Recap:

α-approx. alg.
Approx. factor / ratio

Set Cover
Vertex Cover

f-approx. alg (LP-rounding)
Primal $\leftrightarrow$ Dual

$$\begin{aligned}
\min \quad & c_1x_1+\dots+c_nx_n \\
\text{s.t.} \quad & a_{11}x_1+\dots+a_{1n}x_n \geq b_1 \\
& \vdots \\
& a_{m1}x_1+\dots+a_{mn}x_n \geq b_m \\
& x_1,\dots,x_n \geq 0
\end{aligned}$$

$$\begin{aligned}
\max \quad & b_1y_1+\dots+b_my_m \\
& a_{11}y_1+\dots+a_{m1}y_m \leq c_1 \\
& \vdots \\
& a_{1n}y_1+\dots+a_{mn}y_m \leq c_n \\
& y_1,\dots,y_m \geq 0
\end{aligned}$$

Strong duality:

For any pair $\vec{x},\vec{y}$ of solutions,
$$c_1x_1+\dots+c_nx_n \leq b_1y_1+\dots+b_my_n$$
Weak duality

$\exists$ a pair $\vec{x}^*, \vec{y}^*$ of sol. s.t.
$$c_1x^*+\dots+c_n^*x_n^* = b_1y_1^*+\dots+b_my_n^*$$

Ex:

$$\begin{aligned} \min\ & 7x_1 + x_2 + 5x_3 \\ \text{s.t.}\ & x_1 - x_2 + 3x_3 \geq 10 \\ & 5x_1 + 2x_2 - x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq y_1(x_1 - x_2 + 3x_3) + y_2(5x_1 + 2x_2 - x_3) \\ &= (y_1 + 5y_2)x_1 + (-y_1 + 2y_2)x_2 + (3y_1 - y_2)x_3 \end{aligned}$$

$$\begin{aligned} \max\ & 10y_1 + 6y_2 \\ \text{s.t.}\ & y_1 + 5y_2 \leq 7 \\ & -y_1 + 2y_2 \leq 1 \\ & 3y_1 - y_2 \leq 5 \\ & y_1, y_2 \geq 0 \end{aligned}$$

$$\begin{cases} y_1 = 0 \\ 5x_1 + 2x_2 - x_3 = 6 \\ \\ y_1 + 5y_2 = 7 \\ x_2 = 0 \\ 3y_1 - y_2 = 5 \end{cases}$$

$\Downarrow$

$$\begin{aligned} \overbrace{10y_1}^{0} + 6y_2 &= \overbrace{y_1(x_1 - x_2 + 3x_3)}^{0} + \overbrace{y_2(5x_1 + 2x_2 - x_3)}^{6} \\ &= \underbrace{(y_1 + 5y_2)}_{7}x_1 + \underbrace{(-y_1 + 2y_2)x_2}_{0} + \underbrace{(3y_1 - y_2)}_{5}x_3 \\ &= 7x_1 + \underbrace{x_2}_{0} + 5x_3 \end{aligned}$$

More generally:

$$7x_1 + x_2 + 5x_3 = 10y_1 + 6y_2$$

$$\Updownarrow$$

Complementary Slackness Conditions

$$\begin{cases} x_1 > 0 \Rightarrow y_1 + 5y_2 = 7 \\ x_2 > 0 \Rightarrow -y_1 + 2y_2 = 1 \\ x_3 > 0 \Rightarrow 3y_1 - y_2 = 5 \end{cases} \quad \text{primal c.s.c.}$$

$$\begin{cases} y_1 > 0 \Rightarrow x_1 - x_2 + 3x_3 = 10 \\ y_2 > 0 \Rightarrow 5x_1 + 2x_2 - x_3 = 6 \end{cases} \quad \text{dual c.s.c.}$$

By the Strong Duality Theorem (which we will not prove), there exist solutions fulfilling the c.s.c.

Moreover, if the c.s.c. are „close" to being satisfied, the values of the primal and dual sol. are „close":

Relaxed Complementary Slackness Conditions

$$\begin{cases} x_1 > 0 \Rightarrow y_1 + 5y_2 \geq 7/b \\ x_2 > 0 \Rightarrow -y_1 + 2y_2 \geq 1/b \\ x_3 > 0 \Rightarrow 3y_1 - y_2 \geq 5/b \\ y_1 > 0 \Rightarrow x_1 - x_2 + 3x_3 \leq 10c \\ y_2 > 0 \Rightarrow 5x_1 + 2x_2 - x_3 \leq 6c \end{cases}$$

$$\Downarrow$$

$$7x_1 + x_2 + 5x_3 \leq bc(10y_1 + 6y_2)$$

Ex:

$$\begin{cases} y_1 = 0 \\ 5x_1 + 2x_2 - x_3 \leq 2\cdot 6 \\ y_1 + 5y_2 \geq 7/3 \\ x_2 = 0 \\ 3y_1 - y_2 \geq 5/3 \end{cases}$$

$\Downarrow$

$$2\cdot(\overbrace{10y_1}^{0} + 6y_2) \geq \overbrace{y_1(x_1 - x_2 + 3x_3)}^{=0} + \overbrace{y_2(5x_1 + 2x_2 - x_3)}^{\leq 2\cdot 6y_2}$$

$$= \underbrace{(y_1 + 5y_2)x_1}_{\geq 7/3\, x_1} + \underbrace{(-y_1 + 2y_2)x_2}_{=0} + \underbrace{(3y_1 - y_2)x_3}_{\geq 5/3\, x_3}$$

$$\geq \frac{1}{3}(7x_1 + \underbrace{x_2}_{=0} + 5x_3)$$

$\Downarrow$

$$2\cdot 3(10y_1 + 6y_2) \geq 7x_1 + x_2 + 5x_3$$

Sheet 1

a) LP-formulation of unweighted Vertex Cover

$$\min \sum_{v \in V} x_v$$

$$\text{s.t.} \quad x_u + x_v \geq 1, \qquad (u,v) \in E$$

$$x_v \geq 0, \qquad v \in V$$

b) Dual LP

$$\max \sum_{e \in E} y_e$$

$$\text{s.t.} \quad \sum_{(u,v) \in E} y_{(u,v)} \leq 1, \quad u \in V$$

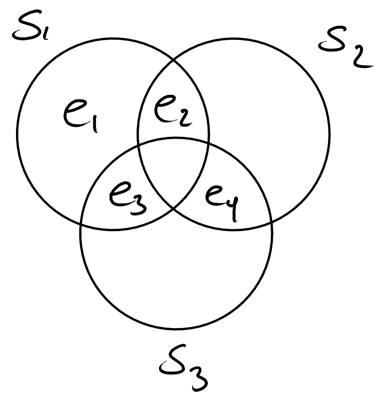
$$y_e \geq 0, \qquad e \in E$$

c) Which combinatorial problem?

Unweighted Matching (Max. Cardinality Matching)

What is the dual of the Set Cover LP?

Ex:



$w_1 = 1$

$w_2 = 2$

$w_3 = 3$

Primal:

$$\begin{aligned} \min \quad & x_1 + 2x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 \geq 1 \\ & x_1 + x_2 \geq 1 \\ & x_1 + x_3 \geq 1 \\ & x_2 + x_3 \geq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

OPT = 3: $x_1 = x_2 = 1$

Dual:

$$\begin{aligned} \max \quad & y_1 + y_2 + y_3 + y_4 \\ \text{s.t.} \quad & y_1 + y_2 + y_3 \leq 1 \\ & y_2 + y_4 \leq 2 \\ & y_3 + y_4 \leq 3 \\ & y_1, y_2, y_3, y_4 \geq 0 \end{aligned}$$

OPT = 3: $y_1 = 1$, $y_4 = 2$ or $y_3 = 1$, $y_4 = 2$

Set Cover Primal

$$\min \sum_{j=1}^{m} x_j w_j$$

$$\text{s.t.} \sum_{j:\, e_i \in S_j} x_j \geq 1, \qquad i = 1, 2, \ldots, n$$

$$x_j \geq 0, \qquad j = 1, 2, \ldots, m$$

Covering problem

Set Cover Dual

$$\max \sum_{i=1}^{n} y_i$$

$$\text{s.t.} \sum_{e_i \in S_j} y_i \leq w_j, \qquad j = 1, 2, \ldots, m$$

$$y_i \geq 0, \qquad i = 1, 2, \ldots, n$$

Packing problem

Recall that the dual is constructed such that the value of any solution to the dual is a lower bound on the value of any solution to the primal:

$$Z_{Primal} \geq Z_{Dual} \qquad \text{(weak duality property)}$$

In fact,

$$Z^*_{Primal} = Z^*_{Dual} \qquad \text{(strong duality property)}$$

Alg. 2 for Set Cover

Solve dual LP

$I' \leftarrow \{ j \mid \sum_{e_i \in S_j} y_i = w_j \}$

In the ex. above,

with $y_1 = 1$, $y_4 = 2$, Alg 2 would choose S_1 and S_2 with a total weight of 3.

with $y_3 = 1$, $y_4 = 2$, Alg. 2 would choose S_1, S_2, ad S_3 with a total weight of 6.

The first solution is optimal, and the latter is a 2-approximation (i.e., an f-approximation).

Alg. 2 is an f-approximation algo.:

If the algo. chooses S_1, S_2, and S_3, the total weight is $W = w_1 + w_2 + w_3$, and

$$w_1 + w_2 + w_3 = (y_1 + y_2 + y_3) + (y_2 + y_4) + (y_3 + y_4),$$

since the algo. chooses exactly those sets that have LHS = RHS.

Since each y_i is present in at most f constraints,

$$W \le f \cdot (y_1 + y_2 + y_3 + y_4) = f \cdot OPT$$

Lemma 1.7

$\boxed{\text{Alg. 2 produces a set cover}}$

<u>Proof:</u>

Assume for the sake of contradiction that some element e_k is not covered by $\{S_j \mid j \in I'\}$.

Then $\sum_{e_i \in S_j} y_i < w_j$ for all S_j containing e_k.

These are exactly the constraints involving y_k.

Thus, none of the constraints involving y_k are tight.

This means that y_k can be increased without violating any constraint.

Since this will increase the value $\sum_{i=1}^{n} y_i$ of the sol., we conclude that

the solution $\vec{y}$ was not optimal. $\square$

Ex:

In the ex. above, assume

$$y_1 = y_4 = 0$$
$$y_2 = y_3 = 1/2$$

Then, only the first constraint is tight, so only S_1 is picked.

$$\begin{aligned} y_1 + y_2 + y_3 \quad &= 1 \\ y_2 \quad + y_4 &= 1/2 < 2 \\ y_3 + y_4 &= 1/2 < 3 \end{aligned}$$

y_4 is not covered, since none of the two constraints involving y_4 are tight.

We can increase y_4 from 0 to $3/2$ without violating any constraints.
This increases the sol. value from 1 to $5/2$.
Thus, the sol. above was not optimal.

This illustrates the idea of the primal-dual alg of Section 1.5 (although this alg. would not start out with the sol $y_2 = y_3 = 1/2$).

We now give a more formal proof that Alg 2 is an f-approximation algo.

Thm 1.8

$\boxed{\text{Alg. 2 is an } f\text{-approx. algo.}}$

Proof:

The correctness follows from Lemma 1.7.

Approx. guarantee:

$$\begin{aligned}
\sum_{j \in I'} w_j &= \sum_{j \in I'} \sum_{e_i \in S_j} y_i \\
&= \sum_{i=1}^{n} \underbrace{\left| \{ j \in I' \mid e_i \in S_j \} \right|}_{\#\text{sets in the sol. containing } e_i} \cdot y_i \\
&\leq \sum_{i=1}^{n} \underbrace{f_i}_{\#\text{sets containing } e_i} \cdot y_i \\
&\leq \sum_{i=1}^{n} f \cdot y_i \\
&= f \cdot OPT
\end{aligned}$$

$\square$

Note that we could also use the relaxed c.s.c. (with $b=1$, $c=f$), since

$$\sum_{j : e_i \in S_j} x_j \leq f, \quad \text{for all } i = 1, 2, \ldots, n$$

Note that, on any instance of Set Cover, $I \subseteq I'$:

Since the LP is solved optimally,

$x_j > 0 \Rightarrow$ constraint j is tight $\Rightarrow$ $j \in I'$.

Thus, $j \in I \Rightarrow x_j \geq \frac{1}{f} \Rightarrow j \in I'$

Thus, Alg. 1 is always at least as good as Alg. 2.

Both Alg. 1 and Alg. 2 rely on solving an LP. In Section 1.5, we will study a more (time) efficient version of Alg. 2.

The crux is to obtain an index set I'', s.t.

- $\bigcup_{j \in I''} S_j$ is a vertex cover
- $\sum_{j \in I''} w_j = \sum_{j \in I''} \sum_{e_i \in S_j} y_i$

without solving an LP.

Section 1.5 : A Primal-Dual Alg. for Set Cover

Alg. 1.1 for Set Cover : Primal-Dual

$I'' \leftarrow \emptyset$

$\vec{y} \leftarrow \vec{0}$

While $\exists e_k \notin \bigcup_{j \in I''} S_j$

  Increase y_k until some constraint, ℓ, becomes tight, i.e., $\sum_{e_i \in S_\ell} y_i = w_\ell$

  $I'' \leftarrow I'' \cup \{\ell\}$

Note that $e_k \in S_\ell$

Thm 1.9

Alg. 1.1 is an f-approx. alg. for Set Cover

Proof:

Alg. 1.1 produces a set cover, since as long as some element is not covered, the corresponding dual constraints are non-tight.

The approx. guarantee follows from the same calculations as in the proof of Thm. 1.8, since $\sum_{j \in I''} w_j = \sum_{j \in I''} \sum_{e_i \in S_j} y_i$

$\square$

In contrast to Alg. 2 from Section 1.4, Alg. 1.1 does not necessarily produce an optimal dual solution:

In the example above, it might do the following.

$y_2 \leftarrow 1$ (S_1 is picked, e_4 still uncovered)

$y_4 \leftarrow 1$ (S_2 is picked)

(This is fine, since the proof of Thm. 1.8 does not use that $\sum y_i = OPT$, only that $\sum y_i \leq OPT$, which is true for <u>any</u> feasible sol. to the dual.)

Section 1.6: A Greedy Algorithm

Alg 1.2 for Set Cover: Greedy

$I \leftarrow \emptyset$

For $j \leftarrow 1$ to m

$\quad \hat{S}_j \leftarrow S_j$ („uncovered" part of S_j)

While $\{S_j \mid j \in I\}$ is not a set cover

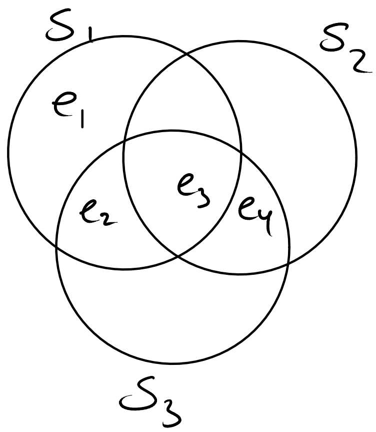
$\quad \ell \leftarrow \arg\min_{j : \hat{S}_j \neq \emptyset} \frac{w_j}{|\hat{S}_j|}$ (S_ℓ: set with smallest cost per uncovered element)

$\quad I \leftarrow I \cup \{\ell\}$

$\quad$ For $j \leftarrow 1$ to m

$\quad\quad \hat{S}_j \leftarrow \hat{S}_j - S_\ell$

Ex:



$w_1 = 12$

$w_2 = 8$

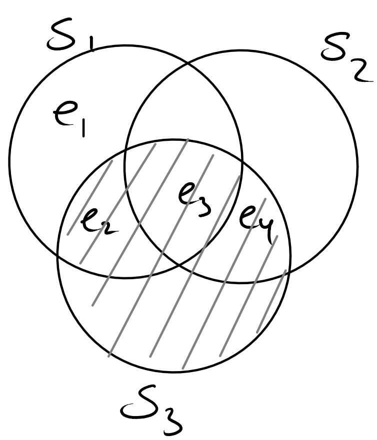
$w_3 = 9$

Let $\alpha_j = \dfrac{w_j}{|\hat{S}_j|}$

$\alpha_1 = \dfrac{12}{3} = 4, \quad \alpha_2 = \dfrac{8}{2} = 4, \quad \alpha_3 = \dfrac{9}{3} = 3$

price per element in first iteration

Pick S_3



$\alpha_1 = \dfrac{12}{1},$

price per element in second iteration

Pick S_1

Done!

$n_1 = 5, \quad n_2 = 1, \quad n_3 = 0$

$\ell = 3$

$n_1 - n_2 = 4, \quad n_3 - n_2 = 1$

Thm 1.11

Alg. 1.2 is an H_n-approx. alg. for Set Cover

Proof:

n_k: # uncovered elements at the beginning of the k'th iteration

OPT has to cover these n_k elements using only sets in $\mathcal{S}_k = \mathcal{S} - \{S_j \mid j \in I\}$, since none of them are contained in $\{S_j \mid j \in I\}$.

Hence, there must be at least one element with a price of at most OPT/n_k. Otherwise, OPT would not be able to cover the n_k elements (and certainly not all n elements) at a cost of only OPT.

Hence, the $n_k - n_{k+1}$ elements covered in iteration k cost at most $(n_k - n_{k+1})\, OPT/n_k$ per element.

Thus, the cost of the set cover produced by the greedy alg. is

$$\sum_{j \in I} w_j \le \sum_{k=1}^{\ell} \frac{n_k - n_{k+1}}{n_k} \, OPT$$

\# iterations

$$= OPT \sum_{k=1}^{\ell} \underbrace{\left(\frac{1}{n_k} + \frac{1}{n_k} + \dots + \frac{1}{n_k} \right)}_{n_k - n_{k+1} \text{ terms}}$$

$$\le OPT \sum_{k=1}^{\ell} \left(\frac{1}{n_k} + \frac{1}{n_k - 1} + \dots + \frac{1}{n_{k+1} + 1} \right)$$

$$= OPT \sum_{s=1}^{n} \frac{1}{s}$$

$$= OPT \cdot H_n \qquad \square$$

Ex from before:

$OPT = w_1 + w_2 = 12 + 8 = 20$

The cost of the greedy alg is

$$w_3 + w_1 = 9 + 12$$

$$= (3+3+3) + 12$$

$$\le \left(\frac{20}{4} + \frac{20}{4} + \frac{20}{4} \right) + \frac{20}{1}$$

$$\le \left(\frac{20}{4} + \frac{20}{3} + \frac{20}{2} \right) + \frac{20}{1}$$

$$= 20 \left(\frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1 \right)$$

Let $g = \max\{ |S_j| \mid S_j \in \mathcal{S} \}$.

Thm 1.12

Alg. 1.2 is an H_g-approx. alg. for Set Cover

<u>Proof</u>: By dual fitting.

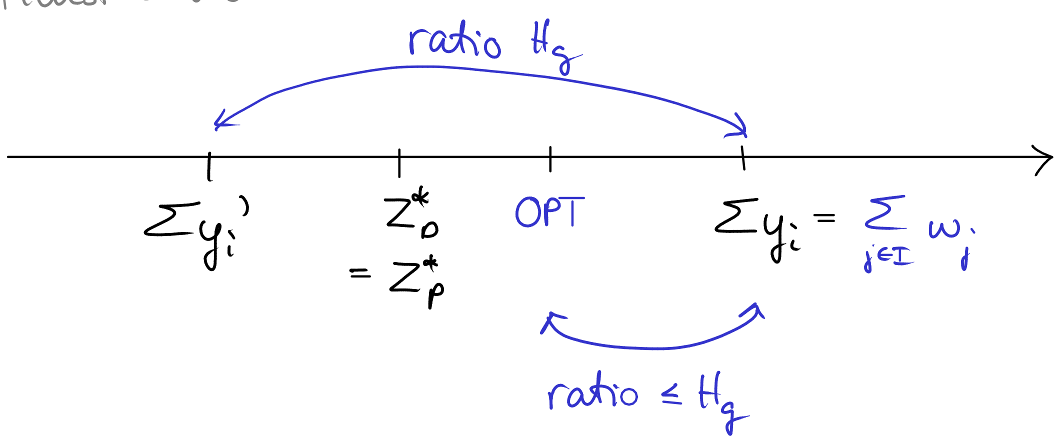
Consider the dual D of the LP for Set Cover.
We will construct an infeasible solution $\vec{y}$ to D s.t.

- $\sum_{j \in I} w_j = \sum_{i=1}^{n} y_i$
- $y_i' = \frac{y_i}{H_g}$, $1 \le i \le n$, is a feasible sol. to D

This will imply that

$$\sum_{j \in I} w_j = \sum y_i = H_g \sum y_i' = H_g Z_D^* = H_g Z_P^* \le H_g \cdot OPT$$

Illustration:



For each i, $1 \le i \le n$, we let

$$y_i = \text{price}(e_i)$$

Then

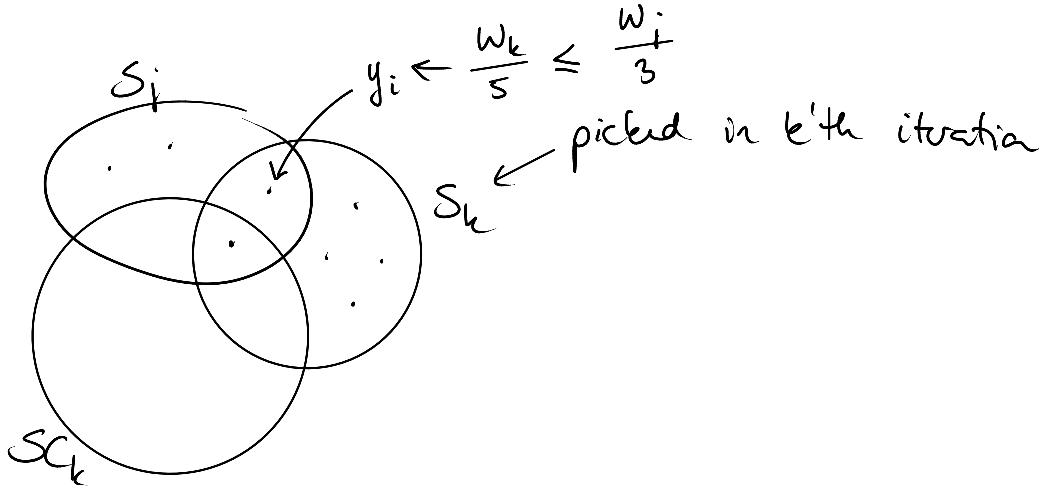
- $\sum_{j \in I} w_j = \sum_{i=1}^{n} y_i$

Hence, we just need to show that

- $\vec{y}$ is feasible,

i.e., show that

$$\sum_{e_i \in S_j} y_i \le w_j, \text{ for each set } S_j \in \mathcal{S}:$$


S_j
$y_i \leftarrow \frac{w_k}{5} \leq \frac{w_j}{3}$
picked in k'th iteration
S_k
SC_k