Section 3.3: The Bin Packing Problem

Last time we discussed simple approx. alg.s
Today we will develop an approximation scheme.

Approximation scheme $\{A_\varepsilon\}$:

1. Transform $I \to I''$:
   a. Remove all items smaller than $\varepsilon/2$. $(I \to I')$
      $\Rightarrow O(\frac{1}{\varepsilon})$ items fit in one bin
   b. Round up sizes of remaining items $(I' \to I'')$
      $\Rightarrow O(\frac{1}{\varepsilon^2})$ different item sizes
2. Do dyn. prg. on I''
   $\Rightarrow A_\varepsilon(I'') = OPT(I'')$
3. Add small items to the packing using First-Fit (or any other Any fit alg.)

Adding small items to the packing (3.)

Lemma 3.10

$$A_\varepsilon(I) \leq \max\left\{A_\varepsilon(I''), \frac{2}{2-\varepsilon}\,\mathrm{size}(I)+1\right\}$$

Proof:

If no extra bin is needed for adding the small items, $A_\varepsilon(I) = A_\varepsilon(I'')$.

Otherwise, all bins, except possibly the last one, are filled to more than $1-\varepsilon/2$.

In this case,

$$A_\varepsilon(I) \leq \left\lceil \frac{\mathrm{size}(I)}{1-\varepsilon/2} \right\rceil \leq \frac{\mathrm{size}(I)}{1-\varepsilon/2} + 1$$

$$= \frac{2}{2-\varepsilon}\,\mathrm{size}(I) + 1$$

$\square$

Rounding scheme (1.b)

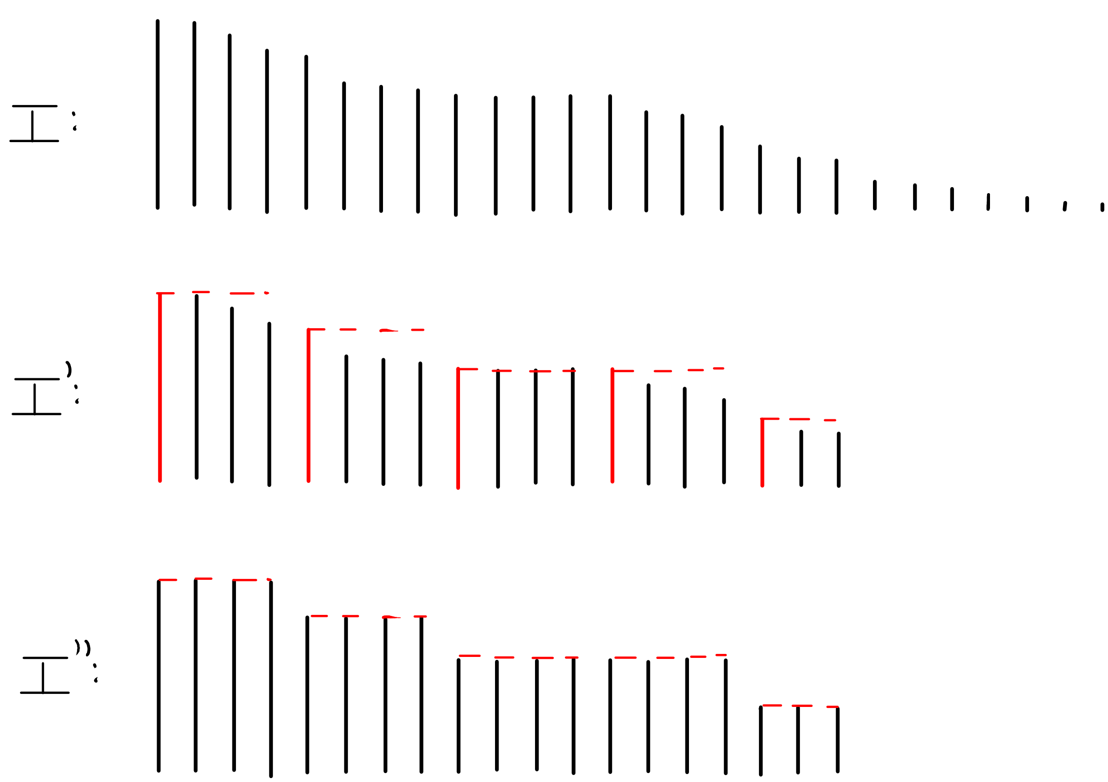
Last time we saw that a rounding scheme similar to the one we used for Knapsack would at best yield an approx. factor of 1.5.
Instead, we will use:

Linear grouping:

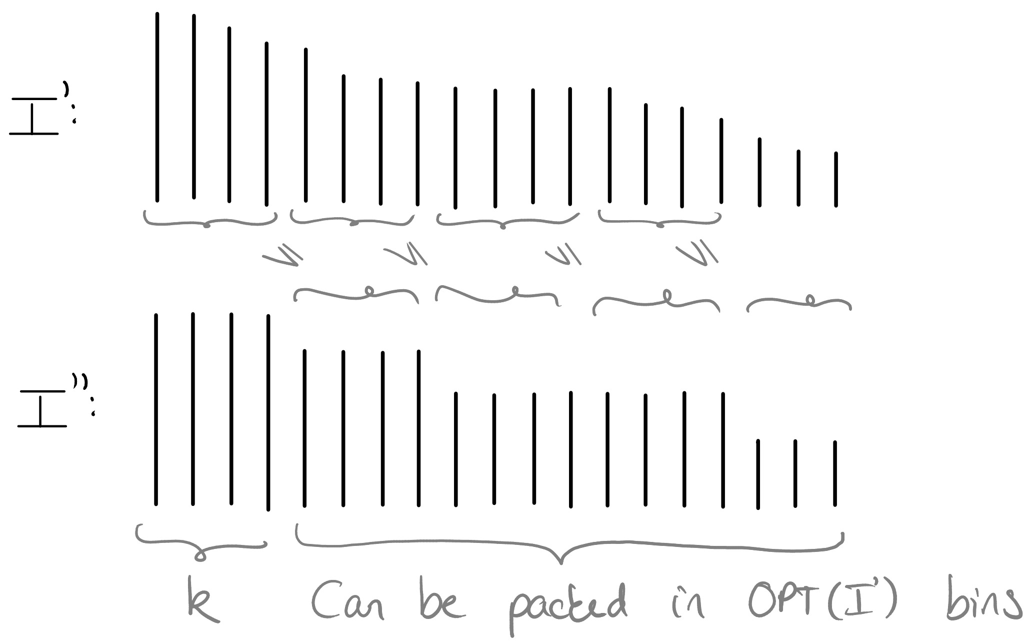
- Sort items in I' by decreasing sizes.
- Partition items in groups of k consecutive items.
  (k will be determined later)
- For each group, round up item sizes to largest size in the group.

The result is called I''.

Ex: $(k=4)$



Each item in the i'th group of I' is at least as large as any item in the (i+1)st group of I'':



Thus, for any packing of I', there is a packing of all but the first group of I'' using the same number of bins.
Since the first group of I'' can be packed in at most k bins, this proves:

Lemma 3.11: $OPT(I'') \le OPT(I') + k$

Packing I'' using dyn. prg. (2.)

We will use the same approach as in Section 3.2:

Since all items in I'' have size at least $\varepsilon/2$, at most $2/\varepsilon$ items fit into each bin.

There are $N \leq \lceil n/k \rceil$ different item sizes $s_1, s_2, \dots, s_N$ in I''.

Hence, any packing of a bin can be represented by a vector $(m_1, m_2, \dots, m_N)$, $m_i \leq 2/\varepsilon$, where m_i is the number of items of size s_i in the bin.
A vector representing the contents of a bin is called a bin configuration.
Let $\mathcal{C}$ be the set of possible bin configurations.
Note that $|\mathcal{C}| < (2/\varepsilon)^N$.

For the dyn. prg. we will use an N-dimensional table B with $n_i + 1$ rows in the i'th dimension, where n_i is the number of items of size s_i in I''.
$B[m_1, m_2, \dots, m_N]$ will be the minimum number of bins required to pack m_i items of size s_i, $1 \leq i \leq N$.

Ex:

$\varepsilon = 0.4$

$I = 0.6,\ 0.5,\ 0.4,\ 0.4,\ 0.3,\ \underbrace{0.1,\ 0.1}_{<\varepsilon/2}$

Choosing $k=3$, we obtain

$I' = \underbrace{0.6,\ 0.5,\ 0.4}, \ \underbrace{0.4,\ 0.3}$

$I'' = 0.6, 0.6, 0.6, 0.4, 0.4$

$s_1 = 0.6, \quad s_2 = 0.4$

$n_1 = 3, \quad n_2 = 2$

$\mathcal{C} = \{(0,1),\ (0,2),\ (1,0),\ (1,1)\}$

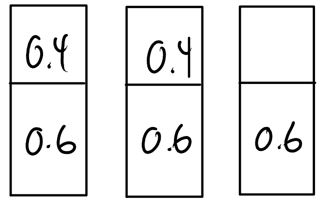
B:

0.6 \ 0.4	0	1	2
0	0	1	1
1	1	1	2
2	2	2	2
3	3	3	3

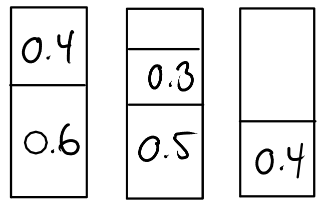
$$B[3,2] = 1 + \min_{(m_1,m_2)\in\mathcal{C}} \{B[3-m_1,\ 2-m_2]\}$$

$$= 1 + \min\{B[3,1],\ B[3,0],\ B[2,2],\ B[2,1]\}$$

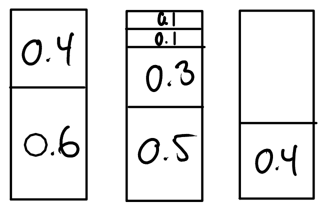
Packing of I'' :



Packing of I':



Packing of I:



Approximation

$$A_\varepsilon(I) \le \max\left\{A_\varepsilon(I''),\ \frac{2}{2-\varepsilon}\,\mathrm{size}(I)+1\right\}, \text{ by Lemma 3.10}$$

$$\le \max\left\{\mathrm{OPT}(I''),\ \frac{2}{2-\varepsilon}\,\mathrm{OPT}(I)+1\right\}, \text{ since}$$

$A_\varepsilon(I'') = \mathrm{OPT}(I'')$ and $\mathrm{OPT} \geq \mathrm{size}(I)$

$$\le \max\left\{\mathrm{OPT}(I')+k,\ \frac{2}{2-\varepsilon}\,\mathrm{OPT}(I)+1\right\}, \text{ by Lemma 3.11}$$

$$\le \max\left\{\mathrm{OPT}(I)+k,\ \frac{2}{2-\varepsilon}\,\mathrm{OPT}(I)+1\right\}, \text{ since } I' \subseteq I$$

$$\frac{2}{2-\varepsilon} \le 1+\varepsilon \iff 2 \le (2-\varepsilon)(1+\varepsilon)$$
$$\iff 2 \le 2+\varepsilon-\varepsilon^2$$
$$\iff \varepsilon \le 1$$

Thus, we just need to choose an appropriate value of k to obtain $k \le \varepsilon \cdot \mathrm{OPT}(I)$:

$$k = \lfloor \varepsilon \cdot \mathrm{Size}(I) \rfloor$$

With this value of k

$$A_\varepsilon(I) \le (1+\varepsilon)\cdot \mathrm{OPT}(I) + 1$$

asymptotic approximation scheme

Running time

$k = \lfloor \varepsilon \cdot \mathrm{size}(I) \rfloor \geq \lfloor \varepsilon \cdot n' \cdot \frac{\varepsilon}{2} \rfloor \geq n' \cdot \frac{\varepsilon^2}{4}$, where $n' = |I'|$,
since all items in I' have size at least $\varepsilon/2$.

$N \leq \lceil \frac{n'}{k} \rceil \leq \lceil \frac{4}{\varepsilon^2} \rceil$

Table size $\leq (n')^N \leq n^N$

Time per entry $O(|\mathcal{C}|) \subseteq O((\frac{2}{\varepsilon})^N)$

Running time $O((\frac{2}{\varepsilon})^N n^N) \subseteq O((\frac{2n}{\varepsilon})^{\lceil 4/\varepsilon^2 \rceil})$

not fully poly. time

Hence, $\{A_\varepsilon\}$ is an
Asymptotic poly. time approx. scheme (APTAS)

This proves:

$$\boxed{\text{Theorem 3.12: } A_\varepsilon \text{ is an APTAS for Bin Packing}}$$

There is no PTAS for Bin Packing:

Theorem 3.8

No approx alg. for Bin Packing has an absolute approx. ratio better than $3/2$, unless $P = NP$.

Proof:

Reduction from Partition Problem (given a set S of integers, can S be partitioned into two sets S_1 and S_2 such that $\sum_{s \in S_1} s = \sum_{s \in S_2} s$?)

Let $B = \sum_{s \in S} s$.

Scale each integer by $\frac{2}{B}$, resulting in a set of numbers with sum 2.
Use these numbers as input for the bin packing problem.

Clearly, at least 2 bins are needed, and 2 bins are sufficient, if and only if the instance of the Partition problem is a yes-instance.

Thus, any Bin Packing alg. with an approx. ratio smaller than $3/2$ will use exactly 2 bins, if and only if the input to the Partition problem is a yes-instance.

$\square$