DM545/DM871

Linear and Integer Programming

Lecture 9
IP Modeling
Formulations, Relaxations

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

1. Formulations
   - Uncapacited Facility Location
   - Alternative Formulations

2. Relaxations

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Uncapacited Facility Location (UFL)

Given:

- depots $N = \{1, \dots, n\}$
- clients $M = \{1, \dots, m\}$
- f_j fixed cost to use depot j
- transport cost for all orders c_{ij}

Task: Which depots to open and which depots serve which client

Variables: $y_j = \begin{cases} 1 & \text{if depot opened} \\ 0 & \text{otherwise} \end{cases}$, x_{ij} fraction of demand of i satisfied by j

Objective:

$$\min \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j$$

Constraints:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i = 1, \dots, m$$

$$\sum_{i \in M} x_{ij} \le m y_j \qquad \forall j \in N$$

Outline

Good and Ideal Formulations

Definition (Formulation)

A polyhedron $P \subseteq \mathbb{R}^{n+p}$ is a formulation for a set $X \subseteq \mathbb{Z}^n \times \mathbb{R}^p$ if and only if $X = P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$

That is, if it does not leave out any of the solutions of the feasible region X.

There are infinite formulations

Definition (Convex Hull)

Given a set $X \subseteq \mathbb{Z}^n$ the convex hull of X is defined as:

$$\mathrm{conv}(X) = \Big\{ \mathrm{x} : \mathrm{x} = \sum_{i=1}^{t} \lambda_i \mathrm{x}^i, \qquad \sum_{i=1}^{t} \lambda_i = 1, \qquad \lambda_i \geq 0, \qquad \text{for } i = 1, \dots, t, \\ \text{for all finite subsets } \{\mathrm{x}^1, \dots, \mathrm{x}^t\} \text{ of } X \Big\}$$

Proposition

$\text{conv}(X)$ is a polyhedron (ie, representable as $Ax \leq b$)

Proposition

Extreme points of $\text{conv}(X)$ all lie in X

Hence:

$$\max\{c^T x : x \in X\} \equiv \max\{c^T x : x \in \text{conv}(X)\}$$

However it might require exponential number of inequalities to describe $\text{conv}(X)$
What makes a formulation better than another?

$$X \subseteq \text{conv}(X) \subseteq P_2 \subset P_1$$

P_2 is better than P_1

Definition

Given a set $X \subseteq \mathbb{R}^n$ and two formulations P_1 and P_2 for X, P_2 is a better formulation than P_1 if $P_2 \subset P_1$

Example

P_1 = UFL with $\sum_{i\in M} x_{ij} \leq my_j \quad \forall j \in N$
P_2 = UFL with $x_{ij} \leq y_j \quad \forall i \in M, j \in N$

$$P_2 \subset P_1$$

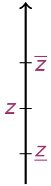
- $P_2 \subseteq P_1$ because summing $x_{ij} \leq y_j$ over $i \in M$ we obtain $\sum_{i\in M} x_{ij} \leq my_j$
- $P_2 \subset P_1$ because there exists a point in P_1 but not in P_2: $m = 6 = 3 \cdot 2 = k \cdot n$

$$\begin{array}{ll} x_{10} = 1, x_{20} = 1, x_{30} = 1, & \sum_i x_{i0} \leq 6y_0 \quad y_0 = 1/2 \\ x_{41} = 1, x_{51} = 1, x_{61} = 1 & \sum_i x_{i1} \leq 6y_1 \quad y_1 = 1/2 \end{array}$$

Outline

Optimality and Relaxation

$$z = \max\{c(x) : x \in X \subseteq \mathbb{Z}^n\}$$



How can we prove that x^* is optimal?
$\overline{z}$ is UB
$\underline{z}$ is LB
stop when $\overline{z} - \underline{z} \leq \epsilon$

- Primal bounds (here lower bounds): every feasible solution gives a primal bound may be easy or hard to find, heuristics
- Dual bounds (here upper bounds): Relaxations

Optimality gap (SCIP):

- If primal and dual bound have opposite signs, the gap is "Infinity".
- If primal and dual bound have the same sign, the gap is

$$\frac{|pb - db|}{\min(|pb|, |db|)}$$

decreases monotonously during the solving process.

Proposition

$$\begin{array}{rll} \textit{Given:} & (IP)\ z & = \max\{c(x) : x \in X \subseteq \mathbb{R}^n\} \\ \textit{a relaxation of it is:} & (RP)\ z^R & = \max\{f(x) : x \in T \subseteq \mathbb{R}^n\} \qquad \textit{if:} \end{array}$$

(i) $X \subseteq T$ *or*

(ii) $f(x) \geq c(x)\ \forall x \in X$

In other terms:

$$\max_{x \in T} f(x) \geq \left\{ \begin{array}{l} \max_{x \in T} c(x) \\ \max_{x \in X} f(x) \end{array} \right\} \geq \max_{x \in X} c(x)$$

- T: candidate solutions;
- $X \subseteq T$ feasible solutions;
- $f(x) \geq c(x)\ \forall x \in X$

Relaxations

How to construct relaxations?

1. $IP : \max\{c^T x : x \in P \cap \mathbb{Z}^n\}, \qquad P = \{x \in \mathbb{R}^n : Ax \leq b\}$
   $LP : \max\{c^T x : x \in P\}$
   Better formulations give better bounds ($P_1 \subseteq P_2$)

   Proposition

   (i) *If a relaxation LP is infeasible, the original problem IP is infeasible.*

   (ii) *Let x^* be optimal solution for LP. If $x^* \in X$ and $f(x^*) = c(x^*)$ then x^* is optimal for IP.*

2. Combinatorial relaxations to easy problems that can be solved rapidly
   Eg: TSP to Assignment problem Eg: Symmetric TSP to 1-tree

3. Lagrangian relaxation

$$
\begin{aligned}
IP: \quad & z = \max\{c^T x : Ax \le b, x \in X \subseteq \mathbb{Z}^n\} \\
LR: \quad & z(u) = \max\{c^T x + u(b - Ax) : x \in X\}
\end{aligned}
$$

$$z(u) \ge z \qquad \forall u \ge 0$$

4. Duality:

Definition

Two problems:

$$z = \max\{c(x) : x \in X\} \qquad w = \min\{w(u) : u \in U\}$$

form a weak-dual pair if $c(x) \le w(u)$ for all $x \in X$ and all $u \in U$.
When $z = w$ they form a strong-dual pair

Proposition

$z = \max\{\mathrm{c}^T\mathrm{x} : A\mathrm{x} \le \mathrm{b}, \mathrm{x} \in \mathbb{Z}^n_+\}$ and $w^{LP} = \min\{\mathrm{u}^T\mathrm{b} : A^T\mathrm{u} \ge \mathrm{c}, \mathrm{u} \in \mathbb{R}^m_+\}$ (ie, dual of linear relaxation) form a weak-dual pair.

Proposition

Let IP and D be weak-dual pair:

(i) *If D is unbounded, then IP is infeasible*

(ii) *If $\mathrm{x}^* \in X$ and $\mathrm{u}^* \in U$ satisfy $c(\mathrm{x}^*) = w(\mathrm{u}^*)$ then x^* is optimal for IP and u^* is optimal for D.*

The advantage is that we do not need to solve an LP like in the LP relaxation to have a bound, any feasible dual solution gives a bound.

Examples

Weak pairs:
Matching: $z = \max\{1^T x : Ax \leq 1, x \in \mathbb{Z}^m_+\}$
V. Covering: $w = \min\{1^T y : A^T y \geq 1, y \in \mathbb{Z}^n_+\}$

Proof: consider LP relaxations, then $z \leq z^{LP} = w^{LP} \leq w$.
(strong when graphs are bipartite)

Weak pairs:
S. Packing: $z = \max\{1^T x : Ax \leq 1, x \in \mathbb{Z}^n_+\}$
S. Covering: $w = \min\{1^T y : A^T y \geq 1, y \in \mathbb{Z}^m_+\}$