DM545/DM871

Linear and Integer Programming

Introduction to Linear Programming
Notation and Modeling

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Outline

1. Course Organization
2. Preliminaries
3. Introduction: Operations Research
   - Resource Allocation
   - Duality

Who is here?

43 in total registered in BlackBoard

DM545 (5 ECTS)
24, who??

- Math-economy (2nd year ?)
- Others?

DM871 (5 ECTS)
19, who??

- Computer Science (Master)
- Applied Mathematics (2nd year ?)
- Others?

Prerequisites

- Programming
- Linear Algebra

Outline

Aims of the course

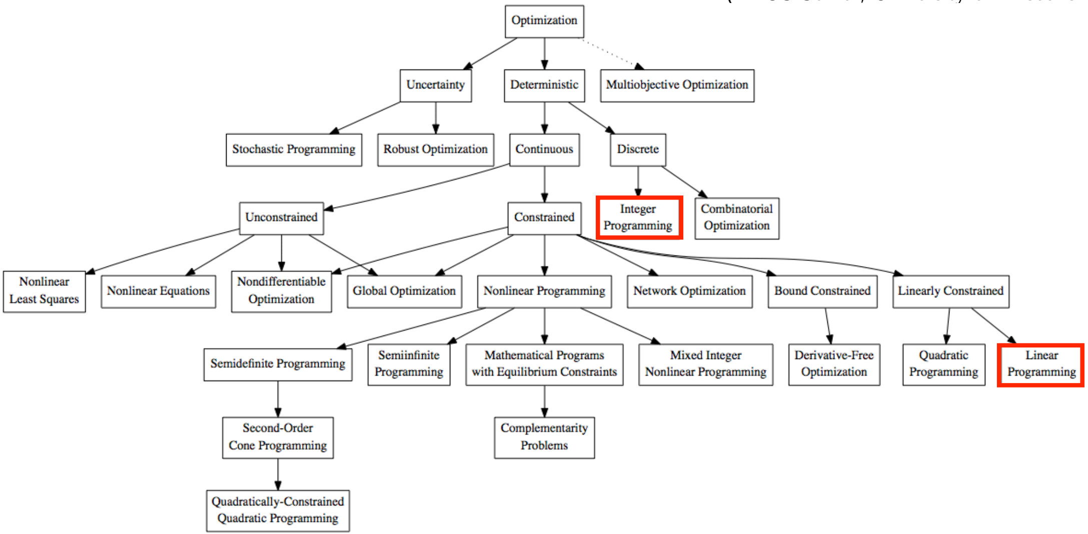
Learn about **mathematical optimization**:

- linear programming (continuous linear optimization)
- integer linear programming (discrete linear optimization)

$\rightsquigarrow$ You will see the theory and apply the tools learned to solve real life problems using computer software (DM871)

Optimization Taxonomy

(NEOS Server, University of Wisconsin)



Contents of the Course (aka Syllabus)

Linear Programming

1 Introduction - Linear Programming, Notation & Modeling
2 Linear Programming, Simplex Method
3 Exception Handling
4 Duality Theory
5 Sensitivity
6 Revised Simplex Method

Integer Linear Programming

7 Modeling Examples, Good Formulations, Relaxations
8 Well Solved Problems
9 Network Optimization Models (Max Flow, Min cost flow, Matching)
10 Cutting Planes & Branch and Bound
11 More on Modeling

Practical Information

Teacher: Marco Chiarandini (www.imada.sdu.dk/~marco/)
Instructor: Peter Bjørn
Sections (hold): H1, M1 — joined

Alternative views of the schedule:

- mitsdu.sdu.dk, SDU Mobile
- Official course description (læserplanen)
- https://dm871.github.io/

Schedule:

- Introductory classes: $\sim$ 28 hours ($\sim$ 14 classes)
- Training classes: $\sim$ 16 hours ($\sim$ 8 classes)

Communication Means

- ItsLearning (BB) $\Leftrightarrow$ External Web Page (WP) (link https://dm871.github.io)
- Announcements in ItsLearning
- Write to Marco (marco@imada.sdu.dk) and to instructor
- Ask peers
- I will hold open the zoom room after the lectures, if you have questions. This part will not be recorded.

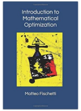
$\leadsto$ It is good to ask questions!!

$\leadsto$ Let me know if you think we should do things differently!

Sources — Reading Material

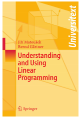
Main references:

[LN] Lecture Notes (continously updated)



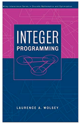
[F] M. Fischetti, Introduction to Mathematical Optimization, Independently published, 2019

Linear Programming:



[MG] J. Matousek and B. Gartner. Understanding and Using Linear Programming. Springer Berlin Heidelberg, 2007

Integer Programming:



[Wo] L.A. Wolsey. Integer programming. John Wiley & Sons, New York, USA, 1998

Others

[HL] Frederick S Hillier and Gerald J Lieberman, Introduction to Operations Research, 9th edition, 2010

Course Material

External Web Page is the main reference for list of contents (ie[1], syllabus, pensum).

It contains:

- slides
- list of topics and references
- exercises
- links
- resources for programming tasks

[1] ie = id est, eg = exempli gratia, wrt = with respect to

Assessment

- Three obligatory 24 h Take-Home Assignments, evaluation by external censor
  - individual !!
  - exercises similar to previous 4 hour written exams
  - style: short answers about calculations and modeling. For DM871, also small programming tasks.
  - (language: Danish and/or English)

- Final grade: overall evaluation but as starting point the average grade rounded up

- Tentative plan:
  - Test 1 in week 8 about weeks 5, 6 and 7
  - Test 2 in week 10 about weeks 8 and 9
  - Test 3 in week 13 about weeks 10, 11 and 12

  Which day? Which time range?

Training Sessions

- Prepare the starred exercises in advance to get out the most
- Try the others (or those unsolved in class) after the session
- Best if carried out in small groups
- Exercises are examples of exam questions (but not only!)

Concepts from Linear Algebra

Linear Algebra:
manipulation of matrices and vectors with some theoretical background

Linear Algebra
- Matrices and vectors - Matrix algebra
- Dot (scalar, Euclidean inner) product
- Geometric insights
- Systems of Linear Equations - Row echelon form, Gaussian elimination
- Matrix inversion and determinants
- Rank and linear dependency

Coding

- gives you the ability to create new and useful artifacts
- allows you to have more control of your world as more and more of it becomes digital
- is just fun.

It can also help you to understand math.

- listening to lectures
- watching an instructor work through a derivation
- working through numerical examples by hand
- learn by doing, interacting with Python.

- Python 3.6+
  PySCIPOpt or Pyomo a Python interface to SCIP Optimization Suite
  (Commercial alternative Gurobi or Cplex $\approx$ 100 000 Dkk)
- ipython, jupyter, jupyterLab (= interactive python)? Or Spyder3 or Atom or Visual Code.

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   - Resource Allocation
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Sets

- A set is a collection of objects. eg.: $A = \{1, 2, 3\}$
- $A = \{n \mid n \text{ is a whole number and } 1 \leq n \leq 3\}$
  ('$|$' reads 'such that')
- $B = \{x \mid x \text{ is a student of this course}\}$
- $x \in A$
  x belongs to A
- set of no members: empty set, denoted $\emptyset$
- if a set S is a (proper) subset of a set T, we write $(S \subset T)$ $T \subseteq S$
  $\{1, 2, 5\} \subset \{1, 2, 4, 5, 6, 30\}$
- for two sets A and B, the union $A \cup B$ is $\{x \mid x \in A \text{ or } x \in B\}$
- for two sets A and B, the intersection $A \cap B$ is $\{x \mid x \in A \text{ and } x \in B\}$
  $\{1, 2, 3, 5\}$ and $B = \{2, 4, 5, 7\}$, then $A \cap B = \{2, 5\}$

Numbers

- set of real numbers: $\mathbb{R}$
- set of natural numbers: $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ (positive integers); $\mathbb{N}_0$ to include zero
- set of all integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$; $\mathbb{Z}_0^+$ only positives and zero
- set of rational numbers: $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$
- set of complex numbers: $\mathbb{C}$
- absolute value (non-negative):

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases}$$

- the set $\mathbb{R}^2$ is the set of ordered pairs (x, y) of real numbers
  (eg, coordinates of a point wrt a pair of axes, the Cartesian plane)

Matrices and Vectors

- A matrix is a rectangular array of numbers or symbols. It can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- An $n \times 1$ matrix is a column vector, or simply a vector:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- the set $\mathbb{R}^n$ is the set of vectors $[x_1, x_2, \ldots, x_n]^T$ of real numbers (eg, coordinates of a point wrt an n-dimensional space, the Euclidean Space)

Basic Algebra

Elementary Algebra: the study of symbols and the rules for manipulating symbols. It differs from arithmetic in the use of abstractions, such as using letters to stand for numbers that are either unknown or allowed to take on many values

- collecting up terms: eg. $2a + 3b - a + 5b = a + 8b$
- multiplication of variables: eg:

$$a(-b) - 3ab + (-2a)(-4b) = -ab - 3ab + 8ab = 4ab$$

- expansion of bracketed terms: eg:

$$\begin{aligned} -(a - 2b) &= -a + 2b, \\ (2x - 3y)(x + 4y) &= 2x^2 - 3xy + 8xy - 12y^2 \\ &= 2x^2 + 5xy - 12y^2 \end{aligned}$$

- $a^r a^s = a^{r+s}, \quad (a^r)^s = a^{rs}, \quad a^{-n} = 1/a^n,$
  $a^{1/n} = x \iff x^n = a, \quad a^{m/n} = (a^{1/n})^m$

Variables

- In Mathematics and Statistics, a variable is an alphabetic character representing a value, which is unknown. They are used in symbolic calculations.
  Commonly given one-character names.

- in contrast, a constant or given or scalar is a known real number

- in contrast, in Computer Science, a variable is a storage location paired with an associated identifier, which contains a value, which may be known or unknown.
  Commonly given long, explanatory names.

Functions

- a function f on a set $\mathcal{X}$ into a set $\mathcal{Y}$ is a rule that assigns a unique element $f(x)$ in S to each element x in $\mathcal{X}$.

$$y = f(x)$$

y dependent variable

x independent variable

- a linear function has only sums and scalar multiplications, that is, for variable $x \in \mathbb{R}$ and scalars $a, b \in \mathbb{R}$:

$$f(x) := ax + b$$

Outline

What is Operations Research?

Operations Research (aka, Management Science, Analytics):
is the discipline that uses a **scientific approach to decision making**.

It seeks to determine how best to design and operate a system,
usually under conditions requiring the allocation of scarce resources,
by means of **mathematics** and **computer science**.

Quantitative methods for planning and analysis.

It encompasses a wide range of problem-solving techniques and methods applied in the pursuit of improved decision-making and efficiency:

- simulation,
- mathematical optimization,
- queueing theory and other stochastic-process models,
- Markov decision processes
- econometric methods,
- data envelopment analysis,
- neural networks,
- expert systems

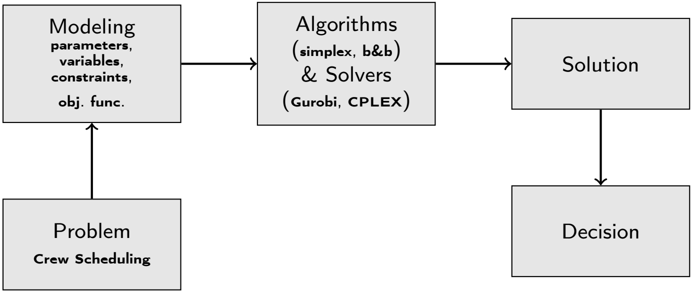
Some Examples ...

- Production Planning and Inventory Control
- Budget Investment
- Blending and Refining
- Manpower Planning
  - Crew Rostering (airline crew, rail crew, nurses)
- Packing Problems
  - Knapsack Problem
- Cutting Problems
  - Cutting Stock Problem
- Routing
  - Vehicle Routing Problem (trucks, planes, trains ...)
- Locational Decisions
  - Facility Location
- Scheduling/Timetabling
  - Examination timetabling/ train timetabling
- .... + many more

Common Characteristics

- Planning decisions must be made
- The problems relate to quantitative issues
  - Cheapest
  - Shortest route
  - Fewest number of people
- Not all plans are feasible - there are constraining rules
  - Limited amount of available resources
- It can be extremely difficult to figure out what to do

OR - The Process?



1. Observe the System
2. Formulate the Problem
3. Formulate Mathematical Model
4. Verify Model
5. Select Alternative
6. Show Results to Company
7. Implementation

Central Idea

Build a mathematical model describing exactly what one wants, and what the "rules of the game" are. However, **what is a mathematical model and how?**

Mathematical Modeling

- Find out exactly what the decision maker needs to know:
  - which investment?
  - which product mix?
  - which job j should a person i do?
- Define Decision Variables of suitable type (continuous, integer, binary) corresponding to the needs
- Formulate Objective Function computing the benefit/cost
- Formulate mathematical Constraints indicating the interplay between the different variables.

Outline

Resource Allocation

In manufacturing industry, factory planning: find the best product mix.

Example

A factory makes two products standard and deluxe. Eg, sleeping beds, yougurt, etc.

A unit of standard gives a profit of 6(k) Dkk.
A unit of deluxe gives a profit of 8(k) Dkk.

The grinding and polishing times in terms of hours per week for a unit of each type of product are given below:

	Standard	Deluxe
(Machine 1) Grinding | Warming	5	10
(Machine 2) Polishing | Cooling	4	4

Grinding capacity: 60 hours per week
Polishing capacity: 40 hours per week
Q: How much of each product, standard and deluxe, should we produce to maximize the profit?

Mathematical Model

Decision Variables

$x_1 \geq 0$ units of product standard
$x_2 \geq 0$ units of product deluxe

Object Function

$\max\ 6x_1 + 8x_2$ maximize profit

Constraints

$$\begin{array}{rcrcll} 5x_1 & + & 10x_2 & \leq & 60 & \text{machine 1 capacity} \\ 4x_1 & + & 4x_2 & \leq & 40 & \text{machine 2 capacity} \end{array}$$

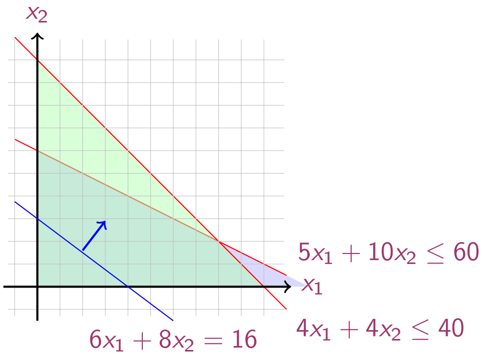
Mathematical Model

Machines/Materials A and B
Products 1 and 2

$$
\begin{array}{lrcrcl}
\max & 6x_1 & + & 8x_2 & & \\
 & 5x_1 & + & 10x_2 & \leq & 60 \\
 & 4x_1 & + & 4x_2 & \leq & 40 \\
 & & & x_1 & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

a_{ij}	1	2	b_i
A	5	10	60
B	4	4	40
c_j	6	8	

Graphical Representation:



Resource Allocation - General Model

Managing a production facility

$j = 1, 2, \ldots, n$	products
$i = 1, 2, \ldots, m$	materials
b_i	units of raw material at disposal
a_{ij}	units of raw material i to produce one unit of product j
σ_j	market price of unit of jth product
ρ_i	prevailing market value for material i
$c_j = \sigma_j - \sum_{i=1}^{m} \rho_i a_{ij}$	profit per unit of product j
x_j	amount of product j to produce

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \ldots & + & c_nx_n & = & z \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
 & \ldots & & & & & & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

Notation

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \ldots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
 & \ldots & & & & & & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & & & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j \\
& \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& x_j \geq 0, \quad j = 1, \ldots, n
\end{aligned}
$$

In Matrix Form

$$
\begin{array}{rrcrcrcccrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & + & \dots & + & c_nx_n & = & z \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
& a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & \leq & b_2 \\
& \dots & & & & & & & & & & \\
& a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
& & & & & & & & & x_1, x_2, \dots, x_n & \geq & 0
\end{array}
$$

$$
\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

$$
\begin{array}{rrcl}
\max & z & = & \mathbf{c}^T\mathbf{x} \\
& A\mathbf{x} & \leq & \mathbf{b} \\
& \mathbf{x} & \geq & 0
\end{array}
$$

Our Numerical Example

$$
\begin{aligned}
\max \ & \sum_{j=1}^{n} c_j x_j \\
& \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& x_j \geq 0, \quad j = 1, \ldots, n
\end{aligned}
$$

$$
\begin{aligned}
\max \ 6x_1 &+ 8x_2 \\
5x_1 &+ 10x_2 \leq 60 \\
4x_1 &+ 4x_2 \leq 40 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max \ & \mathbf{c}^T \mathbf{x} \\
& A\mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq 0
\end{aligned}
$$

$$\mathbf{x} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

$$\max \quad \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

Outline

Duality

Resource Valuation problem: Determine the value of the raw materials on hand such that:
(i) it would be convenient selling and (ii) an outside company would be willing to buy them.

z_i	value of a unit of raw material i
$\sum_{i=1}^{m} b_i z_i$	total expenses for buying or opportunity cost (cost of having instead of selling)
ρ_i	prevailing unit market value of material i
σ_j	prevailing unit product price

Goal: for the outside company to minimize the total expenses;
for the owing company to minimize the lost opportunity cost, ie, minimum amount to accept

$$\min \sum_{i=1}^{m} b_i z_i \tag{1}$$

$$z_i \geq \rho_i, \quad i = 1 \ldots m \tag{2}$$

$$\sum_{i=1}^{m} z_i a_{ij} \geq \sigma_j, \quad j = 1 \ldots n \tag{3}$$

(2) otherwise selling to someone else and (3) otherwise not selling

Let

$$y_i = z_i - \rho_i$$

markup that the company would make by reselling the raw material instead of producing.

$$\begin{aligned}
\min \sum_{i=1}^{m} y_i b_i &+ \cancel{\sum_i \rho_i b_i} \\
\sum_{i=1}^{m} y_i a_{ij} &\geq c_j, \quad j = 1 \dots n \\
y_i &\geq 0, \quad i = 1 \dots m
\end{aligned}$$

Dual Problem

$$\begin{aligned}
\max \sum_{j=1}^{n} c_j x_j & \\
\sum_{j=1}^{n} a_{ij} x_j &\leq b_i, \quad i = 1, \dots, m \\
x_j &\geq 0, \quad j = 1, \dots, n
\end{aligned}$$

Primal Problem