

DM545/DM871
Linear and Integer Programming

Lecture 7
Revised Simplex Method

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Outline

Revised Simplex Method
Efficiency Issues

1. Revised Simplex Method

2. Efficiency Issues

Complexity of single pivot operation in standard simplex:

- entering variable $O(n)$
- leaving variable $O(m)$
- updating the tableau $O(mn)$

Problems with this:

- Time: we are doing operations that are not actually needed
Space: we need to store the whole tableau: $O(mn)$ floating point numbers
- Most problems have sparse matrices (many zeros)
sparse matrices are typically handled efficiently
the standard simplex has the 'Fill in' effect: sparse matrices are lost
- accumulation of Floating Point Errors over the iterations

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Revised Simplex Method

Several ways to improve wrt pitfalls in the previous slide, requires matrix description of the simplex.

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad i = 1..m \\ & x_j \geq 0 \quad j = 1..n \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ & Ax = b \\ & x \geq 0 \\ & A \in \mathbb{R}^{m \times (n+m)} \\ & c \in \mathbb{R}^{(n+m)}, b \in \mathbb{R}^m, x \in \mathbb{R}^{n+m} \end{aligned}$$

$$\max \{ c^T x \mid Ax = b, x \geq 0 \}$$

At each iteration the simplex moves from a basic feasible solution to another.

For each basic feasible solution:

- $B = \{1 \dots m\}$ basis
- $N = \{m+1 \dots m+n\}$
- $A_B = [a_1 \dots a_m]$ basis matrix
- $A_N = [a_{m+1} \dots a_{m+n}]$
- $x_N = 0$
- $x_B \geq 0$

$$\left[\begin{array}{cc|c|c} A_N & A_B & 0 & b \\ \hline c_N^T & c_B^T & 1 & 0 \end{array} \right]$$

$$Ax = A_N x_N + A_B x_B = b$$

$$A_B x_B = b - A_N x_N$$

Basic feasible solution $\iff A_B$ is non-singular

$$x_B = A_B^{-1}b - A_B^{-1}A_N x_N$$

for the objective function:

$$z = c^T x = c_B^T x_B + c_N^T x_N$$

Substituting for x_B from above:

$$\begin{aligned} z &= c_B^T (A_B^{-1} b - A_B^{-1} A_N x_N) + c_N^T x_N = \\ &= c_B^T A_B^{-1} b + (c_N^T - c_B^T A_B^{-1} A_N) x_N \end{aligned}$$

Collecting together:

$$\begin{aligned} x_B &= A_B^{-1} b - A_B^{-1} A_N x_N \\ z &= c_B^T A_B^{-1} b + (c_N^T - c_B^T \underbrace{A_B^{-1} A_N}_{\bar{A}}) x_N \end{aligned}$$

In tableau form, for a basic feasible solution corresponding to B we have:

$$\left[\begin{array}{ccc|c} A_B^{-1} A_N & I & 0 & A_B^{-1} b \\ \hline c_N^T - c_B^T A_B^{-1} A_N & 0 & 1 & -c_B^T A_B^{-1} b \end{array} \right]$$

We do not need to compute all elements of \bar{A}

Example

$$\begin{aligned}\max \quad & x_1 + x_2 \\ -x_1 + x_2 & \leq 1 \\ x_1 & \leq 3 \\ x_2 & \leq 2 \\ x_1, x_2 & \geq 0\end{aligned}$$

Initial tableau

x1	x2	x3	x4	x5	-z	b
-1	1	1	0	0	0	1
1	0	0	1	0	0	3
0	1	0	0	1	0	2
1	1	0	0	0	1	0

$$\begin{aligned}\max \quad & x_1 + x_2 \\ -x_1 + x_2 + x_3 & = 1 \\ x_1 + x_4 & = 3 \\ x_2 + x_5 & = 2 \\ x_1, x_2, x_3, x_4, x_5 & \geq 0\end{aligned}$$

After two iterations

x1	x2	x3	x4	x5	-z	b
1	0	-1	0	1	0	1
0	1	0	0	1	0	2
0	0	1	1	-1	0	2
0	0	1	0	-2	1	3

Basic variables x_1, x_2, x_4 . Non basic: x_3, x_5 . From the [initial tableau](#):

$$A_B = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad A_N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} \quad x_N = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix}$$

$$c_B^T = [1 \ 1 \ 0] \quad c_N^T = [0 \ 0]$$

- **Entering variable:**

in std. we look at tableau, in revised we need to compute: $c_N^T - c_B^T A_B^{-1} A_N$

1. find $y^T = c_B^T A_B^{-1}$ (by solving $y^T A_B = c_B^T$, the latter can be done more efficiently)
2. calculate $c_N^T - y^T A_N$

Step 1:

$$[y_1 \ y_2 \ y_3] \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = [1 \ 1 \ 0]$$

$$y^T A_B = c_B^T$$

$$[1 \ 1 \ 0] \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = [-1 \ 0 \ 2]$$

$$c_B^T A_B^{-1} = y^T$$

Step 2:

$$[0 \ 0] - [-1 \ 0 \ 2] \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = [1 \ -2]$$

$$c_N^T - y^T A_N$$

(Note that they can be computed individually: $c_j - y^T a_j > 0$)

Let's take the first we encounter x_3

- **Leaving variable**

we increase variable by largest feasible amount θ

$$R1: x_1 - x_3 + x_5 = 1$$

$$x_1 = 1 + x_3 \geq 0$$

$$R2: x_2 + 0x_3 + x_5 = 2$$

$$x_2 = 2 \geq 0$$

$$R3: -x_3 + x_4 - x_5 = 2$$

$$x_4 = 2 - x_3 \geq 0$$

$$x_B = x_B^* - A_B^{-1} A_N x_N$$

$$x_B = x_B^* - d\theta$$

d is the column of $A_B^{-1} A_N$ that corresponds to the entering variable, ie, $d = A_B^{-1} a$ where a is the entering column

3. Find θ such that x_B stays positive:

Find $d = A_B^{-1} a$ (by solving $A_B d = a$)

Step 3:

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow d = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow x_B = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \theta \geq 0$$

$$2 - \theta \geq 0 \Rightarrow \theta \leq 2 \rightsquigarrow x_4 \text{ leaves}$$

- So far we have done computations, but now we save the pivoting update. The update of A_B is done by replacing the leaving column by the entering column

$$x_B^* = \begin{bmatrix} x_1 - d_1\theta \\ x_2 - d_2\theta \\ \theta \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \quad A_B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- Many implementations depending on how $y^T A_B = c_B^T$ and $A_B d = a$ are solved. They are in fact solved from scratch.
- many operations saved especially if many variables!
- special ways to call the matrix A from memory
- better control over numerical issues since A_B^{-1} can be recomputed.

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Solving the two Systems of Equations

$A_B x = b$ solved without computing A_B^{-1}
(costly and likely to introduce numerical inaccuracy)

Recall how the inverse is computed:

For a 2×2 matrix the matrix inverse is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For a 3×3 matrix the matrix inverse is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad A^{-1} = \frac{1}{|A|} \begin{bmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}^T$$

Eta Factorization of the Basis

Let $B := A_B$, k th iteration

B_k be the matrix with col p differing from B_{k-1}

Column p is the a column appearing in $B_{k-1}d = a$ solved at 3)

Hence:

$$B_k = B_{k-1}E_k$$

E_k is the **eta matrix** differing from id. matrix in only one column, which is set equal to d

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & & -1 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$

No matter how we solve $y^T B_{k-1} = c_B^T$ and $B_{k-1}d = a$, their update always relies on $B_k = B_{k-1}E_k$ with E_k available. Plus when initial basis by slack variable $B_0 = I$ and $B_1 = E_1, B_2 = E_1 E_2 \dots$:

$$B_k = E_1 E_2 \dots E_k \quad \text{eta factorization}$$

$$\begin{aligned} (((y^T E_1) E_2) E_3) \dots E_k &= c_B^T, \\ (E_1 (E_2 \dots E_k d)) &= a, \end{aligned}$$

$$\begin{aligned} u^T E_4 &= c_B^T, \quad v^T E_3 = u^T, \quad w^T E_2 = v^T, \quad y^T E_1 = w^T \\ E_1 u &= a, \quad E_2 v = u, \quad E_3 w = v, \quad E_4 d = w \end{aligned}$$

Solve the systems $y^T E_1 E_2 E_3 E_4 = [1 \ 2 \ 3]$ and $E_1 E_2 E_3 E_4 d = [1 \ 2 \ 3]^T$ with

$$E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0.5 & 0 \\ 0 & 4 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

We use backward transformation and solve the sequence of linear systems:

$$u^T E_4 = [1 \ 2 \ 3], \quad v^T E_3 = u^T, \quad w^T E_2 = v^T, \quad y^T E_1 = w^T$$

$$u^T \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = [1, 2, 3]$$

Since the eta matrices have always one 1 in two columns then the solution can be read up easily. From the third column we find $u_3 = 3$. From the second column, we find $u_2 = 2$. Substituting in the first column, we find $-0.5u_1 + 3 * 2 + 1 * 3 = 1$, which yields $u_1 = 16$. The next system is:

$$v^T \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = [16, 2, 3]$$

From the first column we get $v_1 = 16$, from the second column $v_2 = 2$ from the last column $v_1 + 3v_2 + v_3 = 3$, which yields $v_3 = -21$. The next:

$$w^T \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = [16, 2, -21]$$

- Solving $y^T B_k = c_B^T$ also called backward transformation (BTRAN)
- Solving $B_k d = a$ also called forward transformation (FTRAN)
- E_i matrices can be stored by only storing the column and the position
- If sparse columns then can be stored in compact mode, ie only nonzero values and their indices

- Tableau method is unstable: computational errors may accumulate. Revised method has a natural control mechanism: we can recompute A_B^{-1} at any time
- Commercial and freeware solvers differ from the way the systems $y^T A_B = c_B^T$ and $A_B d = a$ are resolved

- Dual simplex with steepest descent (largest increase)
- Linear Algebra:
 - Dynamic LU-factorization using Markowitz threshold pivoting (Suhl and Suhl, 1990)
 - sparse linear systems: Typically these systems take as input a vector with a very small number of nonzero entries and output a vector with only a few additional nonzeros.
- Presolve, ie problem reductions: removal of redundant constraints, fixed variables, and other extraneous model elements.
- dealing with degeneracy, stalling (long sequences of degenerate pivots), and cycling:
 - bound-shifting (Paula Harris, 1974)
 - Hybrid Pricing (variable selection): start with partial pricing, then switch to devex (approximate steepest-edge, Harris, 1974)
- A model that might have taken a year to solve 10 years ago can now solve in less than 30 seconds (Bixby, 2002).