Department of Mathematics and Computer Science
University of Southern Denmark, Odense

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Marco Chiarandini

DM545/DM871 – Linear and integer programming

Sheet 7, Week 11, Spring 2021

Exercise 1* MILP Modeling

Shift scheduling. The administrators of a department of a urban hospital have to organize the working shifts of nurses maintaining sufficient staffing to provide satisfactory levels of health care. Staffing requirements at the hospital during the whole day vary from hour to hour and are reported in Table 1.

Hour	Staffing requirement
0 am to 6 am	2
6 am to 8 am	8
8 am to 11 am	5
11 am to 2 pm	7
2 pm to 4 pm	3
4 pm to 6 pm	4
6 pm to 8 pm	6
8 pm to 10 pm	3
10 pm to 12 pm	1

Table 1:

According to union agreements, nurses can work following one of the seven shift patterns in Table 2 each with its own cost.

pattern	Hours of work	total hours	cost
1	0 am to 6 am	6	720 Dkk
2	0 am to 8 am	6	800 Dkk
3	6 am to 2 pm	8	740 Dkk
4	8 am to 4 pm	8	680 Dkk
5	2 pm to 10 pm	8	720 Dkk
6	4 pm to 12 pm	6	780 Dkk
7	6 pm to 12 pm	6	640 Dkk

Table 2:

The department administrators would like to identify the assignment of nurses to working shifts that meets the staffing requirements and minimizes the total cost.

Solution:

Exercise 2*

Consider the following three matrices:

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

For each of them say if it is totally unimodular and justify your answer.

Solution:

$$
\begin{array}{llr}
\min & 720x_1 + 800x_2 + 760x_3 + 680x_4 + 720x_5 + 780x_6 + 640x_7 & \\
0-6: & x_1 + x_2 & \geq 2 \\
6-8: & x_2 + x_3 & \geq 8 \\
8-11: & x_3 + x_4 & \geq 5 \\
11-14: & x_3 + x_4 & \geq 7 \\
14-16: & x_4 + x_5 & \geq 3 \\
16-18: & x_5 + x_6 & \geq 4 \\
18-20: & x_5 + x_6 + x_7 & \geq 6 \\
20-22: & x_5 + x_6 + x_7 & \geq 3 \\
22-24: & x_6 + x_7 & \geq 1 \\
 & x_1, x_2 \dots x_7 \geq 0 \text{ and integer} &
\end{array}
$$

We look for the satisfaction of the conditions of the theorem saw in class. Accordingly, it is *sufficient* for a matrix to be TUM to find a partition of the rows such that the ones with same sign are in different partitions and those with different sign in the same partition.
The first matrix is TUM. The partition is $I_1 = \{1, 4\}$ and $I_2 = \{2, 3\}$.
The second matrix is TUM. The partition is $I_1 = \{1, 2, 3\}$ and $I_2 = \{4\}$.
The third matrix is not TUM. Here the theorem does not apply and there is a submatrix with deteminant -2, hence we cannot be sure that the solutions associated with the matrix will be integer.

Exercise 3*

In class, we proved that the (mininum) vertex covering problem and the (maximum) matching problem are a weak dual pair. Prove that for bipartite graphs they, actually, are a strong dual pair.

Solution:

The formulation of the matching problem is:

$$
\begin{aligned}
\max \quad & \sum_{e \in E} w_e x_e \\
& \sum_{e \in E: v \in e} x_e \leq 1 \qquad \forall v \in V \\
& x_e \in \{0, 1\} \quad \forall e \in E
\end{aligned}
$$

If we take the linear relaxation and make the dual of it we obtain:

$$
\begin{aligned}
\min \quad & \sum_{v \in V} y_v \\
& y_v + y_u \geq w_{uv} \quad \forall u, v \in V, uv \in E \\
& y_v \geq 0 \quad \forall v \in V
\end{aligned}
$$

This latter is the linear relaxation of the vertex cover problem.
Hence the two problems make a weak dual pair. It is not strong, indeed a triangle has optimal vertex cover 2 and optimal matching 1.
In bipartite graphs instead the pair is strong dual. Indeed, the solution of the linear relaxation of the matching problem on bipartite graphs is always integer. The same must old for the dual and hence the gap is closed.
A more formal proof is on page 24-25 of [Wo].

Exercise 4*

Generalized Assignment Problem. Suppose there are n types of tracks available to deliver products to m clients. The cost of track of type i serving client j is c_{ij}. The capacity of track type i is Q_i and the demand of each client is d_j. There are a_i tracks for each type. Formulate an IP model to decide how many tracks of each type are needed to satisfy all clients so that the total cost of doing the deliveries is minimized. If all the input data will be integer, will the solution to the linear programming relaxation be integer?

Solution:

It is good trying to model the problem as a min cost flow problem. However, one can soon realize that this is not possible since we are asked for the number of tracks but we need to take into account a demand and a capacity of products.
We can however write an ILP model:

$$\min \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} \tag{1}$$

$$\sum_{j\in J} x_{ij} \leq a_i \qquad \forall i \in I \tag{2}$$

$$\sum_{i\in I} Q_i x_{ij} \geq d_j \qquad \forall j \in J \tag{3}$$

$$x_{ij} \geq 0 \text{ and integer} \qquad \forall (i,j) \in A \tag{4}$$

The solutions of the linear relaxation are not necessarily integer, becasue this is not a min cost flow model and the matrix is not trivially TUM.

Exercise 5

1. In class we stated that for the uncapaciteted facility location problem there are two formulations:

$$X = \{(x,y) \in \mathbb{R}^m_+ \times \mathbb{B}^1 : \sum_{i=1}^m x_i \leq my, x_i \leq 1 \text{ for } i = 1,\dots,m\}$$

$$P = \{(x,y) \in \mathbb{R}^m_+ \times \mathbb{R}^1 : x_i \leq y \text{ for } i = 1,\dots,m, y \leq 1\}$$

   Prove that the polyhedron $P = \{(x_1,\dots,x_m,y) \in \mathbb{R}^{m+1} : y \leq 1, x_i \leq y \text{ for } i = 1,\dots,m\}$ has integer vertices. [Hint: start by writing the constraint matrix and show that it is TUM.]

   Solution:

2. Consider the following (integer) linear programming problem:

$$\begin{aligned}
\min\ & c_1x_1 + c_2x_2 + c_3x_3 + c_3x_4 \\
& x_3 + x_4 \geq 10 \\
& x_2 + x_3 + x_4 \geq 20 \\
& x_1 + x_2 + x_3 + x_4 \geq 30 \\
& x_2 + x_3 \geq 15 \\
& x_1, x_2, x_3, x_4 \in \mathbb{Z}^+_0
\end{aligned} \tag{5}$$

   The constraint matrix has consecutive 1's in each column. Matrices with consecutive 1's property for each column are totally unimodular. Show that this fact holds for the specific numerical example (5). That is, show first that the constraint matrix of the problem has consecutive 1s in the columns and then that you can transform this matrix into one that you should recognize to be a TUM matrix. [Hint: rewrite the problem in standard form (that is, in equation form) and add a redundant row $\mathbf{0} \cdot \mathbf{x} = 0$ to the set of constraints. Then perform elementary row operations to bring the matrix to a TUM form.]

   Solution:

(a)

$$A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & y \end{array} \\ \begin{bmatrix} & & & & & 1 \\ & & 1 & & & -1 \\ & 1 & & & & -1 \\ 1 & & & & & -1 \\ & & & 1 & & -1 \\ & & & & 1 & -1 \end{bmatrix} \end{array} \begin{array}{c} \\ \leq 1 \\ \leq 0 \\ \leq 0 \\ \leq 0 \\ \leq 0 \\ \leq 0 \end{array}$$

$$A^T \text{ is } \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 1 & -1 & -1 & -1 & -1 \end{bmatrix} \Rightarrow \text{TUM for } I_1 = \text{all} \ldots, \quad I_2 = \emptyset$$

$$\sum_{i \in I_1} a_{ij} = \sum_{i \in I_2} a_{ij} = 0 \quad \forall j$$

b)

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 20 \\ 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 30 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for each $i = 4, 3 .. 1$ subtract ith row with $(i+1)$th row

$$\begin{matrix} 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 10 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 10 \\ -1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & -15 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & -15 \end{matrix}$$

$\Rightarrow$ TUM and min cost flow

3. Use one of the two previous results to show that the *shift scheduling problem* in Exercise 1 of this Sheet can be solved efficiently when formulated as a mathematical programming problem. (You do not need to find numerical results.)

Solution:

c) min $720x_1 + 800x_2 + 740x_3 + 680x_4 + 720x_5 + 780x_6 + 640x_7$

0 - 6: $x_1 + x_2 \geq 2$

6 - 8: $x_2 + x_3 \geq 8$

8 - 11: $x_3 + x_4 \geq 5$

11 - 14: $x_3 + x_4 \geq 7$

14 - 16: $x_4 + x_5 \geq 3$

16 - 18: $x_5 + x_6 \geq 4$

18 - 20: $x_5 + x_6 + x_7 \geq 6$

20 - 22: $x_5 + x_6 + x_7 \geq 3$

22 - 24: $x_6 + x_7 \geq 1$

$x_1, x_2 \ldots x_7 \geq 0$ and integer.

The matrix has consecutive 1's property on cols hence LP relax gives integer results.

Exercise 6* Network Flows: Problem of Representatives

A town has r residents $R_1, R_2, \dots, R_r$; q clubs $C_1, C_2, \dots, C_q$; and p political parties $P_1, P_2, \dots, P_p$. Each resident is a member of at least one club and can belong to exactly one political party. Each club must nominate one of its members to represent it on the town's governing council so that the number of council members belonging to the political party P_k is at most u_k. Is it possible to find a council that satisfies this "balancing" property?
Show how to formulate this problem as a maximum flow problem.

Solution:

[AMO] at library bookshelf pages 170-176.