Techniques: (with Set Cover as an example)

- Solve LP and round solution (Sec. 1.3 + 1.7)
- Primal-dual alg.: combinatorial alg. based on LP formulation (Sec. 1.4 + 1.5)
- Greedy alg. (Sec. 1.6)

# Section 1.2: Set Cover as an LP

**Set Cover**

Input:

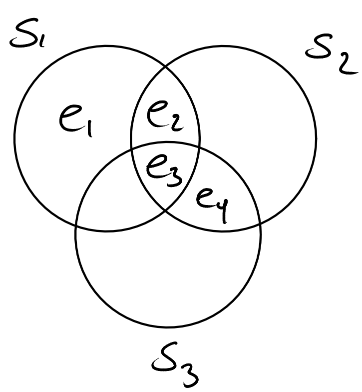
$E = \{e_1, e_2, \ldots, e_n\}$

$\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, where $S_j \subseteq E$ has weight $w_j$.

Objective: Find a cheapest possible subset of $\mathcal{S}$ covering all elements

OPT: value (total weight) of optimum solution

Ex:




$w_1 = 1$

$w_2 = 2$

$w_3 = 3$

$\{S_1, S_2\}$ is a sol. of total weight 3.

This is optimal, so OPT = 3 for this instance of Set Cover.

To cover $e_1$, we need $S_1$

—— " —— $e_2$ —— " —— $S_1$ or $S_2$

—— " —— $e_3$ —— " —— $S_1$, $S_2$ or $S_3$

—— " —— $e_4$ —— " —— $S_2$ or $S_3$

IP-formulation:

$$
\begin{aligned}
\min \quad & x_1 w_1 + x_2 w_2 + x_3 w_3 \\
\text{s.t.} \quad & x_1 \geq 1 \\
& x_1 + x_2 \geq 1 \\
& x_1 + x_2 + x_3 \geq 1 \\
& x_2 + x_3 \geq 1 \\
& x_1, x_2, x_3 \in \{0,1\}
\end{aligned}
$$

More generally:

## IP for Set Cover

$$\min \sum_{j=1}^{m} x_j w_j$$

$$\text{s.t.} \sum_{j: e_i \in S_j} x_j \geq 1, \qquad i = 1, 2, \dots, n$$

$$x_j \in \{0,1\}, \qquad j = 1, 2, \dots, m$$

$Z^*_{IP}$ : optimum solution value, i.e., $Z^*_{IP} = OPT$

## LP-relaxation

$$\min \sum_{j=1}^{m} x_j w_j$$

$$\text{s.t.} \sum_{j: e_i \in S_j} x_j \geq 1, \qquad i = 1, 2, \dots, n$$

$$0 \leq x_j \leq 1, \qquad j = 1, 2, \dots, m$$

redundant

$Z^*_{LP}$ : Optimum solution value

Note that $\boxed{Z^*_{LP} \leq Z^*_{IP} = OPT}$

## Section 1.3: A deterministic rounding algo.

The frequency of an element $e$ is the #sets containing $e$:

$$f_e = |\{ S \in \mathcal{S} \mid e \in S \}|$$

The frequency of an instance of Set Cover:

$$f = \max_{e \in E} \{ f_e \}$$

**Alg. 1 for Set Cover: LP-rounding**

Solve LP

$I \leftarrow \{ j \mid x_j \geq \frac{1}{f} \}$

We prove that Alg 1 produces a set cover (Lemma 1.5) of total weight $\leq f \cdot OPT$ (Thm 1.6)

**Lemma 1.5**

$\{S_j \mid j \in I\}$ is a set cover

Proof:

For each $e_i \in E$, $\sum_{j: e_i \in S_j} x_j \geq 1$.

Since $\sum_{j: e_i \in S_j} x_j$ has at most $f$ terms, at least one of the terms is at least $\frac{1}{f}$.

Thus, there is a set $S_j$ s.t. $e_i \in S_j$ and $x_j \geq \frac{1}{f}$.

This $j$ is included in $I$ $\square$

**Thm 1.6**

Alg. 1 is an $f$-approx. algo. for Set Cover.

Proof:

Correct by Lemma 1.5

Poly, since LP-solving is poly.

Approx. factor $f$:

Each $x_j$ is rounded up to 1, only if it is already at least $\frac{1}{f}$.

Thus, each $x_j$ is multiplied by at most $f$, i.e.,

$$\sum_{j \in I} w_j \leq \sum_{j \in I} f \cdot x_j \cdot w_j \leq \sum_{j=1}^{m} f \cdot x_j \cdot w_j = f \cdot Z_{LP}^* \leq f \cdot OPT$$

$\square$

The Vertex Cover problem is a special case of Set Cover:

## Vertex Cover

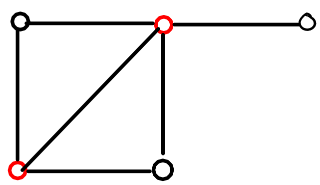
**Input:**

$G = (V, E)$

**Objective:**

Find a min. card. vertex set $C \subseteq V$ s.t. each edge $e \in E$ has at least one endpoint in $C$.

Ex:



With $\mathcal{S} = V$ and $\mathcal{E} = E$, Alg. 1 is a 2-approx. alg. for Vertex Cover.

One of the exercises for Tuesday:
Write down LP for Vertex Cover.

# Section 1.4: The dual LP

What is a dual?

**Ex:**

**P**

$$\boxed{\begin{array}{ll} \min & 7x_1 + x_2 + 5x_3 \\ \text{s.t.} & x_1 - x_2 + 3x_3 \geqslant 10 \\ & 5x_1 + 2x_2 - x_3 \geqslant 6 \\ & x_1, x_2, x_3 \geqslant 0 \end{array}}$$

Primal

$$7x_1 + x_2 + 5x_3 \geqslant x_1 - x_2 + 3x_3 \geqslant 10$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geqslant x_1 - x_2 + 3x_3 + 5x_1 + 2x_2 - x_3 \\ &\geqslant 10 + 6 = 16 \end{aligned}$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geqslant 2(x_1 - x_2 + 3x_3) + 5x_1 + 2x_2 - x_3 \\ &\geqslant 2 \cdot 10 + 6 = 26 \end{aligned}$$

To find a largest possible lower bound on $7x_1 + x_2 + 5x_3$, we should determine $y_1$ and $y_2$ maximizing $10y_1 + 6y_2$, under the constraints that

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\overset{(*)}{\geqslant} y_1 \overbrace{(x_1 - x_2 + 3x_3)}^{\geqslant 10} + y_2 \overbrace{(5x_1 + 2x_2 - x_3)}^{\geqslant 6} \\ &= \underbrace{(y_1 + 5y_2)}_{\leq 7} x_1 + \underbrace{(-y_1 + 2y_2)}_{\leq 1} x_2 + \underbrace{(3y_1 - y_2)}_{\leq 5} x_3 \end{aligned}$$

and $\underbrace{y_1, y_2, y_3 \geqslant 0}_{\text{otherwise}}$

„$\geqslant$" becomes „$\leq$"

necessary to satisfy (*)

Thus, we arrive at the following problem:

**D**

$$\begin{aligned} \max \quad & 10y_1 + 6y_2 \\ \text{s.t.} \quad & y_1 + 5y_2 \le 7 \\ & -y_1 + 2y_2 \le 1 \\ & 3y_1 - y_2 \le 5 \\ & y_1, y_2 \ge 0 \end{aligned}$$

Dual

<u>In general:</u>

**Primal:**

$$\begin{aligned} \min \quad & c_1 x_1 + c_2 x_2 + \dots + c_n x_n \\ \text{s.t.} \quad & a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \ge b_i, \quad i = 1, 2, \dots, m \\ & x_j \ge 0, \quad j = 1, 2, \dots, n \end{aligned}$$

**Dual:**

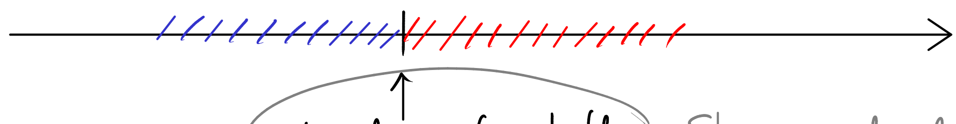
$$\begin{aligned} \max \quad & b_1 y_1 + b_2 y_2 + \dots + b_m y_m \\ \text{s.t.} \quad & a_{1j} y_1 + a_{2j} y_2 + \dots + a_{mj} y_m \le c_1, \quad j = 1, 2, \dots, n \\ & y_i \ge 0, \quad i = 1, 2, \dots, m \end{aligned}$$

Returning to the example above:

The constraints of $D$ ensure that the value of any sol. to $D$ is a lower bound on the value of any sol. to $P$, i.e., for any pair $x, y$ of sol. to $P$ and $D$ resp.,

$$10y_1 + 6y_2 \leq 7x_1 + x_2 + 5x_3$$

Weak duality



opt. value for both problems

Strong duality

Consider again the inequality leading to the constraints of the dual:

$$y_1 = 0 \ \dot\vee \ x_1 - x_2 + 3x_3 = 0 \qquad\qquad y_2 = 0 \ \vee \ 5x_1 + 2x_2 - x_3 = 6$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$7x_1 + x_2 + 5x_3 \overset{(\ast)}{\geqslant} \overbrace{y_1(\underbrace{x_1 - x_2 + 3x_3}_{\geqslant 10})}^{=10y_1} + \overbrace{y_2(\underbrace{5x_1 + 2x_2 - x_3}_{\geqslant 6})}^{=6y_2}$$

$$= \underbrace{(\overbrace{y_1 + 5y_2}^{\leq 7})x_1}_{=7x_1} + \underbrace{(\overbrace{-y_1 + 2y_2}^{\leq 1})x_2}_{=x_2} + \underbrace{(\overbrace{3y_1 - y_2}^{\leq 5})x_3}_{=5x_3}$$

$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Updownarrow$$

$$\underbrace{y_1 + 5y_2 = 7 \ \vee \ x_1 = 0 \qquad -y_1 + 2y = 1 \ \vee \ x_2 = 0 \qquad 3y_1 - y_2 = 5 \ \vee \ x_3 = 0}$$

$$\Updownarrow$$

$(\ast)$ becomes $=$

Thus, (Weak Duality Theorem)

$$7x_1 + x_2 + 5x_3 = 10y_1 + 6y_2$$

$$\Updownarrow$$

Complementary Slackness Conditions

$$\begin{cases} x_1 > 0 \Rightarrow y_1 + 5y_2 = 7 \\ x_2 > 0 \Rightarrow -y_1 + 2y_2 = 1 \\ x_3 > 0 \Rightarrow 3y_1 - y_2 = 5 \end{cases} \text{ primal c.s.c.}$$

$$\begin{cases} y_1 > 0 \Rightarrow x_1 - x_2 + 3x_3 = 10 \\ y_2 > 0 \Rightarrow 5x_1 + 2x_2 - x_3 = 6 \end{cases} \text{ dual c.s.c.}$$

By the Strong Duality Theorem (which we will not prove), there exist solutions fulfilling the c.s.c.

Moreover, if the c.s.c. are „close" to being satisfied, the values of the primal and dual sol. are „close":

Relaxed Complementary Slackness Conditions

$$\begin{cases} x_1 > 0 \Rightarrow y_1 + 5y_2 \geq 7/b \\ x_2 > 0 \Rightarrow -y_1 + 2y_2 \geq 1/b \\ x_3 > 0 \Rightarrow 3y_1 - y_2 \geq 5/b \\ \\ y_1 > 0 \Rightarrow x_1 - x_2 + 3x_3 \leq 10c \\ y_2 > 0 \Rightarrow 5x_1 + 2x_2 - x_3 \leq 6c \end{cases}$$

$$\Downarrow$$

$$7x_1 + x_2 + 5x_3 \leq bc(10y_1 + 6y_2)$$

Proof:

$$(y_1 + 5y_2)x_1 + (-y_1 + 2y_2)x_2 + (3y_1 - y_2)x_3$$

$$\geq \frac{7}{b}x_1 + \frac{1}{b}x_2 + \frac{5}{b}x_3, \text{ by the Primal relaxed c.s.c.}$$

$$= \frac{1}{b}(7x_1 + x_2 + 5x_3)$$

$$\Downarrow$$

$$7x_1 + x_2 + 5x_3 \leq b\big((y_1 + 5y_2)x_1 + (-y_1 + 2y_2)x_2 + (3y_1 - y_2)x_3\big)$$

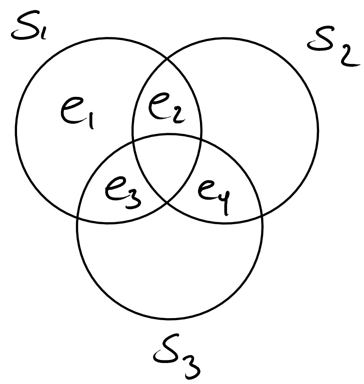
$$= b\big((x_1 - x_2 + 3x_3)y_1 + (5x_1 + 2x_2 - x_3)y_2\big)$$

$$\leq b(10cy_1 + 6cy_2), \text{ by the Dual r.c.s.c.}$$

$$= bc(10y_1 + 6y_2)$$

What is the dual of the Set Cover LP?

Ex:




$w_1 = 1$

$w_2 = 2$

$w_3 = 3$

Primal:

$$
\begin{aligned}
\min \quad & x_1 + 2x_2 + 3x_3 \\
\text{s.t.} \quad & x_1 \geq 1 \\
& x_1 + x_2 \geq 1 \\
& x_1 + x_3 \geq 1 \\
& x_2 + x_3 \geq 1 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

OPT = 3: $x_1 = x_2 = 1$

Dual:

$$
\begin{aligned}
\max \quad & y_1 + y_2 + y_3 + y_4 \\
\text{s.t.} \quad & y_1 + y_2 + y_3 \leq 1 \\
& y_2 + y_4 \leq 2 \\
& y_3 + y_4 \leq 3 \\
& y_1, y_2, y_3, y_4 \geq 0
\end{aligned}
$$

OPT = 3: $y_1 = 1$, $y_4 = 2$ or $y_3 = 1$, $y_4 = 2$

## Set Cover Primal

$$\min \sum_{j=1}^{m} x_j w_j$$
$$\text{s.t.} \sum_{j: e_i \in S_j} x_j \geq 1, \quad i = 1, 2, \ldots, n$$
$$x_j \geq 0, \quad j = 1, 2, \ldots, m$$

Covering problem

## Set Cover Dual

$$\max \sum_{i=1}^{n} y_i$$
$$\text{s.t.} \sum_{e_i \in S_j} y_i \leq w_j, \quad j = 1, 2, \ldots, m$$
$$y_i \geq 0, \quad i = 1, 2, \ldots, n$$

Packing problem

Recall that the dual is constructed such that the value of any solution to the dual is a lower bound on the value of any solution to the primal:

$$Z_{\text{Primal}} \geq Z_{\text{Dual}} \quad (\text{weak duality property})$$

In fact,

$$Z^*_{\text{Primal}} = Z^*_{\text{Dual}} \quad (\text{strong duality property})$$