

## DM545/DM871 – Linear and integer programming

### Sheet 1, Spring 2023 [pdf format]

Exercises with the symbol  $^+$  are to be done at home before the class. Exercises with the symbol  $*$  will be tackled in class and should be at least read at home. The remaining exercises are left for self training after the exercise class.

## Exercises on Modeling System of Linear Equations

### Exercise 1 $^+$

A total of \$6,300 was invested in two accounts. Part was invested in obligations at 0.045 annual interest rate and part was invested in a money market fund at 0.0375 annual interest rate. If the total simple interest for one year was \$267.75, then how much was invested in each account?

### Exercise 2 $^+$

A construction company produces five different products:  $p_1, p_2, p_3, p_4, p_5$ , using five resources: metal, concrete, plastic, water, electricity.

The amount of resources consumed for producing one unit of products is described by the following python dictionaries:

```
p_1={"metal":0, "concrete":1.3, "plastic":0.2, "water":.8, "electricity":.4}
p_2={"metal":0, "concrete":0, "plastic":1.5, "water":.4, "electricity":.3}
p_3={"metal":.25, "concrete":0, "plastic":0, "water":.2, "electricity":.7}
p_4={"metal":0, "concrete":0, "plastic":.3, "water":.7, "electricity":.5}
p_5={"metal":1.5, "concrete":0, "plastic":.5, "water":.4, "electricity":.8}
```

How much metal is consumed if the company decides to produce the following quantities for each product: 10, 9, 12, 13, 11?

Express the calculation asked by the exercise as a linear algebra operation involving matrices and vectors. Then calculate the numerical answer using Python and numpy. (For example, you can transform the dictionary above into a numpy array as follows: `a=np.array(list(p_1.values()))`.)

### Exercise 3 $^+$

Consider again our construction company that produces five different products:  $p_1, p_2, p_3, p_4, p_5$ , using five resources: metal, concrete, plastic, water, electricity.

The amount of resources consumed for producing one unit of products is described by the python dictionaries defined in the previous exercise.

Suppose now that you do not know how much of each product is produced but that you know how much of each material is used. Could you determine from this information the amount produced of each product? Will a solution exist? Will it be unique?

## Exercises on Modeling (Integer) Linear Programming Problems

The exercises that follow ask you to model the problems in linear programming terms. Recall that you have to identify and denote mathematically the:

- i) parameters
- ii) variables (in some cases they might be real and in others integer)

and express as a linear combination of those terms the

iii) objective function

iv) constraints.

The description of the model has to be organized in the following parts:

1. **Notation.** Introduction of the mathematical notation: which symbols denote parameters and variables? which indices are you using and where are they running?

2. **Model.** Mathematical model:

$$\begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ \text{Ax} \leq & \mathbf{b} \\ \mathbf{x} \geq & \mathbf{0} \end{aligned}$$

(Most likely you will express the model in scalar notation and summations. Remember not to leave any index unspecified. Watch out for the quantifiers.)

3. **Explanation.** Explanation of each line of the mathematical model.

Remember to check that every symbol and index in your model is defined.

See Sec. 1.3.2 from the Lecture Notes [LN] for an example of the expected modeling process and the resulting description.

Note that at the tests your answers must be digitalized, hence it is good to start becoming acquainted with different tools to produce text documents containing mathematical notation and graphs. Read the "Instructions for Written Exam" in the Assessment section of the external web page for a list of useful tools for doing this.

Beside these exercises, you could read Chapter 2 and 3 from [F] and Chapter 2 "Examples" from the book [MG]. You find this book downloadable from SDU library system: <http://syddansk.summon.serialssolutions.com/#!/>. Further, the book [Wi] is entirely dedicated to modelling in mathematical programming.

### Exercise 4\* Brewery Problem

A small brewery produces two types of beer: dark and ale, each requiring different combinations of three limited resources: barley, hops, and yeast.

The recipes for dark and ale require different proportions of resources:

Beverage	Barley (pounds)	Hops (ounces)	Yeast (ounces)	Profit (Dkk)
Dark	15	4	20	23
Ale	5	4	35	13
Quantity	480	160	1190	

The task of the small brewery is decide the quantity of each type of beer to produce in order to maximize profit from the sale of the two beers, where the profit for each type is given in the table above.

- Try to guess the optimal answer to the problem by calculating the total profit of some tentative feasible decisions. For example by producing only one type of beer.
- Model the problem as a linear programming problem and provide the graphical representation of it.
- Later the company decides to introduce a third type of beer: light with the following recipe:

Beverage	Barley (pounds)	Hops (ounces)	Yeast (ounces)	Profit (Dkk)
Light	10	2	15	15

Modify the linear programming model from the previous point in order to consider also this new type of beer.

### Exercise 5\* Food manufacture

A chocolate factory produces two types of chocolate bars: Milk & Hazelnuts (M&H) chocolate and Dark (D) chocolate. The price for a pack of M&H and a pack of D is 100 and 160 Dkk, respectively. Each pack of M&H is made of 50 grams of hazelnuts, 60 grams of chocolate, 40 grams of milk and 50 grams of sugar. Each Dark bar contains 150 grams of chocolate, 50 grams of fat and 30 grams of sugar.

The factory has 200 grams of hazelnuts, 1000 grams of chocolate, 250 grams of milk, 300 grams of sugar, and 300 grams of fat left.

Your goal is to determine how many M&H bars and D bars the company should produce to maximize its profit.

- Give the linear program for the problem, using variables  $x_1$  and  $x_2$  and the parameters defined above. Specify (i) the constraints, and (ii) the objective function.
- Graph the feasible region of your linear program. (Tip: look at the “Tools” section in the external web page of the course and find an application that can help you in this task. Those using MacOSx have also the option of the program Grapher).
- Compute the profit at each vertex of the feasible region and report the best solution.

### Exercise 6\* Market Survey

An advertising company has to carry out a market survey in order to launch a new product. The survey has to be carried out by telephone, contacting a significant sample of people composed as shown in Table 1:

type of people	married women	unmarried women	married men	unmarried men
number	$\geq 150$	$\geq 110$	$\geq 120$	$\geq 100$

Table 1:

Calls can be made in the morning (with an operating cost for the company of 1 Euro/call) and in the evening (with an operating cost of 1.6 Euro/call). The average percentage of people reached is as shown in Table 2.

Responder	Morning	Evening
married women	30%	30%
unmarried women	10%	20%
married men	10%	30%
unmarried men	10%	15%
nobody	40%	5%

Table 2:

As it can be seen, evening calls are more expensive, but allow the company to reach a greater number of people: only 5% of the calls is “in vain”. The aim is to minimize the total cost of the calls to be made (morning/evening) so as to reach a significant sample of people. (Problem taken from ref. [F].)

### Exercise 7\* Optimal Blending

The Metalco Company wants to blend a new alloy (metal) made by 40 percent tin, 35 percent zinc, and 25 percent lead from 5 available alloys having the following properties:

Property	Alloy				
	1	2	3	4	5
Percentage of tin	60	25	45	20	50
Percentage of zinc	10	15	45	50	40
Percentage of lead	30	60	10	30	10
Cost (DKK/Kg)	77	70	88	84	94

The objective is to determine the proportions of these alloys that should be blended to produce the new alloy at a minimum cost. Formulate a linear programming model for this problem. [Problem from ref. HL]

To help you in the task we start introducing the mathematical notation that will be used for the model. Let  $J = \{1, 2, \dots, 5\}$  indexed by  $j$  be the set of alloys and  $I = \{\text{tin, zinc, lead}\}$  indexed by  $i$  be the set of metals. Let  $a_{ij}$  be the fixed parameters that determine the percentage amount of metal  $i$  in alloy  $j$ . Let  $c_j$  be the cost of alloy  $j$  in Dkk per Kg. The problem asks to determine the proportions of the alloys to blend to obtain the new alloy with the properties of 40% tin, 35% zinc and 25% lead. Let's call these last parameters  $b_i, i \in I$  and the proportion of each alloy to blend with respect to the new alloy by  $y_j \geq 0$ . Specify the constraints and the objective function using the mathematical terms introduced.

[Units of measure offer a way to test the correctness of the model. In particular, the proportion  $y_j$  can be seen as the amount of alloy in Kg per amount of new alloy. Let  $\alpha$  be the amount in Kg of the new alloy, and let  $x_j$  be the amount in Kg of each alloy from  $I$ . Then  $y_j = x_j/\alpha$  and it is an adimensional quantity.]

## Exercise 8

A cargo plane has three compartments for storing cargo: front, center and back. These compartments have capacity limits on both *weight* and *space*, as summarized below:

Compartment	Weight Capacity (Tons)	Space Capacity (Cubic meters)
Front	12	7000
Center	18	9000
Rear	10	5000

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of the weight capacity of that compartment to maintain the balance of the airplane.

The following four cargos have been offered for shipment on an upcoming flight as space is available:

Cargo	Weight(Tons)	Volume (Cubic meters/Tons)	Profit
1	20	500	320
2	16	700	400
3	25	600	360
4	13	400	290

Any portion of these cargoes can be accepted. The objective is to determine how much (if any) of each cargo should be accepted and how to distribute each among the compartments to maximize the total profit for the flight. Formulate a linear programming model for this problem. [Problem from ref. HL]

## Exercise 9

A small airline flies between three cities: Copenhagen, Aarhus, and Odense. They offer several flights but, for this problem, let us focus on the Friday afternoon flight that departs from Copenhagen, stops in Odense, and continues to Aarhus. There are three types of passengers:

- (a) Those traveling from Copenhagen to Odense.
- (b) Those traveling from Odense to Aarhus.
- (c) Those traveling from Copenhagen to Aarhus.

The aircraft is a small commuter plane that seats 30 passengers. The airline offers three fare classes:

- (a) Y class: full coach.
- (b) B class: nonrefundable.
- (c) M class: nonrefundable, 3-week advanced purchase.

Ticket prices, which are largely determined by external influences (i.e., train and bus competitors), have been set and advertised as follows:

Based on past experience, demand forecasters at the airline have determined the following upper bounds on the number of potential customers in each of the 9 possible origin-destination/fare-class combinations:

	Copenhagen–Odense	Odense–Aarhus	Copenhagen–Aarhus
Y	300	160	360
B	220	130	280
M	100	80	140

  

	Copenhagen–Odense	Odense–Aarhus	Copenhagen–Aarhus
Y	4	8	3
B	8	13	10
M	22	20	18

The goal is to decide how many tickets from each of the 9 origin/destination/ fare-class combinations to sell. The constraints are that the plane cannot be overbooked on either of the two legs of the flight and that the number of tickets made available cannot exceed the forecast maximum demand. The objective is to maximize the revenue.

Formulate this problem as a linear programming problem.

### Exercise 10

In this exercise we study the application of linear programming to an area of statistics, namely, regression.

Consider a set of  $m = 9$  measurements: 28, 62, 80, 84, 86, 86, 92, 95, 98. A way to summarize these data is by their mean

$$\bar{x} = \frac{1}{m} \sum_{i=1}^m x_i$$

An alternative way is by the median, ie, the measurement that is worse than half of the other scores and better than the other half.

There is a close connection between these statistics and optimization. Show that the mean is the measure that minimizes the sum of squared deviation between the data points and itself and that the median minimizes the sum of the absolute values of the differences between each data point and itself.

Consider now a set of points on a two-dimensional space  $S = (x_1, y_1), \dots, (x_m, y_m)$ . The points are measurements of a response variable given some control variable, for example, blood pressure given the weight of a person. The points hint at a linear dependency between the variables representing the two dimensions. We may assume a random fluctuation around the right value and hence the following regression model:

$$y = ax + b + \epsilon$$

Specifically, for our set of points  $S$  we have

$$y_i = ax_i + b + \epsilon_i, \quad i = 1, \dots, m$$

We thus want to find a line that best fits the measured points. That is, we wish to determine the (unknown) numbers  $a$  and  $b$ . There is no unique criterion to formulate the desire that a given line “best fits” the points. The task can be achieved by minimizing, in some sense, the vector  $\epsilon$ . As for the mean and median, we can consider minimizing either the sum of the squares of the  $\epsilon_i$ 's or the sum of the absolute values of the  $\epsilon_i$ 's. These concepts are formalized in measure theory by the so called  $L^p$ -norm ( $1 \leq p < \infty$ ). For the vector  $\epsilon$ :

$$\epsilon_p = \left( \sum_i \epsilon_i^p \right)^{1/p}$$

The method of least squares, which is perhaps the most popular, corresponds to  $L^2$ -norm. The minimization of  $L^2$ -norm for the vector  $\epsilon$  in the variables  $a$  and  $b$  has a closed form solution that you may have encountered in the statistics courses. This method needs not always to be the most suitable, however. For instance, if a few exceptional points are measured with very large error, they can influence the resulting line a great deal. Just as the median gives a more robust estimate of a collection of numbers than the means, the  $L^1$  norm is less sensitive to outliers than least square regression is. The problem is to solve the following minimization problem:

$$\operatorname{argmin}_{a,b} \sum_i |\epsilon_i| = \operatorname{argmin}_{a,b} \sum_i |ax_i + b - y_i|$$

Unlike for least square regression, there is no explicit formula for the solution of the  $L^1$ -regression problem. However the problem can be formulated as a linear programming problem. Show how this can be done.

The regression via  $L^\infty$ -norm corresponds to solving the problem:

$$\operatorname{argmin} \max_{i=1}^n |ax_i + b - y_i|$$

This problem can also be solved by linear programming. Show how to formulate the problem as a linear program.

## Exercise 11 Dynamic Input-Output Model

This problem is the dynamic version of [Leontief's Input-Output model](#). A treatment of the model from the point of view of LP is available in Sec. 5.2 of [Wi] (the book is available as ebook from SDU).

An economy consists of three industries: coal, steel and transport. Each unit produced by one of the industries (a unit will be taken as DKK 1's worth of value of production) requires inputs from possibly its own industry as well as other industries. The required inputs and the manpower requirements (also measured in DKK) are given in the Table 3. There is a time lag in the economy so that output in year  $t + 1$  requires an input in year  $t$ .

Inputs (year $t$ )	Outputs (year $t + 1$ ), production		
	Coal	Steel	Transport
Coal	0.1	0.5	0.4
Steel	0.1	0.1	0.2
Transport	0.2	0.1	0.2
Manpower	0.6	0.3	0.2

Table 3:

Output from an industry may also be used to build productive capacity for itself or other industries in future years. The inputs required to give unit increases (capacity for DKK 1's worth of extra production) in productive capacity are given in Table 4. Input from an industry in year  $t$  results in a (permanent) increase in productive capacity in year  $t + 2$ .

Inputs (year $t$ )	Outputs (year $t + 2$ ), productive capacity		
	Coal	Steel	Transport
Coal	0.0	0.7	0.9
Steel	0.1	0.1	0.2
Transport	0.2	0.1	0.2
Manpower	0.4	0.2	0.1

Table 4:

Stocks of goods may be held from year to year. At present (year 0) the stocks and productive capacities (per year) are given in Table 5 (in DKKm). There is a limited yearly manpower capacity of DKK 470m.

	Year 0	
	Stocks	Productive capacity
Coal	150	300
Steel	80	350
Transport	100	280

Table 5:

It is wished to investigate different possible growth patterns for the economy over the next five years. In particular it is desirable to know the growth patterns which would result from pursuing the following objectives:

- Maximizing total productive capacity at the end of the five years while meeting an exogenous consumption requirement of DKK 60m of coal DKK 60m of steel and DKK 30m of transport in every year (apart from year 0).

- (ii) Maximizing total production (rather than productive capacity) in the fourth and fifth years, but ignoring exogenous demand in each year.
- (iii) Maximizing the total manpower requirement (ignoring the manpower capacity limitation) over the period while meeting the yearly exogenous demands of (i).