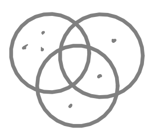
Set Cover - recap.

LP - relax: $\min \sum_{j=1}^{m} x_j w_j$

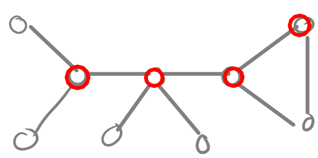
$$\text{s.t.} \sum_{j:\, e_i \in S_j} x_j \geq 1, \qquad 1 \leq i \leq n$$

$$x_j \geq 0, \qquad 1 \leq j \leq m$$

Deterministic rounding: f - approx. alg.

Eks:  $f = 2$

Vertex Cover

Ex: 

Exercises

Recap ctd.:

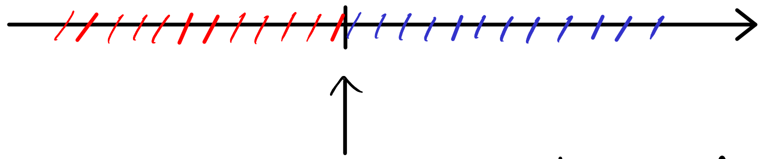
Dual LP: $\max \sum_{i=1}^{n} y_i$

$$\text{s.t.} \sum_{e_i \in S_j} y_i \leq w_j, \qquad 1 \leq j \leq m$$

$$y_i \geq 0, \qquad 1 \leq i \leq n$$

For <u>any</u> pair $\vec{x}, \vec{y}$ to the primal / dual problems:

$$\sum_{i=1}^{n} y_i \leq \sum_{j=1}^{m} x_j w_j \qquad \text{(weak duality)}$$



optimum objective value
for both problems

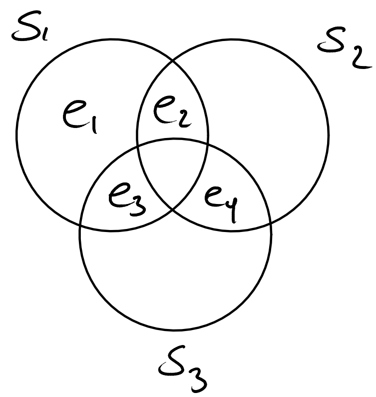
(strong duality)

(Relaxed) Complementary Slackness Conditions:

Primal c.s.c. $\quad x_j > 0 \Rightarrow \sum_{e_i \in S_j} y_i = w_j \quad \left(\geq \frac{1}{b} w_j\right), \quad 1 \leq j \leq m$

Dual c.s.c. $\quad y_i > 0 \Rightarrow \sum_{j : e_i \in S_j} x_j = 1 \quad (\leq c), \quad 1 \leq i \leq n$

Ex:



$w_1 = 1$

$w_2 = 2$

$w_3 = 3$

Primal:

$$
\begin{aligned}
\min \quad & x_1 + 2x_2 + 3x_3 \\
\text{s.t.} \quad & x_1 \geq 1 \\
& x_1 + x_2 \geq 1 \\
& x_1 + x_3 \geq 1 \\
& x_2 + x_3 \geq 1 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

OPT = 3 :

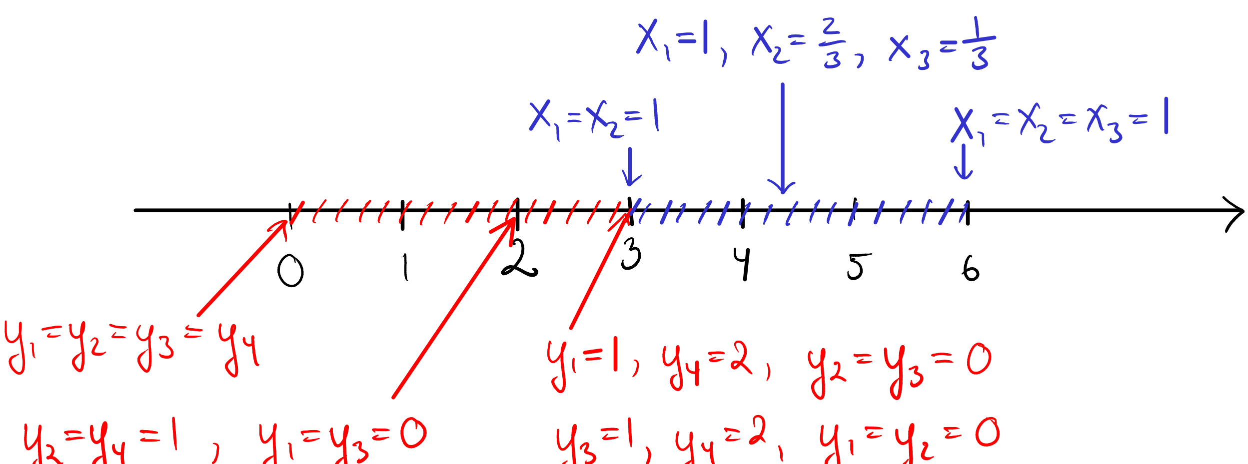
$x_1 = x_2 = 1$

Dual:

$$
\begin{aligned}
\max \quad & y_1 + y_2 + y_3 + y_4 \\
\text{s.t.} \quad & y_1 + y_2 + y_3 \leq 1 \\
& y_2 + y_4 \leq 2 \\
& y_3 + y_4 \leq 3 \\
& y_1, y_2, y_3, y_4 \geq 0
\end{aligned}
$$

OPT = 3 :

$y_1 = 1$, $y_4 = 2$ or $y_3 = 1$, $y_4 = 2$



Alg. 2 for Set Cover

$\vec{y}^* \leftarrow$ opt. sol. to dual LP

$I' \leftarrow \{ j \mid \sum_{e_i \in S_j} y_i = w_j \}$

In the ex. above:

with $y_1^* = 1$, $y_4^* = 2$, Alg 2 would choose S_1 and S_2 with a total weight of 3.

with $y_3^* = 1$, $y_4^* = 2$, Alg. 2 would choose S_1, S_2, and S_3 with a total weight of 6.

The first solution is optimal, and the latter is a 2-approximation (i.e., an f-approximation).

Alg. 2 is an f-approximation algo.:

If the algo. chooses S_1, S_2, and S_3, the total weight is $W = w_1 + w_2 + w_3$, and

$$w_1 + w_2 + w_3 = (y_1^* + y_2^* + y_3^*) + (y_2^* + y_4^*) + (y_3^* + y_4^*),$$

since the algo. chooses exactly those sets that have LHS = RHS.

Since each y_i is present in at most f constraints,

$$\begin{aligned} W &\le f \cdot (y_1^* + y_2^* + y_3^* + y_4^*) \\ &= f \cdot Z^*_{dual} \\ &\le f \cdot Z^*_{primal}, \quad \text{by the weak duality property} \\ &= f \cdot OPT \end{aligned}$$

Lemma 1.7

Alg. 2 produces a set cover

Proof:

Assume for the sake of contradiction that some element e_k is not covered by $\{S_j \mid j \in I'\}$.

Then $\sum_{e_i \in S_j} y_i < w_j$ for all S_j containing e_k.

These are exactly the constraints involving y_k.

Thus, none of the constraints involving y_k are tight.

This means that y_k can be increased without violating any constraint.

Since this will increase the value $\sum_{i=1}^{n} y_i$ of the sol., we conclude that the solution $\vec{y}$ was not optimal. $\square$

Ex:

In the ex. above, assume

$y_1 = y_2 = y_3 = 0$
$y_4 = 2$

Then, only the second constraint is tight, so only S_2 is picked:

$$\begin{aligned} y_1 + y_2 + y_3 \quad\quad &= 0 < 1 \\ y_2 \quad\quad + y_4 &= 2 \\ y_3 + y_4 &= 2 < 3 \end{aligned}$$

e_4 is not covered, since none of the two constraints involving y_4 are tight.

We can increase y_3 from 0 to 1 without violating any constraints
(Then two other constrainsts become tight.)
This increases the sol. value from 2 to 3.
Thus, the sol. above was not optimal.

Or we could increase y_1 from 0 to 1.
Then only the first constraint becomes tight, resulting in an optimal solution.

This illustrates the idea of the primal-dual alg of Section 1.5.

We now give a more formal proof that Alg 2 is an f-approximation algo.

Thm 1.8

$\boxed{\text{Alg. 2 is an } f\text{-approx. algo.}}$

Proof:

The correctness follows from Lemma 1.7.

Approx. guarantee:

$$\sum_{j\in I'} w_j = \underbrace{\sum_{j\in I'} \sum_{e_i\in S_j}}_{y_i^* \text{ appears once for each set in the sol. containing } e_i} y_i^*$$

$$= \sum_{i=1}^{n} \underbrace{\left| \{ j\in I' \mid e_i\in S_j \} \right|}_{\#\text{sets in the sol. containing } e_i} \cdot y_i^*$$

$$\leq \sum_{i=1}^{n} \underbrace{f_{e_i}}_{\#\text{sets containing } e_i} \cdot y_i^*$$

$$\leq \sum_{i=1}^{n} f \cdot y_i^*$$

$$= f \cdot Z^*_{dual}$$

$$\leq f \cdot Z^*_{primal}, \text{ by the weak duality property}$$

$$\leq f \cdot OPT$$

□

Note that for proving the above theorem, we could also use the relaxed C.S.C. (with $b=1$, $c=f$), since

$$\sum_{j: e_i \in S_j} x_j \le f, \quad \text{for all } i=1,2,\dots,n$$

Note that, on any instance of Set Cover, $I \subseteq I'$:

Since the LP is solved optimally,

$x_j > 0 \Rightarrow$ constraint j is tight $\Rightarrow j \in I'$.

Thus, $j \in I \Rightarrow x_j \geq \frac{1}{f} \Rightarrow j \in I'$

Thus, Alg. 1 is always at least as good as Alg. 2.

Both Alg. 1 and Alg. 2 rely on solving an LP (optimally). In Section 1.5, we will study a more time efficient alg.

The key observation is that in the proof of Thm 1.8, we did not need the fact that $\vec{y}^*$ is optimal, since $Z_{dual} \leq Z^*_{primal}$, for any feasible dual solution.

Thus, the crux is to obtain an index set I'' s.t.

- $\bigcup_{j \in I''} S_j$ is a vertex cover
- $\sum_{j \in I''} w_j = \sum_{j \in I''} \sum_{e_i \in S_j} y_i$, for some feasible sol. $\vec{y}$ to the dual LP

without solving an LP optimally.

Section 1.5 : A Primal-Dual Alg. for Set Cover

Alg. 1.1 for Set Cover : Primal-Dual

$I'' \leftarrow \emptyset$

$\vec{y} \leftarrow \vec{0}$

While $\exists e_k \notin \bigcup_{j \in I''} S_j$

  Increase y_k until some constraint, ℓ, becomes tight, i.e., $\sum_{e_i \in S_\ell} y_i = w_\ell$

$I'' \leftarrow I'' \cup \{\ell\}$

Note that $e_k \in S_\ell$

Thm 1.9

Alg. 1.1 is an f-approx. alg. for Set Cover

<u>Proof</u>:

Alg. 3 produces a set cover, since as long as some element is not covered, the corresponding dual constraints are non-tight.

The approx. guarantee follows from the same calculations as in the proof of Thm. 1.8, since

$$\sum_{j \in I''} w_j = \sum_{j \in I''} \sum_{e_i \in S_j} y_i \leq f \cdot Z_{dual} \leq f \cdot Z^*_{dual}$$

$\square$

In contrast to Alg. 2 from Section 1.4, Alg. 1.1 does not necessarily produce an optimal dual solution:

In the example above, it might do the following.

$y_2 \leftarrow 1$ (S_1 is picked, e_4 still uncovered)

$y_4 \leftarrow 1$ (S_2 is picked)

(This is fine, since the proof of Thm. 1.8 does not use that $\sum y_i = OPT$, only that $\sum y_i \leq OPT$, which is true for <u>any</u> feasible sol. to the dual.)