

DM545/DM871
Linear and Integer Programming

Lecture 9
IP Modeling
Formulations, Relaxations

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

1. Formulations

- Uncapacited Facility Location
- Alternative Formulations

2. Relaxations

1. Formulations

Uncapacited Facility Location
Alternative Formulations

2. Relaxations

1. Formulations

Uncapacited Facility Location

Alternative Formulations

2. Relaxations

Uncapacitated Facility Location (UFL)

Given:

- depots $N = \{1, \dots, n\}$
- clients $M = \{1, \dots, m\}$
- f_j fixed cost to use depot j
- transport cost for all orders c_{ij}

Task: Which depots to open and which depots serve which client

Variables: $y_j = \begin{cases} 1 & \text{if depot opened} \\ 0 & \text{otherwise} \end{cases}$, x_{ij} fraction of demand of i satisfied by j

Objective:

$$\min \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j$$

Constraints:

$$\sum_{j=1}^n x_{ij} = 1$$

$$\forall i = 1, \dots, m$$

$$\sum_{i \in M} x_{ij} \leq m y_j$$

$$\forall j \in N$$

1. Formulations

Uncapacited Facility Location
Alternative Formulations

2. Relaxations

Definition (Formulation)

A polyhedron $P \subseteq \mathbb{R}^{n+p}$ is a **formulation** for a set $X \subseteq \mathbb{Z}^n \times \mathbb{R}^p$ if and only if $X = P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$

That is, if it does not leave out any of the solutions of the feasible region X .

There are **infinite** formulations

Definition (Convex Hull)

Given a set $X \subseteq \mathbb{Z}^n$ the **convex hull** of X is defined as:

$$\text{conv}(X) = \left\{ x : x = \sum_{i=1}^t \lambda_i x^i, \quad \sum_{i=1}^t \lambda_i = 1, \quad \lambda_i \geq 0, \quad \text{for } i = 1, \dots, t, \right. \\ \left. \text{for all finite subsets } \{x^1, \dots, x^t\} \text{ of } X \right\}$$

Proposition

$\text{conv}(X)$ is a polyhedron (ie, representable as $Ax \leq b$)

Proposition

Extreme points of $\text{conv}(X)$ all lie in X

Hence:

$$\max\{c^T x : x \in X\} \equiv \max\{c^T x : x \in \text{conv}(X)\}$$

However it might require exponential number of inequalities to describe $\text{conv}(X)$

What makes a formulation better than another?

$$X \subseteq \text{conv}(X) \subseteq P_2 \subset P_1$$

P_2 is better than P_1

Definition

Given a set $X \subseteq \mathbb{R}^n$ and two formulations P_1 and P_2 for X , P_2 is a better formulation than P_1 if $P_2 \subset P_1$

Example

 $P_1 = \text{UFL with } \sum_{i \in M} x_{ij} \leq my_j \quad \forall j \in N$ $P_2 = \text{UFL with } x_{ij} \leq y_j \quad \forall i \in M, j \in N$

$$P_2 \subset P_1$$

- $P_2 \subseteq P_1$ because summing $x_{ij} \leq y_j$ over $i \in M$ we obtain $\sum_{i \in M} x_{ij} \leq my_j$
- $P_2 \subset P_1$ because there exists a point in P_1 but not in P_2 : $m = 6 = 3 \cdot 2 = k \cdot n$

$$x_{10} = 1, x_{20} = 1, x_{30} = 1,$$

$$x_{41} = 1, x_{51} = 1, x_{61} = 1$$

$$\sum_i x_{i0} \leq 6y_0 \quad y_0 = 1/2$$

$$\sum_i x_{i1} \leq 6y_1 \quad y_1 = 1/2$$

1. Formulations

- Uncapacited Facility Location
- Alternative Formulations

2. Relaxations

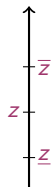
$$z = \max\{c(x) : x \in X \subseteq \mathbb{Z}^n\}$$

How can we prove that x^* is optimal?

\bar{z} is UB

\underline{z} is LB

stop when $\bar{z} - \underline{z} \leq \epsilon$



- **Primal bounds** (here lower bounds): every feasible solution gives a primal bound, it may be easy or hard to find, heuristics
- **Dual bounds** (here upper bounds): Relaxations

Optimality gap (SCIP):

- If primal and dual bound have opposite signs, the gap is "Infinity".
- If primal and dual bound have the same sign, the gap is

$$\frac{|pb - db|}{\min(|pb|, |db|)}$$

decreases monotonously during the solving process.

Proposition

Given: $(IP) \ z = \max\{c(x) : x \in X \subseteq \mathbb{R}^n\}$
a relaxation of it is: $(RP) \ z^R = \max\{f(x) : x \in T \subseteq \mathbb{R}^n\}$ if:

- (i) $X \subseteq T$ or
- (ii) $f(x) \geq c(x) \ \forall x \in X$

In other terms:

$$\max_{x \in T} f(x) \geq \left\{ \begin{array}{l} \max_{x \in T} c(x) \\ \max_{x \in X} f(x) \end{array} \right\} \geq \max_{x \in X} c(x)$$

- T : candidate solutions;
- $X \subseteq T$ feasible solutions;
- $f(x) \geq c(x) \ \forall x \in X$

How to construct relaxations?

1. $IP : \max\{c^T x : x \in P \cap \mathbb{Z}^n\}, \quad P = \{x \in \mathbb{R}^n : Ax \leq b\}$
 $LP : \max\{c^T x : x \in P\}$

Better formulations give better bounds ($P_1 \subseteq P_2$)

Proposition

- (i) *If a relaxation LP is infeasible, the original problem IP is infeasible.*
- (ii) *Let x^* be optimal solution for LP. If $x^* \in X$ and $f(x^*) = c(x^*)$ then x^* is optimal for IP.*

2. **Combinatorial relaxations** to easy problems that can be solved rapidly
Eg: TSP to Assignment problem Eg: Symmetric TSP to 1-tree

3. Lagrangian relaxation

$$IP : \quad z = \max\{c^T x : Ax \leq b, x \in X \subseteq \mathbb{Z}^n\}$$

$$LR : \quad z(u) = \max\{c^T x + u(b - Ax) : x \in X\}$$

$$z(u) \geq z \quad \forall u \geq 0$$

4. Duality:

Definition

Two problems:

$$z = \max\{c(x) : x \in X\} \quad w = \min\{w(u) : u \in U\}$$

form a **weak-dual pair** if $c(x) \leq w(u)$ for all $x \in X$ and all $u \in U$.

When $z = w$ they form a **strong-dual pair**

Proposition

$z = \max\{c^T x : Ax \leq b, x \in \mathbb{Z}_+^n\}$ and $w^{LP} = \min\{u^T b : A^T u \geq c, u \in \mathbb{R}_+^m\}$
(ie, dual of linear relaxation) form a weak-dual pair.

Proposition

Let IP and D be weak-dual pair:

- (i) If D is unbounded, then IP is infeasible
- (ii) If $x^* \in X$ and $u^* \in U$ satisfy $c(x^*) = w(u^*)$ then x^* is optimal for IP and u^* is optimal for D .

The advantage is that we do not need to solve an LP like in the LP relaxation to have a bound, any feasible dual solution gives a bound.

Weak pairs:

Matching: $z = \max\{1^T x : Ax \leq 1, x \in \mathbb{Z}_+^m\}$

V. Covering: $w = \min\{1^T y : A^T y \geq 1, y \in \mathbb{Z}_+^n\}$

Proof: consider LP relaxations, then $z \leq z^{LP} = w^{LP} \leq w$.
(strong when graphs are bipartite)

Weak pairs:

S. Packing: $z = \max\{1^T x : Ax \leq 1, x \in \mathbb{Z}_+^n\}$

S. Covering: $w = \min\{1^T y : A^T y \geq 1, y \in \mathbb{Z}_+^m\}$