

DM545/DM871 – Linear and integer programming

Sheet 0, Spring 2022

Solution:

Included.

Exercise 1

Consider the matrices:

$$A = \begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix}$$

In each part compute the given expression. Where the computation is not possible explain why.

1. $D + E$
2. $D - E$
3. $5A$
4. $2B - C$
5. $2(D + 5E)$
6. $(C^T B)A^T$
7. $2\text{tr}(AB)$
8. $\det(E)$

Solution:

Taken by a former student: andrm17.

1) $D + E$

$$D = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} + E = \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix} = \begin{bmatrix} -2+0 & 1+3 & 8+0 \\ 3+(-5) & 0+1 & 2+1 \\ 4+7 & -6+6 & 3+2 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 8 \\ -2 & 1 & 3 \\ 11 & 0 & 5 \end{bmatrix}$$

2) $D - E$

$$D = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} - E = \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix} = \begin{bmatrix} -2-0 & 1-3 & 8-0 \\ 3-(-5) & 0-1 & 2-1 \\ 4-7 & -6-6 & 3-2 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 8 \\ 8 & -1 & 1 \\ -3 & -12 & 1 \end{bmatrix}$$

3) $5A$

$$5 \cdot \begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 5 \cdot 2 & 5 \cdot 0 \\ 5 \cdot -4 & 5 \cdot 6 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ -20 & 30 \end{bmatrix}$$

4) $2B - C$

$$2 \cdot \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot -7 & 2 \cdot 2 \\ 2 \cdot 5 & 2 \cdot 3 & 2 \cdot 0 \end{bmatrix} - \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 4 \\ 10 & 6 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix}$$

We cannot perform a subtraction with the two matrices since they do not have the same dimensions.

5) $2(D + 5E)$

$$2 \cdot \left(\begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} + 5 \cdot \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix} \right) = 2 \cdot \left(\begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} + \begin{bmatrix} 5 \cdot 0 & 5 \cdot 3 & 5 \cdot 0 \\ 5 \cdot -5 & 5 \cdot 1 & 5 \cdot 1 \\ 5 \cdot 7 & 5 \cdot 6 & 5 \cdot 2 \end{bmatrix} \right)$$

$$2 \cdot \left(\begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 15 & 0 \\ -25 & 5 & 5 \\ 35 & 30 & 10 \end{bmatrix} \right) = 2 \cdot \left(\begin{bmatrix} -2+0 & 1+15 & 8+0 \\ 3+(-25) & 0+5 & 2+5 \\ 4+35 & -6+30 & 3+10 \end{bmatrix} \right)$$

$$2 \cdot \begin{bmatrix} -2 & 16 & 8 \\ -22 & 5 & 7 \\ 39 & 24 & 13 \end{bmatrix} = \begin{bmatrix} 2 \cdot -2 & 2 \cdot 16 & 2 \cdot 8 \\ 2 \cdot -22 & 2 \cdot 5 & 2 \cdot 7 \\ 2 \cdot 39 & 2 \cdot 24 & 2 \cdot 13 \end{bmatrix} = \begin{bmatrix} -4 & 32 & 16 \\ -44 & 10 & 14 \\ 78 & 48 & 26 \end{bmatrix}$$

6) $(C^T B)A^T$

$$\left(\begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix}^T \cdot \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix}^T = \left(\begin{bmatrix} 4 & -3 & 2 \\ 9 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 2 & -4 \\ 0 & 6 \end{bmatrix}$$

We cannot perform this multiplication since C^T doesn't have the same number of columns as B has rows.

7) $2tr(AB)$

$$2tr \left(\begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix} \right) = 2tr \left(\begin{bmatrix} 2 \cdot 1 + 0 \cdot 5 & 2 \cdot -7 + 0 \cdot 3 & 2 \cdot 2 + 0 \cdot 0 \\ -4 \cdot 1 + 6 \cdot 5 & -4 \cdot -7 + 6 \cdot 3 & -4 \cdot 2 + 6 \cdot 0 \end{bmatrix} \right)$$

$$2tr \left(\begin{bmatrix} 2 & -14 & 4 \\ 26 & 46 & -8 \end{bmatrix} \right)$$

Trace is not defined for non-square matrices.

8) $\det(E)$

We're using cofactor expansion to get the determinant.

$$\det \left(\begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix} \right) = 0 \cdot \begin{bmatrix} 1 & 1 \\ 6 & 2 \end{bmatrix} - 3 \cdot \begin{bmatrix} -5 & 1 \\ 7 & 2 \end{bmatrix} - 0 \cdot \begin{bmatrix} -5 & 1 \\ 7 & 6 \end{bmatrix}$$

we can discard the zeroes and we're then left with:

$$-3 \cdot (-5 \cdot 2 - 7 \cdot 1) = -3 \cdot (-10 - 7) = -3 \cdot -17 = 51$$

Exercise 2

Consider the following system of linear equations in the variables $x, y, z \in \mathbb{R}$.

$$\begin{aligned} -2y + 3z &= 3 \\ 3x + 6y - 3z &= -2 \\ -3x - 8y + 6z &= 5 \end{aligned}$$

1. Write the augmented matrix of this system.
2. Reduce this matrix to row echelon form by performing a sequence of elementary row operations.
3. Solve the system and write its general solution in parametric form.

Solution:

```
import numpy as np

# The augmented matrix
AA = np.array([[ 0, -2, 3, 3],
               [ 3, 6, -3, -2],
               [-3, -8, 6, 5]])

In [30]: import sympy as sy
```

```

...: # np.linalg.solve(A,b)
...:
...: sy.Matrix(AA).rref()
...:
Out[30]:
(Matrix([
[1, 0, 2, 7/3],
[0, 1, -3/2, -3/2],
[0, 0, 0, 0]]), (0, 1))

```

Hence, the solution is:

$$\mathbf{x} = \begin{bmatrix} 7/3 - 2t \\ -3/2 + 3/2t \\ t \end{bmatrix} = \begin{bmatrix} 7/3 \\ -3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 3/2 \\ 1 \end{bmatrix} t \quad t \in \mathbb{R}$$

Exercise 3

Consider the following matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}.$$

1. Find M^{-1} by performing row operations on the matrix $[M \mid I]$.
2. Is it possible to express M as a product of elementary matrices? Explain why or why not.

Solution:

```

import numpy as np

M = np.array([[ 1, 0, 1],[-1,1,0],[2,2,2]])
MM = np.concatenate([M,np.identity(3)],axis=1)

import sympy as sy
sy.Matrix(MM).rref()

```

```

(Matrix([
[1, 0, 0, -1.0, -1.0, 0.5],
[0, 1, 0, -1.0, 0, 0.5],
[0, 0, 1, 2.0, 1.0, -0.5]]), (0, 1, 2))

```

Yes, it is possible. Since the matrix is invertible we have shown above that we can go from an identity matrix to M^{-1} and consequently also to M from an identity matrix with elementary row operations. Elementary row operations can be expressed as products between elementary matrices.

Exercise 4

1. Given the point $[3, 2]$ and the vector $[-1, 0]$ find the vector and parametric (Cartesian) equation of the line containing the point and parallel to the vector.
2. Find the vector and parametric (Cartesian) equations of the plane in \mathbb{R}^3 that passes through the origin and is orthogonal to $\mathbf{v} = [3, -1, -6]$.

Solution:

The vector equation: Let $[3, 2]^T = \mathbf{p}$ and $[-1, 0]^T = \mathbf{v}$. Any point \mathbf{x} on the line can be expressed as:

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad \forall t \in \mathbb{R}$$

We can derive the Cartesian equation by eliminating t from the equation above:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

From the first coordinate: $x_1 = 3 - t$ and from the second: $x_2 = 2$. The Cartesian equation is $x_2 = 2$ since x_1 is free to get any value.

The plane through the origin orthogonal to $\mathbf{v} = [3, -1, -6]$ is given by:

$$\mathbf{x}^T \mathbf{v} = 0$$

That is: $3x_1 - x_2 - 6x_3 = 0$.

Exercise 5

Write a one line description of the methods you know to compute the inverse of a square matrix.

Solution:

- using cofactors for the adjoint matrix and dividing by the determinant
- by row reduction of $[A|I]$

Exercise 6

Calculate by hand the inverse of

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

and check your result with the function `numpy.linalg.inv`.

Solution:

$$\begin{bmatrix} 0. & 1. & -1.5 \\ 1. & -3. & 4. \\ 0. & 0. & 0.5 \end{bmatrix}$$

Exercise 7

Use Cramer's rule to express the solution of the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

Solution:

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is the matrix obtained from A by replacing the i th column with the vector \mathbf{b} .

$$x_1 = -4.5 \quad x_2 = 14 \quad x_3 = 1.5$$

Exercise 8

Given two points in the Cartesian plane \mathbb{R}^2 , $A = (1, 2)$ and $B = (3, 4)$ write the vector parametric equation and the Cartesian equation of the line that passes through them. Express the vector equation as an affine combination of the two points.

Solution:

The vector equation is an affine combination of the two points:

$$\mathbf{x} = [1, 2]^T + t([3, 4]^T - [1, 2]^T), \forall t \in \mathbb{R}^2$$

To find a, b, c such that $ax + by + c = 0$ describes the line we can rewrite the equation above as

$$[x, y]^T = [1, 2]^T + t([1, 2]^T), \forall t \in \mathbb{R}^2$$

we then eliminate t by substituting in the two equations.

Exercise 9

Express the segment in \mathbb{R}^2 between the points $A = (1, 2)$ and $B = (3, 4)$ as a convex combination of its extremes.

Solution:

$$\{\alpha[1, 2]^T + \beta[3, 4]^T \mid \alpha, \beta \in \mathbb{R}, \alpha, \beta \geq 0, \alpha + \beta = 1\}$$

Exercise 10

Write a generic vector parametric equation and a generic Cartesian equation of a plane in \mathbb{R}^3 .

Solution:

$$\begin{aligned} \mathbf{x} &= \mathbf{p} + s\mathbf{v} + t\mathbf{w}, \quad \mathbf{x}, t \in \mathbb{R} \\ ax + by + cz + d &= 0 \end{aligned}$$

Exercise 11

Write a generic Cartesian equation of an hyperplane in \mathbb{R}^n that does not pass through the origin.

Solution:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Exercise 12

Prove that the following vectors in \mathbb{R}^3 linearly independent?

$$\begin{aligned} &- [6, 9, 5]^T \\ &- [5, 5, 7]^T \\ &- [2, 0, 7]^T \end{aligned}$$

Solution:

We need to solve homogeneous system $A\mathbf{x} = \mathbf{0}$, where the matrix A has the three vectors forming its columns. The matrix A has $\det(A) \neq 0$ and rank 3. Therefore the only solution to the homogeneous system is the trivial solution $\mathbf{0}$. The three column vectors are therefore linearly independent.

```
A=np.array([[6, 9, 5],[5, 5, 7],[2, 0, 7]]).T
np.linalg.det(A)
np.linalg.matrix_rank(A)
```