R}^n$ is an hyperplane if $\exists\, \mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$:

  $$G = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} = \alpha\}$$

- $H \subseteq \mathbb{R}^n$ is an halfspace if $\exists\, \mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$:

  $$H = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} \leq \alpha\}$$

  ($\mathbf{a}^T\mathbf{x} = \alpha$ is a supporting hyperplane of H)

Definitions

- a set $S \subset \mathbb{R}^n$ is a polyhedron if $\exists\, m \in \mathbb{Z}^+, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$:

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\} = \bigcap_{i=1}^{m} \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_{i,\cdot}\mathbf{x} \leq b_i\}$$

  i.e., a polyhedron $P \neq \mathbb{R}^n$ is determined by finitely many halfspaces

- a polyhedron P is a polytope if it is bounded: $\exists\, B \in \mathbb{R}, B > 0$:

$$P \subseteq \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq B\}$$

  ($\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} x_j^2}$ is the Euclidean norm of the vector $\mathbf{x} \in \mathbb{R}$)

- A set of vectors is a polytope if it is the convex hull of finitely many vectors.

Linear Programming Problem

Input: a matrix $A \in \mathbb{R}^{m \times n}$ and column vectors $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$

Task:

1. decide that $\{\mathbf{x} \in \mathbb{R}^n; A\mathbf{x} \leq \mathbf{b}\}$ is empty (prob. infeasible), or
2. find a column vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c}^T\mathbf{x}$ is max, or
3. decide that for all $\alpha \in \mathbb{R}$ there is an $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{c}^T\mathbf{x} > \alpha$ (prob. unbounded)

1. $F = \emptyset$
2. $F \neq \emptyset$ and $\exists$ solution
    1. one solution
    2. infinite solutions
3. $F \neq \emptyset$ and $\nexists$ solution

Fundamental Theorem of LP

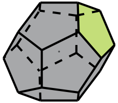
Theorem (Fundamental Theorem of Linear Programming)

Given:

$$\min\{\mathbf{c}^T\mathbf{x} \mid \mathbf{x} \in P\} \ \ \textit{where}\ P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$$

If P is a bounded polyhedron and not empty and $\mathbf{x}^*$ is an optimal solution to the problem, then:

- $\mathbf{x}^*$ *is an extreme point (vertex) of P, or*
- $\mathbf{x}^*$ *lies on a face $F \subset P$ of optimal solutions*



Proof idea:

- assume $\mathbf{x}^*$ not a vertex of P then $\exists$ a ball around it still in P. Show that a point in the ball has better cost
- if $\mathbf{x}^*$ is not a vertex then it is a convex combination of vertices. Show that all points are also optimal.

Implications:

- the optimal solution is at the intersection of supporting hyperplanes.
- hence finitely many possibilities
- solution method: write all inequalities as equalities and solve all $\binom{m}{n}$ systems of linear equalities (n # variables, m # equality constraints)
- for each point we then need to check if feasible and if best in cost.
- each system is solved by Gaussian elimination
- Stirling approximation:

$$\binom{2m}{m} \approx \frac{4^m}{\sqrt{\pi m}} \text{ as } m \to \infty$$

Standard Form

Every LP problem can be converted in the standard form:

$$\begin{aligned} \max\ & \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\in \mathbb{R}^n \end{aligned}$$

$$\mathbf{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m$$

- if equations, then put two constraints, $\mathbf{a}x \leq b$ and $\mathbf{a}x \geq b$
- if $\mathbf{a}x \geq b$ then $-\mathbf{a}x \leq -b$
- if $\min \mathbf{c}^T\mathbf{x}$ then $\max(-\mathbf{c}^T\mathbf{x})$

and then be put in equational standard form:

$$\boxed{\begin{aligned} \max\ & \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \\ \mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n, & A \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m \end{aligned}}$$

1. "$=$" constraints
2. $\mathbf{x} \geq 0$ nonnegativity constraints
3. $(\mathbf{b} \geq 0)$
4. max

Transformation to Std Form

Every LP problem can be transformed in eq. std. form

1. introduce slack variables (or surplus)

$$\begin{array}{rcrcrcr} 5x_1 & + & 10x_2 & + & x_3 & = & 60 \\ 4x_1 & + & 4x_2 & + & x_4 & = & 40 \end{array}$$

2. if $x_1 \gtreqless 0$ then $\begin{array}{l} x_1 = x_1' - x_1'' \\ x_1' \geq 0 \\ x_1'' \geq 0 \end{array}$

3. $(b \geq 0)$

4. $\min c^T x \equiv \max(-c^T x)$

LP in $m \times n$ converted into LP with at most $(m + 2n)$ variables and m equations (n # original variables, m # constraints)

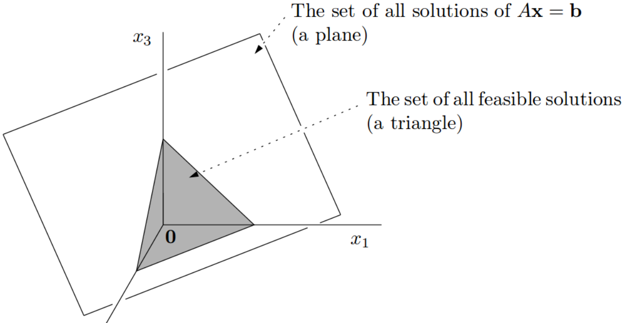
Geometry of LP in Eq. Std. Form

$$\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

From linear algebra:

- the set of solutions of $A\mathbf{x} = \mathbf{b}$ is an affine space (hyperplane not passing through the origin).
- $\mathbf{x} \geq \mathbf{0}$ nonegative orthant (octant in $\mathbb{R}^3$)

In $\mathbb{R}^3$:



- $A\mathbf{x} = \mathbf{b}$ is a system of equations that we can solve by Gaussian elimination

- Elementary row operations of $\left[A \mid \mathbf{b}\right]$ do not affect set of feasible solutions
  - multiplying all entries in some row of $\left[A \mid \mathbf{b}\right]$ by a nonzero real number λ
  - replacing the ith row of $\left[A \mid b\right]$ by the sum of the ith row and jth row for some $i \neq j$

- Let n' be the number of vars in eq. std. form.

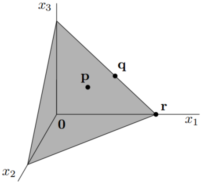
  we assume $n' \geq m$ and $\mathrm{rank}(\left[A \mid \mathbf{b}\right]) = \mathrm{rank}(A) = m$

  ie, rows of A are linearly independent
  otherwise, remove linear dependent rows

Outline

Basic Feasible Solutions

Basic feasible solutions are the vertices of the feasible region:



More formally:
Let $B = \{1 \dots m\}$, $N = \{m+1 \dots n+m = n'\}$ be subsets partitioning the columns of A: A_B be made of columns of A indexed by B:

Definition

$\mathbf{x} \in \mathbb{R}^n$ is a basic feasible solution of the program $\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ for an index set B if:

- $x_j = 0$ forall $j \notin B$
- the square matrix A_B is nonsingular, ie, all columns indexed by B are lin. indep.
- $\mathbf{x}_B = A_B^{-1}\mathbf{b}$ is nonnegative, ie, $\mathbf{x}_B \geq 0$ (feasibility)

We call x_j for $j \in B$ basic variables and remaining variables nonbasic variables.

Theorem

A basic feasible solution is uniquely determined by the set B.

Proof:

$$\begin{aligned} A\mathbf{x} = & A_B\mathbf{x}_B + A_N\mathbf{x}_N = b \\ & \mathbf{x}_B + A_B^{-1}A_N\mathbf{x}_N = A_B^{-1}b \\ & \mathbf{x}_B = A_B^{-1}b \end{aligned}$$

A_B is nonsingular hence one solution

Note: we call B a (feasible) basis

Extreme points and basic feasible solutions are geometric and algebraic manifestations of the same concept:

Theorem

Let P be a (convex) polyhedron from LP in std. form. For a point $v \in P$ the following are equivalent:

(i) *v is an extreme point (vertex) of P*

(ii) *v is a basic feasible solution of LP in eq. std. form*

Theorem

Let $LP = \max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ be feasible and bounded, then there is an optimal solution that is a basic feasible solution.

Proof. consequence of previous theorem and fundamental theorem of linear programming
However, an optimal solution does not need to be basic, eg, $\max\{x_1 + x_2 \mid x_1 + x_2 \leq 1\}$.

Solution by Enumeration

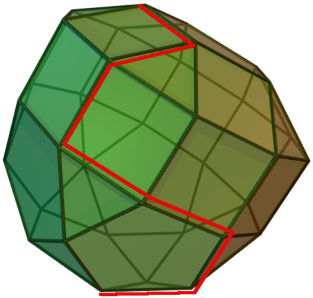
- examine all basic solutions.
- there are finitely many: $\binom{m+n}{m}$.
- however, if $n = m$ then $\binom{2m}{m} \approx 4^m$.

Simplex Method

1. find a solution that is at the intersection of some n hyperplanes

2. try systematically to produce the other points by exchanging one hyperplane with another

3. check optimality, proof provided by duality theory



Dual Problem

Dual variables $\mathbf{y}$ in one-to-one correspondence with the constraints:

Primal problem:

$$\begin{aligned} \max \quad z &= \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad w &= \mathbf{b}^T\mathbf{y} \\ A^T\mathbf{y} &\geq \mathbf{c} \\ \mathbf{y} &\geq \mathbf{0} \end{aligned}$$

Duality Derivation: Bounding approach

$$\begin{array}{rrcrcrcl}
z^* = \max & 4x_1 & + & x_2 & + & 3x_3 & & \\
& x_1 & + & 4x_2 & & & \leq & 1 \\
& 3x_1 & + & x_2 & + & x_3 & \leq & 3 \\
& & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

a feasible solution is a lower bound but how good?
By tentatives:

$$(x_1, x_2, x_3) = (1, 0, 0) \rightsquigarrow z^* \geq 4$$
$$(x_1, x_2, x_3) = (0, 0, 3) \rightsquigarrow z^* \geq 9$$

What about upper bounds?

$$\begin{array}{ccrcrcrcl}
& & 2 \cdot (\; x_1 & + & 4x_2 & &) & \leq & 2 \cdot 1 \\
& & + 3 \cdot (\; 3x_1 & + & x_2 & + & x_3) & \leq & 3 \cdot 3 \\
\hline
4x_1 + x_2 + 3x_3 & \leq & & & 11x_1 + 11x_2 + 3x_3 & & & \leq & 11 \\
\\
\mathbf{c}^T\mathbf{x} & \leq & & & \mathbf{y}^T A\mathbf{x} & & & \leq & \mathbf{y}^T\mathbf{b}
\end{array}$$

multipliers $y_1, y_2 \geq 0$ that preserve sign of inequality

$$\begin{array}{rcl}
y_1 \cdot (\ x_1 + 4x_2 \qquad\quad) & \leq & y_1(1) \\
y_2 \cdot (\ 3x_1 + x_2 + \qquad x_3) & \leq & y_2(3) \\
\hline
(y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2 x_3 & \leq & y_1 + 3y_2
\end{array}$$

Coefficients

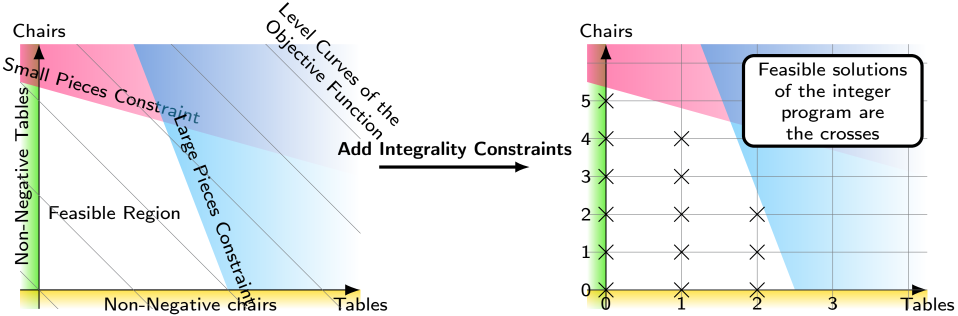
$$\begin{array}{rcrcl}
y_1 & + & 3y_2 & \geq & 4 \\
4y_1 & + & y_2 & \geq & 1 \\
 & & y_2 & \geq & 3
\end{array}$$

$z = 4x_1 + x_2 + 3x_3 \leq (y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2 x_3 \leq y_1 + 3y_2$ then to attain the best upper bound:

$$\begin{array}{lrcrcl}
\min & y_1 & + & 3y_2 & & \\
 & y_1 & + & 3y_2 & \geq & 4 \\
 & 4y_1 & + & y_2 & \geq & 1 \\
 & & & y_2 & \geq & 3 \\
 & & & y_1, y_2 & \geq & 0
\end{array}$$

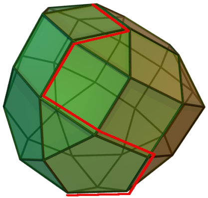
Outline

Integrality Requirements

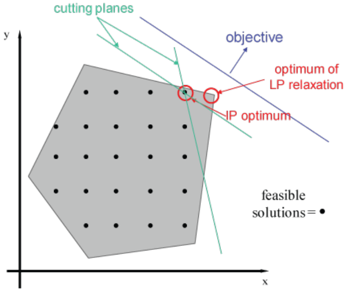


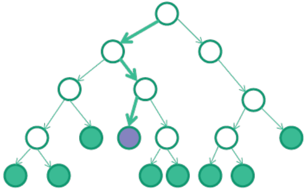
Solution Approaches

Linear Programming:



Integer Programming:





Each node in branch-and-bound is a new MIP

Outline

1. Economic Models
   - Leontief Input Output Models
   - Production Planning
   - Duality — Economic Interpretation
2. Other Applications
3. Geometrical Interpretation
4. Theoretical Background
   - Basic Feasible Solutions
5. Integer Programming