DM865 – Heuristics & Approximation Algorithms

(Marco) (here)

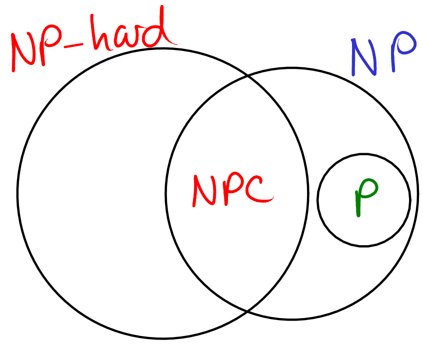
Combinatorial problems:

First 2 weeks → Traveling Salesman (TSP)

- MAX SAT
- Set Cover
- Knapsack
- Bin packing
- Scheduling

decision version $\in$ NPC

Polynomial algorithm:
algo. with running time $O(n^c)$, for some constant c.



P: The set of decision problems that allow for a poly. algo.

NP: A problem belongs to NP, if solutions can be verified in poly. time.

If <u>any</u> NP-hard problem has a poly. algo., then <u>all</u> problems in NPC have poly. algo.s.

$\underbrace{\text{Optimal solutions}}_{(1)}\ \underbrace{\text{in poly. time}}_{(2)}\ \underbrace{\text{for all instances}}_{(3)}$

- Choose two! ((2) & (3))

Section 1.1

An approximation algorithm comes with a performance guarantee:

Def 1.1: α-approximation algorithm

An α-approximation algorithm for an optimization problem P is a poly. time algo. ALG s.t. for any instance I of P,

- $\frac{ALG(I)}{OPT(I)} \leq \alpha$, if P is a <u>minimization</u> problem
- $\frac{ALG(I)}{OPT(I)} \geq \alpha$, if P is a <u>maximization</u> problem

Thus, for max. problems, $0 \leq \alpha \leq 1$, and, for min. problems, $\alpha \geq 1$.

The approximation factor / approximation ratio is
- the smallest possible α (for min. problems)
- the largest possible α (for max. problems)

More precisely, the approx. factor R is

$$R = \inf \left\{ \alpha \;\middle|\; \forall I : \frac{ALG(I)}{OPT(I)} \leq \alpha \right\} \text{ for min. problems}$$

$$R = \sup \left\{ \alpha \;\middle|\; \forall I : \frac{ALG(I)}{OPT(I)} \geq \alpha \right\} \text{ for max. problems}$$

We will cover the rest of Section 1.1 later.

Section 2.4: TSP

The Traveling Salesman Problem (TSP)

Input: Weighted complete graph G

$c_{ij} = c_{ji}, \quad i,j \in V$

$c_{ii} = 0, \quad i \in V$

$c_{ij} \geq 0, \quad i,j \in V$

Output: $\underbrace{\text{Hamiltonian cycle}}_{\text{Cycle visiting each vertex exactly once.}}$ of min. total weight

Decision version of TSP:

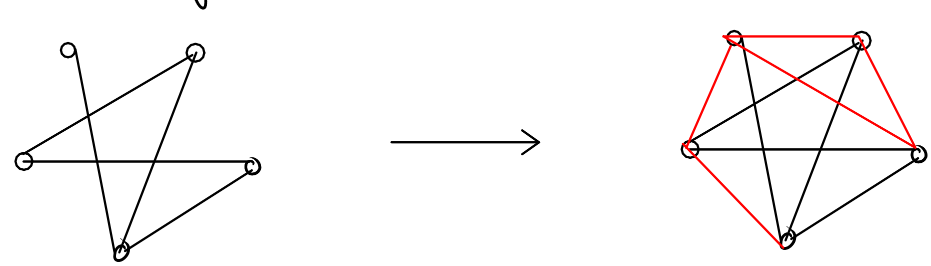
Does there exist a tour of cost $\leq X$?

Decision version of TSP is NP-hard

Reduction from Hamilton Cycle:

Ham. cycle

TSP



$c_{ij} = 1$

$c_{ij} = 2$

$\exists$ ham. cycle $\iff$ $\exists$ tour of cost n

Even worse, no approximation guarantee possible:

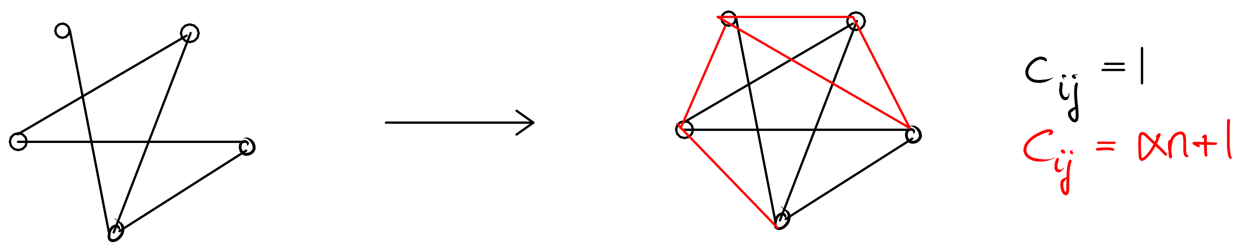
Theorem 2.9

$\forall \alpha > 1$, $\nexists$ α-approx alg. for TSP (unless $P = NP$)

Proof: Reduction from Hamilton Cycle:

Ham. cycle

TSP



$\exists$ ham. cycle $\iff$ $\exists$ tour of cost n

$\iff$ α-approx. alg. gives tour of cost $\leq \alpha n$

$\iff$ α-approx. alg. returns a tour with no red edges, i.e., a tour of cost n.

$\square$

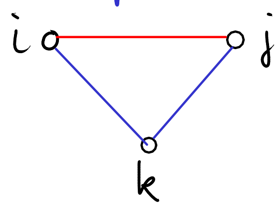
Note: The proof does not require α to be a constant. In fact, it could be 2^n, or <u>any</u> function computable in poly. time.

Thus, we will only consider a special case of TSP:

Metric TSP :

The edge weights satisfy the triangle inequality:

$$c_{ij} \leq c_{ik} + c_{kj}, \quad \text{for all } i,j,k \in V$$



For metric TSP, the proof of Thm 2.9 does not work (the max. possible cost of the red edges would be 2).

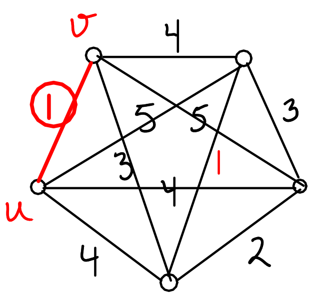
For Metric TSP, we will consider three algorithms:

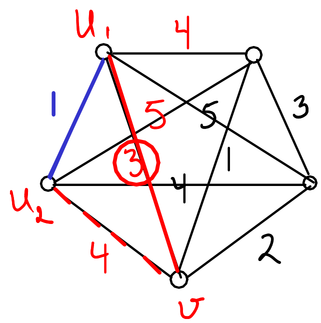
The Nearest Addition algorithm	2-approx.
The Double Tree algorithm	2-approx.
Christofide's Algorithm	3/2-approx

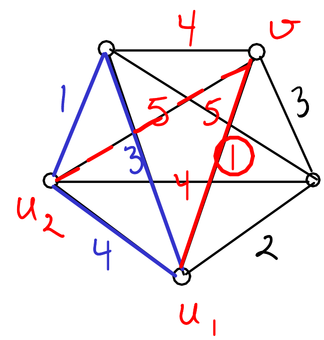
Nearest Addition (NA)

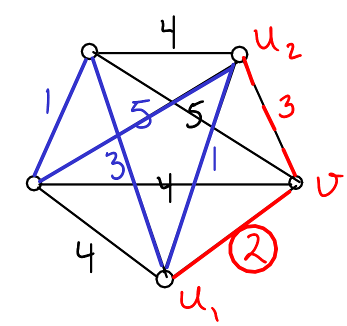
$u, v \leftarrow$ two nearest neighbors in V
Tour $\leftarrow \langle u, v, u \rangle$
For $i \leftarrow 1$ to $n-2$
    $v \leftarrow$ nearest neighbour of Tour
    $u_1 \leftarrow$ nearest neighbor of v in Tour
    $u_2 \leftarrow$ u_1's successor in Tour
    Add v to Tour between u_1 and u_2

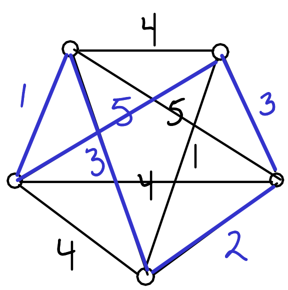
Ex:



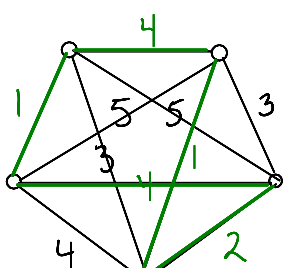








$$C_{NA} = 1 + 3 + 2 + 3 + 5 = 14$$

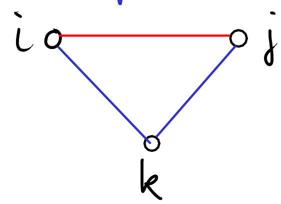


$$C_{OPT} \leq 1 + 4 + 1 + 2 + 4 = 12$$

Metric TSP:

The edge weights satisfy the triangle inequality:

$$c_{ij} \leq c_{ik} + c_{kj}, \quad \text{for all } i,j,k \in V$$



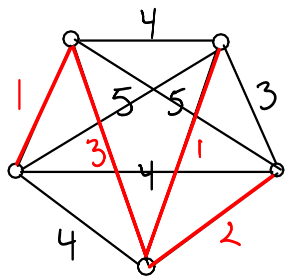
For Metric TSP, Nearest Neighbor is a 2-approx. alg.:

We will prove that

$$\boxed{\begin{array}{ll} (1) & C_{NA} \leq 2 \cdot c(MST) \\ (2) & c(MST) \leq C_{OPT} \end{array}}$$

(Lemma 2.10)

(1): The solid red edges are exactly those chosen by Prim's Algorithm:

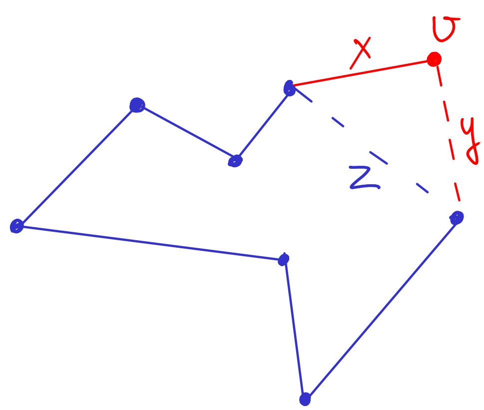


$C = 1 + 3 + 1 + 2 = 7$

Thus, the total cost C of these edges is that of a minimum spanning tree:

$$\boxed{C = c(MST)}$$

Adding a new vertex v to the tour, we add two edges and delete one:



$y \leq z$ by the Δ-ineq.

Adding v costs

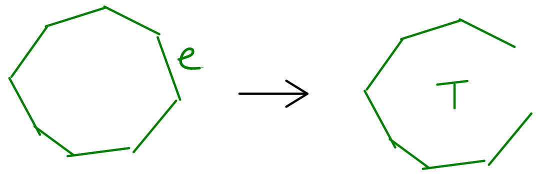
$$x + y - z \leq x + (x + z) - z = 2x$$

where x is the cost of Prim's Alg. in this step

Thus,

$$\boxed{C_{NA} \leq 2C = 2c(MST)}$$

(2): Deleting any edge from a tour, we get a spanning tree:



For any spanning tree T obtained by deleting an edge e from an optimal tour,

$$\boxed{c_{OPT}} \geq c(T), \quad \text{since } w(e) \geq 0$$

$$\geq \boxed{c(MST)}$$

Now,

$$(1) \,\&\, (2) \implies c_{NA} \leq 2\,c(MST) \leq 2\,c_{OPT}$$

This proves:

Theorem 2.11

For Metric TSP, Nearest Addition is a 2-approx. alg.