Section 3.2: Makespan Scheduling - A PTAS

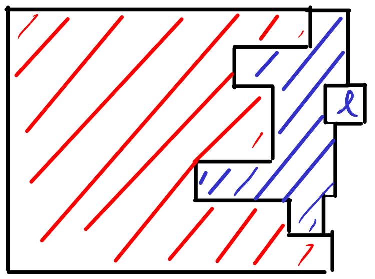
Idea for PTAS:

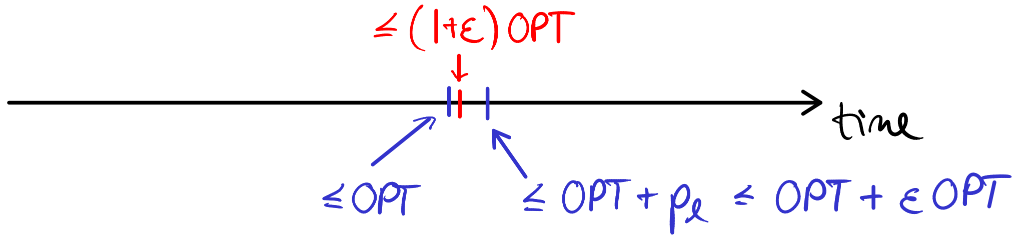
Partition the jobs into two sets (long and short jobs):

$$\overbrace{\underbrace{p_1, p_2, \ldots, p_x,}_{}}^{>\varepsilon \cdot OPT} \ \overbrace{\underbrace{p_{x+1}, \ldots, p_n}_{}}^{\leq \varepsilon \cdot OPT}$$

Schedule using rounding and dyn. prg. as for bin packing

Then use LPT




$\leq (1+\varepsilon) OPT$
time
$\leq OPT$
$\leq OPT + p_\ell \leq OPT + \varepsilon OPT$

We will derive a family of algorithms with an algorithm, B_k, for each $k \in \mathbb{Z}^+$. $(\varepsilon = \frac{1}{k})$

How to identify long / short jobs when we don't know OPT?

We need the short jobs to be $\leq \varepsilon \cdot OPT$ to ensure the approx. factor. For this purpose, we could use any lower bound on OPT, like P/m.

But we also need the long jobs to be $\geq \varepsilon \cdot OPT$ to ensure the approx. factor as well as the running time.

Scheduling the long jobs:

(1) „Guess" an optimal makespan T

(2) The long jobs are those longer than T/k^2. Round down each job size to the nearest multiple of T/k^2.

(3) Use dyn. prg. to check whether optimal makespan $\leq T$ for rounded long jobs.

Do binary search for T on the interval $[L, U]$, where

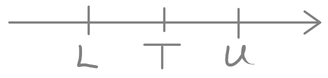
$$L = \max\left\{\left\lceil \frac{P}{m} \right\rceil, p_{max}\right\}$$

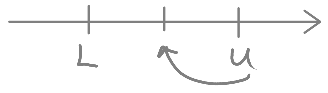
$$U = \left\lfloor \frac{P - p_{max}}{m} + p_{max} \right\rfloor = \left\lfloor \frac{P + (m-1)\, p_{max}}{m} \right\rfloor$$

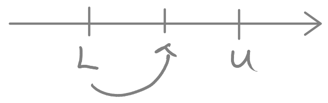
$B_k(I)$

```
$L \leftarrow \max\{\lceil \frac{P}{m} \rceil, p_{max}\}$;   $U \leftarrow \lceil \frac{P + (m-1)\,p_{max}}{m} \rceil$
While $L \neq U$
    $T \leftarrow \frac{1}{2}\lceil L+U \rceil$
    $I' \leftarrow \{ \text{job } j \in I \mid p_j > T/k \}$ // Update set of long jobs
    $I'' \leftarrow I'$ with each job size rounded down to
            nearest multiple of $T/k^2$
    Use dyn. prg. to pack $I''$ in bins of size $T$
    If #bins $\leq m$
        $U \leftarrow T$
    else
        $L \leftarrow T+1$
$S'' \leftarrow$ schedule of $I''$ corresponding to the packing
        found by dyn. prg.
$S' \leftarrow$ schedule of $I'$ corresponding to $S''$
$S \leftarrow$ schedule of $I$ obtained by adding
        short jobs to $S'$ using LPT
```

Binary search for T







Dyn. prg. as for bin packing:

S'' places $\leq k$ jobs on each machine:

Each long job has size $\geq T/k$

Since T/k is a multiple of T/k^2, each job in I'' also has size $\geq T/k$.

There are $\leq k^2$ different job sizes in I'', since no job is longer than T.

Hence, the configuration of a machine can be represented by a vector $(s_1, s_2, \dots, s_{k^2})$, where $0 \leq s_i \leq k$.

Thus, $|\mathcal{C}| \leq (k+1)^{k^2}$.

Table (B):

$\leq k^2$ dimensions (one for each size in I'')

n_i rows in dimension i ($n_i = \#$ items of size i in I'')

$$\boxed{B(n_1, \dots, n_{k^2}) = 1 + \min_{\vec{s} \in \mathcal{C}} \{ B(n_1 - s_1, \dots, n_{k^2} - s_{k^2}) \}}$$

Running time:

# table entries: $O(n^{k^2})$

Time per entry: $|\mathcal{C}| \leq (k+1)^{k^2}$

# iterations of while loop: $\log(U-L) \leq \log(p_{max})$

Total time: $O(n^{k^2} (k+1)^{k^2} \log(p_{max}))$

Approximation ratio:

When B_k terminates the while loop,

$$\text{makespan}(S'') = T = \text{OPT}(I)$$

S'' places $\leq k$ jobs on each machine:

Each long job has size $\geq T/k$

Since T/k is a multiple of T/k^2, each job in I'' also has size $\geq T/k$.

Since each of the $\leq k$ jobs on a machine loses less than T/k^2 in the rounding,

$$\begin{aligned} \text{makespan}(S') &< \text{makespan}(S'') + k \cdot \frac{T}{k^2} \\ &= T + \frac{T}{k} \\ &= \left(1+\frac{1}{k}\right) \text{OPT}(I'') \\ &\leq \left(1+\frac{1}{k}\right) \text{OPT}(I) \end{aligned}$$

Thus, if the last job to finish is a long job,

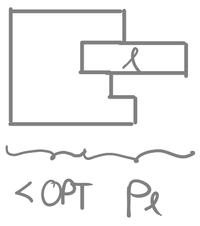
$$B_k(I) < \left(1+\frac{1}{k}\right)\text{OPT}(I).$$

Otherwise, the last job to finish has $p_\ell \leq \frac{T}{k} \leq \frac{\text{OPT}(I)}{k}$.

Hence,

$$B_k(I) < \text{OPT}(I) + p_\ell \leq \left(1+\frac{1}{k}\right)\text{OPT}$$

By the same argument as in the analysis of LS:



Thus, in both cases, $B_k(I) < \left(1+\frac{1}{k}\right)\text{OPT}$.

$\boxed{\text{Theorem 3.7 : } \{B_k\} \text{ is a PTAS}}$

<u>Proof</u>:

B_k achieves an approx. factor of $1+\varepsilon$ with running time

$$O\left(\left(\left(\tfrac{1}{\varepsilon}+1\right)n\right)^{\left(\frac{1}{\varepsilon}\right)^2} \cdot n \cdot \log(p_{max})\right).$$

If $\varepsilon \in O(1)$, this is poly. in the input size, since it takes $\geq \log(p_{max})$ bits to represent the job sizes. $\square$

$\{B_k\}$ is <u>not</u> a FPTAS, since the running time is exponential in $\frac{1}{\varepsilon}$.

Note that we did not expect a FPTAS, since the problem is <u>strongly</u> NP-complete...

The problem is strongly NP-complete, meaning that even the special case where $\exists$ polynomial q s.t. $p_{max} \leq q(n)$, for all input instances, is NP-complete.

This implies that $\boxed{\nexists \text{FPTAS, unless } P = NP}$

Assume to the contrary that $\exists$FPTAS for the problem, i.e.,

$\forall \varepsilon > 0 : \exists$ $(1+\varepsilon)$-approx alg. A_ε with running time poly. in n and $\frac{1}{\varepsilon}$.

Consider the special case of the problem where $\exists$ polynomial q s.t. $p_{max} \leq q(n)$, for all instances. In this case, $P \leq n \cdot q(n) \equiv p(n)$.

For $\varepsilon = \frac{1}{p(n)}$,

- $\frac{1}{\varepsilon}$ is poly. in n, so the running time of A_ε is poly. in n.

- $A_\varepsilon(I) \leq \left(1 + \frac{1}{p(n)}\right) \cdot OPT(I)$, for any input I

  $< OPT(I) + 1$, since $OPT(I) < P \leq p(n)$

  Thus, since $A_\varepsilon(I)$ is integer, $A_\varepsilon(I) = OPT(I)$.

If $P \neq NP$, this contradicts the fact that the problem is strongly NP-complete.