

DM545/DM871
Linear and Integer Programming

Lecture
Cutting Planes

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1. Cutting Plane Algorithms

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Valid Inequalities

- IP: $z = \max\{\mathbf{c}^T \mathbf{x} : \mathbf{x} \in X\}$, $X = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{Z}_+^n\}$
- Proposition: $\text{conv}(X) = \{\mathbf{x} : \tilde{A}\mathbf{x} \leq \tilde{\mathbf{b}}, \mathbf{x} \geq \mathbf{0}\}$ is a polyhedron
- LP: $z = \max\{\mathbf{c}^T \mathbf{x} : \tilde{A}\mathbf{x} \leq \tilde{\mathbf{b}}, \mathbf{x} \geq \mathbf{0}\}$ would be the best formulation
- Key idea: try to approximate the best formulation.

Definition (Valid inequalities)

$\mathbf{a}\mathbf{x} \leq \mathbf{b}$ is a **valid inequality** for $X \subseteq \mathbb{R}^n$ if $\mathbf{a}\mathbf{x} \leq \mathbf{b} \forall \mathbf{x} \in X$

Which are useful inequalities? and how can we find them?
How can we use them?

Example: Pre-processing

- $X = \{(x, y) : x \leq 999y; \quad 0 \leq x \leq 5, \quad y \in \mathbb{B}^1\}$

$$x \leq 5y$$

- $X = \{x \in \mathbb{Z}_+^n : 13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72\}$

$$2x_1 + 2x_2 + x_3 + x_4 \geq \frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq \frac{72}{11} = 6 + \frac{6}{11}$$

$$2x_1 + 2x_2 + x_3 + x_4 \geq 7$$

- Capacitated facility location:

$$\sum_{i \in M} x_{ij} \leq b_j y_j \quad \forall j \in N$$

$$x_{ij} \leq b_j y_j$$

$$\sum_{j \in N} x_{ij} = a_i \quad \forall i \in M$$

$$x_{ij} \leq a_i$$

$$x_{ij} \geq 0, \quad y_j \in \mathbb{B}^n$$

$$x_{ij} \leq \min\{a_i, b_j\} y_j$$

Chvátal-Gomory cuts

- $X \in P \cap \mathbb{Z}_+^n$, $P = \{\mathbf{x} \in \mathbb{R}_+^n : A\mathbf{x} \leq \mathbf{b}\}$, $A \in \mathbb{R}^{m \times n}$
- $\mathbf{u} \in \mathbb{R}_+^m$, $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ columns of A

CG procedure to construct valid inequalities

$$1) \quad \sum_{j=1}^n \mathbf{u} \mathbf{a}_j x_j \leq \mathbf{u} \mathbf{b} \quad \text{valid: } \mathbf{u} \geq \mathbf{0}$$

$$2) \quad \sum_{j=1}^n \lfloor \mathbf{u} \mathbf{a}_j \rfloor x_j \leq \mathbf{u} \mathbf{b} \quad \text{valid: } \mathbf{x} \geq \mathbf{0} \text{ and } \sum \lfloor \mathbf{u} \mathbf{a}_j \rfloor x_j \leq \sum \mathbf{u} \mathbf{a}_j x_j$$

$$3) \quad \sum_{j=1}^n \lfloor \mathbf{u} \mathbf{a}_j \rfloor x_j \leq \lfloor \mathbf{u} \mathbf{b} \rfloor \quad \text{valid for } X \text{ since } \mathbf{x} \in \mathbb{Z}^n$$

Theorem

by applying this CG procedure a finite number of times every valid inequality for X can be obtained

However not all the constraints generated by $\mathbf{u} \in \mathbb{R}_+^m$ are tightenings.

- $X \in P \cap \mathbb{Z}_+^n$
- a family of valid inequalities $\mathcal{F} : \mathbf{a}^T \mathbf{x} \leq b, (\mathbf{a}, b) \in \mathcal{F}$ for X
- we do not find them all a priori, only interested in those close to optimum

Cutting Plane Algorithm

Init.: $t = 0, P^0 = P$

Iter. t : Solve $\bar{z}^t = \max\{\mathbf{c}^T \mathbf{x} : \mathbf{x} \in P^t\}$
let \mathbf{x}^t be an optimal solution
if $\mathbf{x}^t \in \mathbb{Z}^n$ stop, \mathbf{x}^t is opt to the IP
if $\mathbf{x}^t \notin \mathbb{Z}^n$ solve separation problem for \mathbf{x}^t and \mathcal{F}
if (\mathbf{a}^t, b^t) is found with $\mathbf{a}^t \mathbf{x}^t > b^t$ that cuts off \mathbf{x}^t

$$P^{t+1} = P \cap \{\mathbf{x} : \mathbf{a}^i \mathbf{x} \leq b^i, i = 1, \dots, t\}$$

else stop (P^t is in any case an improved formulation)

Gomory's fractional cutting plane algorithm

Cutting plane algorithm + Chvátal-Gomory cuts

- $\max\{\mathbf{c}^T \mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0, \mathbf{x} \in \mathbb{Z}^n\}$
- Solve LPR to optimality

$$\left[\begin{array}{c|c|c|c|c} I & \bar{A}_N = A_B^{-1} A_N & 0 & \bar{b} & \\ \hline \bar{c}_B & \bar{c}_N (\leq 0) & 1 & -\bar{d} & \end{array} \right]$$

$$x_{B_u} = \bar{b}_u - \sum_{j \in N} \bar{a}_{uj} x_j, \quad u = 1..m$$

$$z = \bar{d} + \sum_{j \in N} \bar{c}_j x_j$$

- If basic optimal solution to LPR is not integer then \exists some row u : $\bar{b}_u \notin \mathbb{Z}^1$.
The Chvátal-Gomory cut applied to this row is:

$$x_{B_u} + \sum_{j \in N} \lfloor \bar{a}_{uj} \rfloor x_j \leq \lfloor \bar{b}_u \rfloor$$

(B_u is the index in the basis B corresponding to the row u)

(cntd)

- Eliminating $x_{B_u} = \bar{b}_u - \sum_{j \in N} \bar{a}_{uj} x_j$ in the CG cut we obtain:

$$\sum_{j \in N} \underbrace{(\bar{a}_{uj} - \lfloor \bar{a}_{uj} \rfloor)}_{0 \leq f_{uj} < 1} x_j \geq \underbrace{\bar{b}_u - \lfloor \bar{b}_u \rfloor}_{0 < f_u < 1}$$

$$\sum_{j \in N} f_{uj} x_j \geq f_u$$

$f_u > 0$ or else u would not be row of fractional solution. It implies that x^* in which $x_N^* = 0$ is cut out!

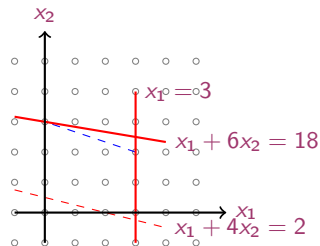
(theoretically it terminates after a finite number of iterations, but in practice not successful.)

Example

$$\begin{aligned}
 \max \quad & x_1 + 4x_2 \\
 \text{s.t.} \quad & x_1 + 6x_2 \leq 18 \\
 & x_1 \leq 3 \\
 & x_1, x_2 \geq 0 \\
 & x_1, x_2 \text{ integer}
 \end{aligned}$$

	x1	x2	x3	x4	-z	b
	1	6	1	0	0	18
	1	0	0	1	0	3
	1	4	0	0	1	0

	x1	x2	x3	x4	-z	b
	0	1	1/6	-1/6	0	15/6
	1	0	0	1	0	3
	0	0	-2/3	-1/3	1	-13



$x_2 = 5/2, x_1 = 3$
 Optimum, not integer

- We take the first row: $| \quad | \quad 0 \quad | \quad 1 \quad | \quad 1/6 \quad | \quad -1/6 \quad | \quad 0 \quad | \quad 15/6 \quad |$

- CG cut $\sum_{j \in N} f_{uj} x_j \geq f_u \rightsquigarrow \frac{1}{6} x_3 + \frac{5}{6} x_4 \geq \frac{1}{2}$

- Let's see that it leaves out x^* : from the CG proof:

$$\begin{array}{rcl} 1/6 (x_1 + 6x_2 \leq 18) & & \\ 5/6 (x_1 \leq 3) & & \\ \hline x_1 + x_2 \leq 3 + 5/2 = 5.5 & & \end{array}$$

since x_1, x_2 are integer $x_1 + x_2 \leq 5$

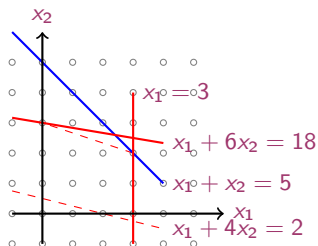
- Let's see how it looks in the space of the original variables: from the first tableau:

$$x_3 = 18 - 6x_2 - x_1$$

$$x_4 = 3 - x_1$$

$$\frac{1}{6}(18 - 6x_2 - x_1) + \frac{5}{6}(3 - x_1) \geq \frac{1}{2} \quad \rightsquigarrow \quad x_1 + x_2 \leq 5$$

- Graphically:



- Let's continue:

	x_1	x_2	x_3	x_4	x_5	$-z$	b
	0	0	-1/6	-5/6	1	0	-1/2
	0	1	1/6	-1/6	0	0	5/2
	1	0	0	1	0	0	3
	0	0	-2/3	-1/3	0	1	-13

We need to apply dual-simplex
(will always be the case, why?)

ratio rule: $\min\{|\frac{c_j}{a_{ij}}| : a_{ij} < 0\}$

- After the dual simplex iteration:

	x1	x2	x3	x4	x5	-z	b
	0	0	1/5	1	-6/5	0	3/5
	0	1	1/5	0	-1/5	0	13/5
	1	0	-1/5	0	6/5	0	12/5
	0	0	-3/5	0	-2/5	1	-64/5

- In the space of the original variables:

$$4(18 - x_1 - 6x_2) + (5 - x_1 - x_2) \geq 2$$

$$x_1 + 5x_2 \leq 15$$

• ...

We can choose any of the three rows.

Let's take the third: CG cut:

$$\frac{4}{5}x_3 + \frac{1}{5}x_5 \geq \frac{2}{5}$$

