

DM545/DM871
Linear and Integer Programming

Lecture 6
More on Duality

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

Derivation
Dual Simplex
Sensitivity Analysis

1. Derivation
 Lagrangian Duality
2. Dual Simplex
3. Sensitivity Analysis

Summary

- Derivation:
 1. economic interpretation
 2. bounding
 3. multipliers
 4. recipe
 5. Lagrangian
- Theory:
 - Symmetry
 - Weak duality theorem
 - Strong duality theorem
 - Complementary slackness theorem
- Dual Simplex
- Sensitivity Analysis, Economic interpretation

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Lagrangian Duality

Relaxation: if a problem is hard to solve then find an easier problem resembling the original one that provides information in terms of bounds. Then, search for the strongest bounds.

$$\begin{aligned} \min \quad & 13x_1 + 6x_2 + 4x_3 + 12x_4 \\ & 2x_1 + 3x_2 + 4x_3 + 5x_4 = 7 \\ & 3x_1 + \quad + 2x_3 + 4x_4 = 2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

We wish to reduce to a problem easier to solve, ie:

$$\begin{aligned} \min \quad & c_1x_1 + c_2x_2 + \dots + c_nx_n \\ & x_1, x_2, \dots, x_n \geq 0 \end{aligned}$$

solvable by inspection: if $c < 0$ then $x = +\infty$, if $c \geq 0$ then $x = 0$.

measure of violation of the constraints:

$$\begin{aligned} & 7 - (2x_1 + 3x_2 + 4x_3 + 5x_4) \\ & 2 - (3x_1 + \quad + 2x_3 + 4x_4) \end{aligned}$$

We relax these measures in obj. function with **Lagrangian multipliers** y_1, y_2 .

We obtain a family of problems:

$$PR(y_1, y_2) = \min_{x_1, x_2, x_3, x_4 \geq 0} \left\{ \begin{array}{l} 13x_1 + 6x_2 + 4x_3 + 12x_4 \\ +y_1(7 - 2x_1 - 3x_2 - 4x_3 - 5x_4) \\ +y_2(2 - 3x_1 - 2x_3 - 4x_4) \end{array} \right\}$$

1. for all $y_1, y_2 \in \mathbb{R} : \text{opt}(PR(y_1, y_2)) \leq \text{opt}(P)$
2. $\max_{y_1, y_2 \in \mathbb{R}} \{\text{opt}(PR(y_1, y_2))\} \leq \text{opt}(P)$

PR is easy to solve.

(It can be also seen as a proof of the weak duality theorem)

$$PR(y_1, y_2) = \min_{x_1, x_2, x_3, x_4 \geq 0} \left\{ \begin{array}{l} (13 - 2y_2 - 3y_2) x_1 \\ + (6 - 3y_1) x_2 \\ + (4 - 2y_2) x_3 \\ + (12 - 5y_1 - 4y_2) x_4 \\ + 7y_1 + 2y_2 \end{array} \right\}$$

if coeff. of x is < 0 then bound is $-\infty$ then LB is useless

$$(13 - 2y_2 - 3y_2) \geq 0$$

$$(6 - 3y_1) \geq 0$$

$$(4 - 2y_2) \geq 0$$

$$(12 - 5y_1 - 4y_2) \geq 0$$

If they all hold then we are left with $7y_1 + 2y_2$ because all go to 0.

$$\max 7y_1 + 2y_2$$

$$2y_2 + 3y_2 \leq 13$$

$$3y_1 \leq 6$$

$$+ 2y_2 \leq 4$$

$$5y_1 + 4y_2 \leq 12$$

General Formulation

$$\begin{array}{ll} \min & z = \mathbf{c}^T \mathbf{x} & \mathbf{c} \in \mathbb{R}^n \\ & A\mathbf{x} = \mathbf{b} & A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m \\ & \mathbf{x} \geq \mathbf{0} & \mathbf{x} \in \mathbb{R}^n \end{array}$$

$$\max_{\mathbf{y} \in \mathbb{R}^m} \left\{ \min_{\mathbf{x} \in \mathbb{R}_+^n} \{ \mathbf{c}^T \mathbf{x} + \mathbf{y}^T (\mathbf{b} - A\mathbf{x}) \} \right\}$$

$$\max_{\mathbf{y} \in \mathbb{R}^m} \left\{ \min_{\mathbf{x} \in \mathbb{R}_+^n} \{ (\mathbf{c}^T - \mathbf{y}^T A) \mathbf{x} + \mathbf{y}^T \mathbf{b} \} \right\}$$

$$\begin{array}{ll} \max & \mathbf{b}^T \mathbf{y} \\ & A^T \mathbf{y} \leq \mathbf{c} \\ & \mathbf{y} \in \mathbb{R}^m \end{array}$$

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Dual Simplex

- Dual simplex (Lemke, 1954): apply the simplex method to the dual problem and observe what happens in the primal tableau:

$$\begin{aligned}\max\{c^T x \mid Ax \leq b, x \geq 0\} &= \min\{b^T y \mid A^T y \geq c^T, y \geq 0\} \\ &= -\max\{-b^T y \mid -A^T y \leq -c^T, y \geq 0\}\end{aligned}$$

- We obtain a new algorithm for the primal problem: the dual simplex
It corresponds to the primal simplex applied to the dual

Primal simplex on primal problem:

1. pivot > 0
2. col c_j with wrong sign
3. row: $\min \left\{ \frac{b_i}{a_{ij}} : a_{ij} > 0, i = 1, \dots, m \right\}$

Dual simplex on primal problem:

1. pivot < 0
2. row $b_i < 0$
(condition of feasibility)
3. col: $\min \left\{ \left| \frac{c_j}{a_{ij}} \right| : a_{ij} < 0, j = 1, 2, \dots, n + m \right\}$
(least worsening solution)

1. (primal) simplex on primal problem (the one studied so far)
2. Now: dual simplex on primal problem \equiv primal simplex on dual problem
(implemented as dual simplex, understood as primal simplex on dual problem)

Uses of 1.:

- The dual simplex can work better than the primal in some cases.
Eg. since running time in practice between $2m$ and $3m$, then if $m = 99$ and $n = 9$ then better the dual
- Infeasible start
Dual based Phase I algorithm (Dual-primal algorithm)

Dual Simplex for Phase I

Example

Primal:

$$\begin{aligned} \max \quad & -x_1 - x_2 \\ & -2x_1 - x_2 \leq 4 \\ & -2x_1 + 4x_2 \leq -8 \\ & -x_1 + 3x_2 \leq -7 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Initial tableau

	x1	x2	w1	w2	w3	-z	b
	-2	-1	1	0	0	0	4
	-2	4	0	1	0	0	-8
	-1	3	0	0	1	0	-7
	-1	-1	0	0	0	1	0

infeasible start

- x_1 enters, w_2 leaves

Dual:

$$\begin{aligned} \min \quad & 4y_1 - 8y_2 - 7y_3 \\ & -2y_1 - 2y_2 - y_3 \geq -1 \\ & -y_1 + 4y_2 + 3y_3 \geq -1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

- Initial tableau ($\min by \equiv -\max -by$)

	y1	y2	y3	z1	z2	-p	b
	2	2	1	1	0	0	1
	1	-4	-3	0	1	0	1
	-4	8	7	0	0	1	0

feasible start (thanks to $-x_1 - x_2$)

- y_2 enters, z_1 leaves

- x_1 enters, w_2 leaves

	x_1	x_2	w_1	w_2	w_3	$-z$	b
	0	-5	1	-1	0	0	12
	1	-2	0	-0.5	0	0	4
	0	1	0	-0.5	1	0	-3
	0	-3	0	-0.5	0	1	4

- w_2 enters, w_3 leaves (note that we kept $c_j < 0$, ie, optimality)

	x_1	x_2	w_1	w_2	w_3	$-z$	b
	0	-7	1	0	-2	0	18
	1	-3	0	0	-1	0	7
	0	-2	0	1	-2	0	6
	0	-4	0	0	-1	1	7

- y_2 enters, z_1 leaves

	y_1	y_2	y_3	z_1	z_2	$-p$	b
	1	1	0.5	0.5	0	0	0.5
	5	0	-1	2	1	0	3
	-4	0	3	-12	0	1	-4

- y_3 enters, y_2 leaves

	y_1	y_2	y_3	z_1	z_2	$-p$	b
	2	2	1	1	0	0	1
	7	2	0	3	1	0	3
	-18	-6	0	-7	0	1	-7

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Economic Interpretation

$$\begin{aligned}
 \max \quad & 5x_0 + 6x_1 + 8x_2 \\
 & 6x_0 + 5x_1 + 10x_2 \leq 60 \\
 & 8x_0 + 4x_1 + 4x_2 \leq 40 \\
 & 4x_0 + 5x_1 + 6x_2 \leq 50 \\
 & x_0, x_1, x_2 \geq 0
 \end{aligned}$$

final tableau:

$$\begin{array}{c|ccccccc}
 & x_0 & x_1 & x_2 & s_1 & s_2 & s_3 & -z & b \\
 \hline
 & & 0 & 1 & & 0 & & & 5/2 \\
 & & 1 & 0 & & 0 & & & 7 \\
 & & 0 & 0 & & 1 & & & 2 \\
 \hline
 & -1/5 & 0 & 0 & -1/5 & 0 & -1 & & -62
 \end{array}$$

- Which are the values of variables, the reduced costs, the shadow prices (or marginal prices), the values of dual variables?
- If one slack variable > 0 then overcapacity: $s_2 = 2$ then the second constraint is not tight
- How many products can be produced at most? at most m
- How much more expensive a product not selected should be?
look at reduced costs: $c_j + \pi a_j > 0$
- What is the value of extra capacity of manpower? In +1 out +1/5

Economic Interpretation

Game: Suppose two economic operators:

- P owns the factory and produces goods
- D is in the market buying and selling raw material and resources
- D asks P to close and sell him all resources
- P considers if the offer is convenient
- D wants to spend least possible
- y are prices that D offers for the resources
- $\sum y_i b_i$ is the amount D has to pay to have all resources of P
- $\sum y_i a_{ij} \geq c_j$ total value to make $j >$ price per unit of product
- P either sells all resources $\sum y_i a_{ij}$ or produces product j (c_j)
- without \geq there would not be negotiation because P would be better off producing and selling
- ▶ at optimality the situation is indifferent (strong th.)
- ▶ resource 2 that was not totally utilized in the primal has been given value 0 in the dual.
(complementary slackness th.) Plausible, since we do not use all the resource, likely to place not so much value on it.
- ▶ for product 0 $\sum y_i a_{ij} > c_j$ hence not profitable producing it. (complementary slackness th.)

Instead of solving each modified problem from scratch, exploit results obtained from solving the original problem.

$$\max\{\mathbf{c}^T \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\} \quad (*)$$

(I) changes to coefficients of objective function: $\max\{\tilde{\mathbf{c}}^T \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\}$ (primal)

\mathbf{x}^* of (*) remains feasible hence we can restart the simplex from \mathbf{x}^*

(II) changes to RHS terms: $\max\{\mathbf{c}^T \mathbf{x} \mid A\mathbf{x} = \tilde{\mathbf{b}}, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\}$ (dual)

\mathbf{x}^* optimal feasible solution of (*)

basic sol $\bar{\mathbf{x}}$ of (II): $\bar{\mathbf{x}}_N = \mathbf{x}_N^*$, $A_B \bar{\mathbf{x}}_B = \tilde{\mathbf{b}} - A_N \bar{\mathbf{x}}_N$

$\bar{\mathbf{x}}$ is dual feasible and we can start the dual simplex from there. If $\tilde{\mathbf{b}}$ differs from \mathbf{b} only slightly it may be we are already optimal.

(III) introduce a new variable:

(primal)

$$\begin{aligned} \max \quad & \sum_{j=1}^6 c_j x_j \\ & \sum_{j=1}^6 a_{ij} x_j = b_i, \quad i = 1, \dots, 3 \\ & l_j \leq x_j \leq u_j, \quad j = 1, \dots, 6 \\ & [x_1^*, \dots, x_6^*] \text{ feasible} \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{j=1}^7 c_j x_j \\ & \sum_{j=1}^7 a_{ij} x_j = b_i, \quad i = 1, \dots, 3 \\ & l_j \leq x_j \leq u_j, \quad j = 1, \dots, 7 \\ & [x_1^*, \dots, x_6^*, 0] \text{ feasible} \end{aligned}$$

(IV) introduce a new constraint:

(dual)

$$\begin{aligned} & \sum_{j=1}^6 a_{4j} x_j = b_4 \\ & \sum_{j=1}^6 a_{5j} x_j = b_5 \\ & l_j \leq x_j \leq u_j \quad j = 7, 8 \end{aligned}$$

$$\begin{aligned} & [x_1^*, \dots, x_6^*] \text{ optimal} \\ & [x_1^*, \dots, x_6^*, x_7^*, x_8^*] \text{ feasible} \\ & x_7^* = b_4 - \sum_{j=1}^6 a_{4j} x_j^* \\ & x_8^* = b_5 - \sum_{j=1}^6 a_{5j} x_j^* \end{aligned}$$

Examples

(I) Variation of reduced costs:

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

The last tableau gives the possibility to estimate the effect of variations

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	$1/5$	$-1/4$	0	2
x_1	1	0	$-1/5$	$1/2$	0	8
	0	0	$-2/5$	-1	1	-64

For a variable in basis the perturbation goes unchanged in the red. costs. Eg:

$$\max (6 + \delta)x_1 + 8x_2 \implies \bar{c}_1 = 1(6 + \delta) - \frac{2}{5} \cdot 5 - 1 \cdot 4 = \delta$$

then need to bring in canonical form and hence δ changes the obj value.

For a variable not in basis, if it changes the sign of the reduced cost \implies worth bringing in basis \implies the δ term propagates to other columns

(II) Changes in RHS terms

$$\begin{array}{c|cccccc}
 & x_1 & x_2 & x_3 & x_4 & -z & b \\
 \hline
 x_3 & 5 & 10 & 1 & 0 & 0 & 60 + \delta \\
 x_4 & 4 & 4 & 0 & 1 & 0 & 40 + \epsilon \\
 \hline
 & 6 & 8 & 0 & 0 & 1 & 0
 \end{array}$$

$$\begin{array}{c|cccccc}
 & x_1 & x_2 & x_3 & x_4 & -z & b \\
 \hline
 x_2 & 0 & 1 & 1/5 & -1/4 & 0 & 2 + 1/5\delta - 1/4\epsilon \\
 x_1 & 1 & 0 & -1/5 & 1/2 & 0 & 8 - 1/5\delta + 1/2\epsilon \\
 \hline
 & 0 & 0 & -2/5 & -1 & 1 & -64 - 2/5\delta - \epsilon
 \end{array}$$

(It would be more convenient to augment the second. But let's take $\epsilon = 0$.)

If $60 + \delta \implies$ all RHS terms change and we must check feasibility

Which are the multipliers for the first row? $k_1 = \frac{1}{5}$, $k_2 = -\frac{1}{4}$, $k_3 = 0$

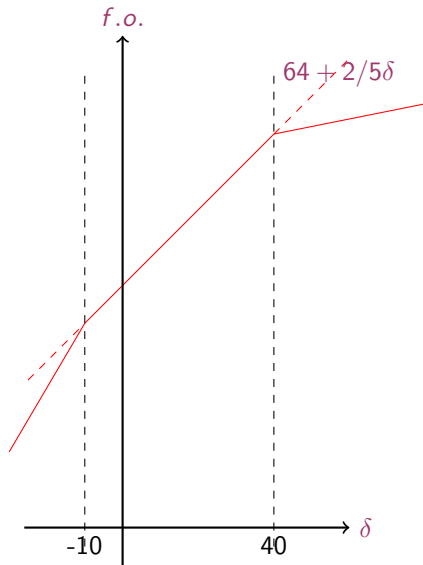
I: $1/5(60 + \delta) - 1/4 \cdot 40 + 0 \cdot 0 = 12 + \delta/5 - 10 = 2 + \delta/5$

II: $-1/5(60 + \delta) + 1/2 \cdot 40 + 0 \cdot 0 = -60/5 + 20 - \delta/5 = 8 - 1/5\delta$

Risk that RHS becomes negative

Eg: if $\delta = -10 \implies$ tableau stays optimal but not feasible \implies apply dual simplex

Graphical Representation



(III) Add a variable

$$\begin{aligned} \max \quad & 5x_0 + 6x_1 + 8x_2 \\ & 6x_0 + 5x_1 + 10x_2 \leq 60 \\ & 8x_0 + 4x_1 + 4x_2 \leq 40 \\ & x_0, x_1, x_2 \geq 0 \end{aligned}$$

Reduced cost of x_0 ? $c_j + \sum \pi_i a_{ij} = +1 \cdot 5 - \frac{2}{5} \cdot 6 + (-1)8 = -\frac{27}{5}$

To make worth entering in basis:

- increase its cost
- decrease the technological coefficient in constraint II: $5 - 2/5 \cdot 6 - a_{20} > 0$

(IV) Add a constraint

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & 5x_1 + 6x_2 \leq 50 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Final tableau not in canonical form, need to iterate with dual simplex

	x_1	x_2	x_3	x_4	x_5	$-z$	b
x_2	0	1	$1/5$	$-1/4$		0	2
x_1	1	0	$-1/5$	$1/2$		0	8
	0	0	$5/5$	$6/4$	1	0	-2
	0	0	$-2/5$	-1	0	1	-64

(V) change in a technological coefficient:

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	$10 + \delta$	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

- first effect on its column
- then look at c
- finally look at b

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	$(10 + \delta)1/5 + 4(-1/4)$	$1/5$	$-1/4$	0	2
x_1	1	$(10 + \delta)(-1/5) + 4(1/2)$	$-1/5$	$1/2$	0	8
	0	$-2/5\delta$	$-2/5$	-1	1	-64

Relevance of Sensitivity Analysis

- The dominant application of LP is mixed integer linear programming.
- In this context it is extremely important being able to begin with a model instantiated in one form followed by a sequence of problem modifications
 - row and column additions and deletions,
 - variable fixings

interspersed with resolves

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