

DM872
Math Optimization at Work

More on Modeling

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Outline

Modeling with IP, BIP, MIP

1. Modeling with IP, BIP, MIP

Integer Linear Programming

Modeling with IP, BIP, MIP

Linear Objective
Linear Constraints
but! integer variables

$$\begin{aligned} \max \quad & c^T x \\ Ax \leq & b \\ x \geq & 0 \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ Ax \leq & b \\ x \geq & 0 \\ x \quad & \text{integer} \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ Ax \leq & b \\ x \in & \{0, 1\}^n \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x + h^T y \\ Ax + Gy \leq & b \\ x \geq & 0 \\ y \geq & 0 \\ y \quad & \text{integer} \end{aligned}$$

Linear Programming
(LP)

Binary Integer Program
(BIP)
0/1 Integer Programming

Mixed Integer Linear
Programming (MILP)

$$\begin{aligned} \max \quad & f(x) \\ g(x) \leq & b \\ x \geq & 0 \end{aligned}$$

Non-linear Programming (NLP)

Outline

Modeling with IP, BIP, MIP

1. Modeling with IP, BIP, MIP

Iterate:

1. define parameters
2. define variables
3. use variables to express objective function
4. use variables to express constraints

Examples:

- problems with discrete input/output (knapsack, factory planning)
- problems with logical conditions
- combinatorial problems (sequencing, allocation, transport, assignment, partitioning)
- network problems

continuous quantities	$\in \mathbb{R}^n$
discrete quantities	$\in \mathbb{Z}^n$
decision variables	$\in \mathbb{B}^n$
indicator/auxiliary variables (for logical conditions)	$\in \mathbb{B}^n$
special ordered sets	$\in \mathbb{B}^n$
incidence vector of S	$\in \mathbb{B}^n$

Assignment

$$\max_{\sigma} \left\{ \sum_i c_{i,\sigma(i)} \mid \sigma : I \rightarrow J \right\}$$

Traveling Salesman Problem (TSP)

$$\min_{\pi} \left\{ \sum_i c_{i,\pi(i)} \mid \pi : \{1..n\} \rightarrow \{1..n\} \text{ and } \pi \text{ is a circuit} \right\}$$

Mathematical description
but not in linear form

Combinatorial Optimization Problem (COP)

$$\min_{S \subseteq N} \left\{ \sum_{j \in S} c_j \mid S \in \mathcal{F} \right\}$$

Objective function and/or constraints do not appear to be linear?

- Absolute values
- Minimize the largest function value
- Maximize the smallest function value
- Constraints include variable division
- Constraints are either/or
- A variable must take one of several candidate values

Modeling: Absolute Values

Modeling with IP, BIP, MIP

$$\min \sum_{i=1}^n |f_i(\mathbf{x})|$$

$$\mathbf{x} \in \mathbb{R}^q$$

$$\begin{aligned}\min \quad & \sum_{i=1}^n z_i \\ \text{s.t.} \quad & z_i \geq f_i(\mathbf{x}) \quad i = 1..n \\ & z_i \geq -f_i(\mathbf{x}) \quad i = 1..n \\ & z_i \in \mathbb{R} \quad \quad i = 1..n \\ & \mathbf{x} \in \mathbb{R}^q\end{aligned}$$

n additional variables and $2n$ additional constraints.

$$\begin{aligned}\min \quad & \sum_{i=1}^n (z_i^+ + z_i^-) \\ \text{s.t.} \quad & f_i(\mathbf{x}) = z_i^+ - z_i^- \quad i = 1..n \\ & z_i^+, z_i^- \geq 0 \quad \quad i = 1..n \\ & \mathbf{x} \in \mathbb{R}^q\end{aligned}$$

$2n$ additional variables and n additional constraints.

Minimize the largest of a number of function values:

$$\min \quad \max\{f_1(\mathbf{x}), \dots, f_n(\mathbf{x})\}$$

- Introduce an auxiliary variable z :

$$\min \quad z$$

$$\text{s. t. } f_1(\mathbf{x}) \leq z$$

$$f_2(\mathbf{x}) \leq z$$

Modeling: Divisions

Constraints include variable division:

- Constraint of the form

$$\frac{a_1x + a_2y + a_3z}{d_1x + d_2y + d_3z} \leq b$$

- Rearrange:

$$a_1x + a_2y + a_3z \leq b(d_1x + d_2y + d_3z)$$

which gives:

$$(a_1 - bd_1)x + (a_2 - bd_2)y + (a_3 - bd_3)z \leq 0$$

Later we will see linear-fractional programming transformed into equivalent linear programs
(Charnes and Cooper)

Modeling: “Either/Or Constraints”

Modeling with IP, BIP, MIP

In conventional mathematical models, the solution must satisfy all constraints.
Suppose that your constraints are “either/or”:

$$a_1x_1 + a_2x_2 \leq b_1 \quad \text{or}$$

$$d_1x_1 + d_2x_2 \leq b_2$$

Introduce new variable $y \in \{0, 1\}$ and a large number M :

$$a_1x_1 + a_2x_2 \leq b_1 + My \quad \text{if } y = 0 \text{ then this is active}$$

$$d_1x_1 + d_2x_2 \leq b_2 + M(1 - y) \quad \text{if } y = 1 \text{ then this is active}$$

Modeling: “Either/Or Constraints”

Modeling with IP, BIP, MIP

Binary integer programming allows to model alternative choices:

- Eg: 2 feasible regions, ie, disjunctive constraints, not possible in LP.
introduce y auxiliary binary variable and M , a big number:

$$Ax \leq b + My$$

if $y = 0$ then this is active

$$A'x \leq b' + M(1 - y)$$

if $y = 1$ then this is active

Modeling: “Either/Or Constraints”

Generally:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1m}x_m \leq d_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2m}x_m \leq d_2$$

⋮

$$a_{m1}x_1 + a_{N2}x_2 + a_{N3}x_3 + \dots + a_{Nm}x_m \leq d_N$$

Exactly K of the N constraints must be satisfied.

Introduce binary variables y_1, y_2, \dots, y_N and a large number M

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1m}x_m \leq d_1 + My_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2m}x_m \leq d_2 + My_2$$

⋮

$$a_{m1}x_1 + a_{N2}x_2 + a_{N3}x_3 + \dots + a_{Nm}x_m \leq d_N + My_N$$

$$y_1 + y_2 + \dots + y_N = N - K$$

K of the y -variables are 0, so K constraints must be satisfied

Modeling: “Either/Or Constraints”

Modeling with IP, BIP, MIP

At least $K \leq N$ of $\sum_{j=1}^n a_{ij}x_j \leq b_i, i = 1, \dots, N$ must be satisfied

introduce $y_i, i = 1, \dots, N$ auxiliary binary variables

$$\sum_{j=1}^n a_{ij}x_j \leq b_i + My_i, \quad i = 1..N$$

$$\sum_i y_i \leq N - K$$

Modeling: “Possible Constraint Values”

Modeling with IP, BIP, MIP

A constraint must take on one of N given values:

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_mx_m = d_1 \text{ or}$$

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_mx_m = d_2 \text{ or}$$

⋮

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_mx_m = d_N$$

Introduce binary variables y_1, y_2, \dots, y_N :

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_mx_m = d_1y_1 + d_2y_2 + \dots + d_Ny_N$$

$$y_1 + y_2 + \dots + y_N = 1$$

Logical Conditions

x binary

y integer

z continuous

Linking constraints $z \in \mathbb{R}, x \in \mathbb{B}$

$$\text{if } z = 0 \text{ then } x = 0, \text{ if } z > 0 \text{ then } x = 1 \iff z - Mx \leq 0$$

$$x = 1 \implies z \geq m \iff z - mx \geq 0$$

Logical conditions and 0 – 1 variables

$$X_1 \vee X_2 \iff x_1 + x_2 \geq 1$$

$$X_1 \wedge X_2 \iff x_1 = 1, x_2 = 1$$

$$\neg X_1 \iff x_1 = 0 \text{ or } (1 - x_1 = 1)$$

$$X_1 \rightarrow X_2 \iff x_1 - x_2 \leq 0$$

$$X_1 \leftrightarrow X_2 \iff x_1 - x_2 = 0$$

Logical Relationships

$$z = x \vee y$$

	$z \times y$
$z \geq x$	0 0 0
$z \geq y$	1 0 1
$z \leq x + y$	1 1 0
	1 1 1

$z = x \oplus y$ (or $z = x \text{ xor } y$, or $z = x <> y$, i.e. z is 1 if x and y are different (and z is zero if they are the same).

	$z \times y$
$z \geq x - y$	0 0 0
$z \geq y - x$	1 0 1
$z \leq x + y$	1 1 0
$z \leq 2 - x - y$	0 1 1

$$z = xy \text{ (or } z = x \wedge y\text{)}$$

	$z \times y$
$z \leq x$	0 0 0
$z \leq y$	0 0 1
$z \geq x + y - 1$	0 1 0
	1 1 1

See also: <http://yetanothermathprogrammingconsultant.blogspot.dk/2016/02/>

Examples

- $(X_A \vee X_B) \rightarrow (X_C \vee X_D \vee X_E)$

$$x_A + x_B \geq 1$$

$$x_A + x_B \geq 1 \implies x = 1$$

$$x_A + x_B - 2x \leq 0$$

$$x_C + x_D + x_E \geq 1$$

$$x = 1 \implies x_C + x_D + x_E \geq 1$$

$$x_C + x_D + x_E \geq x$$

- Disjunctive constraints in scheduling (see few slides back)
job i must either precede or follow job j

- Constraint: $x_1 x_2 = 0$

- 1) replace $x_1 x_2$ by x_3

- 2) $x_3 = 1 \iff x_1 = 1, x_2 = 1$

$$-x_1 + x_3 \leq 0$$

$$-x_2 + x_3 \leq 0$$

$$x_1 + x_2 - x_3 \leq 1$$

- $z \cdot x, \quad z \in \mathbb{R}, x \in \mathbb{B}$

1) replace zx by z_1

2) impose:

$$x = 0 \iff z_1 = 0$$

$$x = 1 \iff z_1 = z$$

$$z_1 - Mx \leq 0$$

$$-z + z_1 \leq 0$$

$$z - z_1 + Mx \leq M$$

- Special ordered sets of type 1/2 (for continuous or integer vars):

SOS1: set of vars within which (exactly/at most) one must be non-zero

SOS2: set of vars within which at most two can be non-zero. The two variables must be adjacent in the ordering

- separable programming and piecewise linear functions (next 5 slides)

Separable Programming

- Separable functions: sum of functions of single variables:

$$x_1^2 + 2x_2 + e^{x_3} \quad \text{YES}$$

$$x_1x_2 + \frac{x_2}{x_1 + 1} + x_3 \quad \text{NO}$$

(actually, some non-separable can also be made separable:

- x_1x_2 by y
- relate y to x_1 and x_2 by:

$$\log y = \log x_1 + \log x_2$$

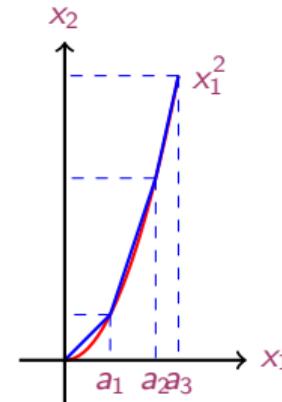
needs care if x_1 and x_2 close to zero.)

- non-linear separable functions can be approximated by piecewise linear functions
(valid for both constraints and objective functions)

Convex Non-linear Functions

- We can model convex non-linear functions by piece-wise linear functions and LP

$$\begin{aligned} \min \quad & x_1^2 - 4x_1 - 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 4 \\ & 2x_1 + x_2 \leq 5 \\ & -x_1 + 4x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$



- LP Formulation

$$\begin{aligned} x &= \lambda_0 a_0 + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 \\ y &= \lambda_0 f(a_0) + \lambda_1 f(a_1) + \lambda_2 f(a_2) + \lambda_3 f(a_3) \\ \sum_{i=0}^3 \lambda_i &= 1 \\ \lambda_i &\geq 0 \quad i = 0, \dots, 3 \\ \text{at most two adjacent } \lambda_i \text{ can be non zero} & \quad (*) \end{aligned}$$

- To model (*) which are SOS2 we would need binary indicator variables and hence BIP as in next slide.
- However since the problem is convex, an optimal solution lies on the borders of the functions and hence we can skip introducing the binary variables and relax (*)

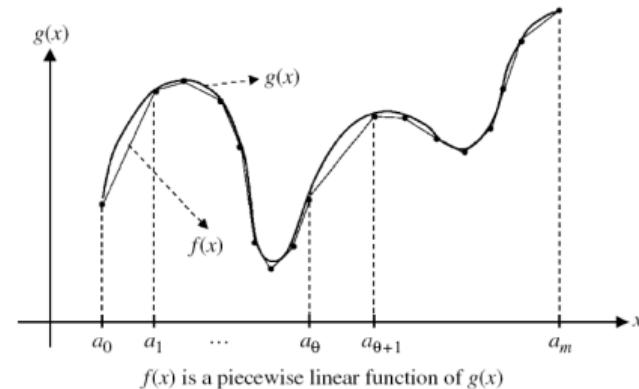
Non-convex Functions

Modeling with IP, BIP, MIP

Piece-wise Linear Functions

- non-convex functions require indicator variables and IP formulation

$$g(x) = \sum_j g_j(x) \quad g_j \text{ non linear}$$



- approximated by $f(x)$ piecewise linear in the disjoint intervals $[a_i, b_i]$
- convex hull formulation (convex combination of points)

$$\bigcup_{i \in I} \begin{pmatrix} x = \lambda_i a_i + \mu_i b_i \\ y = \lambda_i f(a_i) + \mu_i f(b_i) \\ \lambda_i + \mu_i = 1 \quad \lambda_i, \mu_i \geq 0 \end{pmatrix}$$

Remember how we modeled disjunctive polyhedra...

(cntd)

- using indicator variables δ s we obtain the BIP formulation:

$$x = \sum_{i \in I} (\lambda_i a_i + \mu_i b_i)$$

$$y = \sum_{i \in I} (\lambda_i f(a_i) + \mu_i f(b_i))$$

$$\lambda_i + \mu_i = \delta_i \quad \forall i \in I$$

$$\sum_{i \in I} \delta_i = 1$$

$$\lambda_i, \mu_i \geq 0 \quad \forall i \in I$$

$$\delta_i \in \{0, 1\} \quad \forall i \in I$$

the δ s are SOS1.

Good/Bad Models

- Number of variables: sometimes it may be advantageous increasing if they are used in search tree.

$0 - 1$ var have specialized algorithms for preprocessing and for branch and bound. Hence a large number solved efficiently. Good using.

Binary expansion:

$$0 \leq y \leq u$$

$$y = x_0 + 2x_1 + 4x_2 + 8x_3 + \dots + 2^r x_r \quad r = \log_2 u$$

- Making explicit good variables for branching:

$$\sum_j a_j x_j \leq b$$

$$\sum_j a_j x_j + u = b$$

u may be a good variable to branch (u is relaxed in LP but must be integer as well)

- Symmetry breaking:
Eg machine maintenance (in FPMM) $y_j \in \mathbb{Z}$ vs $x_j \in \mathbb{B}$
- Difficulty of LP models depends on number of constraints:

$$\min \sum_t |a_t z_t - b_t|$$

$$\max \sum_t z'_t$$

$$z'_t \geq a_t z_t - b_1$$

$$z'_t \geq b_t - a_t z_t$$

$$\max \sum_t z_t^+ - z_t^-$$

$$z_t^+ - z_t^- = a_t z_t - b_t$$

more variables but less
constraints

- With IP it might be instead better increasing the number of constraints.
- Make big M as small as possible in IP (reduces feasible region possibly fitting it to convex hull).

- Units of measure: check them!
 - all data should be scaled to stay in $0.1 - 10$
 - some software does this automatically but better to have control over things
- Write few lines of text describing what the equations express and which are the variables, give examples on the problem modeled.
- Try the model on small simple example that can be checked by hand.
- Be diffident of infeasibility and unboundedness, double check.
- Estimate the potential size.
 - If IP problem large and no structure then it might be hard.
 - If TUM then solvable with very large size
 - If other structure, eg, packing, covering also solvable with large size
- Check the output of the solver and understand what is happening
- If all fails resort to heuristics