

DM545/DM871

Linear and Integer Programming

# Introduction to Linear Programming Notation and Modeling

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# Outline

Course Organization  
Preliminaries  
Introduction: Operations Research

1. Course Organization

2. Preliminaries

3. Introduction: Operations Research

Resource Allocation

Duality

# Who is here?

58 in total registered in ItsLearning

## **DM545 (5 ECTS)**

30, who??

- Math-economy  
(2nd year)
- Others?

### Prerequisites

- Programming
- Linear Algebra

## **DM871 (5 ECTS)**

28, who??

- Computer Science  
(Master)
- Mathematics  
(Bachelor/Master ? )
- Applied Mathematics  
(Bachelor/Master)
- Data Science (Master)
- Others?

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## 2. Preliminaries

## 3. Introduction: Operations Research

- Resource Allocation
- Duality

# Aims of the course

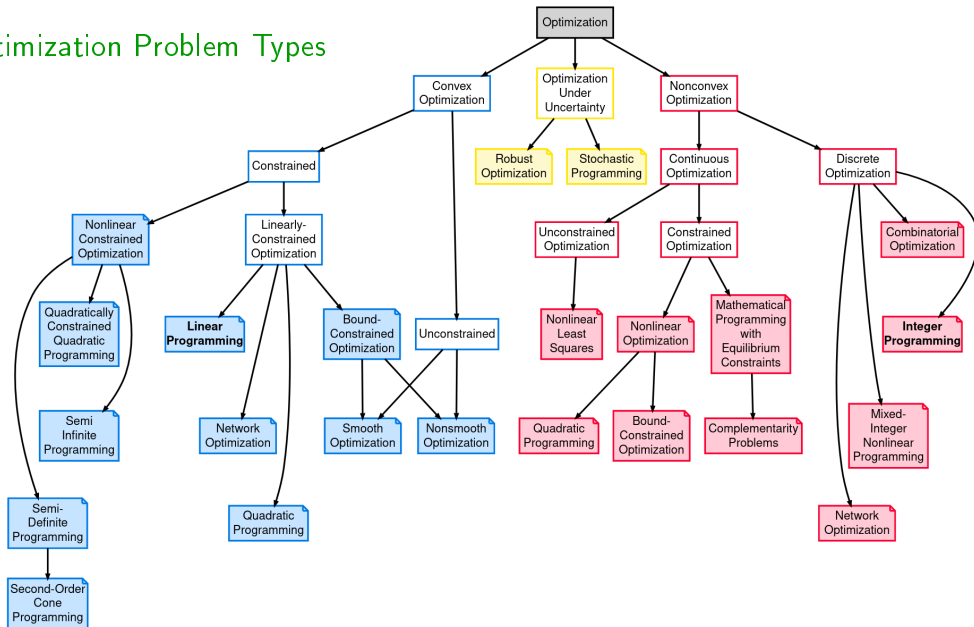
Learn about **mathematical optimization**:

- linear programming (continuous linear optimization)
- integer linear programming (discrete linear optimization)

**Optimization** is an important tool in **making decisions** and in analyzing physical systems. In mathematical terms, an **optimization problem** is the problem of finding the **best solution** from among the set of all **feasible solutions**.

The first step in the optimization process is constructing an appropriate **mathematical model**

# Optimization Problem Types



# Contents of the Course (aka Syllabus)

## Linear Programming

- 1 Introduction - Linear Programming, Notation & Modeling
- 2 Linear Programming, Simplex Method
- 3 Exception Handling
- 4 Duality Theory
- 5 Sensitivity
- 6 Revised Simplex Method

## Integer Linear Programming

- 7 Modeling Examples, Good Formulations, Relaxations
- 8 Well Solved Problems
- 9 Network Optimization Models (Max Flow, Min cost flow, Matching)
- 10 Cutting Planes & Branch and Bound
- 11 More on Modeling

# Practical Information

Teacher: Marco Chiarandini ([imada.sdu.dk/~marco/](mailto:imada.sdu.dk/~marco/))  
Instructor: Frederik Lund Andersen (M1),  
Juraj Marcibál (H1)

Alternative views of the schedule:

- [mitsdu.sdu.dk](https://mitsdu.sdu.dk), SDU Mobile
- Official course description (læserplanen)
- <https://dm871.github.io/>

Schedule (8 weeks):

- Introductory classes: 28 hours (14 classes)
- Training classes: 16 hours (8 classes)



# Communication Means

- ItsLearning  $\Leftrightarrow$  External Web Page  
(link <https://dm871.github.io>)
- **Announcements** in ItsLearning
- Write to Marco ([marco@imada.sdu.dk](mailto:marco@imada.sdu.dk)) or to instructor
- Ask peers

~> It is good to ask questions!!

~> Let me know if you think we should do things differently!

# Sources — Reading Material

## Main references:

[LN] Lecture Notes (continously updated)

[F] M. Fischetti, [Introduction to Mathematical Optimization](#),  
Independently published, 2019



## Linear Programming:

[MG] J. Matousek and B. Gartner. Understanding and Using Linear Programming.  
Springer Berlin Heidelberg, 2007



## Integer Programming:

[Wo] L.A. Wolsey. Integer programming. John Wiley & Sons, New York, USA, 1998



## Others

[HL] Frederick S Hillier and Gerald J Lieberman, Introduction to Operations Research, 9th edition,  
2010

External Web Page is the main reference for list of contents (ie<sup>1</sup>, syllabus, pensum).

It contains:

- slides
- list of topics and references
- exercises
- links
- resources for programming tasks

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<sup>1</sup>ie = id est, eg = exempli gratia, wrt = with respect to

- Two obligatory [24 h Take-Home Assignments](#), evaluation by internal censor
  - individual !!
  - exercises similar to previous 4 hour written exams
  - style: short answers about calculations and modeling. Differences between DM871 and DM545
  - (language: Danish and/or English)
- Final grade: overall evaluation but as starting point the average grade rounded up
- Tentative plan:
  - Test 1 (about weeks 5, 6, 7, 8) in week 9 or 10
  - Test 2 (about weeks 8, 9, 10, 11, 12) in week 13 Thursday 30/3 or week 14 Monday 3/4

Which day? Which time range? (Pool coming in ItsLearning)

# Training Sessions

- Do the plus exercises in advance
- Prepare the starred exercises in advance to get out the most from the session
- Try the others unsolved in class after the session
- Best if carried out in small groups
- Exercises are examples of exam questions (but not only!)

# Concepts from Linear Algebra

Linear Algebra:

manipulation of matrices and vectors with some theoretical background

## Linear Algebra

- Matrices and vectors - Matrix algebra

- Dot (scalar, Euclidean inner) product

- Geometric insights

- Systems of Linear Equations - Row echelon form, Gaussian elimination

- Matrix inversion and determinants

- Rank and linear dependency

- gives you the ability to create new and useful artifacts
- allows you to have more control of your world as more and more of it becomes digital
- is just fun.

It can also help you to [understand math](#).

- listening to lectures
- watching an instructor work through a derivation
- working through numerical examples by hand
- learn [by doing](#), [interacting with Python](#).
- Python 3.8+  
Pyomo or PySCIPOpt a Python interface to SCIP Optimization Suite  
(Commercial alternative Gurobi or Cplex  $\approx$  100 000 Dkk)
- ipython, jupyter, jupyterLab (= interactive python)? Or Visual Code or Spyder3.

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Resource Allocation  
Duality



# Sets

- A **set** is a collection of objects. eg.:  $A = \{1, 2, 3\}$
- $A = \{n \mid n \text{ is a whole number and } 1 \leq n \leq 3\}$   
( $\mid$  reads 'such that')
- $B = \{x \mid x \text{ is a student of this course}\}$
- $x \in A$   
 $x$  belongs to  $A$
- set of no members: **empty set**, denoted  $\emptyset$
- if a set  $S$  is a **(proper) subset** of a set  $T$ , we write  $(S \subset T) \quad T \supseteq S$   
 $\{1, 2, 5\} \subset \{1, 2, 4, 5, 6, 30\}$
- for two sets  $A$  and  $B$ , the **union**  $A \cup B$  is  $\{x \mid x \in A \text{ or } x \in B\}$
- for two sets  $A$  and  $B$ , the **intersection**  $A \cap B$  is  $\{x \mid x \in A \text{ and } x \in B\}$   
 $A = \{1, 2, 3, 5\}$  and  $B = \{2, 4, 5, 7\}$ , then  $A \cap B = \{2, 5\}$

# Numbers

- set of real numbers:  $\mathbb{R}$
- set of natural numbers:  $\mathbb{N} = \{1, 2, 3, 4, \dots\}$  (positive integers);  $\mathbb{N}_0$  to include zero
- set of all integers:  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ ;  $\mathbb{Z}_0^+$  only positives and zero
- set of rational numbers:  $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$
- set of complex numbers:  $\mathbb{C}$
- absolute value (non-negative):

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases}$$

- the set  $\mathbb{R}^2$  is the set of ordered pairs  $(x, y)$  of real numbers  
(eg, coordinates of a point wrt a pair of axes, the [Cartesian plane](#))

# Matrices and Vectors

- A **matrix** is a rectangular array of numbers or symbols. It can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- An  $n \times 1$  matrix is a **column vector**, or simply a vector:

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- the set  $\mathbb{R}^n$  is the set of vectors  $[x_1, x_2, \dots, x_n]^T$  of real numbers (eg, coordinates of a point wrt an  $n$ -dimensional space, the **Euclidean Space**)

# Basic Algebra

**Elementary Algebra**: the study of symbols and the rules for manipulating symbols. It differs from **arithmetic** in the use of abstractions, such as using letters to stand for numbers that are either unknown or allowed to take on many values

- collecting up terms: eg.  $2a + 3b - a + 5b = a + 8b$
- multiplication of variables: eg:

$$a(-b) - 3ab + (-2a)(-4b) = -ab - 3ab + 8ab = 4ab$$

- expansion of bracketed terms: eg:

$$\begin{aligned} -(a - 2b) &= -a + 2b, \\ (2x - 3y)(x + 4y) &= 2x^2 - 3xy + 8xy - 12y^2 \\ &= 2x^2 + 5xy - 12y^2 \end{aligned}$$

- $a^r a^s = a^{r+s}$ ,  $(a^r)^s = a^{rs}$ ,  $a^{-n} = 1/a^n$ ,  
 $a^{1/n} = x \iff x^n = a$ ,  $a^{m/n} = (a^{1/n})^m$

- In Mathematics and Statistics, a **variable** is an alphabetic character representing a **value**, which is unknown. They are used in **symbolic** calculations.  
Commonly given one-character names.
- in contrast, a **parameter** or **constant** or **given** is a known real number
- in contrast, in **Computer Science**, a **variable** is a storage location paired with an associated identifier, which contains a value that may be known or unknown.  
Commonly given long, explanatory names.

- a **function**  $f$  on a set  $\mathcal{X}$  into a set  $\mathcal{Y}$  is a rule that assigns a **unique** element  $f(x)$  in  $\mathcal{Y}$  to each element  $x$  in  $\mathcal{X}$ .

$$y = f(x)$$

$y$  dependent  
variable

$x$  independent  
variable

- a **linear function** has only sums and scalar multiplications, that is, for variable  $x \in \mathbb{R}$  and scalars  $a, b \in \mathbb{R}$ :

$$f(x) := ax + b$$

(actually the one above is an affine function)

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# What is Operations Research?

**Operations Research** (aka, Management Science, Analytics):  
is the discipline that uses a **scientific approach to decision making**.

It seeks to determine how best to design and operate a system,  
usually under conditions requiring the allocation of scarce resources,  
by means of **mathematics** and **computer science**.

## Quantitative methods for planning and analysis.

It encompasses a wide range of problem-solving techniques and methods applied in the pursuit of improved decision-making and efficiency:

- simulation,
- **mathematical optimization**,
- queueing theory and other stochastic-process models,
- Markov decision processes
- econometric methods,
- data envelopment analysis,
- neural networks,
- expert systems



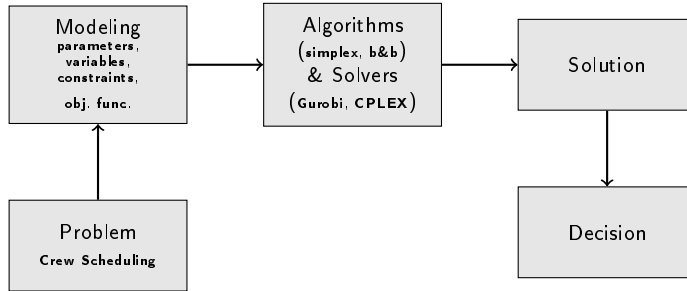
## Some Examples ...

- Production Planning and Inventory Control
- Budget Investment
- Blending and Refining
- Manpower Planning
  - Crew Rostering (airline crew, rail crew, nurses)
- Packing Problems
  - Knapsack Problem
- Cutting Problems
  - Cutting Stock Problem
- Routing
  - Vehicle Routing Problem (trucks, planes, trains ...)
- Locational Decisions
  - Facility Location
- Scheduling/Timetabling
  - Examination timetabling/ train timetabling
- .... + many more

# Common Characteristics

- Planning decisions must be made
- The problems relate to quantitative issues
  - Cheapest
  - Shortest route
  - Fewest number of people
- Not all plans are feasible - there are constraining rules
  - Limited amount of available resources
- It can be extremely difficult to figure out what to do

# OR - The Process?



1. Observe the System
2. Formulate the Problem
3. Formulate Mathematical Model
4. Verify Model
5. Select Alternative
6. Show Results to Company
7. Implementation

## Central Idea

Build a mathematical model describing exactly what one wants, and what the “rules of the game” are. However, **what is a mathematical model and how?**

# Mathematical Modeling

- Find out exactly what the decision maker needs to know:
  - which investment?
  - which product mix?
  - which job  $j$  should a person  $i$  do?
- Define **Parameters**, that is the given, known elements of the problem.
- Define **Decision Variables** of suitable type (continuous, integer, binary) corresponding to the needs
- Formulate **Objective Function** computing the benefit/cost
- Formulate mathematical **Constraints** indicating the interplay between the different variables.

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# Resource Allocation

In manufacturing industry, **factory planning**: find the best product mix.

## Example

A factory makes two products **standard** and **deluxe**. Eg, yougurt, sleeping beds, etc.

A unit of **standard** gives a profit of 6(k) Dkk.

A unit of **deluxe** gives a profit of 8(k) Dkk.

The warming|grinding and cooling|polishing times in terms of hours per week for a **unit** of each type of product are given below:

	Standard	Deluxe
(Machine 1) Warming Grinding	5	10
(Machine 2) Cooling Polishing	4	4

Warming|Grinding capacity: 60 hours per week

Cooling|Polishing capacity: 40 hours per week

**Q:** How much of each product, **standard** and **deluxe**, should we produce to maximize the profit?

# Mathematical Model

## Decision Variables

$x_1 \geq 0$  units of product standard

$x_2 \geq 0$  units of product deluxe

## Object Function

$\max 6x_1 + 8x_2$  maximize profit

## Constraints

$5x_1 + 10x_2 \leq 60$  machine 1 capacity

$4x_1 + 4x_2 \leq 40$  machine 2 capacity

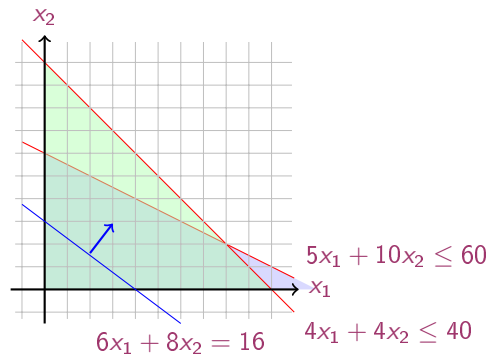
# Mathematical Model

Machines/Materials A and B  
Products 1 and 2

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ \text{s.t.} \quad & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

$a_{ij}$	1	2	$b_i$
A	5	10	60
B	4	4	40
$c_j$	6	8	

Graphical Representation:





# Resource Allocation – General Model

Managing a production facility

$j = 1, 2, \dots, n$  products

$i = 1, 2, \dots, m$  materials

$b_i$  units of raw material at disposal

$a_{ij}$  units of raw material  $i$  to produce one unit of product  $j$

$\sigma_j$  market price of unit of  $j$ th product

$\rho_i$  prevailing market value for material  $i$

$c_j = \sigma_j - \sum_{i=1}^m \rho_i a_{ij}$  profit per unit of product  $j$

$x_j$  amount of product  $j$  to produce

$$\begin{aligned}
 &\max \quad c_1 x_1 + c_2 x_2 + c_3 x_3 + \dots + c_n x_n = z \\
 &\text{subject to} \quad a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots + a_{1n} x_n \leq b_1 \\
 &\quad \quad \quad a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots + a_{2n} x_n \leq b_2 \\
 &\quad \quad \quad \dots \\
 &\quad \quad \quad a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \dots + a_{mn} x_n \leq b_m \\
 &\quad \quad \quad x_1, x_2, \dots, x_n \geq 0
 \end{aligned}$$

# Notation

$$\begin{array}{ll}\max & c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n = z \\ \text{s.t.} & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n \leq b_2 \\ & \dots \\ & a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \dots, x_n \geq 0\end{array}$$

$$\begin{array}{ll}\max & \sum_{j=1}^n c_jx_j \\ & \sum_{j=1}^n a_{ij}x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n\end{array}$$

# In Matrix Form

$$\begin{aligned} \max \quad & c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n = z \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n \leq b_2 \\ & \dots \\ & a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \dots, x_n \geq 0 \end{aligned}$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{aligned} \max \quad & z = c^T x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

# Our Numerical Example

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$$x \in \mathbb{R}^n, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

$$\max \quad \begin{bmatrix} 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

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# Duality

**Resource Valuation problem:** Determine the value of the raw materials on hand such that:  
(i) it would be convenient selling and (ii) an outside company would be willing to buy them.

$z_i$  value of a unit of raw material  $i$

$\sum_{i=1}^m b_i z_i$  total expenses for buying (or opportunity cost, cost of having instead of selling)

$\rho_i$  prevailing unit market value of material  $i$

$\sigma_j$  prevailing unit product price

Goal: for the outside company to minimize the total expenses;  
(for the owing company the minimum amount of opportunity cost to accept for selling)

$$\min \sum_{i=1}^m b_i z_i \tag{1}$$

$$z_i \geq \rho_i, \quad i = 1 \dots m \tag{2}$$

$$\sum_{i=1}^m z_i a_{ij} \geq \sigma_j, \quad j = 1 \dots n \tag{3}$$

(2) otherwise selling to someone else and (3) otherwise not selling

Let

$$y_i = z_i - \rho_i$$

markup that the company would make by reselling the raw material instead of producing.

$$\begin{aligned} \min \quad & \sum_{i=1}^m y_i b_i + \sum_i \cancel{\rho_i b_i} \\ & \sum_{i=1}^m y_i a_{ij} \geq c_j, \quad j = 1 \dots n \\ & y_i \geq 0, \quad i = 1 \dots m \end{aligned}$$

Dual Problem

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

Primal Problem