Section 5.4: Randomized rounding

In Section 5.3 we saw that biasing the prob. of setting each variable true resulted in a better approx. guarantee.
The approximation ratio can be further improved by allowing a different bias for each variable.
We will develop an LP-formulation of the problem.

For each clause, C_j, we define:
P_j : the set of indices of variables that occur positively in C_j.
N_j : ——— " ——— negatively —"—
Then, C_j can be written as

$$\bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \overline{x_i}$$

If $y_i = 0$ corresponds to $x_i \equiv F$ and $y_i = 1$ corresponds to $x_i \equiv T$, then C_j is true, iff

$$\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq 1$$

This leads to the following IP-formulation:

IP_ϕ:

$$\max \sum_{j=1}^{m} z_j w_j$$

subject to

$$\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1-y_i) \geq z_j, \quad 1 \leq j \leq m$$

$$y_i \in \{0,1\}, \quad 1 \leq i \leq n$$

$$z_j \in \{0,1\}, \quad 1 \leq j \leq m$$

Let LP_ϕ be the LP-relaxation of IP_ϕ, i.e.,

LP_ϕ:

$$\max \sum_{j=1}^{m} z_j w_j$$

subject to

$$\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1-y_i) \geq z_j, \quad 1 \leq j \leq m$$

$$0 \leq y_i \leq 1, \quad 1 \leq i \leq n$$

$$0 \leq z_j \leq 1, \quad 1 \leq j \leq m$$

Clearly, $Z^*_{LP_\phi} \geq Z^*_{IP_\phi} = OPT$

value of opt. sol. to LP_ϕ

value of opt. sol. to IP_ϕ

value of opt. sol. to corresponding MAXSAT problem

RandRounding (ϕ)

$(\vec{y^*}, \vec{z^*}) \leftarrow$ opt. sol. to LP_ϕ
For $i \leftarrow 1$ to n
    Set x_i true with prob. y_i^*

The approx. ratio of RandRounding is at least $1-\frac{1}{e} \approx 0.632$.

For proving this, we will use the following two facts:

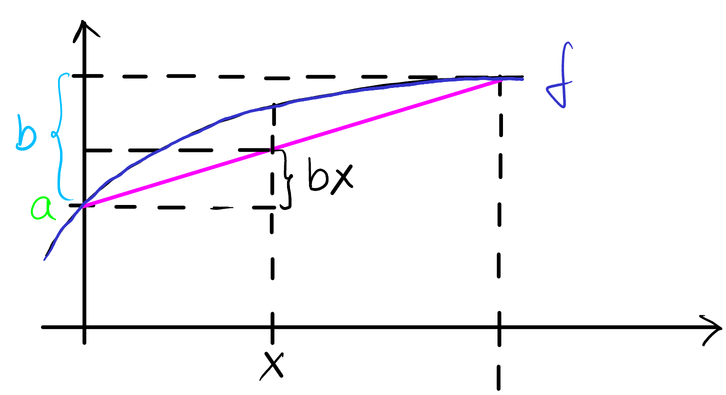
Fact 5.8 (Arithmetic-geometric mean inequality):
For any $a_1, a_2, \dots, a_k \geq 0$,

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq \frac{1}{k}\sum_{i=1}^{k} a_i$$

A function f is concave on an interval I, if $f''(x) \leq 0$ for any $x \in I$. (the slope is nonincreasing)

Fact 5.9:

$$\left.\begin{array}{l} f \text{ is concave on } [0,1] \\ f(0) = a, \quad f(1) = a+b \end{array}\right\} \Rightarrow f(x) \geq a+bx, \text{ for } x \in [0,1]$$



$\boxed{\text{Theorem 5.10: RandRounding is a } (1-\tfrac{1}{e})\text{-approx. alg}}$

Proof:

For $1 \le j \le m$, let p_j be the probability that C_j is satisfied, and let $\overline{p_j} = 1 - p_j$.

Our goal is to show that $p_j \geq (1-\frac{1}{e}) z_j^*$.

This will establish the approx. factor, since $OPT \le \sum_{j=1}^{m} z_j^* w_j$

$$\overline{p_j} = \prod_{i \in P_j} (1 - y_i^*) \prod_{i \in N_j} y_i^*$$

$$\leq \left(\frac{1}{\ell_j} \left(\sum_{i \in P_j} (1-y_i^*) + \sum_{i \in N_j} y_i^* \right) \right)^{\ell_j}, \text{ by Fact 5.8}$$

$$= \left(\frac{1}{\ell_j} \left(|P_j| - \sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - 1 + y_i^*) \right) \right)^{\ell_j}$$

$$= \left(\frac{1}{\ell_j} \left(|P_j| - \sum_{i \in P_j} y_i^* + |N_j| - \sum_{i \in N_j} (1-y_i^*) \right) \right)^{\ell_j}$$

$$= \left(1 - \frac{1}{\ell_j} \left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1-y_i^*) \right) \right)^{\ell_j}, \text{ since } |P_j| + |N_j| = \ell_j$$

$$\leq \left(1 - \frac{z_j^*}{\ell_j} \right)^{\ell_j}, \text{ since } (\vec{y}^*, \vec{z}^*) \text{ is a solution to } LP_\phi$$

Thus, $p_j \geq 1 - \left(1 - \frac{z_j^*}{\ell_j} \right)^{\ell_j} \equiv f(z_j^*)$

which is a concave function of z_j^*:

$$f'(z_j^*) = -\ell_j \left(1 - \frac{z_j^*}{\ell_j}\right)^{\ell_j - 1} \cdot \left(-\frac{1}{\ell_j}\right) = \left(1 - \frac{z_j^*}{\ell_j}\right)^{\ell_j - 1}$$

$$f''(z_j^*) = (\ell_j - 1)\left(1 - \frac{z_j^*}{\ell_j}\right)^{\ell_j - 2} \cdot \left(-\frac{1}{\ell_j}\right) = \underbrace{\left(\frac{1}{\ell_j} - 1\right)}_{\leq 0} \underbrace{\left(1 - \frac{z_j^*}{\ell_j}\right)^{\ell_j - 2}}_{\geq 0}$$

$$\leq 0$$

Note that

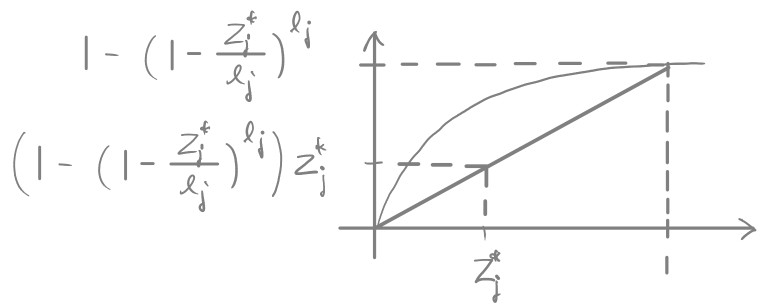
$$f(0) = 1 - \left(1 - \frac{0}{\ell_j}\right)^{\ell_j} = 1 - 1 = 0$$

$$f(1) = 1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j} =$$

Thus, using Fact 5.9 with $a = f(0)$ and $b = f(1) - f(0)$,

$$p_j \geq 1 - \left(1 - \frac{z_j^*}{\ell_j}\right)^{\ell_j}$$

$$\geq \left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) z_j^*$$

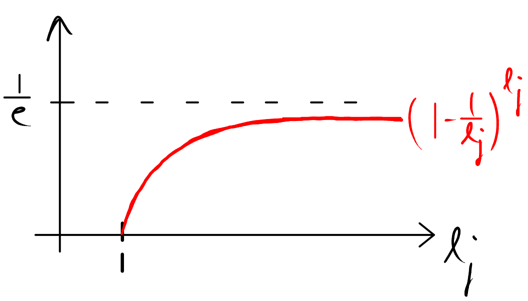


Hence,

$$E[\text{Rand Rounding}] = \sum_{j=1}^{m} p_j w_j$$

$$\geq \sum_{j=1}^{m} \left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) z_j^* w_j$$

$$\geq \left(1 - \frac{1}{e}\right) \cdot \underbrace{\sum_{j=1}^{m} z_j^* w_j}_{= Z^*_{LP_\phi} \geq OPT}$$



$\square$

Note that

$$\boxed{\text{Rand Rounding can be derandomized exactly like Rand and Rand}_p}$$

Section 5.5 : Choosing the better of two solutions

Combining the alg.s of Sections 5.1 and 5.4 gives a better approx. factor than using any one of them separately. This is because they have different worst-case inputs:

Rand satisfies clause C_j with prob. $p_R = 1 - \left(\frac{1}{2}\right)^{\ell_j}$.

RandRounding satisfies C_j with prob. $p_{RR} \geq \left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) z_j^*$.

While p_R increases with ℓ_j, the lower bound on p_{RR} decreases with ℓ_j.

BestOfTwo(ϕ)

$\vec{x}_R \leftarrow$ Rand(ϕ)
$\vec{x}_{RR} \leftarrow$ RandRounding(ϕ)
If $w(\phi, \vec{x}_R) \geq w(\phi, \vec{x}_{RR})$
    Return $\vec{x}_R$
Else
    Return $\vec{x}_{RR}$

Note that

BestOfTwo is derandomized by using the derandomized versions of Rand and RandRounding.