DM545/DM871

Linear and Integer Programming

Lecture 7

Revised Simplex Method

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Outline

Motivation

Complexity of single pivot operation in standard simplex:

- entering variable $O(n)$
- leaving variable $O(m)$
- updating the tableau $O(mn)$

Problems with this:

- Time: we are doing operations that are not actually needed
  Space: we need to store the whole tableau: $O(mn)$ floating point numbers
- Most problems have sparse matrices (many zeros)
  sparse matrices are typically handled efficiently
  the standard simplex has the 'Fill in' effect: sparse matrices are lost
- accumulation of Floating Point Errors over the iterations

Outline

1. Revised Simplex Method

2. Efficiency Issues

Revised Simplex Method

Several ways to improve wrt pitfalls in the previous slide, requires matrix description of the simplex.

$$\begin{aligned} \max \ & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1..m \\ & x_j \geq \ 0 \quad j = 1..n \end{aligned}$$

$$\begin{aligned} \max \ & \mathrm{c}^T \mathrm{x} \\ & A\mathrm{x} = \mathrm{b} \\ & \mathrm{x} \geq 0 \\ & A \in \Re^{m \times (n+m)} \\ & \mathrm{c} \in \Re^{(n+m)}, \mathrm{b} \in \Re^{m}, \mathrm{x} \in \Re^{n+m} \end{aligned}$$

$$\max\{\mathrm{c}^T \mathrm{x} \mid A\mathrm{x} = \mathrm{b}, \mathrm{x} \geq 0\}$$

At each iteration the simplex moves from a basic feasible solution to another.

For each basic feasible solution:

- $B = \{1 \dots m\}$ basis
- $N = \{m + 1 \dots m + n\}$
- $A_B = [\mathrm{a}_1 \dots \mathrm{a}_m]$ basis matrix
- $A_N = [\mathrm{a}_{m+1} \dots \mathrm{a}_{m+n}]$

- $\mathrm{x}_N = 0$
- $\mathrm{x}_B \geq 0$

$$\left[\begin{array}{c:c:c:c} A_N & A_B & 0 & b \\ \hdashline c_N^T & c_B^T & 1 & 0 \end{array}\right]$$

$$\begin{aligned} A\mathrm{x} &= A_N \mathrm{x}_N + A_B \mathrm{x}_B = \mathrm{b} \\ A_B \mathrm{x}_B &= \mathrm{b} - A_N \mathrm{x}_N \end{aligned}$$

Basic feasible solution $\iff$ A_B is non-singular

$$\mathrm{x}_B = A_B^{-1}\mathrm{b} - A_B^{-1} A_N \mathrm{x}_N$$

for the objective function:

$$z = c^T x = c_B^T x_B + c_N^T x_N$$

Substituting for x_B from above:

$$\begin{aligned} z &= c_B^T (A_B^{-1} b - A_B^{-1} A_N x_N) + c_N^T x_N = \\ &= c_B^T A_B^{-1} b + (c_N^T - c_B^T A_B^{-1} A_N) x_N \end{aligned}$$

Collecting together:

$$\begin{aligned} x_B &= A_B^{-1} b - A_B^{-1} A_N x_N \\ z &= c_B^T A_B^{-1} b + (c_N^T - c_B^T \underbrace{A_B^{-1} A_N}_{\bar{A}}) x_N \end{aligned}$$

In tableau form, for a basic feasible solution corresponding to B we have:

$$\left[\begin{array}{c:c:c:c} A_B^{-1} A_N & I & 0 & A_B^{-1} b \\ \hdashline c_N^T - c_B^T A_B^{-1} A_N & 0 & 1 & -c_B^T A_B^{-1} b \end{array}\right]$$

We do not need to compute all elements of $\bar{A}$

Example

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & -x_1 + x_2 \le 1 \\ & x_1 \le 3 \\ & x_2 \le 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & -x_1 + x_2 + x_3 = 1 \\ & x_1 + x_4 = 3 \\ & x_2 + x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

Initial tableau

$x1$	$x2$	$x3$	$x4$	$x5$	$-z$	b
−1	1	1	0	0	0	1
1	0	0	1	0	0	3
0	1	0	0	1	0	2
1	1	0	0	0	1	0

After two iterations

$x1$	$x2$	$x3$	$x4$	$x5$	$-z$	b
1	0	−1	0	1	0	1
0	1	0	0	1	0	2
0	0	1	1	−1	0	2
0	0	1	0	−2	1	3

Basic variables x_1, x_2, x_4. Non basic: x_3, x_5. From the initial tableau:

$$A_B = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad A_N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} \qquad x_N = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix}$$

$$c_B^T = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \qquad c_N^T = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

- **Entering variable**:
  in std. we look at tableau, in revised we need to compute: $c_N^T - c_B^T A_B^{-1} A_N$
  1. find $y^T = c_B^T A_B^{-1}$ (by solving $y^T A_B = c_B^T$, the latter can be done more efficiently)
  2. calculate $c_N^T - y^T A_N$

Step 1:

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \qquad \mathrm{y}^T A_B = \mathrm{c}_B^T$$

$$\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix} \qquad \mathrm{c}_B^T A_B^{-1} = \mathrm{y}^T$$

Step 2:

$$\begin{bmatrix} 0 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \end{bmatrix} \qquad \mathrm{c}_N^T - \mathrm{y}^T A_N$$

(Note that they can be computed individually: $c_j - \mathrm{y}^T \mathrm{a}_j > 0$)
Let's take the first we encounter x_3

- **Leaving variable**
  we increase variable by largest feasible amount θ

$$\begin{aligned} \text{R1: } & x_1 - x_3 + x_5 = 1 \\ \text{R2: } & x_2 + 0x_3 + x_5 = 2 \\ \text{R3: } & -x_3 + x_4 - x_5 = 2 \end{aligned}$$

$$\begin{aligned} x_1 &= 1 + x_3 \geq 0 \\ x_2 &= 2 \geq 0 \\ x_4 &= 2 - x_3 \geq 0 \end{aligned}$$

$$\mathrm{x}_B = \mathrm{x}_B^* - A_B^{-1} A_N \mathrm{x}_N$$
$$\mathrm{x}_B = \mathrm{x}_B^* - \mathrm{d}\theta$$

d is the column of $A_B^{-1} A_N$ that corresponds to the entering variable, ie, $\mathrm{d} = A_B^{-1}\mathrm{a}$ where a is the entering column

3. Find θ such that x_B stays positive:
   Find $\mathrm{d} = A_B^{-1}\mathrm{a}$ (by solving $A_B \mathrm{d} = \mathrm{a}$)

Step 3:

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \implies \mathrm{d} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \implies \mathrm{x}_B = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \theta \geq 0$$

$2 - \theta \geq 0 \implies \theta \leq 2 \rightsquigarrow x_4$ leaves

- So far we have done computations, but now we save the pivoting update. The update of A_B is done by replacing the leaving column by the entering column

$$x_B^* = \begin{bmatrix} x_1 - d_1\theta \\ x_2 - d_2\theta \\ \theta \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \qquad A_B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- Many implementations depending on how $\mathrm{y}^T A_B = \mathrm{c}_B^T$ and $A_B\mathrm{d} = \mathrm{a}$ are solved. They are in fact solved from scratch.
- many operations saved especially if many variables!
- special ways to call the matrix A from memory
- better control over numerical issues since A_B^{-1} can be recomputed.

Outline

1. Revised Simplex Method

2. Efficiency Issues

Solving the two Systems of Equations

$A_B x = b$ solved without computing A_B^{-1}
(costly and likely to introduce numerical inaccuracy)

Recall how the inverse is computed:

For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the matrix inverse is

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the matrix inverse is

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}^T$$

Eta Factorization of the Basis

Let $B := A_B$, kth iteration
B_k be the matrix with col p differing from B_{k-1}
Column p is the $\mathbf{d}$ column appearing in $B_{k-1}\mathbf{d} = \mathbf{a}$ solved at 3)
Hence:

$$B_k = B_{k-1}E_k$$

E_k is the eta matrix differing from id. matrix in only one column

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & & -1 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$

No matter how we solve $\mathbf{y}^T B_{k-1} = \mathbf{c}_B^T$ and $B_{k-1}\mathbf{d} = \mathbf{a}$, their update always relays on $B_k = B_{k-1}E_k$ with E_k available.
Plus when initial basis by slack variable $B_0 = I$ and $B_1 = E_1, B_2 = E_1E_2 \cdots$:

$$B_k = E_1E_2 \ldots E_k \qquad \text{eta factorization}$$

$$((((\mathbf{y}^T E_1)E_2)E_3)\cdots)E_k = \mathbf{c}_B^T, \qquad \mathbf{u}^T E_4 = \mathbf{c}_B^T,\ \mathbf{v}^T E_3 = \mathbf{u}^T,\ \mathbf{w}^T E_2 = \mathbf{v}^T,\ \mathbf{y}^T E_1 = \mathbf{w}^T$$

$$(E_1(E_2\cdots E_k\mathbf{d})) = \mathbf{a}, \qquad E_1\mathbf{u} = \mathbf{a},\ E_2\mathbf{v} = \mathbf{u},\ E_3\mathbf{w} = \mathbf{v},\ E_4\mathbf{d} = \mathbf{w}$$

Exercise

Solve the systems $y^T E_1 E_2 E_3 E_4 = [1\ 2\ 3]$ and $E_1 E_2 E_3 E_4 d = [1\ 2\ 3]^T$ with

$$E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0.5 & 0 \\ 0 & 4 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

We use backward transformation and solve the sequence of linear systems:

$$\mathrm{u}^T E_4 = [1\ 2\ 3], \quad \mathrm{v}^T E_3 = \mathrm{u}^T, \quad \mathrm{w}^T E_2 = \mathrm{v}^T, \quad \mathrm{y}^T E_1 = \mathrm{w}^T$$

$$\mathrm{u}^T \begin{bmatrix} -0.5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = [1, 2, 3]$$

Since the eta matrices have always one 1 in two columns then the solution can be read up easily. From the third column we find $u_3 = 3$. From the second column, we find $u_2 = 2$. Substituting in the first column, we find $-0.5u_1 + 3 * 2 + 1 * 3 = 1$, which yields $u_1 = 18$. The next syestem is:

$$\mathrm{v}^T \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = [18, 2, 3]$$

From the first column we get $v_1 = 18$, from the second column $v_2 = 2$ from the last column $v_3 = 3/24$. The next:

$$\mathrm{w}^T \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = [18, 2, 3/24]$$

...

- Solving $y^T B_k = c_B^T$ also called backward transformation (BTRAN)
- Solving $B_k d = a$ also called forward transformation (FTRAN)

- E_i matrices can be stored by only storing the column and the position
- If sparse columns then can be stored in compact mode, ie only nonzero values and their indices

More on LP

- Tableau method is unstable: computational errors may accumulate. Revised method has a natural control mechanism: we can recompute A_B^{-1} at any time
- Commercial and freeware solvers differ from the way the systems $y^T A_B = c_B^T$ and $A_B d = a$ are resolved

Efficient Implementations

- Dual simplex with steepest descent (largest increase)
- Linear Algebra:
  - Dynamic LU-factorization using Markowitz threshold pivoting (Suhl and Suhl, 1990)
  - sparse linear systems: Typically these systems take as input a vector with a very small number of nonzero entries and output a vector with only a few additional nonzeros.
- Presolve, ie problem reductions: removal of redundant constraints, fixed variables, and other extraneous model elements.
- dealing with degeneracy, stalling (long sequences of degenerate pivots), and cycling:
  - bound-shifting (Paula Harris, 1974)
  - Hybrid Pricing (variable selection): start with partial pricing, then switch to devex (approximate steepest-edge, Harris, 1974)
- A model that might have taken a year to solve 10 years ago can now solve in less than 30 seconds (Bixby, 2002).