

DM545/DM871
Linear and Integer Programming

Lecture
Cutting Planes

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

1. Cutting Plane Algorithms

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Valid Inequalities

- IP: $z = \max\{c^T x : x \in X\}$, $X = \{x : Ax \leq b, x \in \mathbb{Z}_+^n\}$
- Proposition: $\text{conv}(X) = \{x : \tilde{A}x \leq \tilde{b}, x \geq 0\}$ is a polyhedron
- LP: $z = \max\{c^T x : \tilde{A}x \leq \tilde{b}, x \geq 0\}$ would be the best formulation
- $\tilde{a}x \leq \tilde{b}$ facet defining inequalities
- Key idea: try to approximate the best formulation.

Definition (Valid inequalities)

$ax \leq b$ is a valid inequality for $X \subseteq \mathbb{R}^n$ if $ax \leq b \forall x \in X$

Which are useful inequalities? and how can we find them? How can we use them?

Example: Pre-processing

- $X = \{(x, y) : x \leq 999y; \quad 0 \leq x \leq 5, \quad y \in \mathbb{B}^1\}$

$$x \leq 5y$$

- $X = \{x \in \mathbb{Z}_+^4 : 13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72\}$

$$2x_1 + 2x_2 + x_3 + x_4 \geq \frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq \frac{72}{11} = 6 + \frac{6}{11}$$

$$2x_1 + 2x_2 + x_3 + x_4 \geq 7$$

- Capacitated facility location:

$$\sum_{i \in M} x_{ij} \leq b_j y_j \quad \forall j \in N$$

$$x_{ij} \leq b_j y_j$$

$$\sum_{j \in N} x_{ij} = a_i \quad \forall i \in M$$

$$x_{ij} \leq a_i$$

$$x_{ij} \geq 0, \quad y_j \in B^n$$

$$x_{ij} \leq \min\{a_i, b_j\} y_j$$

Converting Weak to Strong MIP Formulations

Strong formulations \equiv better, tighter formulations

Detection possible from the log output of a solver.

Possible actions:

1. Add cuts to existing models

- Combining constraints
- Using a graph representation (clique cuts)
- Using a disjunctive approach

\rightsquigarrow Many found automatically by the solver in pre-solver phase

2. (Change the model)

3. (Change the algorithm, eg, column generation)

(Lazy) constraints \neq cuts

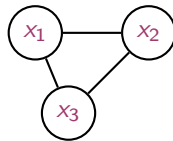
Add cuts to the existing model

$$\begin{aligned}
 &\text{maximize } x_1 + x_2 + x_3 \\
 &\text{subject to } x_1 + x_2 \leq 1 \\
 &\quad \quad \quad x_2 + x_3 \leq 1 \\
 &\quad \quad \quad x_1 + x_3 \leq 1 \\
 &\quad \quad \quad x_i \in \{0, 1\} \quad i = 1, 2, 3
 \end{aligned}$$

Combine and round constraints:

$$\begin{aligned}
 2x_1 + 2x_2 + 2x_3 &\leq 3 \\
 x_1 + x_2 + x_3 &\leq \frac{3}{2} \\
 x_1 + x_2 + x_3 &\leq 1
 \end{aligned}$$

Create a conflict graph; at most one binary in a clique can be 1



$$x_1 + x_2 + x_3 \leq 1$$

Chvátal-Gomory cuts

- $X \in P \cap \mathbb{Z}_+^n$, $P = \{x \in \mathbb{R}_+^n : Ax \leq b\}$, $A \in \mathbb{R}^{m \times n}$
- $u \in \mathbb{R}_+^m$, $\{a_1, a_2, \dots, a_n\}$ columns of A

CG procedure to construct valid inequalities

$$1) \quad \sum_{j=1}^n u^T a_j x_j \leq u^T b \quad \text{valid: } u \geq 0$$

$$2) \quad \sum_{j=1}^n \lfloor u^T a_j \rfloor x_j \leq u^T b \quad \text{valid: } x \geq 0 \text{ and } \sum \lfloor u^T a_j \rfloor x_j \leq \sum u^T a_j x_j$$

$$3) \quad \sum_{j=1}^n \lfloor u^T a_j \rfloor x_j \leq \lfloor u^T b \rfloor \quad \text{valid for } X \text{ since } x \in \mathbb{Z}^n$$

Theorem

by applying this CG procedure a finite number of times every valid inequality for X can be obtained

However not all the constraints generated by $u \in \mathbb{R}_+^m$ are tightenings.

- $X \in P \cap \mathbb{Z}_+^n$
- a family of valid inequalities $\mathcal{F} : a^T x \leq b, (a, b) \in \mathcal{F}$ for X
- we do not find them all a priori, only interested in those close to optimum

Cutting Plane Algorithm

Init.: $t = 0, P^0 = P$

Iter. t : Solve $\bar{z}^t = \max\{c^T x : x \in P^t\}$

let x^t be an optimal solution

if $x^t \in \mathbb{Z}^n$ stop, x^t is opt to the IP

if $x^t \notin \mathbb{Z}^n$ solve separation problem for x^t and \mathcal{F}

if (a^t, b^t) is found with $a^t x^t > b^t$ that cuts off x^t

$$P^{t+1} = P \cap \{x : a^i x \leq b^i, i = 1, \dots, t\}$$

else stop (P^t is in any case an improved formulation)

Gomory's fractional cutting plane algorithm

Cutting plane algorithm + Chvátal-Gomory cuts

- $\max\{c^T x : Ax = b, x \geq 0, x \in \mathbb{Z}^n\}$
- Solve LPR to optimality

$$\left[\begin{array}{c|c|c|c|c} I & \bar{A}_N = A_B^{-1} A_N & 0 & \bar{b} & \\ \hline \bar{c}_B & \bar{c}_N (\leq 0) & 1 & -\bar{d} & \end{array} \right]$$

$$x_{B_u} = \bar{b}_u - \sum_{j \in N} \bar{a}_{uj} x_j, \quad u = 1..m$$

$$z = \bar{d} + \sum_{j \in N} \bar{c}_j x_j$$

- If basic optimal solution to LPR is not integer then \exists some row u : $\bar{b}_u \notin \mathbb{Z}^1$.
The Chvátal-Gomory cut applied to this row is:

$$x_{B_u} + \sum_{j \in N} \lfloor \bar{a}_{uj} \rfloor x_j \leq \lfloor \bar{b}_u \rfloor$$

(B_u is the index in the basis B corresponding to the row u)

(cntd)

- Eliminating $x_{B_u} = \bar{b}_u - \sum_{j \in N} \bar{a}_{uj} x_j$ in the CG cut we obtain:

$$\sum_{j \in N} \underbrace{(\bar{a}_{uj} - \lfloor \bar{a}_{uj} \rfloor)}_{0 \leq f_{uj} < 1} x_j \geq \underbrace{\bar{b}_u - \lfloor \bar{b}_u \rfloor}_{0 < f_u < 1}$$

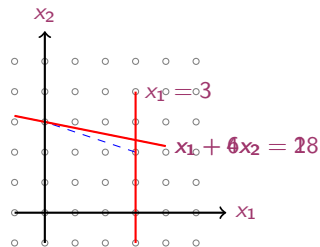
$$\sum_{j \in N} f_{uj} x_j \geq f_u$$

$f_u > 0$ or else u would not be row of fractional solution. It implies that x^* in which $x_N^* = 0$ is cut out!

(theoretically it terminates after a finite number of iterations, but in practice not successful.)

Example

$$\begin{aligned}
 \max \quad & x_1 + 4x_2 \\
 \text{s.t.} \quad & x_1 + 6x_2 \leq 18 \\
 & x_1 \leq 3 \\
 & x_1, x_2 \geq 0 \\
 & x_1, x_2 \text{ integer}
 \end{aligned}$$



	x1	x2	x3	x4	-z	b
	1	6	1	0	0	18
	1	0	0	1	0	3
	1	4	0	0	1	0

	x1	x2	x3	x4	-z	b
	0	1	1/6	-1/6	0	15/6
	1	0	0	1	0	3
	0	0	-2/3	-1/3	1	-13

$x_2 = 5/2, x_1 = 3$
 Optimum, not integer

- We take the first row: $| \quad | \quad 0 \quad | \quad 1 \quad | \quad 1/6 \quad | \quad -1/6 \quad | \quad 0 \quad | \quad 15/6 \quad |$

- CG cut $\sum_{j \in N} f_{uj} x_j \geq f_u \rightsquigarrow \frac{1}{6} x_3 + \frac{5}{6} x_4 \geq \frac{1}{2}$

- Let's verify that it is a CG cut:

$$\begin{array}{r} 1/6 (x_1 + 6x_2 \leq 18) \\ 5/6 (x_1 \leq 3) \\ \hline x_1 + x_2 \leq 3 + 5/2 = 5.5 \end{array}$$

since x_1, x_2 are integer $x_1 + x_2 \leq 5$. And it leaves out x^* .

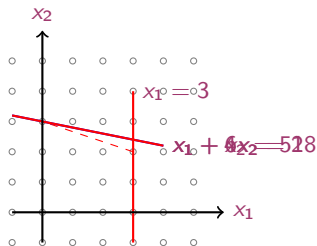
- Let's see how it looks in the space of the original variables: from the first tableau:

$$x_3 = 18 - 6x_2 - x_1$$

$$x_4 = 3 - x_1$$

$$\frac{1}{6}(18 - 6x_2 - x_1) + \frac{5}{6}(3 - x_1) \geq \frac{1}{2} \quad \rightsquigarrow \quad x_1 + x_2 \leq 5$$

- Graphically:



- Let's continue:

	x_1	x_2	x_3	x_4	x_5	$-z$	b
---	+	+	+	+	+	+	+
	0	0	-1/6	-5/6	1	0	-1/2
	0	1	1/6	-1/6	0	0	5/2
	1	0	0	1	0	0	3
---	+	+	+	+	+	+	+
	0	0	-2/3	-1/3	0	1	-13

We need to apply dual-simplex
(will always be the case, why?)

ratio rule: $\min\{|\frac{c_j}{a_{ij}}| : a_{ij} < 0\}$

- After the dual simplex iteration:

	x1	x2	x3	x4	x5	-z	b
	0	0	1/5	1	-6/5	0	3/5
	0	1	1/5	0	-1/5	0	13/5
	1	0	-1/5	0	6/5	0	12/5
	0	0	-3/5	0	-2/5	1	-64/5

- In the space of the original variables:

$$4(18 - x_1 - 6x_2) + (5 - x_1 - x_2) \geq 2$$

$$x_1 + 5x_2 \leq 15$$

• ...

We can choose any of the three rows.

Let's take the third: CG cut:

$$\frac{4}{5}x_3 + \frac{1}{5}x_5 \geq \frac{2}{5}$$

