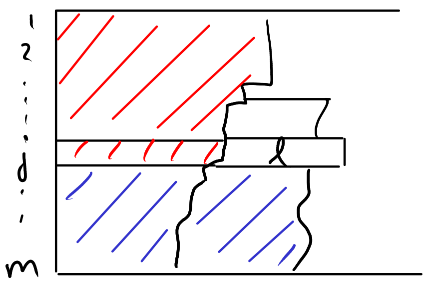
Exercise 2.2:

If $p_\ell > \frac{1}{3} \cdot OPT$, LPT produces an optimal schedule.

Proof:

$n \leq 2m$



Assume for the sake of contradiction $p_j + p_\ell > OPT$, ie., there is a schedule with makespan $< p_j + p_\ell$.

In that schedule no two jobs from are combined, and job ℓ is not combined with a job from .

Furthermore, no job in can be combined with a job in , since the jobs in are at least as large as job ℓ.

Thus, the j jobs in must be scheduled on separate machine, and they cannot be combined with job j or any of the $2(m-j)$ job in .

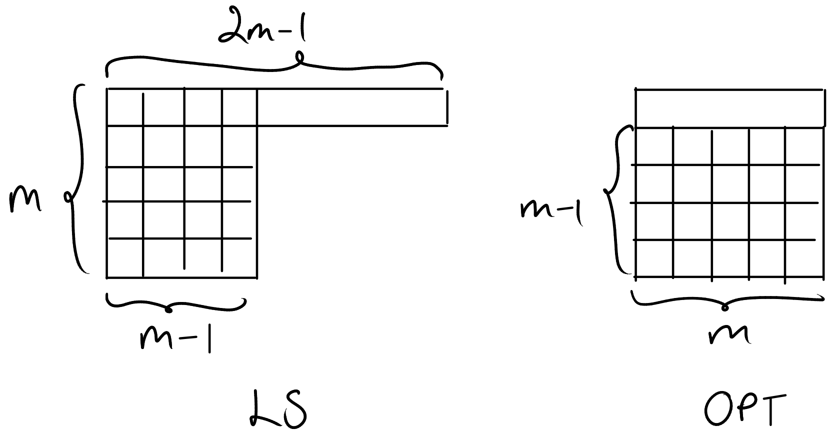
This gives a total of $2(m-j)+1$ jobs that must be scheduled on $2(m-j)$ machines.

Thus, there must be a machine with at least three jobs, and hence, a total load of more than $3 \cdot \frac{1}{3} \cdot OPT = OPT$. $\lightning$

<u>Exercise</u>: Give an instance I, where $LS(I) = (2 - \frac{1}{m}) \cdot OPT$.

$m(m-1) \times 1$

$1 \times m$



Section 3.2: Makespan Scheduling - A PTAS

Longest Processing Time:
From the proof of Thm 2.7 and Exercise 2.2, we learned that if the job l to finish last has length $p_l > \frac{1}{3} \cdot OPT$, then $LPT = OPT$.
Otherwise, $LPT \le OPT + p_l$.

Idea for PTAS:

Partition the jobs into two sets:

$$\underbrace{\overbrace{p_1, p_2, \ldots, p_x}^{> \varepsilon \cdot OPT}}_{\text{Schedule optimally,}}, \underbrace{\overbrace{p_{x+1}, \ldots, p_n}^{\le \varepsilon \cdot OPT}}_{\text{Use LPT}}$$

if $m, x \in O(1)$.
Otherwise, use
dyn. prg. as for bin packing

We will derive a family of algorithms with an algorithm for each $k \in \mathbb{Z}^+$

Let $P = \sum_{j=1}^{n} p_j$ (as before).

Job j is short, if $p_j \le \frac{P}{km}$, i.e., if it is at most $1/k$ of the average machine load.

Otherwise, it is long.

Algorithm:

Schedule long jobs first.

Then, add short jobs using LPT.

$\#$long jobs $< km$

Hence, $\#$schedules of long jobs $< m^{km}$

(choose one of m machines for each job).

Thus, if $k, m \in O(1)$, we can find an optimal schedule for the long jobs in time $O(1)$.

Otherwise, we can round job sizes and do dyn. prg. as for the bin packing problem.

The alg. will be poly. in m, but not in k.

Thus, the algorithm will be a PTAS, not an FPTAS.

Idea for the long jobs:

(1) „Guess" an optimal makespan T

(2) Round each job size down to the nearest multiple of $\frac{T}{k^2}$

(3) Use dyn. prg. to check whether $\exists$ schedule of makespan $\leq T$ for the rounded jobs.

If not, then $OPT > T$ (for the rounded job sizes, and hence, for the original job sizes)

Do binary search for T in the interval $[L, U]$, where

$$L = \max\{\lceil P/m \rceil, p_{max}\}$$

$$U = \left\lceil \frac{P - p_{max}}{m} + p_{max} \right\rceil = \left\lceil \frac{P + (m-1)p_{max}}{m} \right\rceil$$

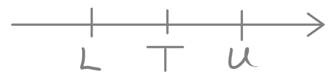
$B_k(I)$

$L \leftarrow \max\left\{ \left\lceil \frac{P}{m} \right\rceil, p_{max} \right\}$

$U \leftarrow \left\lceil \frac{P + (m-1)\,p_{max}}{m} \right\rceil$

While $L \neq U$

  $T \leftarrow \left\lceil \frac{1}{2}(L+U) \right\rceil$

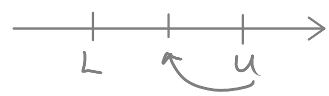


  $I_\ell \leftarrow \{ \text{job } j \in I \mid p_j > T/k \}$ // long jobs

  $I_\ell' \leftarrow I_\ell$ with each job length rounded down to nearest multiple of T/k^2

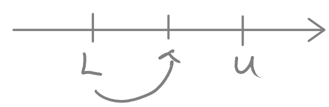
  If $\exists$ schedule S' of I_ℓ' s.t. makespan$(S') \leq T$

    $U \leftarrow T$



  Else

    $L \leftarrow T$



$S \leftarrow$ schedule of I_ℓ corresponding to S'.

Add the short jobs to S, using LPT.

Approximation ratio:

When B_k terminates the while loop,

$$\text{makespan}(S') = T = OPT(I_\ell')$$

Each job j in I_ℓ has $p_j > \frac{T}{k}$.
Since $\frac{T}{k}$ is a multiple of $\frac{T}{k^2}$, each job j in I_ℓ' has $p_j \geq \frac{T}{k}$.
Thus, S' has at most k jobs on each machine.
Hence,

$$\begin{aligned}\text{makespan}(S) &< \text{makespan}(S') + k \cdot \frac{T}{k^2} \\ &= T + \frac{T}{k} = \left(1+\frac{1}{k}\right)T \\ &= \left(1+\frac{1}{k}\right) OPT(I_\ell') \\ &\leq \left(1+\frac{1}{k}\right) OPT(I_\ell) \\ &\leq \left(1+\frac{1}{k}\right) OPT(I)\end{aligned}$$

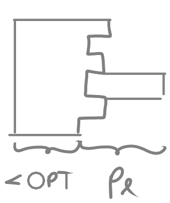
each of the at most k jobs on a machine is rounded down by less than T/k^2.

Thus, if the last job ℓ to finish belongs to I_ℓ
$B_k(I) = \text{makespan}(S) < \left(1+\frac{1}{k}\right) OPT(I)$.

Otherwise, $p_\ell \leq \frac{T}{k} \leq \frac{OPT}{k}$

Hence, $B_k(I) < OPT(I) + p_\ell \leq \left(1+\frac{1}{k}\right) OPT(I)$

By the same arguments as in the proof that LPT is a $4/3$-approx. alg.:



since job ℓ is a short job

Thus, in both cases,

$$B_k(I) < \left(1+\frac{1}{k}\right) OPT(I)$$

Running time:

Dyn. prg. as for bin packing:

$\leq k$ jobs on each machine

$\leq k^2$ different job sizes.

Hence, the configuration of a machine can be represented by a vector $(s_1, s_2, \dots, s_{k^2})$, where $0 \leq s_i \leq k$.

$\leq (k+1)^{k^2}$ possible conf.

$$OPT(n_1, n_2, \dots, n_{k^2}) = 1 + \min_{\vec{s} \in \mathcal{C}} \{ n_1 - s_1, \dots, n_{k^2} - s_{k^2} \}$$

$\mathcal{C}$: set of possible conf.

Dyn. prg. table:

$\begin{cases} k^2 \text{ dimensions (one for each rounded job size)} \\ k+1 \text{ entries in each dimension} \end{cases}$

$\Downarrow$

$k^2(k+1) = O(k^3)$ entries in the table.

Time per entry: $|\mathcal{C}| \leq (k+1)^{k^2} = O(k^{k^2})$

Total time: $O(k^{k^2+3})$.

iterations of while loop $\leq \log U \leq \log P$

Total time: $O(k^{k^2+3} \log P)$

$$\boxed{\text{Theorem 3.7: } B_k \text{ is a PTAS}}$$

<u>Proof</u>: B_k achieves an approx. ratio of $1+\varepsilon$ with running time $O\left(\left(\frac{1}{\varepsilon}\right)^{\left(\frac{1}{\varepsilon}\right)^2+3} \cdot \log P\right)$.

If ε is a constant, this is poly. in the input size, since it takes at least $\log P$ bits to write the processing time in binary. $\square$

Note that we did not expect a FPTAS:

The problem is strongly NP-hard, meaning that even if $p_{max} \le q(n)$, for some polynomium q, the problem is still NP-hard.

This implies that $\nexists$ FPTAS, unless $P = NP$:

Assume to the contrary that $\exists$ FPTAS A_k with relative error $\frac{1}{k}$.

Then, with $k = \lceil 2nq(n) \rceil$,

$$
\begin{aligned}
A_k &\le \left\lfloor \left(1+\tfrac{1}{k}\right) OPT \right\rfloor, \quad \lfloor\ \rfloor \text{ since proc. times are integers} \\
&= \left\lfloor OPT + \frac{OPT}{k} \right\rfloor \\
&\le \left\lfloor OPT + \tfrac{1}{2} \right\rfloor, \quad \text{since } OPT \le n p_{max} \le n q(n) \\
&= OPT
\end{aligned}
$$

Thus, with $k = \lceil 2nq(n) \rceil$, A_k produces an optimal schedule in time poly. in n and $q(n)$, i.e., poly. in n.