# DM545/DM871 – Linear and integer programming

## Sheet 4, Spring 2021

Starred exercises are relevant for the tests.

**Solution:**
Included.

### Exercise 1* Sensitivity Analysis and Revised Simplex

A furniture-manufacturing company can produce four types of product using three resources.

- A bookcase requires three hours of work, one unit of metal, and four units of wood and it brings in a net profit of 19 Euro.
- A desk requires two hours of work, one unit of metal and three units of wood, and it brings in a net profit of 13 Euro.
- A chair requires one hour of work, one unit of metal and three units of wood and it brings in a net profit of 12 Euro.
- A bedframe requires two hours of work, one unit of metal, and four units of wood and it brings in a net profit of 17 Euro.
- Only 225 hours of labor, 117 units of metal and 420 units of wood are available per day.

In order to decide how much to make of each product so as to maximize the total profit, the managers solve the following LP problem

$$\begin{aligned} \max \quad & 19x_1 + 13x_2 + 12x_3 + 17x_4 \\ & 3x_1 + 2x_2 + x_3 + 2x_4 \leq 225 \\ & x_1 + x_2 + x_3 + x_4 \leq 117 \\ & 4x_1 + 3x_2 + 3x_3 + 4x_4 \leq 420 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The final tableau has $x_1, x_3$ and $x_4$ in basis. With the help of a computational environment such as Python for carrying out linear algebra operations, address the following points:

a) Write $A_B$, $A_N$, $A_B^{-1}A_N$, the final simplex tableau and verify that the solution is indeed optimal.

**Solution:**

The initial tableau is:

```
|-----+-----+-----+-----+-----+-----+-----+----+-----|
| x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | x_7 | -z |   b |
|-----+-----+-----+-----+-----+-----+-----+----+-----|
|   3 |   2 |   1 |   2 |   1 |   0 |   0 |  0 | 225 |
|   1 |   1 |   1 |   1 |   0 |   1 |   0 |  0 | 117 |
|   4 |   3 |   3 |   4 |   0 |   0 |   1 |  0 | 420 |
|  19 |  13 |  12 |  17 |   0 |   0 |   0 |  1 |   0 |
|-----+-----+-----+-----+-----+-----+-----+----+-----|
```

We know that there will be 3 variables in basis. The text of the problem tells us which these 3 variables are: 1, 3, 4. Hence,

$$A_B = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 4 & 3 & 4 \end{bmatrix} \qquad A_N = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

We can calculate $A_B^{-1}A_N$ in Python or in R:

```
> B=matrix(c(3,1,2,1,1,1,4,3,4),byrow=TRUE,ncol=3)
> B1=solve(B)
> B%*%B1  # check to make sure it is correct!
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1
> N=matrix(c( 2, 1, 0, 0, 1, 0, 1, 0, 3, 0, 0, 1),ncol=4,byrow=TRUE)
> B1%*%N
     [,1] [,2] [,3] [,4]
[1,]    1    1    2   -1
[2,]    1    0    4   -1
[3,]   -1   -1   -5    2
> cN=c(13,0,0,0)
> cB=c(19,12,17)
> cN-cB%*%B1%*%N
     [,1] [,2] [,3] [,4]
[1,]   -1   -2   -1   -3
> cB%*%B1%*%c(225,117,420)
> 1827
```

This code gives us:

$$\bar{A} = A_B^{-1}A_N = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 0 & 4 & -1 \\ -1 & -1 & -5 & 2 \end{bmatrix} \qquad x_B^* = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = A_B^{-1}b = \begin{bmatrix} 39 \\ 48 \\ 30 \end{bmatrix}$$

$$\bar{c}_N = \begin{bmatrix} \bar{c}_2 & \bar{c}_5 & \bar{c}_6 & \bar{c}_7 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 & -3 \end{bmatrix}$$

and we can write the final tableau as:

```
|-----+-----+-----+-----+-----+-----+-----+----+-------|
| x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | x_7 | -z |     b |
|-----+-----+-----+-----+-----+-----+-----+----+-------|
|   1 |   1 |   0 |   0 |   1 |   2 |  -1 |  0 |    39 |
|   0 |   1 |   1 |   0 |   0 |   4 |  -1 |  0 |    48 |
|   0 |  -1 |   0 |   1 |  -1 |  -5 |   2 |  0 |    30 |
|   0 |  -1 |   0 |   0 |  -2 |  -1 |  -3 |  1 | -1827 |
|-----+-----+-----+-----+-----+-----+-----+----+-------|
```

Since all reduced costs are negative then the tableau and the corresponding solution are optimal.

b) What is the increase in price (reduced cost) that would make product $x_2$ worth to be produced?

**Solution:**

The increase in price of a quantity strictly larger than 1 would make the product 2 worth being produced. Indeed, let $c_2' = c_2 + \delta$ be the new price. We know that the coefficient in the objective function goes in the reduced cost calculation multiplied by 1. Hence, to have a positive reduced cost we have:

$$-1 + \delta > 0 \qquad \Longrightarrow \qquad \delta > 1$$

We could also recalcuate the reduced cost from scratch using the multipliers $\pi$: $c_2' + \sum_{i=1}^{3} \pi_i a_{i2}$. The value of $\pi_i$ are read from the final tableau and they correspond to the reduced costs of the slack variables, ie, $(-2, -1, -3)$.

c) What is the marginal value (shadow price) of an extra hour of work or amount of metal and wood?

**Solution:**

The marginal values are the values of the dual variable $y_1, y_2, y_3$. From the strong duality theorem, we know that $y_i = -\pi_i = -\bar{c}_{n+i}$, $i = 1..m$. Hence, $\mathbf{y} = (2, 1, 3)$.

An extra hour of work has marginal value of 2, that is, having one unit more of work would improve the revenue by 2. For the other two resources the marginal values are 1 and 3, respectively.

We can cross check these conclusions: by the complementary slackness theorem, the fact that all three dual variables are strictly positive indicates that all three constraints in the primal are active≡tight≡ binding. Hence, it makes sense to have that an increase in the capacity of those constraints implies an increase in the profit. The conclusion that all three constraints are tight can be also reached by the fact that the slack variables are 0 in the final tableau. If some constraint was not tight, then the marginal value of the corresponding resource would be zero since an increase in its capacity does not imply an immediate improvement in total profit.

d) Are all resources totally utilized, i.e. are all constraints "binding", or is there slack capacity in some of them? Answer this question in the light of the complementary slackness theorem.

**Solution:**

Since all dual variables are strictly larger than zero, then all constraints are binding. Indeed for the complementary slackness theorem, we have that:

$$\left(b_i - \sum_{j=1}^{n} a_{ij}x_j^*\right) y_i^* = 0, \quad i = 1, \ldots, m$$

e) From the economical interpretation of the dual why product $x_2$ is not worth producing? What is its imputed cost?

**Solution:**

It is not worth producing 2 because $\sum_i y_i a_{i2} > c_2$, that is, we are better off selling the raw materials to produce the product. Indeed $y_i$ is the price of one unit of resource $i$ and $a_{i2}$ is the amount of $i$ necessary to produce 2.

$$\sum_i y_i a_{i2} = 2 * (2) + 1 * (1) + 3 * (3) = 14 > 13$$

Solve the following variations:

1. The net profit brought in by each desk increases from 13 Euro to 15 Euro.

**Solution:**

We saw earlier that if the price of product 2 increases by more than 1 then the reduced cost becomes positive and it enters the basis. We can iterate the revised simplex as follows:

Step 1 and 2 to determine the entering varible are already done in the point a) above.

We need to do Step 3 to determine the leaving variable: we need to find the constraint that limit the increase of $x_2$, $theta$. We solve first $A_B d = a$ in $d$. Here, $\mathbf{a}$ is the column of the matrix $A$ (augmented with the slack variables) from the initial tableau corresponding to the entering variable $x_2$. We use the inverse of $A_B$ calculated earlier in a) above in R:

```
> B1%*%c(2,1,3)
     [,1]
[1,]    1
[2,]    1
[3,]   -1
```

that is

$$\mathbf{d} = A_B^{-1}\mathbf{a} = A_B^{-1}\begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$$

Then the new solutioon $x_B$ is derived from the old one by means of $\mathbf{d}$ and the increase $\theta$:

$$x_B = \begin{bmatrix}x_1\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}39\\48\\30\end{bmatrix} - \begin{bmatrix}1\\1\\-1\end{bmatrix}\theta \geq 0$$

The increase $\theta$ must be such that the value of the variables still remains feasible, ie, $x_i \geq 0$. Hence $\theta \leq 39$ and the leaving variable is $x_1$, since it is the one that goes to zero. The new solutions is

$$x_B = \begin{bmatrix}x_2\\x_3\\x_4\end{bmatrix} == \begin{bmatrix}39\\48\\30\end{bmatrix} - \begin{bmatrix}1\\1\\-1\end{bmatrix}\theta = \begin{bmatrix}39-39\\48-39\\30+39\end{bmatrix} = \begin{bmatrix}0\\9\\69\end{bmatrix}$$

and the objective value:

```
> c=c(19,15,12,17)
> c%*%c(0,39,9,69)
     [,1]
[1,] 1866
```

2. The availability of metal increases from 117 to 125 units per day

**Solution:**

This is a change in the RHS term of constraint 2. The optimality of the current solution does not change, since all reduced costs stay negative, but we need to check if we are still feasible. We need to look at the final tablea and recompute the $b$ of all constraints. We can do this with $A_B^{-1}b$:

```
> b=c(225,125,420)
> B1%*%b
     [,1]
[1,]   55
[2,]   80
[3,]  -10
```

The last cosntraint becomes negative, hence we need to iterate with the dual simplex.

3. The company may also produce coffee tables, each of which requires three hours of work, one unit of metal, two units of wood and bring in a net profit of 14 Euro.

**Solution:**

We need to check if the reduced cost of the new variable would become positive by computing $c_0 + \sum_i \pi_i a_{ij}$:

```
> 14-3*2-1*1-2*3
[1] 1
```

which is positive, hence we need to iterate as done in point 1).

4. The number of chairs produced must be at most five times the numbers of desks

**Solution:**

This corresponds to introduce a new constraint: $x_3 \leq 5x_2$. In the new standard form we have a new slack variable $x_8$. Adding the constraint in the tableau and bringing back the tableau in canonical standard form we observe that a RHS term becomes negative. Hence, we need to iterate with the dual simplex. After on iteration with the dual simplex, the final tableau becomes:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 4/3 & -5/6 & 1/6 & 0 & 31 \\ 0 & 1 & 0 & 0 & 0 & 2/3 & -1/6 & -1/6 & 0 & 8 \\ 0 & 0 & 0 & 1 & -1 & -13/3 & 11/6 & -1/6 & 0 & 38 \\ 0 & 0 & 1 & 0 & 0 & 10/3 & -5/6 & 1/6 & 0 & 40 \\ 0 & 0 & 0 & 0 & -2 & -1/3 & -19/6 & -1/6 & 1 & -1819 \end{bmatrix}$$

If after the introduction of the constraint the current solution had stayed feasible then we would have needed to check whether its was also optimal. We can either repeat the steps done at part 1 above to compute the new reduced costs or we can include the new row in the final tableau and proceed to put the tableau in canonical form. Then we look at the value of the reduced costs.

## Exercise 2

Solve the systems $\mathbf{y}^T E_1E_2E_3E_4 = [1\ 2\ 3]$ and $E_1E_2E_3E_4\mathbf{d} = [1\ 2\ 3]^T$ with

$$E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0.5 & 0 \\ 0 & 4 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

**Solution:**

This exercise is to show that the two systems can be solved quite easily. Let's take first $\mathbf{y}^T E_1E_2E_3E_4 = [1\ 2\ 3]$, we use the backward transformation and solve the sequence of linear systems:

$$\mathbf{u}^T E_4 = [1\ 2\ 3], \quad \mathbf{v}^T E_3 = \mathbf{u}^T, \quad \mathbf{w}^T E_2 = \mathbf{v}^T, \quad \mathbf{y}^T E_1 = \mathbf{w}^T$$

$$\mathbf{u}^T \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = [1, 2, 3]$$

Since the eta matrices have always one 1 in two columns then the solution can be read up easily. From the third column we find $u_3 = 3$. From the second column, we find $u_2 = 2$. Substituting in the first column, we find $-0.5u_1 + 3 * 2 + 1 * 3 = 1$, which yields $u_1 = 18$. The next syestem is:

$$\mathbf{v}^T \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = [18, 2, 3]$$

From the first column we get $v_1 = 18$, from the second column $v_2 = 2$ from the last column $v_3 = 3/24$. The next:

$$\mathbf{w} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = [18, 2, 3/24]$$

...

## Exercise 3 Factory Planning and Machine Maintenance

A firm makes seven products $1, \ldots, 7$ on the following machines: 4 grinders, 2 vertical drills, 3 horizontal drills, 1 borer, and 1 planer.

Each product yields a certain contribution to the profit (defined as selling price minus cost of raw materials expressed in Euro/unit). These quantities (in Euro/unit) together with the production times (hours/unit) required on each process are given below.

| product | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| profit | 10 | 6 | 8 | 4 | 11 | 9 | 3 |
| grinding | 0.5 | 0.7 | 0 | 0 | 0.3 | 0.2 | 0.5 |
| vdrill | 0.1 | 0.2 | 0 | 0.3 | 0 | 0.6 | 0 |
| hdrill | 0.2 | 0 | 0.8 | 0 | 0 | 0 | 0.6 |
| boring | 0.05 | 0.03 | 0 | 0.07 | 0.1 | 0 | 0.08 |
| planning | 0 | 0 | 0.01 | 0 | 0.05 | 0 | 0.05 |

In the first month (January) and the five subsequent months certain machines will be down for maintenance. These machines will be:

| | |
|---|---|
| January | 1 grinder |
| February | 2 hdrill |
| March | 1 borer |
| April | 1 vdrill |
| May | 1 grinder |
| May | 1 vdrill |
| June | 1 planer |
| June | 1 hdrill |

There are marketing limitations on each product in each month. That is, in each month the amount sold for each product cannot exceed these values:

| product | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| January | 500 | 1000 | 300 | 300 | 800 | 200 | 100 |
| February | 600 | 500 | 200 | 0 | 400 | 300 | 150 |
| March | 300 | 600 | 0 | 0 | 500 | 400 | 100 |
| April | 200 | 300 | 400 | 500 | 200 | 0 | 100 |
| May | 0 | 100 | 500 | 100 | 1000 | 300 | 0 |
| June | 500 | 500 | 100 | 300 | 1100 | 500 | 60 |

It is possible to store products in a warehouse. The capacity of the storage is 100 units per product type per month. The cost is 0.5 Euro per unit of product per months. There are no stocks in the first month but it is desired to have a stock of 50 of each product type at the end of June.
The factory works 6 days a week with two shifts of 8 hours each day. (It can be assumed that each month consists of 24 working days.)
The factory wants to determine a production plan, that is, the quantity to produce, sell and store in each month for each product, that maximizes the total profit.

**Task 1** Model the factory planning problem for the month of January as an LP problem.

**Solution:**

The problem is taken from the book [Wi].
The problem is also one of Gurobi Examples:
http://www.gurobi.com/resources/examples/factory-planning-I
There is also a video: https://youtu.be/vnLc_3VnVcw?t=32m51s
You find the solutions also in this document.

**Solution:**

The objective is to find the optimum "product mix" subject to the production capacity and the marketing limitations. If storage of single products is not allowed, the model for January can be formulated as follows. Let the real variables $x_i$ represent the quantities of product $i$ to be made. Let GR, VD, HD, BR and PL stand for, respectively, grinding, vertical drilling, horizontal drilling, boring and planing. Let the total working hours for each machine be $8 * 2 * 24 = 384$.

$$
\begin{array}{rcccccccccccccl}
\max & 10x_1 & + & 6x_2 & + & 8x_2 & + & 4x_4 & + & 11x_5 & + & 9x_6 & + & 3x_7 & \\
GR: & 0.5x_1 & + & 0.7x_2 & + & & + & & + & 0.3x_5 & + & 0.2x_6 & + & 0.5x_7 & \leq 1152 \\
VD: & 0.1x_1 & + & 0.2x_2 & & & + & 0.3x_4 & & & + & 0.6x_6 & & & \leq 768 \\
HD: & 0.2x_1 & & & + & 0.8x_3 & & & & & & & + & 0.6x_7 & \leq 1152 \\
BR: & 0.05x_1 & + & 0.03x_2 & & & + & 0.07x_4 & + & 0.1x_5 & & & + & 0.08x_7 & \leq 384 \\
PL: & & & & & 0.01x_3 & & & + & 0.05x_5 & & & + & 0.05x_7 & \leq 384 \\
\multicolumn{15}{r}{x_1 \leq 500,\ x_2 \leq 1000,\ x_3 \leq 300,\ x_4 \leq 300,\ x_5 \leq 800,\ x_6 \leq 200,\ x_7 \leq 100}
\end{array}
$$

The single-period problems for the other months would be similar apart from different market bounds, and different capacity figures for the different types of machine.
The matrix has no special structure, the coefficients are not just $\{-1, 1, 0\}$ as in a TUM matrix and non zeros can appear everywhere. The matrix is not necessarily sparse.

**Task 2** Model the multi-period (from January to June) factory planning problem as an LP problem. Use mathematical notation and indicate in general terms how many variables and how many constraints your model has.

**Solution:**

It is necessary to distinguish for each month the quantities of each product manufactured from the quantities sold and held over in storage. These quantities must be represented by different variables. Let the quantities of product $i$ manufactured, sold, and held over in successive months $t$ be represented by variables $x_{it}, s_{it}, h_{it}$, $t = 1, \dots, 6$.
A convenient way to represent the link between these variables is shown in Figure 1. Hence, the mass balance constraints to be imposed are:

$$h_{i,t-1} + x_{it} = s_{it} + h_{it}$$

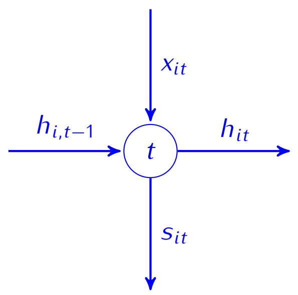


Figure 1: Mass balance constraint at each time period.

Initially (month 0), there is nothing held in stock but finally (month 6) there are (at least) 50 of each product held. This relation involving product 1 gives rise to the following constraints:

$$\begin{array}{llll}
 & x_{11} - s_{11} & -h_{11} & = 0 \\
h_{11} & +x_{12} - s_{12} & -h_{12} & = 0 \\
h_{12} & +x_{13} - s_{13} & -h_{13} & = 0 \\
h_{13} & +x_{14} - s_{14} & -h_{14} & = 0 \\
h_{14} & +x_{15} - s_{15} & -h_{15} & = 0 \\
h_{15} & +x_{16} - s_{16} & & = 50
\end{array}$$

Similar constraints must be specified for the other six products. It may be more convenient to define also variables $h_{16}, h_{26}$, etc, and fix them at the value 50.
The general model is:

$$\max \sum_{i=1}^{7} \sum_{t=1}^{6} p_i s_{it} - \sum_{i=1}^{7} \sum_{t=1}^{6} f h_{it} \tag{1}$$
$$\sum_{i} a_{ij} x_{it} \leq 384(c_j - m_{j,t}) \qquad j \in \{GR, VD, HD, BR, PL\}, t = 1 \dots, 6 \tag{2}$$
$$h_{i,t-1} + x_{it} - s_{it} - h_{it} = 0 \qquad i = 1, \dots, 7; t = 1, \dots, 6 \tag{3}$$
$$s_{it} \leq u_{it} \qquad i = 1, \dots, 7; t = 1, \dots, 6 \tag{4}$$
$$h_{it} \leq 100 \qquad i = 1, \dots, 7; t = 1, \dots, 6 \tag{5}$$
$$s_{it}, x_{it}, h_{it} \geq 0 \qquad i = 1, \dots, 7; t = 1, \dots, 6 \tag{6}$$
$$h_{i0} = 0, h_{i6} = 50 \qquad i = 1, \dots, 7 \tag{7}$$

In the objective function (1) the "selling" variables are given the appropriate "unit profit" $p_i$ and the "holding" variables the coefficients of $f = 0.5$. Constraints (2) are the resource constraints where $c_m$ is the capacity for each resource $m$. Constraints (3) are the mass balance constraints described above and constraints (4) are the marketing limitations where $u_{it}$ are product upper bounds.
The resulting model has the following dimensions:

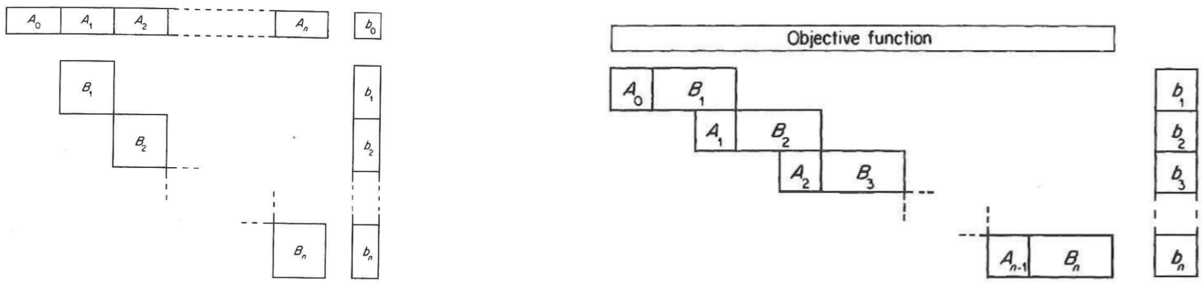


Figure 2: On the left a block angular structure and on the right a staircase structure

| | |
|---|---|
| $6 \times 7 = 42$ | manufacturing variables |
| $6 \times 7 = 42$ | selling variables |
| $6 \times 7 = 42$ | holding variables |
| Total 126 | variables |
| $6 \times 5 = 30$ | capacity constraints |
| $6 \times 7 = 42$ | monthly linking constraints |
| $6 \times 7 = 42$ | marketing limitations |
| $6 \times 7 = 42$ | holding quantity constraints |
| Total 156 | constraints |

We typically do not count positivity constraints, as those are standard.

If we present the problem in a diagrammatic form we obtain the illustration on the left of Figure 2.
The matrix is not apparently TUM. It has however a *block angular structure*. A *block angular structure* is made by common rows and blocks in diagonal representing submodels. In our case the common rows are the linking equality constraints of mass balance while the submodels are the per period production planning as the one seen in Task 1. Clearly, a matrix with *block angular structure* without common constraints could be decomposed and each submodel solved separately. Nevertheless advanced techniques exist to handle efficiently problems with block angular structure. A typical problem with this structure often used in examples is the multi-commodity flow problem (we will see this in one of the next classes).
Another type of structure which may arise in multi-period models is the *staircase* structure which is illustrated in Figure 2, right. In fact a staircase structure such as this could be converted into a block angular structure. If alternate "steps" such as $(A_0, B_1)$, $(A_2, B_3)$ were treated as subproblem constraints and the intermidiate "steps" as common rows we would have a block angular structure.