Implicit Euler for a Coupled Drying Model
Newton Gauss Seidel Versus Fixed Point Iteration

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Abstract

This report studies a coupled two state drying model with moisture content and temperature. The time integration uses the implicit Euler method. The nonlinear system at each time step is solved using two strategies. The first strategy is a fixed point iteration. The second strategy is a Newton method where the linear Newton system is solved using Gauss Seidel. The work compares accuracy and computational cost using tables and figures. The observed convergence order is close to one which matches the expected first order behavior of implicit Euler.

1 Model

We solve a coupled ordinary differential equation system for moisture content $M(t)$ and temperature $T(t)$. The parameters are

$$k_1 = 0.08 \qquad k_2 = 0.15 \qquad k_3 = 0.6 \qquad \alpha = 0.05 \qquad T_a = 60 \qquad M_{eq} = 0.12$$

and the dynamics are

$$\frac{dM}{dt} = -k_1(M - M_{eq})\exp\big(\alpha(T - T_a)\big)$$

$$\frac{dT}{dt} = k_2(T_a - T) - k_3\frac{dM}{dt}$$

The initial condition is

$$M(0) = 0.6 \qquad T(0) = 20$$

and the time interval is $t \in [0, 50]$.

2 Implicit Euler discretization

Let $t_{n+1} = t_n + h$. Implicit Euler computes $y_{n+1} = (M_{n+1}, T_{n+1})$ from

$$y_{n+1} = y_n + hf(y_{n+1}, t_{n+1})$$

where f is the right hand side of the system. We define the residual

$$F(y_{n+1}) = y_{n+1} - y_n - hf(y_{n+1}, t_{n+1})$$

and each time step requires solving

$$F(y_{n+1}) = 0$$

3 Nonlinear solvers

3.1 Fixed point iteration

The fixed point iteration uses

$$y^{(k+1)} = y_n + hf\big(y^{(k)}, t_{n+1}\big)$$

and stops when the update becomes smaller than a tolerance.

3.2 Newton Gauss Seidel

Newton iteration uses the linearization

$$J(y^{(k)})\Delta^{(k)} = -F(y^{(k)}) \qquad y^{(k+1)} = y^{(k)} + \Delta^{(k)}$$

where J is the Jacobian matrix of F. The Jacobian entries use the given closed form expressions. Let

$$E = \exp\big(\alpha(T - T_a)\big)$$

then

$$J_{11} = 1 + hk_1E \qquad J_{12} = hk_1\alpha(M - M_{eq})E$$

$$J_{21} = -hk_3k_1E \qquad J_{22} = 1 + hk_2 + hk_3k_1\alpha(M - M_{eq})E$$

The linear system for $\Delta^{(k)}$ is solved using Gauss Seidel iterations.

4 Reference solution and error metrics

A reference solution is computed using the Newton Gauss Seidel solver with a very small step size

$$h_{ref} = 0.001$$

The test solutions use step sizes

$$h \in \{0.20, 0.10, 0.05, 0.01\}$$

Errors are computed by interpolating the reference solution onto the test grid. We report the discrete L^2 error

$$\|e\|_2 = \sqrt{\frac{1}{N}\sum_{n=0}^{N} e_n^2}$$

and the maximum error

$$\|e\|_\infty = \max_{0\leq n\leq N} |e_n|$$

The observed order between two step sizes h_i and h_{i+1} is

$$p = \frac{\log(e_i/e_{i+1})}{\log(h_i/h_{i+1})}$$

Implicit Euler is expected to produce first order convergence so the order should be close to one.

5 Results

5.1 Trajectories and phase portrait

Figure 1 shows $M(t)$ and $T(t)$ for both solvers at $h = 0.1$. The curves overlap closely which indicates that both nonlinear solvers converge to the same implicit Euler solution. Figure 2 shows the phase portrait T versus M.

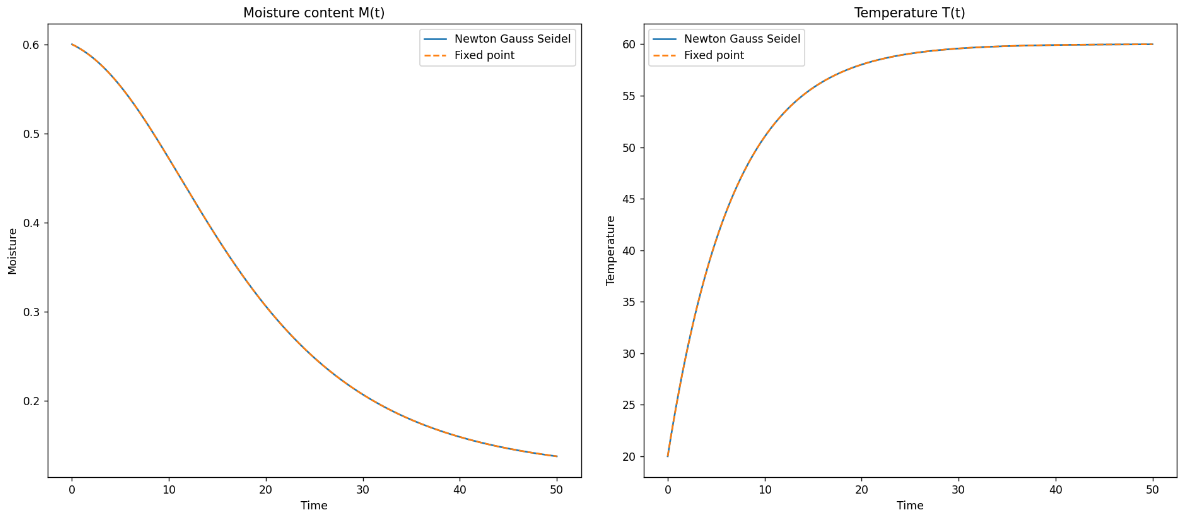

Figure 1: Moisture content and temperature trajectories at step size $h = 0.1$.

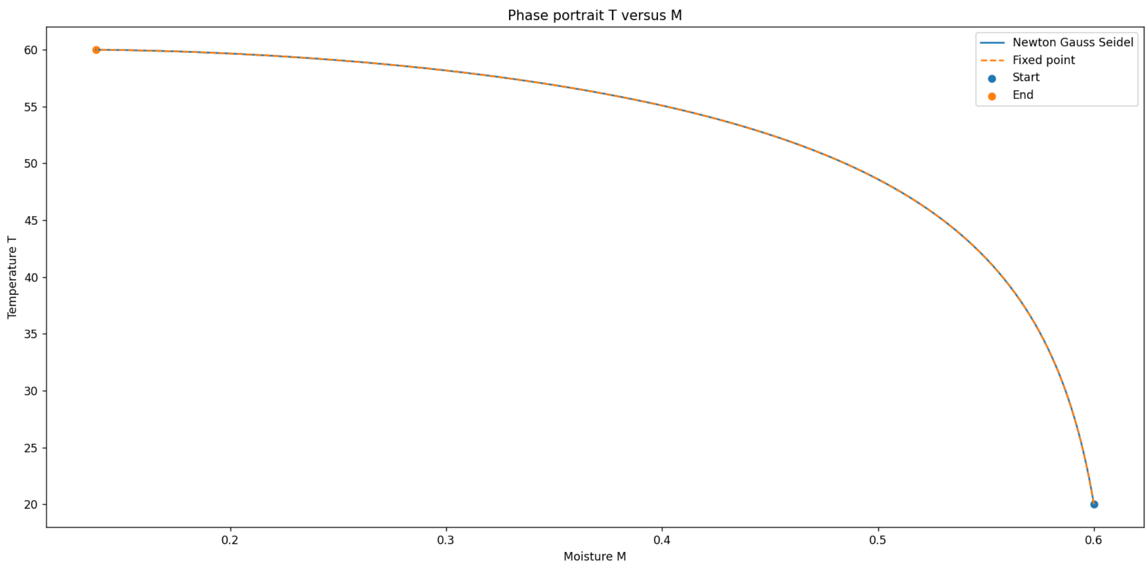

Figure 2: Phase portrait of temperature versus moisture at step size $h = 0.1$.

5.2 Iteration behavior and convergence behavior

Figure 3 compares iteration counts per time step. Figure 4 shows the total Gauss Seidel effort inside Newton. Figure 5 shows residual norms versus iteration for selected time steps.

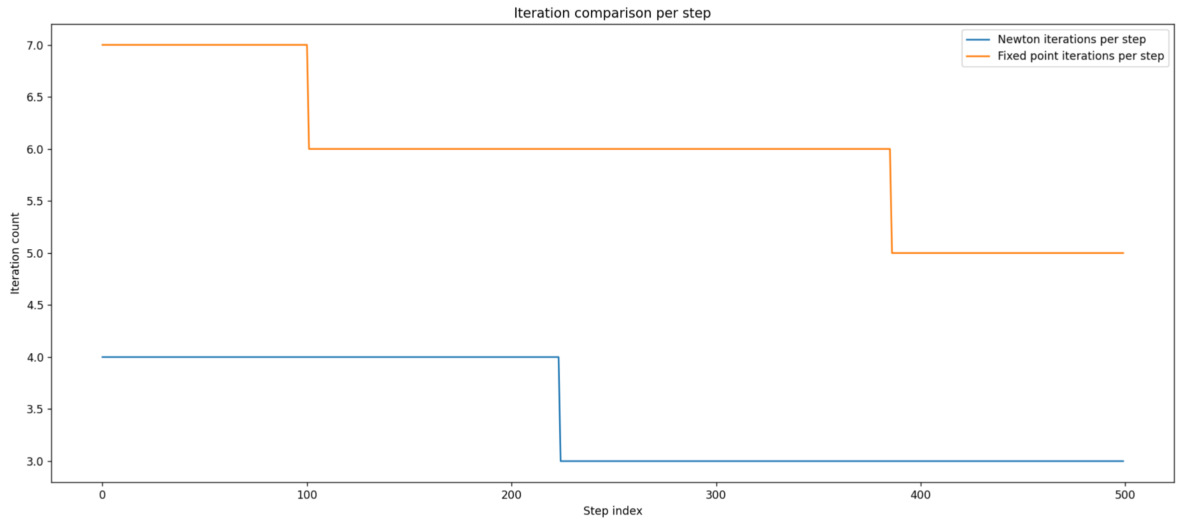

Figure 3: Iteration comparison per step for Newton and fixed point.

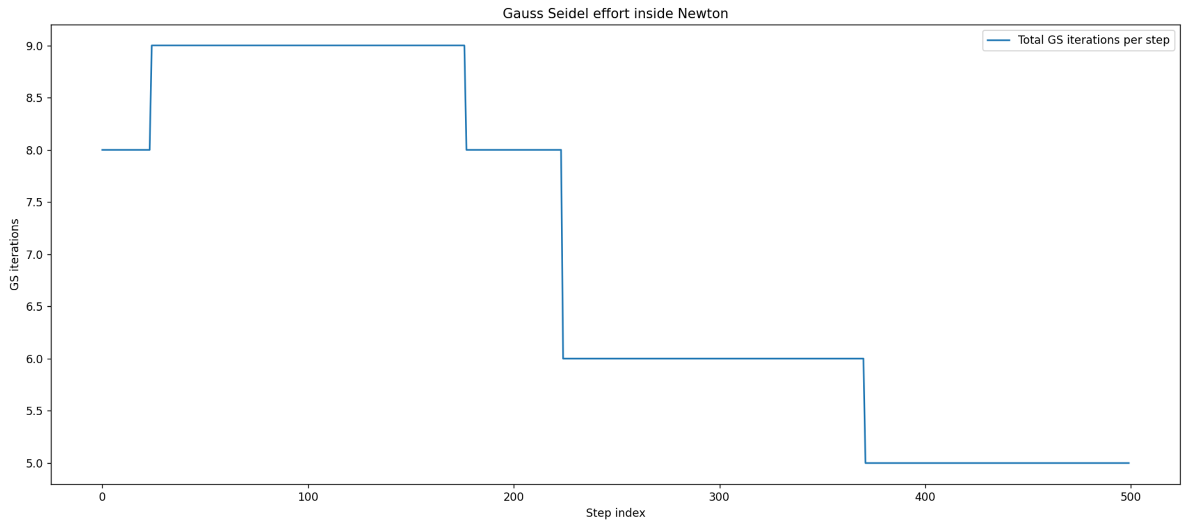

Figure 4: Total Gauss Seidel iterations per step inside Newton.

5.3 Convergence order

Figure 6 and Figure 7 present log log error versus step size. The slope one reference line matches the expected first order behavior.

5.4 Cost accuracy trade off

Figure 8 compares CPU time and moisture error for multiple step sizes.

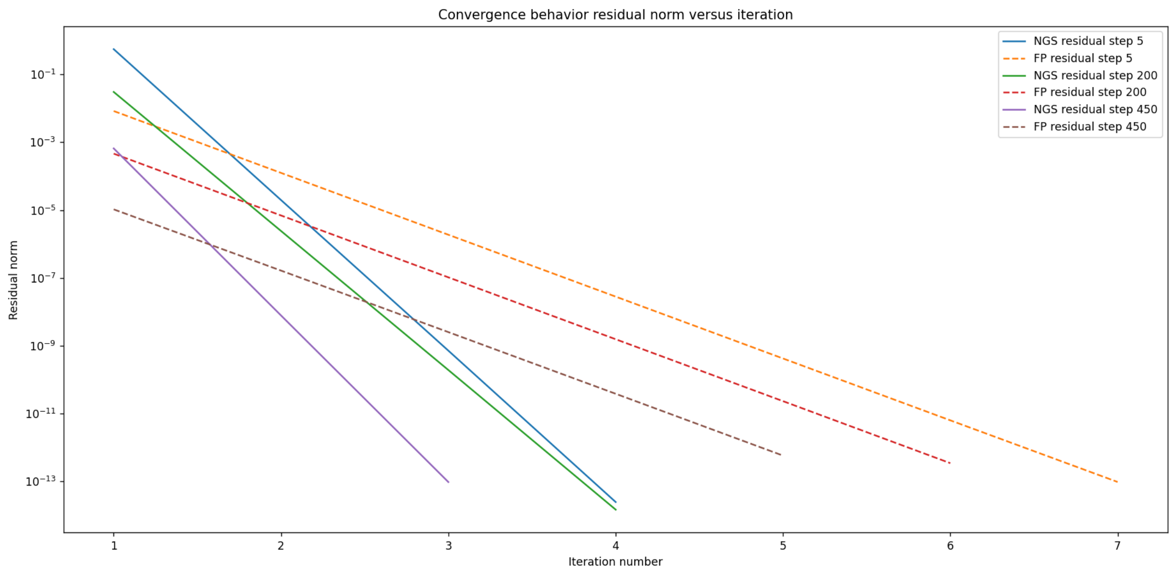

Figure 5: Residual norm versus iteration for selected steps.

5.5 Tables

Table 1 reports performance at $h = 0.1$. The fixed point method uses fewer total iterations but has higher time per iteration. Newton Gauss Seidel has lower time per iteration due to faster convergence in the nonlinear solve.

Table 1: Solver performance comparison at $h = 0.1$.

Metric	Newton Gauss Seidel	Fixed point
Avg Newton iter per step	3.448	N A
Avg GS iter per Newton step	2.014	N A
Total GS iterations	3472	N A
Avg fixed point iter per step	N A	5.974
Total iterations	5196	2987
CPU time in seconds	0.038219	0.041554
Time per iteration in ms	0.0074	0.0139

The reference solution uses $h_{ref} = 0.001$ with runtime 3.1697 seconds.

6 Discussion

Both nonlinear solvers produce the same implicit Euler trajectories at $h = 0.1$. The convergence study shows first order accuracy with observed orders close to one for both moisture and temperature. The iteration plots show that Newton Gauss Seidel needs fewer nonlinear iterations per step compared with fixed point. The cost accuracy plot supports that smaller step sizes reduce error while increasing runtime. The efficiency table reports Newton Gauss Seidel as the winner under the used winner rule.

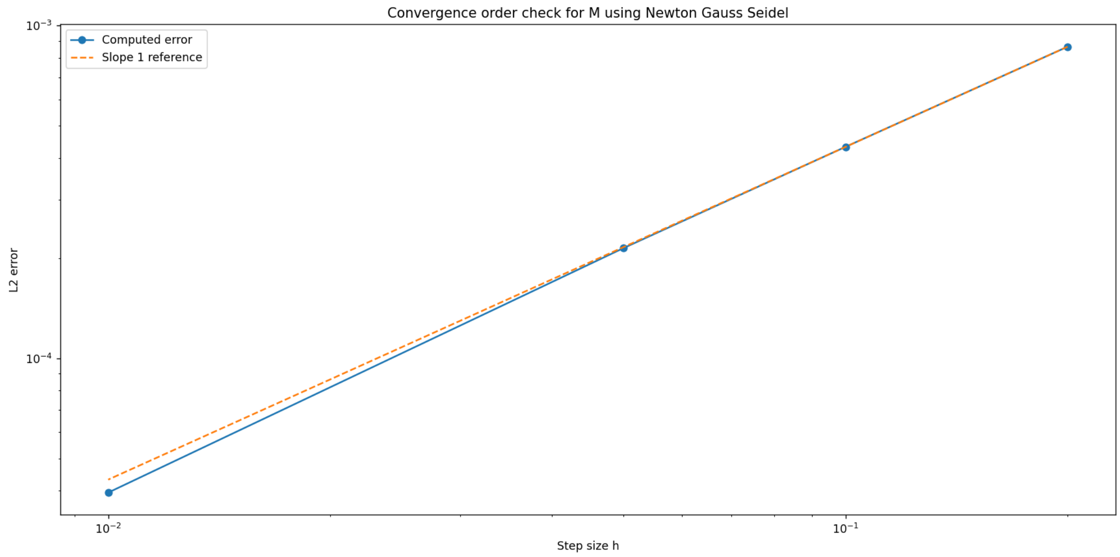

Figure 6: Convergence order check for moisture using Newton Gauss Seidel.

Table 2: Accuracy analysis for Newton Gauss Seidel.

h	L^2 error M	L^2 error T	max error M	max error T	order M	order T
0.20	$8.646 \cdot 10^{-4}$	$1.081 \cdot 10^{-1}$	$1.300 \cdot 10^{-3}$	$2.171 \cdot 10^{-1}$	-	-
0.10	$4.325 \cdot 10^{-4}$	$5.405 \cdot 10^{-2}$	$6.501 \cdot 10^{-4}$	$1.087 \cdot 10^{-1}$	1.00	1.00
0.05	$2.147 \cdot 10^{-4}$	$2.682 \cdot 10^{-2}$	$3.227 \cdot 10^{-4}$	$5.396 \cdot 10^{-2}$	1.01	1.01
0.01	$3.952 \cdot 10^{-5}$	$4.937 \cdot 10^{-3}$	$5.941 \cdot 10^{-5}$	$9.935 \cdot 10^{-3}$	1.05	1.05

7 Conclusion

Implicit Euler for the coupled drying model shows first order convergence in both state variables. Newton Gauss Seidel and fixed point iteration both succeed in solving the nonlinear implicit step. Newton Gauss Seidel reduces nonlinear iteration counts and achieves a lower time per iteration. Fixed point iteration is competitive for larger step sizes but becomes more expensive as accuracy requirements increase. The numerical results are consistent with the expected behavior of implicit Euler.

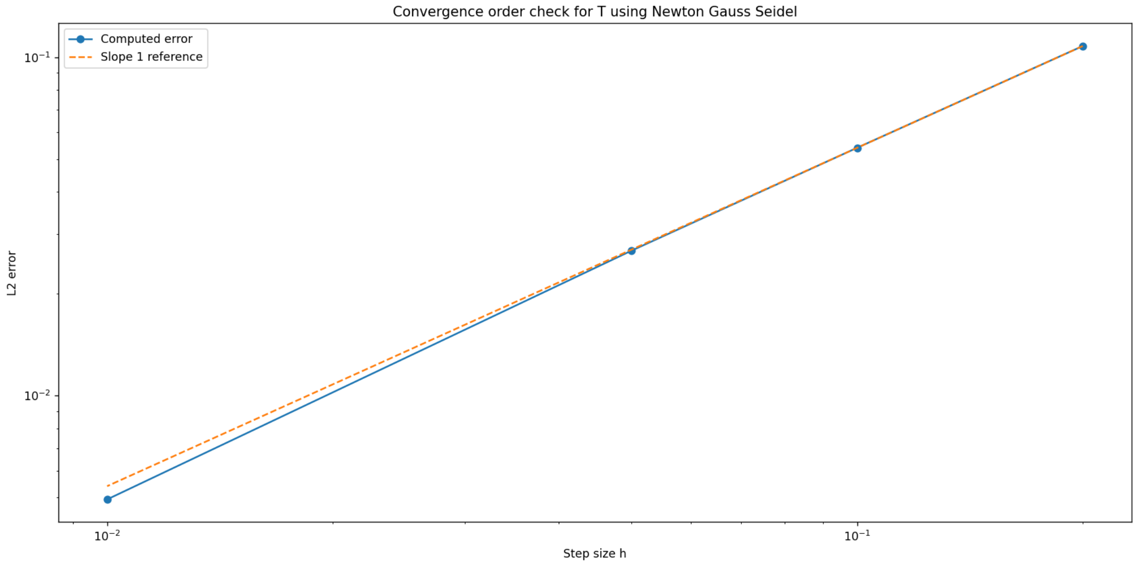

Figure 7: Convergence order check for temperature using Newton Gauss Seidel.

Table 3: Accuracy analysis for Fixed point.

h	L^2 error M	L^2 error T	max error M	max error T	order M	order T
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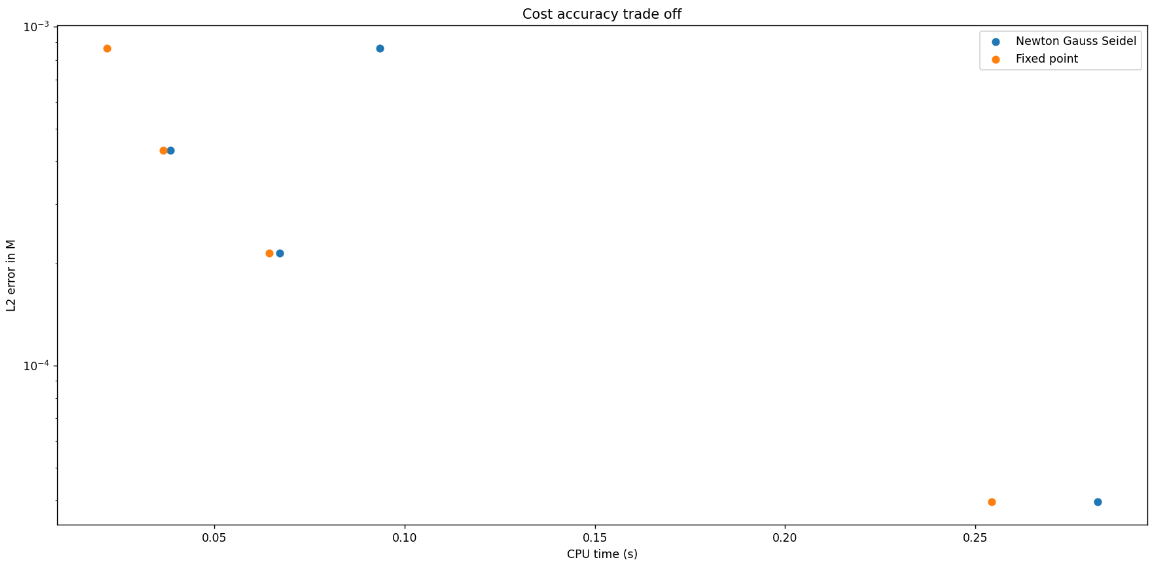

Figure 8: Cost accuracy trade off using L^2 error in moisture versus CPU time.

Table 4: Computational efficiency.

h	NGS time	FP time	NGS error M	FP error M	winner
0.20	0.093478	0.021750	$8.646 \cdot 10^{-4}$	$8.646 \cdot 10^{-4}$	NGS
0.10	0.038360	0.036525	$4.325 \cdot 10^{-4}$	$4.325 \cdot 10^{-4}$	NGS
0.05	0.067134	0.064374	$2.147 \cdot 10^{-4}$	$2.147 \cdot 10^{-4}$	NGS
0.01	0.282165	0.254268	$3.952 \cdot 10^{-5}$	$3.952 \cdot 10^{-5}$	NGS