

DM561  
Linear Algebra with Applications

## Linear Programming

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# Outline

## 1. Economic Models

Leontief Input Output Models

Production Planning

Duality — Economic Interpretation

## 2. Other Applications

## 3. Geometrical Interpretation

## 4. Theoretical Background

Basic Feasible Solutions

## 5. Integer Programming

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# Closed Model

- Economic system consisting of a finite number of industries,  $1, 2, \dots, k$ .
- Over some fixed period of time, each industry produces output of some good or service that is completely utilized in a predetermined manner by the  $k$  industries.
- Find suitable prices to be charged for these  $k$  outputs so that for each industry, total expenditures equal total income.
- Such a price structure represents an equilibrium position for the economy.

## Closed Model

Three homeowners – a carpenter, an electrician, and a plumber – agree to make repairs in their three homes. They agree to work a total of 10 days each according to the following schedule:

	Work Performed by		
	Carpenter	Electrician	Plumber
Days of Work in Home of Carpenter	2	1	6
Days of Work in Home of Electrician	4	5	1
Days of Work in Home of Plumber	4	4	3

For tax purposes, they must report and pay each other a reasonable daily wage, even for the work each does on his or her own home.

Their normal daily wages are about \$100, but they agree to adjust their respective daily wages so that each homeowner will come out even—that is, so that the total amount paid out by each is the same as the total amount each receives.

What should be the **prices** of their work?

# Closed Model

- $$\begin{aligned}2p_1 + p_2 + p_3 &= 10p_1 \\4p_1 + 5p_2 + p_3 &= 10p_2 \\4p_1 + 4p_2 + 3p_3 &= 10p_3\end{aligned}$$
- $(I - A)\mathbf{p} = \mathbf{p}$ . If  $\det(I - A) \neq 0$  then non trivial solution. Moreover, it can be shown that for exchange matrices  $A$  that are stochastic matrices the solution  $\mathbf{p}$  is such that its elements are non-negative and if  $A^m$  are positive for all  $m$  positive integer then all  $\mathbf{p}$  entries are positive.

# Open Model

- Consider a market with  $n$  industries producing  $n$  different commodities.
- The market is interdependent, meaning that each industry requires input from the other industries and possibly even its own commodity.
- In addition, there is an outside demand for each commodity that has to be satisfied.
- We wish to determine the amount of output of each industry which will satisfy all demands exactly; that is, both the demands of the other industries and the outside demand.



# Open Model

- Let  $n = 3$  and let  $a_{ij}$  indicate the amount of commodity  $i$ ,  $i = 1, 2, 3$  necessary to produce one unit of commodity  $j$ ,  $j = 1, 2, 3$ .
- $a_{ij}$  are given in monetary terms:  
 $a_{ij}$  cost of the commodity  $i$  necessary to produce one unit profit of commodity  $j$ .  
 $\leadsto$  Hence, we will assume that  $a_{ij} \geq 0$   
Example: to produce an amount of commodity  $j$  worth 100 dkk, one needs an amount of commodity  $i$  worth 30 dkk.  
  
For the sake of simplicity we scale all these values such that the profit of each commodity is 1 unity of currency.
- $d_i$  be the demand of commodity  $i$  expressed in units of currency.

For each commodity, the outside demand is covered by the production of the commodity after the subtraction of the amount of commodity that has to go in the other industries and the amount that has to go in the same industry.

Hence:

$$x_i - \sum_{j=1}^n a_{ij}x_j = d_i$$

For  $n = 3$  we have:

$$x_1 - a_{11}x_1 - a_{12}x_2 - a_{13}x_3 = d_1$$

$$x_2 - a_{21}x_1 - a_{22}x_2 - a_{23}x_3 = d_2$$

$$x_3 - a_{31}x_1 - a_{32}x_2 - a_{33}x_3 = d_3$$

In matrix terms:

$$I\mathbf{x} - A\mathbf{x} = \mathbf{d} \quad \text{or} \quad (I - A)\mathbf{x} = \mathbf{d}$$

which is a system of linear equations. To make sense the solution  $\mathbf{x}$  must be non-negative. It can be shown that under the conditions expressed above the solution to the system is unique and non-negative.

# Open Model

- Unique non-negative solution for  $\mathbf{x}$  if and only if  $(I - A)^{-1}$  exists and  $(I - A)^{-1} \geq 0$ .
- The matrix  $A$  such that  $(I - A)^{-1}$  exists and  $(I - A)^{-1} \geq 0$  is called **productive**.
- The matrix  $A$  is productive  $\iff$  there exists  $\mathbf{x} \geq 0$  such that  $\mathbf{x} > A\mathbf{x}$   
 $\iff \sum_{j=1}^n a_{ij} < 1$  (row sums)  
 that is, there is some production plan such that each industry produces (monetarily) more than it consumes.
- A matrix is productive  $\iff \sum_{i=1}^m a_{ij} < 1$  (column sums)  
 that is, the  $j$ th industry is **profitable** if the total value of the outputs of all  $m$  industries needed to produce one unit of value of output of the industry  $j$  is less than one.

## A More General I/O Model

- $(I - A)^{-1}\mathbf{x} = \mathbf{d}$
- Let  $\tilde{A} = (I - A)^{-1} \implies \tilde{A}\mathbf{x} = \mathbf{d}$
- We can relax this assumption to say that net production of commodities must be greater or equal to demand:

$$\tilde{A}\mathbf{x} \leq \mathbf{d}$$

- The total cost of the economy is  $\sum_j x_j$ , hence:

$$\begin{array}{lll} \min & \mathbf{1}^T \mathbf{x} \\ & \tilde{A}\mathbf{x} \geq \mathbf{d} \\ & \mathbf{x} \geq 0 \end{array}$$

# Decision Support Tools

- So far we considered a full economic system (country, region) and the decision making from the point of view of a planning Government

I/O Models and Linear Systems of Equations

- Now, let's consider the Planning of Activities by a single Firm.

A firm have can produce multifunctional outputs in many different ways. The planning problem is characterized by a large number of feasible ways of providing the same output.

Eg: Supply chain management, logistics, production scheduling

Linear Programming

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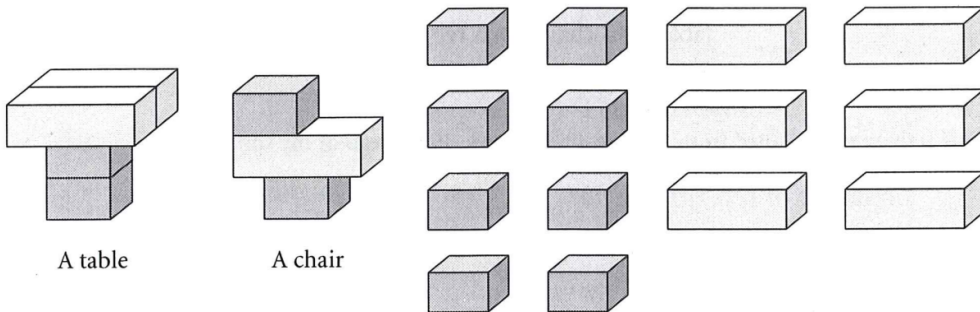
## 5. Integer Programming

# Production Planning

Suppose a company produces only **tables** and **chairs**.

A table is made of 2 large Lego pieces and 2 small pieces, while a chair is made of 1 large and 2 small pieces.

The resources available are 8 small and 6 large pieces.



The profit for a table is 1600 dkk and for a chair 1000 dkk. What product mix maximizes the company's profile using the available resources?

# Mathematical Model

	Tables	Chairs	Capacity
Small Pieces	2	2	8
Large Pieces	2	1	6
Profit	16	10	

## Decision Variables

$x_1 \geq 0$  units of small pieces

$x_2 \geq 0$  units of large pieces

## Object Function

$\max 16x_1 + 10x_2$  maximize profit

## Constraints

$2x_1 + 2x_2 \leq 8$  small pieces capacity

$2x_1 + x_2 \leq 6$  large pieces capacity



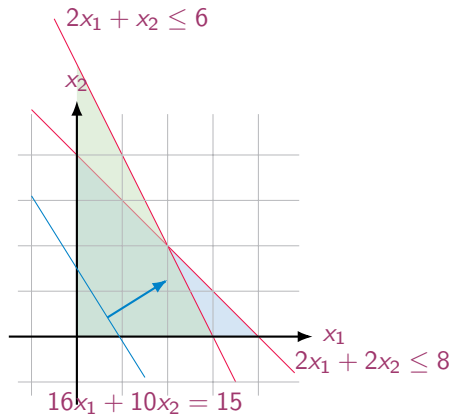
# Mathematical Model

Materials A and B  
Products 1 and 2

$$\begin{aligned} \max \quad & 16x_1 + 10x_2 \\ & 2x_1 + 2x_2 \leq 8 \\ & 2x_1 + x_2 \leq 6 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

$a_{ij}$	1	2	$b_i$
A	2	2	8
B	2	1	6
$c_j$	16	10	

Graphical Representation:



# Resource Allocation - General Model

Managing a production facility

$1, 2, \dots, n$  products

$1, 2, \dots, m$  materials

$b_i$  units of raw material at disposal

$a_{ij}$  units of raw material  $i$  to produce one unit of product  $j$

$\sigma_j$  market price of unit of  $j$ th product

$\rho_i$  prevailing market value for material  $i$

$c_j = \sigma_j - \sum_{i=1}^n \rho_i a_{ij}$  profit per unit of product  $j$

$x_j$  amount of product  $j$  to produce

$$\begin{aligned}
 \max \quad & c_1 x_1 + c_2 x_2 + c_3 x_3 + \dots + c_n x_n = z \\
 \text{subject to} \quad & a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots + a_{1n} x_n \leq b_1 \\
 & a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots + a_{2n} x_n \leq b_2 \\
 & \dots \\
 & a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \dots + a_{mn} x_n \leq b_m \\
 & x_1, x_2, \dots, x_n \geq 0
 \end{aligned}$$

# Notation

$$\begin{aligned} \max \quad & c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n = z \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n \leq b_2 \\ & \dots \\ & a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \dots, x_n \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

## In Matrix Form

$$\begin{aligned}
 \max \quad & c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n = z \\
 \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n \leq b_1 \\
 & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n \leq b_2 \\
 & \dots \\
 & a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n \leq b_m \\
 & x_1, x_2, \dots, x_n \geq 0
 \end{aligned}$$

$$c^T = [c_1 \ c_2 \ \dots \ c_n]$$

$$\begin{aligned}
 \max \quad & z = c^T x \\
 & Ax \leq b \\
 & x \geq 0
 \end{aligned}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{31} & a_{32} & \dots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

## Our Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

$$\begin{aligned} \max \quad & 16x_1 + 10x_2 \\ & 2x_1 + 2x_2 \leq 8 \\ & 2x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$$x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

$$\max \quad \begin{bmatrix} 16 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

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# Duality

**Resource Valuation problem:** Determine the value of the raw materials on hand such that:

(i) it would be convenient selling and (ii) an outside company would be willing to buy them.

- $z_i$  value of a unit of raw material  $i$
- $\sum_{i=1}^m b_i z_i$  total expenses for buying (or opportunity cost, cost of having instead of selling)
- $\rho_i$  prevailing unit market value of material  $i$
- $\sigma_j$  prevailing unit product price

Goal: for the outside company to minimize the total expenses;

(for the owing company the minimum amount of opportunity cost to accept for selling)

$$\min \sum_{i=1}^m b_i z_i \tag{1}$$

$$z_i \geq \rho_i, \quad i = 1 \dots m \tag{2}$$

$$\sum_{i=1}^m z_i a_{ij} \geq \sigma_j, \quad j = 1 \dots n \tag{3}$$

(2) otherwise selling to someone else and (3) otherwise not selling

Let

$$y_i = z_i - \rho_i$$

markup that the company would make by reselling the raw material instead of producing.

$$\begin{aligned} \min \quad & \sum_{i=1}^m y_i b_i + \cancel{\sum_i \rho_i b_i} \\ & \sum_{i=1}^m y_i a_{ij} \geq c_j, \quad j = 1 \dots n \\ & y_i \geq 0, \quad i = 1 \dots m \end{aligned}$$

Dual Problem

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

Primal Problem



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# The Diet Problem (Blending Problems)

- Select a set of foods that will satisfy a set of daily nutritional requirement at minimum cost.
- Motivated in the 1930s and 1940s by US army.
- Formulated as a **linear programming problem** by George Stigler
- First **linear programming problem** solved
- (programming intended as planning not computer code)

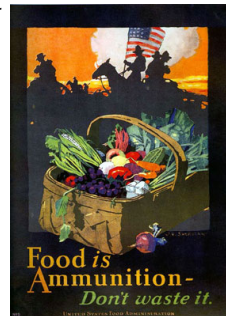
min cost/weight

subject to nutrition requirements:

eat enough but not too much of Vitamin A

eat enough but not too much of Sodium

eat enough but not too much of Calories



# The Diet Problem

Suppose there are:

- 3 foods available: corn, milk, and bread,
- there are restrictions on the number of calories (between 2000 and 2250) and the amount of Vitamin A (between 5000 and 50 000)

Food	Corn	2% Milk	Wheat bread
Vitamin A	107	500	0
Calories	72	121	65
Cost per serving	\$0.18	\$0.23	\$0.05

# The Mathematical Model

## Parameters (given data)

$F$  = set of foods

$N$  = set of nutrients

$a_{ij}$  = amount of nutrient  $j$  in food  $i$ ,  $\forall i \in F, \forall j \in N$

$c_i$  = cost per serving of food  $i$ ,  $\forall i \in F$

$F_{\min i}$  = minimum number of required servings of food  $i$ ,  $\forall i \in F$

$F_{\max i}$  = maximum allowable number of servings of food  $i$ ,  $\forall i \in F$

$N_{\min j}$  = minimum required level of nutrient  $j$ ,  $\forall j \in N$

$N_{\max j}$  = maximum allowable level of nutrient  $j$ ,  $\forall j \in N$

## Decision Variables

$x_i$  = number of servings of food  $i$  to purchase/consume,  $\forall i \in F$

# The Mathematical Model

**Objective Function:** Minimize the total cost of the food

$$\text{Minimize } \sum_{i \in F} c_i x_i$$

**Constraint Set 1:** For each nutrient  $j \in N$ , at least meet the minimum required level

$$\sum_{i \in F} a_{ij} x_i \geq N_{\min j}, \quad \forall j \in N$$

**Constraint Set 2:** For each nutrient  $j \in N$ , do not exceed the maximum allowable level.

$$\sum_{i \in F} a_{ij} x_i \leq N_{\max j}, \quad \forall j \in N$$

**Constraint Set 3:** For each food  $i \in F$ , select at least the minimum required number of servings

$$x_i \geq F_{\min i}, \quad \forall i \in F$$

**Constraint Set 4:** For each food  $i \in F$ , do not exceed the maximum allowable number of servings.

$$x_i \leq F_{\max i}, \quad \forall i \in F$$

# The Mathematical Model

system of equalities and inequalities

$$\begin{aligned} \min \quad & \sum_{i \in F} c_i x_i \\ \sum_{i \in F} a_{ij} x_i & \geq N_{\min j}, \quad \forall j \in N \\ \sum_{i \in F} a_{ij} x_i & \leq N_{\max j}, \quad \forall j \in N \\ x_i & \geq F_{\min i}, \quad \forall i \in F \\ x_i & \leq F_{\max i}, \quad \forall i \in F \end{aligned}$$

# Budget Allocation

- A company has six different **opportunities** to invest money.
- Each opportunity requires a certain investment over a period of 6 years or less.

Expected Investment Cash Flows and Net Present Value							
	Opp. 1	Opp. 2	Opp. 3	Opp. 4	Opp. 5	Opp. 6	Budget
Year 1	-\$5.00	-\$9.00	-\$12.00	-\$7.00	-\$20.00	-\$18.00	\$45.00
Year 2	-\$6.00	-\$6.00	-\$10.00	-\$5.00	\$6.00	-\$15.00	\$30.00
Year 3	-\$16.00	\$6.10	-\$5.00	-\$20.00	\$6.00	-\$10.00	\$20.00
Year 4	\$12.00	\$4.00	-\$5.00	-\$10.00	\$6.00	-\$10.00	\$0.00
Year 5	\$14.00	\$5.00	\$25.00	-\$15.00	\$6.00	\$35.00	\$0.00
Year 6	\$15.00	\$5.00	\$15.00	\$75.00	\$6.00	\$35.00	\$0.00
NPV	\$8.01	\$2.20	\$1.85	\$7.51	\$5.69	\$5.93	

- The company has an investment budget that needs to be met for each year.
- It also has the wish of investing in those opportunities that maximize the combined **Net Present Value** (NPV) after the 6th year.

## Digression: What is the Net Present Value?

- $P$ : value of the original payment presently due
- the debtor wants to delay the payment for  $t$  years,
- let  $r$  be the market rate of return that the creditor would obtain from a similar investment asset
- the future value of  $P$  is  $F = P(1 + r)^t$

Viceversa, consider the task of finding:

- the present value  $P$  of \$100 that will be received in five years, or equivalently,
- which amount of money today will grow to \$100 in five years when subject to a constant discount rate.

Assuming a 5% per year interest rate, it follows that

$$P = \frac{F}{(1 + r)^t} = \frac{\$100}{(1 + 0.05)^5} = \$78.35.$$



# Budget Allocation

Net Present Value calculation:

for each opportunity we calculate the NPV at time zero (the time of decision) as:

$$P_0 = \sum_{t=1}^5 \frac{F_t}{(1 + 0.05)^t}$$

Expected Investment Cash Flows and Net Present Value							
	Opp. 1	Opp. 2	Opp. 3	Opp. 4	Opp. 5	Opp. 6	Budget
Year 1	-\$5.00	-\$9.00	-\$12.00	-\$7.00	-\$20.00	-\$18.00	\$45.00
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Year 3	-\$16.00	\$6.10	-\$5.00	-\$20.00	\$6.00	-\$10.00	\$20.00
Year 4	\$12.00	\$4.00	-\$5.00	-\$10.00	\$6.00	-\$10.00	\$0.00
Year 5	\$14.00	\$5.00	\$25.00	-\$15.00	\$6.00	\$35.00	\$0.00
Year 6	\$15.00	\$5.00	\$15.00	\$75.00	\$6.00	\$35.00	\$0.00
NPV	\$8.01	\$2.20	\$1.85	\$7.51	\$5.69	\$5.93	

# Budget Allocation - Mathematical Model

- Let  $B_t$  be the budget available for investments during the years  $t = 1..5$ .
- Let  $a_{tj}$  be the cash flow for opportunity  $j$  and  $c_j$  its NPV
- Task: choose a set of opportunities such that the budget is never exceeded and the expected return is maximized. Consider both the case of indivisible and divisible opportunities.

Variables  $x_j = 1$  if opportunity  $j$  is selected and  $x_j = 0$  otherwise,  $j = 1..6$

Objective

$$\max \sum_{j=1}^6 c_j x_j$$

Constraints

$$\sum_{j=1}^6 a_{tj} x_j + B_t \geq 0 \quad \forall t = 1..5$$

# Linear Programming

## The Syntax of a Linear Programming Problem

objective func.	$\max / \min$	$\mathbf{c}^T \mathbf{x}$	$\mathbf{c} \in \mathbb{R}^n$
constraints	s.t.	$A\mathbf{x} \begin{matrix} \geq \\ \leq \\ = \end{matrix} \mathbf{b}$ $\mathbf{x} \geq \mathbf{0}$	$A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$ $\mathbf{x} \in \mathbb{R}^n, \mathbf{0} \in \mathbb{R}^n$

Essential features: continuity, linearity (proportionality and additivity), certainty of parameters

- Any vector  $\mathbf{x} \in \mathbb{R}^n$  satisfying all constraints is a **feasible solution**.
- Each  $\mathbf{x}^* \in \mathbb{R}^n$  that gives the best possible value for  $\mathbf{c}^T \mathbf{x}$  among all feasible  $\mathbf{x}$  is an **optimal solution** or **optimum**
- The value  $\mathbf{c}^T \mathbf{x}^*$  is the **optimum value**

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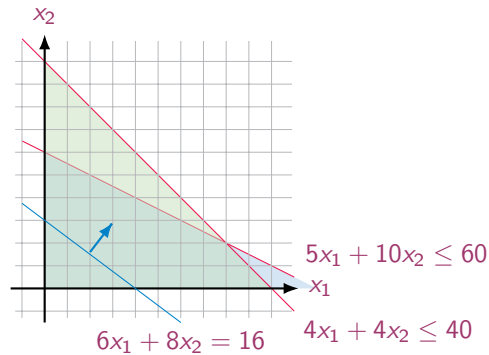
# Mathematical Model

Machines/Materials A and B  
Products 1 and 2

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

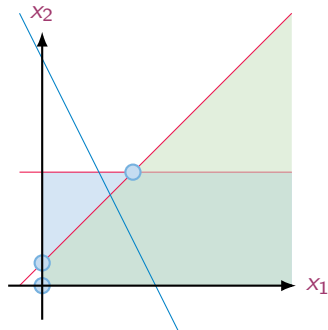
$a_{ij}$	1	2	$b_i$
A	5	10	60
B	4	4	40
$c_j$	6	8	

Graphical Representation:



# Unboundedness

$$\begin{array}{ll}\max & 2x_1 + x_2 \\ & x_2 \leq 5 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0\end{array}$$



# Infinite Solutions

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\mathbf{x} = \sum_i \alpha_i \mathbf{x}_i$$

$$\alpha_i \geq 0$$

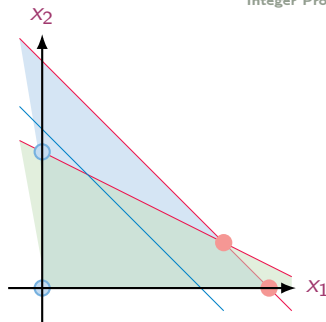
$$\sum_i \alpha_i = 1$$

$$\mathbf{x}_1^T = [8, 2, 0, 0]$$

$$\mathbf{x}_2^T = [10, 0, 10, 0]$$

$$\alpha_1 = \alpha$$

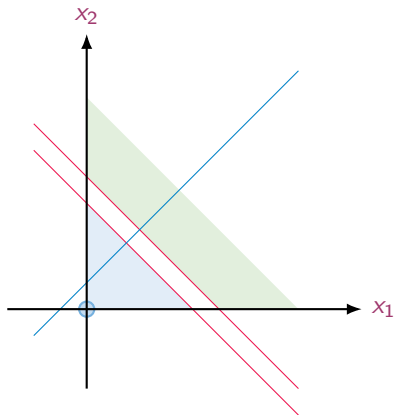
$$\alpha_2 = 1 - \alpha$$



$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} 8 \\ 2 \\ 0 \\ 0 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} 10 \\ 0 \\ 10 \\ 0 \end{bmatrix}$$

# Infeasibility

$$\begin{array}{llll} \max & x_1 & - & x_2 \\ & x_1 & + & x_2 \leq 2 \\ & 2x_1 & + & 2x_2 \geq 5 \\ & x_1, x_2 & \geq & 0 \end{array}$$





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  - Production Planning
  - Duality — Economic Interpretation
2. Other Applications
3. Geometrical Interpretation
4. Theoretical Background
  - Basic Feasible Solutions
5. Integer Programming

# Definitions

- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$  closed interval  
 $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$  open interval
- linear combination

$$\begin{aligned} \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k &\in \mathbb{R}^n \\ \boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_k]^T &\in \mathbb{R}^k \end{aligned} \quad \mathbf{x} = \lambda_1 \mathbf{v}_1 + \dots + \lambda_k \mathbf{v}_k = \sum_{i=1}^k \lambda_i \mathbf{v}_i$$

moreover:

$$\boldsymbol{\lambda} \geq \mathbf{0}$$

$$\boldsymbol{\lambda}^T \mathbf{1} = 1$$

$$\boldsymbol{\lambda} \geq \mathbf{0} \quad \text{and} \quad \boldsymbol{\lambda}^T \mathbf{1} = 1$$

conic combination

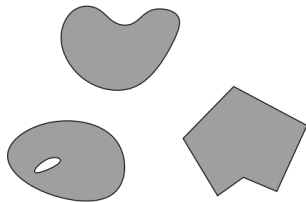
affine combination

convex combination

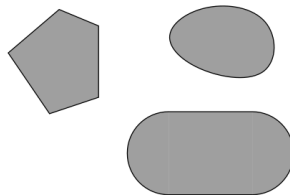
$$\left( \sum_{i=1}^k \lambda_i = 1 \right)$$

# Definitions

- **convex set**: if  $\mathbf{x}, \mathbf{y} \in S$  and  $0 \leq \lambda \leq 1$  then  $\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in S$



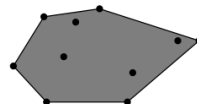
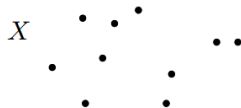
nonconvex



convex

# Definitions

- For a set of points  $S \subseteq \mathbb{R}^n$ 
  - $\text{lin}(S)$  linear hull (span)
  - $\text{cone}(S)$  conic hull
  - $\text{aff}(S)$  affine hull
  - $\text{conv}(S)$  convex hull



the convex hull of  $X$

$$\text{conv}(X) = \{ \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \dots + \lambda_n \mathbf{x}_n \mid \mathbf{x}_i \in X, \lambda_1, \dots, \lambda_n \geq 0 \text{ and } \sum_i \lambda_i = 1 \}$$

# Definitions

- $G \subseteq \mathbb{R}^n$  is an **hyperplane** if  $\exists \mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$  and  $\alpha \in \mathbb{R}$ :

$$G = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T \mathbf{x} = \alpha\}$$

- $H \subseteq \mathbb{R}^n$  is an **halfspace** if  $\exists \mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$  and  $\alpha \in \mathbb{R}$ :

$$H = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T \mathbf{x} \leq \alpha\}$$

( $\mathbf{a}^T \mathbf{x} = \alpha$  is a supporting hyperplane of  $H$ )

# Definitions

- a set  $S \subset \mathbb{R}^n$  is a **polyhedron** if  $\exists m \in \mathbb{Z}^+, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$ :

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\} = \bigcap_{i=1}^m \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_{i,\cdot} \mathbf{x} \leq b_i\}$$

i.e., a polyhedron  $P \neq \mathbb{R}^n$  is determined by finitely many halfspaces

- a polyhedron  $P$  is a **polytope** if it is bounded:  $\exists B \in \mathbb{R}, B > 0$ :

$$P \subseteq \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq B\}$$

( $\|\mathbf{x}\| = \sqrt{\sum_{j=1}^n x_j^2}$  is the Euclidean norm of the vector  $\mathbf{x} \in \mathbb{R}$ )

- A set of vectors is a **polytope** if it is the convex hull of finitely many vectors.

# Linear Programming Problem

**Input:** a matrix  $A \in \mathbb{R}^{m \times n}$  and column vectors  $\mathbf{b} \in \mathbb{R}^m$ ,  $\mathbf{c} \in \mathbb{R}^n$

**Task:**

1. decide that  $\{\mathbf{x} \in \mathbb{R}^n; A\mathbf{x} \leq \mathbf{b}\}$  is empty (prob. infeasible), or
2. find a column vector  $\mathbf{x} \in \mathbb{R}^n$  such that  $A\mathbf{x} \leq \mathbf{b}$  and  $\mathbf{c}^T \mathbf{x}$  is max, or
3. decide that for all  $\alpha \in \mathbb{R}$  there is an  $\mathbf{x} \in \mathbb{R}^n$  with  $A\mathbf{x} \leq \mathbf{b}$  and  $\mathbf{c}^T \mathbf{x} > \alpha$  (prob. unbounded)

1.  $F = \emptyset$
2.  $F \neq \emptyset$  and  $\exists$  solution
  1. one solution
  2. infinite solutions
3.  $F \neq \emptyset$  and  $\nexists$  solution

# Fundamental Theorem of LP

## Theorem (Fundamental Theorem of Linear Programming)

Given:

$$\min\{\mathbf{c}^T \mathbf{x} \mid \mathbf{x} \in P\} \text{ where } P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$$

If  $P$  is a bounded polyhedron and not empty and  $\mathbf{x}^*$  is an optimal solution to the problem, then:

- $\mathbf{x}^*$  is an extreme point (vertex) of  $P$ , or
- $\mathbf{x}^*$  lies on a face  $F \subset P$  of optimal solutions



Proof idea:

- assume  $\mathbf{x}^*$  not a vertex of  $P$  then  $\exists$  a ball around it still in  $P$ . Show that a point in the ball has better cost
- if  $\mathbf{x}^*$  is not a vertex then it is a convex combination of vertices. Show that all points are also optimal.



## Implications:

- the optimal solution is at the intersection of supporting hyperplanes.
- hence finitely many possibilities
- solution method: write all inequalities as equalities and solve all  $\binom{m}{n}$  systems of linear equalities ( $n$  # variables,  $m$  # equality constraints)
- for each point we then need to check if feasible and if best in cost.
- each system is solved by Gaussian elimination
- Stirling approximation:

$$\binom{2m}{m} \approx \frac{4^m}{\sqrt{\pi m}} \text{ as } m \rightarrow \infty$$

# Standard Form

Every LP problem can be converted in the **standard form**:

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

$$\mathbf{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

- if equations, then put two constraints,  
 $\mathbf{a}\mathbf{x} \leq b$  and  $\mathbf{a}\mathbf{x} \geq b$
- if  $\mathbf{a}\mathbf{x} \geq b$  then  $-\mathbf{a}\mathbf{x} \leq -b$
- if  $\min \mathbf{c}^T \mathbf{x}$  then  $\max(-\mathbf{c}^T \mathbf{x})$

and then be put in **equational standard form**:

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$$\mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

1. “=” constraints
2.  $\mathbf{x} \geq \mathbf{0}$  nonnegativity constraints
3. ( $\mathbf{b} \geq \mathbf{0}$ )
4. max

# Transformation to Std Form

Every LP problem can be transformed in eq. std. form

1. introduce **slack variables** (or surplus)

$$5x_1 + 10x_2 + x_3 = 60$$

$$4x_1 + 4x_2 + x_4 = 40$$

2. if  $x_1 \geq 0$  then  $x_1 = x'_1 - x''_1$   
 $x'_1 \geq 0$   
 $x''_1 \geq 0$

3. ( $b \geq 0$ )

4.  $\min c^T x \equiv \max(-c^T x)$

LP in  $m \times n$  converted into LP with at most  $(m + 2n)$  variables and  $m$  equations ( $n \neq$  original variables,  $m \neq$  constraints)

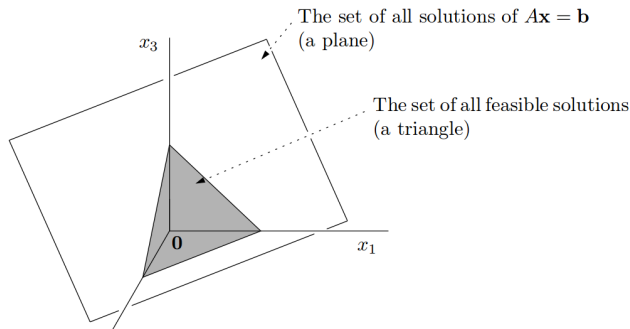
# Geometry of LP in Eq. Std. Form

$$\max\{\mathbf{c}^T \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

In  $\mathbb{R}^3$ :

From linear algebra:

- the set of solutions of  $A\mathbf{x} = \mathbf{b}$  is an affine space (hyperplane not passing through the origin).
- $\mathbf{x} \geq \mathbf{0}$  nonnegative orthant (octant in  $\mathbb{R}^3$ )



- $Ax = b$  is a system of equations that we can solve by Gaussian elimination
- Elementary row operations of  $[A \mid b]$  do not affect set of feasible solutions
  - multiplying all entries in some row of  $[A \mid b]$  by a nonzero real number  $\lambda$
  - replacing the  $i$ th row of  $[A \mid b]$  by the sum of the  $i$ th row and  $j$ th row for some  $i \neq j$
- Let  $n'$  be the number of vars in eq. std. form.

we assume  $n' \geq m$  and  $\text{rank}([A \mid b]) = \text{rank}(A) = m$

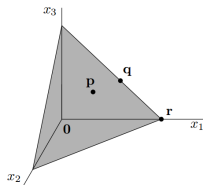
ie, rows of  $A$  are linearly independent  
otherwise, remove linear dependent rows

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# Basic Feasible Solutions

Basic feasible solutions are the vertices of the feasible region:



More formally:

Let  $B = \{1 \dots m\}$ ,  $N = \{m+1 \dots n+m = n'\}$  be subsets partitioning the columns of  $A$ :  $A_B$  be made of columns of  $A$  indexed by  $B$ :

## Definition

$\mathbf{x} \in \mathbb{R}^n$  is a **basic feasible solution** of the program  $\max\{\mathbf{c}^T \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$  for an index set  $B$  if:

- $x_j = 0$  for all  $j \notin B$
- the square matrix  $A_B$  is nonsingular, ie, all columns indexed by  $B$  are lin. indep.
- $\mathbf{x}_B = A_B^{-1} \mathbf{b}$  is nonnegative, ie,  $\mathbf{x}_B \geq \mathbf{0}$  (feasibility)

We call  $x_j$  for  $j \in B$  **basic variables** and remaining variables **nonbasic variables**.

### Theorem

A **basic feasible solution** is uniquely determined by the set  $B$ .

Proof:

$$Ax = A_B x_B + A_N x_N = b$$

$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$

$$x_B = A_B^{-1} b$$

$A_B$  is nonsingular hence one solution

Note: we call  $B$  a **(feasible) basis**



Extreme points and basic feasible solutions are geometric and algebraic manifestations of the same concept:

### Theorem

Let  $P$  be a (convex) polyhedron from LP in std. form. For a point  $v \in P$  the following are equivalent:

- (i)  $v$  is an extreme point (vertex) of  $P$
- (ii)  $v$  is a basic feasible solution of LP in eq. std. form

### Theorem

Let  $LP = \max\{c^T x \mid Ax = b, x \geq 0\}$  be feasible and bounded, then there is an optimal solution that is a basic feasible solution.

Proof. consequence of previous theorem and fundamental theorem of linear programming  
However, an optimal solution does not need to be basic, eg,  $\max\{x_1 + x_2 \mid x_1 + x_2 \leq 1\}$ .

- Idea for solution method:
- examine all basic solutions.
- There are finitely many:  $\binom{m+n}{m}$ .
- However, if  $n = m$  then  $\binom{2m}{m} \approx 4^m$ .

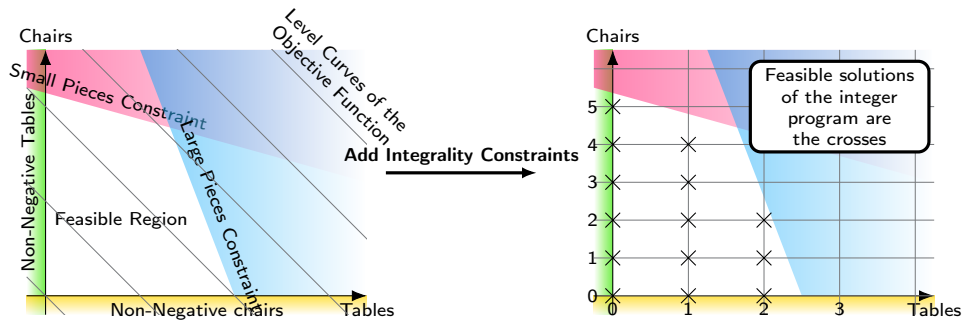
# Simplex Method

1. find a solution that is at the intersection of some  $n$  hyperplanes
2. try systematically to produce the other points by exchanging one hyperplane with another
3. check optimality, proof provided by duality theory

# Outline

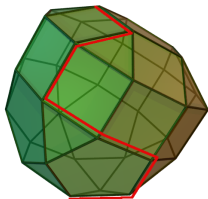
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# Integrality Requirements

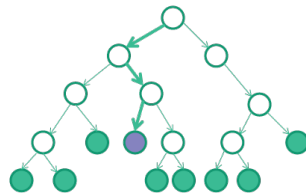
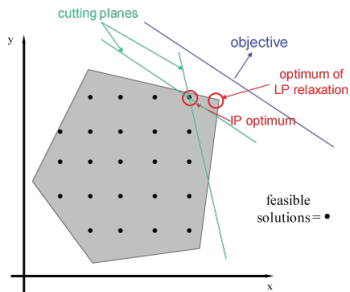


# Solution Approaches

## Linear Programming:



## Integer Programming:



Each node in branch-and-bound is a new MIP

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