

DM545/DM871
Linear and Integer Programming

Lecture 5
Duality

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

1. Derivation and Motivation

2. Theory

Outline

1. Derivation and Motivation

2. Theory

Dual variables y in one-to-one correspondence with the constraints:

Primal problem:

$$\begin{aligned} \max \quad & z = c^T x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual Problem:

$$\begin{aligned} \min \quad & w = b^T y \\ & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

Bounding approach

$$\begin{aligned} z^* = \max \quad & 4x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 + 4x_2 \leq 1 \\ & 3x_1 + x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

a feasible solution is a **lower bound** but how good?

By tentatives:

$$(x_1, x_2, x_3) = (1, 0, 0) \rightsquigarrow z^* \geq 4$$

$$(x_1, x_2, x_3) = (0, 0, 3) \rightsquigarrow z^* \geq 9$$

What about **upper bounds**?

$$\begin{array}{rcl} 2 \cdot (x_1 + 4x_2) & \leq & 2 \cdot 1 \\ + 3 \cdot (3x_1 + x_2 + x_3) & \leq & 3 \cdot 3 \\ \hline 4x_1 + x_2 + 3x_3 & \leq & 11x_1 + 11x_2 + 3x_3 \leq 11 \end{array}$$
$$c^T x \leq y^T A x \leq y^T b$$

Hence $z^* \leq 11$. Is this the best upper bound we can find?

multipliers $y_1, y_2 \geq 0$ that preserve sign of inequality

$$\begin{array}{rcl} y_1 \cdot (x_1 + 4x_2) & \leq & y_1(1) \\ y_2 \cdot (3x_1 + x_2 + x_3) & \leq & y_2(3) \\ \hline (y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2x_3 & \leq & y_1 + 3y_2 \end{array}$$

Coefficients

$$\begin{array}{rcl} y_1 + 3y_2 & \geq & 4 \\ 4y_1 + y_2 & \geq & 1 \\ y_2 & \geq & 3 \end{array}$$

$z = 4x_1 + x_2 + 3x_3 \leq (y_1 + 3y_2)x_1 + (4y_1 + y_2)x_2 + y_2x_3 \leq y_1 + 3y_2$ then to attain the best upper bound:

$$\begin{array}{rcl} \min & y_1 + 3y_2 \\ y_1 + 3y_2 & \geq & 4 \\ 4y_1 + y_2 & \geq & 1 \\ y_2 & \geq & 3 \\ y_1, y_2 & \geq & 0 \end{array}$$

Multipliers Approach

$$\begin{array}{l} \pi_1 \\ \vdots \\ \pi_m \\ \pi_{m+1} \end{array} \left[\begin{array}{cccc|cccc|c|c} a_{11} & a_{12} & \dots & a_{1n} & a_{1,n+1} & a_{1,n+2} & \dots & a_{1,m+n} & 0 & b_1 \\ \vdots & \vdots & & \ddots & & & & & & \\ \pi_m & a_{m1} & a_{m2} & \dots & a_{mn} & a_{m,n+1} & a_{m,n+2} & \dots & a_{m,m+n} & 0 & b_m \\ \hline \pi_{m+1} & c_1 & c_2 & \dots & c_n & 0 & 0 & \dots & 0 & 1 & 0 \end{array} \right]$$

Working columnwise, since at optimum $\bar{c}_k \leq 0$ for all $k = 1, \dots, n + m$:

$$\left\{ \begin{array}{l} \pi_1 a_{11} + \pi_2 a_{21} + \dots + \pi_m a_{m1} + \pi_{m+1} c_1 \leq 0 \\ \vdots \\ \pi_1 a_{1n} + \pi_2 a_{2n} + \dots + \pi_m a_{mn} + \pi_{m+1} c_n \leq 0 \\ \pi_1 a_{1,n+1}, \quad \pi_2 a_{2,n+1}, \dots, \pi_m a_{m,n+1} \leq 0 \\ \vdots \\ \pi_1 a_{1,n+m}, \quad \pi_2 a_{2,n+m}, \dots, \pi_m a_{m,n+m} \leq 0 \\ \pi_{m+1} = 1 \\ \pi_1 b_1 + \pi_2 b_2 + \dots + \pi_m b_m (\leq 0) \end{array} \right.$$

(from the last row, $\max \pi b$)

$$\begin{array}{rcl} \max & \pi_1 b_1 + \pi_2 b_2 \dots + \pi_m b_m & \\ & \pi_1 a_{11} + \pi_2 a_{21} \dots + \pi_m a_{m1} \leq -c_1 & \\ & \vdots & \vdots \\ & \pi_1 a_{1n} + \pi_2 a_{2n} \dots + \pi_m a_{mn} \leq -c_n & \\ & \pi_1, \pi_2, \dots, \pi_m \leq 0 & \end{array}$$

$$y = -\pi$$

$$\begin{array}{rcl} \max & -y_1 b_1 + -y_2 b_2 \dots + -y_m b_m & \\ & -y_1 a_{11} + -y_2 a_{21} \dots + -y_m a_{m1} \leq -c_1 & \\ & \vdots \quad \ddots & \vdots \\ & -y_1 a_{1n} + -y_2 a_{2n} \dots + -y_m a_{mn} \leq -c_n & \\ & -y_1, -y_2, \dots - y_m \leq 0 & \end{array}$$

$$\begin{array}{ll}\min & w = b^T y \\ & A^T y \geq c \\ & y \geq 0\end{array}$$

Example

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ \text{s.t.} \quad & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{cases} 5\pi_1 + 4\pi_2 + 6\pi_3 \leq 0 \\ 10\pi_1 + 4\pi_2 + 8\pi_3 \leq 0 \\ 1\pi_1 + 0\pi_2 + 0\pi_3 \leq 0 \\ 0\pi_1 + 1\pi_2 + 0\pi_3 \leq 0 \\ 0\pi_1 + 0\pi_2 + 1\pi_3 = 1 \\ 60\pi_1 + 40\pi_2 \end{cases}$$

$$y_1 = -\pi_1 \geq 0$$

$$y_2 = -\pi_2 \geq 0$$

Duality Recipe

	Primal linear program	Dual linear program
Variables	x_1, x_2, \dots, x_n	y_1, y_2, \dots, y_m
Matrix	A	A^T
Right-hand side	\mathbf{b}	\mathbf{c}
Objective function	$\max \mathbf{c}^T \mathbf{x}$	$\min \mathbf{b}^T \mathbf{y}$
Constraints	i th constraint has \leq \geq $=$	$y_i \geq 0$ $y_i \leq 0$ $y_i \in \mathbb{R}$
	$x_j \geq 0$ $x_j \leq 0$ $x_j \in \mathbb{R}$	j th constraint has \geq \leq $=$

1. Derivation and Motivation

2. Theory

The dual of the dual is the primal:

Primal problem:

$$\begin{aligned}\max \quad & z = c^T x \\ & Ax \leq b \\ & x \geq 0\end{aligned}$$

Let's put the dual in the standard form

Dual problem:

$$\begin{aligned}\min \quad & b^T y \equiv -\max -b^T y \\ & -A^T y \leq -c \\ & y \geq 0\end{aligned}$$

Dual Problem:

$$\begin{aligned}\min \quad & w = b^T y \\ & A^T y \geq c \\ & y \geq 0\end{aligned}$$

Dual of Dual:

$$\begin{aligned}-\min \quad & -c^T x \\ & -Ax \geq -b \\ & x \geq 0\end{aligned}$$

Weak Duality Theorem

As we saw the dual produces upper bounds. This is true in general:

Theorem (Weak Duality Theorem)

Given:

$$(P) \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

$$(D) \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

for any feasible solution x of (P) and any feasible solution y of (D):

$$c^T x \leq b^T y$$

Proof:

From (D) $c_j \leq \sum_{i=1}^m y_i a_{ij} \forall j$ and from (P) $\sum_{j=1}^n a_{ij} x_j \leq b_i \forall i$

From (D) $y_i \geq 0$ and from (P) $x_j \geq 0$

$$\sum_{j=1}^n c_j x_j \leq \sum_{j=1}^n \left(\sum_{i=1}^m y_i a_{ij} \right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) y_i \leq \sum_{i=1}^m b_i y_i$$

Strong Duality Theorem

Due to Von Neumann and Dantzig 1947 and Gale, Kuhn and Tucker 1951.

Theorem (Strong Duality Theorem)

Given:

$$(P) \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

$$(D) \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

exactly one of the following occurs:

1. *(P) and (D) are both infeasible*
2. *(P) is unbounded and (D) is infeasible*
3. *(P) is infeasible and (D) is unbounded*
4. *(P) has feasible solution, then let an optimal be: $x^* = [x_1^*, \dots, x_n^*]$
(D) has feasible solution, then let an optimal be: $y^* = [y_1^*, \dots, y_m^*]$, then:*

$$c^T x^* = b^T y^*$$

Proof:

- all other combinations of 3 possibilities (Optimal, Infeasible, Unbounded) for (P) and 3 for (D) are ruled out by weak duality theorem.
- we use the simplex method. (Other proofs independent of the simplex method exist, eg, Farkas Lemma and convex polyhedral analysis)
- The last row of the final tableau will give us

$$\begin{aligned}
 z &= z^* + \sum_{k=1}^{n+m} \bar{c}_k x_k = z^* + \sum_{j=1}^n \bar{c}_j x_j + \sum_{i=1}^m \bar{c}_{n+i} x_{n+i} \\
 &= z^* + \bar{c}_B x_B + \bar{c}_N x_N
 \end{aligned} \tag{*}$$

In addition, $z^* = \sum_{j=1}^n c_j x_j^*$ (c_j , original values) because optimal value

- We define $y_i^* = -\bar{c}_{n+i}$, $i = 1, 2, \dots, m$
- We claim that $(y_1^*, y_2^*, \dots, y_m^*)$ is a dual feasible solution satisfying $c^T x^* = b^T y^*$.

- Let's verify the claim:

We substitute in (*): i) $z = \sum_{j=1}^n c_j x_j$; ii) $\bar{c}_{n+i} = -y_i^*$; and iii) $x_{n+i} = b_i - \sum_{j=1}^n a_{ij} x_j$ for $i = 1, 2, \dots, m$ ($n+i$ are the slack variables)

$$\begin{aligned} \sum_{j=1}^n c_j x_j &= z^* + \sum_{j=1}^n \bar{c}_j x_j - \sum_{i=1}^m y_i^* \left(b_i - \sum_{j=1}^n a_{ij} x_j \right) \\ &= \left(z^* - \sum_{i=1}^m y_i^* b_i \right) + \sum_{j=1}^n \left(\bar{c}_j + \sum_{i=1}^m a_{ij} y_i^* \right) x_j \end{aligned}$$

This must hold for every (x_1, x_2, \dots, x_n) hence:

$$\begin{aligned} z^* &= \sum_{i=1}^m b_i y_i^* & \implies y^* \text{ satisfies } c^T x^* = b^T y^* \\ c_j &= \bar{c}_j + \sum_{i=1}^m a_{ij} y_i^*, j = 1, 2, \dots, n \end{aligned}$$

Since $\bar{c}_k \leq 0$ for every $k = 1, 2, \dots, n + m$:

$$\bar{c}_j \leq 0 \rightsquigarrow c_j - \sum_{i=1}^m y_i^* a_{ij} \leq 0 \rightsquigarrow$$

$$\sum_{i=1}^m y_i^* a_{ij} \geq c_j \quad j = 1, 2, \dots, n$$

$$\bar{c}_{n+i} \leq 0 \rightsquigarrow y_i^* = -\bar{c}_{n+i} \geq 0,$$

$$i = 1, 2, \dots, m$$

$\Rightarrow y^*$ is also dual feasible solution

Complementary Slackness Theorem

Theorem (Complementary Slackness)

A feasible solution x^* for (P)

A feasible solution y^* for (D)

Necessary and sufficient conditions for optimality of both:

$$\left(c_j - \sum_{i=1}^m y_i^* a_{ij} \right) x_j^* = 0, \quad j = 1, \dots, n$$

If $x_j^* \neq 0$ then $\sum y_i^* a_{ij} = c_j$ (no surplus)

If $\sum y_i^* a_{ij} > c_j$ then $x_j^* = 0$

Proof:

$$z^* = c^T x^* \leq y^* A x^* \leq b^T y^* = w^*$$

Hence from strong duality theorem:

$$c x^* - y^* A x^* = 0$$

In scalars

$$\sum_{j=1}^n \underbrace{\left(c_j - \sum_{i=1}^m y_i^* a_{ij} \right)}_{\leq 0} \underbrace{x_j^*}_{\geq 0} = 0$$

Hence each term must be $= 0$

Proof in scalar form:

$$c_j x_j^* \leq \left(\sum_{i=1}^m a_{ij} y_i^* \right) x_j^* \quad j = 1, 2, \dots, n \quad \text{from feasibility in D}$$

$$\left(\sum_{j=1}^n a_{ij} x_j^* \right) y_i^* \leq b_i y_i^* \quad i = 1, 2, \dots, m \quad \text{from feasibility in P}$$

Summing in j and in i :

$$\sum_{j=1}^n c_j x_j^* \leq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i^* \right) x_j^* = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j^* \right) y_i^* \leq \sum_{i=1}^m b_i y_i^*$$

For the strong duality theorem the left hand side is equal to the right hand side and hence all inequalities become equalities.

$$\sum_{j=1}^n \underbrace{\left(c_j - \sum_{i=1}^m y_i^* a_{ij} \right)}_{\leq 0} \underbrace{x_j^*}_{\geq 0} = 0$$

- Derivation:
 - Economic interpretation
 - Bounding Approach
 - Multiplier Approach
 - Recipe
 - Lagrangian Multipliers Approach (next time)
- Theory:
 - Symmetry
 - Weak Duality Theorem
 - Strong Duality Theorem
 - Complementary Slackness Theorem