Take-Home Test 1
Linear and Integer Programming
(DM545)

Department of Mathematics and Computer Science
University of Southern Denmark

Friday, February 28, 2020, 9:00 –
Saturday, February 29, 2020, 8:59

This is the first of a series of three tests that consitute the exam of the course. The test consists of a number of tasks divided into subtasks. The answers must be collected in a unique PDF document and are to be handed in electronically in Blackboard under SDU Assignment (http://e-learn.sdu.dk).

- **The test is individual. You are not allowed to collaborate by any means with other persons**.
- **In the PDF document make sure that you start a new page for every SUBTASK and you write which SUBTASK you are addressing.** Use the Latex and Word templates provided in BlackBoard.
- You can write your answers in Danish or in English.
- *Remember to justify all your statements!* It is not sufficient to present an answer, you must show how you found it. You may refer to results from the lecture notes, the slides or the books listed at the course web page. References to other books (outside the course material) or to internet links are not accepted as valid answers to a task.
- You are allowed to use tools such as Python to assist you in calculations. If you report source code in Python or other languages, you must also report the output it produces when executed.
- Make sure you take security copies of your documents while the test is in progress. It is your own responsibility in case of technical issues.
- Tools and tutorials for typessetting your answers are available from the Public Web Page:
  Assessment → Instructions for the take-home assignments.
- The test consists of 5 tasks distributed on 6 pages.
- The contribution to the final evaluation of each task, if carried out correctly, is given at the beginning of each task as a list of points for each subtask. The maximum total score is 100.

Task 1 Modeling (20 points)

Choose **one** of the following two cases and give its linear programming formulation.

Subtask 1.a Traffic Light Control

Automobile traffic from three highways, H1, H2, and H3, must stop and wait for a green light before exiting to a toll road. The tolls are 40 kr, 50 kr, and 60 kr for cars exiting from H1, H2, and H3, respectively. The flow rates from H1, H2, and H3 are 550, 650, and 450 cars per hour. The traffic light cycle may not exceed 2.2 minutes, and the green light on any highway must be at least 22 seconds. The yellow light is on for 10 seconds. The toll gate can handle a maximum of 500 cars per hour. Assuming that no cars move on yellow, and that clearly when a highway has the green signal the other two have the red, determine the optimal green time interval for the three highways that will maximize toll gate revenue per traffic cycle.

Subtask 1.b Leveling the Terrain for a New Highway

The Highway Department is planning a new 10-km highway on uneven terrain as shown by the profile in Figure 1. The width of the construction terrain is approximately 50 meters. To simplify the situation, the terrain profile can be replaced by a step function as shown in the figure. Using heavy machinery, earth removed from high terrain is pulled with effort to fill low areas. There are also two burrow pits, I and II, located at the ends of the 10-km stretch from which additional earth can be pulled, if needed. Pit I has a capacity of 15 000 cubic meters and pit II a capacity of 11 000 cubic meters. The costs of removing earth from pits I and II are, respectively, 15 kr and 19 kr per cubic meter.
The transportation cost per cubic meter per kilometer is 1.5 kr, and the cost of using heavy machinery to load pulling trucks is 2 kr per cubic meter. This means that a cubic meter extracted from pit I and transported for 1 km will cost a total of $15 * 1 + (1.5 + 2) * 1 = 18.5$ kr and a cubic meter pulled 1 km from a hill to a fill area will cost $2 + 1.5 = 3.5$ kr. Develop a minimum cost plan for leveling the 10-km stretch.

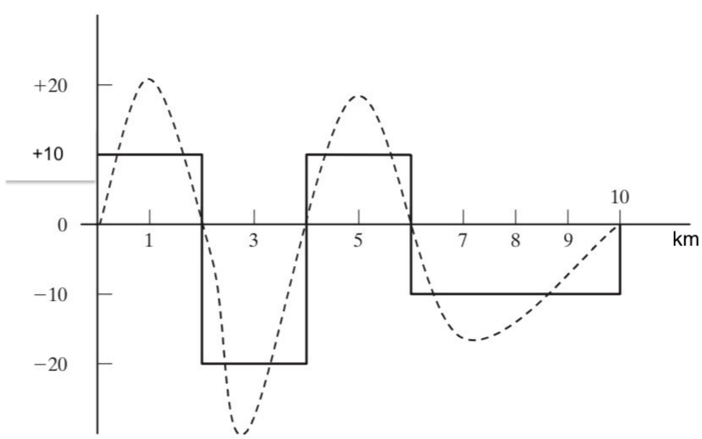

Figure 1

Task 2 Solution by Inspection

Subtask 2.a

Solve the following problem by inspection and justify your answer.

$$
\begin{aligned}
\max \quad & 4x_1 + 5x_2 - 7x_3 \\
\text{s.t.} \quad & -x_1 - x_2 + x_3 \leq 2 \\
& -5x_1 + 10x_3 \leq 10 \\
& 0 \leq x_1 \leq 5 \\
& -1 \leq x_2 \leq +1 \\
& -2 \leq x_3 \leq +2
\end{aligned}
$$

Task 3 Simplex (10 + 15 + 10 Points)

Consider the following linear programming problem (LP):

$$\begin{aligned} \text{maximize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 8 \\ & 2x_1 - 7x_2 \geq -20 \\ & x_1 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(Throughout this task use fraction mode for numeric computations.)

Subtask 3.a

- Write (LP) in equational standard form
- Indicate the corresponding basic solution (give the value of each variable in the form obtained) and its value.
- Argue whether the basic solution from the previous point is feasible and if it is optimal. Justify your answers.

Subtask 3.b

Perform one iteration of the simplex algorithm using the *largest increase* pivot rule.

Subtask 3.c

After another iteration the tableau looks as in Figure 2:

```
+-----+-----+------+------+-----+-----+--------+
|x1   |x2   |s1    |s2    |s3   |z    |b       |
+-----+-----+------+------+-----+-----+--------+
| 1   |  0  | 7/11 |-2/11 |0    |0    |16/11   |
|0    |1    |2/11  |1/11  |0    |0    |36/11   |
|0    |0    |-7/11 |2/11  |1    |0    |39/11   |
|0    |0    |-13/11|-1/11 |0    |1    |-124/11 |
+-----+-----+------+------+-----+-----+--------+
```

Figure 2: The tableau of Subtask 3.c.

Write:

- the corresponding basic solution
- the objective function value
- argue that the solution is optimal.
- the optimal solution of the dual problem (without solving the dual problem).

Task 4 Simplex (10 + 10 Points)

Subtask 4.a

Consider the following linear programming problem:

$$
\begin{array}{llcccccl}
\min & z = 3x_1 & + & 2x_2 & + & 7x_3 & & \\
\text{s.t.} & -x_1 & + & x_2 & & & & = 10 \\
 & 2x_1 & - & x_2 & + & x_3 & & \geq 10 \\
 & & & & & x_1, x_2, x_3 & & \geq 0
\end{array}
$$

Write the initial tableau, state the optimality conditions and conclude whether the tableau is optimal.

Subtask 4.b

Consider the following tableau for a (different) linear programming problem in maximization form:

x1	x2	x3	x4	x5	x6	-z	b
0	0	0	1	1	0	0	0
0	1	1	2	0	-1	0	30
1	0	1	1	0	-1	0	20
0	0	2	-7	0	5	1	-120

If the next iteration is done with the largest coefficient pivoting rule what would you conclude? What would the conclusion be applying other pivoting rules?

Task 5 Duality (15 Points)

Consider the following linear programming problem:

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 3x_2 \\
\text{subject to} \quad & x_1 + 2x_2 \leq 8 \\
& 2x_1 - 7x_2 \geq -20 \\
& x_1 \leq 5 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Subtask 5.a

Derive the dual with one of the methods seen in the course.