The University Examination Timetabling Problem with Uncertain Timeslot Capacity: A Two-stage Stochastic Programming Approach

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Uncapacitated Examination Timetabling (UETT, 1996)

- **Data**:
  - enrolment matrix (exams $\times$ students)
  - no rooms
- **Constraint**:
  - no exams with students in common in the same period
- **Objective**:
  - Students should have exams at a minimum distance
  - *Proximity costs*:
    distances ≤ 5 are penalized with "exponential" weights ($\times$ student)

Capacitated Examination Timetabling (CETT)

- **Data**:
  - enrolment matrix (exams $\times$ students)
  - number of rooms available
- **Constraint**:
  - no exams with students in common in the same period
- **Objective**:
  - Students should have exams at a minimum distance
  - *Proximity costs*:
    distances ≤ 5 are penalized with "exponential" weights ($\times$ student)

CETT: IP formulation

Input data

- T: timeslots, E: exams, S: students
- $n_{e,e'}$: number of students enrolled both in exam e and e'
- $[e, e'] \in C$: set of pairs of exams in conflict ($n_{e,e'} > 0$)
- B: number of available rooms

Decision variables

$$y_{e,t} := \begin{cases} 1 & \text{if exam } e \in E \text{ is scheduled on timeslot } t \in T \\ 0 & \text{otherwise} \end{cases}$$

$$u^{i}_{e,e'} := \begin{cases} 1 & \text{if exams } [e, e'] \in C \text{ are scheduled at distance } i = 1, \ldots, 5 \\ 0 & \text{otherwise} \end{cases}$$

CETT: IP formulation

$$\min \quad \sum_{i=1}^{5} \sum_{[e,e'] \in C} 2^{(5-i)} n_{e,e'} u^i_{e,e'} \tag{1}$$

subject to

$$\sum_{t=1}^{T} y_{e,t} = 1 \qquad e \in E \tag{2}$$

$$y_{e,t} + y_{e',t} \leq 1 \qquad [e,e'] \in C, t = 1, \ldots, T \tag{3}$$

$$y_{e,t} + y_{e',t+i} \leq 1 + u^i_{e,e'} \qquad [e,e'] \in C, i = 1, \ldots, 5, t = 1, \ldots, T - i \tag{4}$$

$$\sum_{e \in E} y_{e,t} \leq B, \qquad t = 1, \ldots, T \tag{5}$$

$$y_{e,t} \in \{0,1\} \qquad e \in E, t = 1, \ldots, T \tag{6}$$

$$u^i_{e,e'} \in \{0,1\} \qquad [e,e'] \in C, i = 1, \ldots, 5.$$

Uncertainty in timeslot capacity

After the exam calendar is released,**unexpected events** may occur

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some **rooms** are **no longer available** in some periods

$\Downarrow$

the original schedule is **unfeasible**!

Stochastic CETT under uncertain timeslot capacity

Stochastic CETT under uncertain timeslot capacity

$\tilde{B}_t(\xi) :=$ stochastic variable representing the loss of capacity of scheduled exams, with ξ random vector

$$\min_{\boldsymbol{y},\boldsymbol{u},\xi} \quad \sum_{i=1}^{5} \sum_{[e,e'] \in C} 2^{(5-i)} n_{e,e'} u^i_{e,e'}$$

subject to

$$\begin{aligned}
\sum_{t=1}^{T} y_{e,t} &= 1 && e \in E \\
y_{e,t} + y_{e',t} &\leq 1 && [e,e'] \in C, t = 1, \ldots, T \\
y_{e,t} + y_{e',t+i} &\leq 1 + u^i_{e,e'} && [e,e'] \in C, i = 1, \ldots, 5, t = 1, \ldots, T - i \\
\sum_{e \in E} y_{e,t} &\leq B - \tilde{B}_t(\xi), && t = 1, \ldots, T \\
y_{e,t} &\in \{0,1\} && e \in E, t = 1, \ldots, T \\
u^i_{e,e'} &\in \{0,1\} && [e,e'] \in C, i = 1, \ldots, 5.
\end{aligned}$$

Two-stage Stochastic Programming paradigm

First stage

- prescheduling of exams to timeslots

Second stage: recourse actions

- rescheduling an exam to a **later** timeslot
- relocate an exam to a **spot-market** room in the same timeslot

Two-stage SP formulation: First stage problem

$$\min_{\boldsymbol{y},\boldsymbol{u}} \quad \sum_{i=1}^{5} \sum_{[e,e'] \in C} 2^{(5-i)} n_{e,e'} u^{i}_{e,e'} + \mathbb{E}_{\xi}[Q(\boldsymbol{y}, \boldsymbol{u}, \xi)] \tag{7}$$

subject to

Constraints (2)–(6)

$\mathbb{E}_{\xi}[Q(\boldsymbol{y}, \boldsymbol{u}, \xi)] :=$ expectation of the **recourse function** Q, i.e. the optimal solution value of the *second stage problem*

Two-stage SP formulation: Second stage problem

After the realization of the uncertainty:

$$\tilde{y}_{e,t} := \begin{cases} 1 & \text{if exam } e \in E \text{ is } \textbf{rescheduled} \text{ on timeslot } t \in T \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{u}^{i}_{e,e'} := \begin{cases} 1 & \text{if exams } [e, e'] \in C \text{ are } \textbf{rescheduled} \text{ at distance } i = 1, \ldots, 5 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{w}_{e,t} := \begin{cases} 1 & \text{exam } e \text{ is relocated to a } \textbf{spot-market} \text{ room in timeslot } t \in T \\ 0 & \text{otherwise} \end{cases}$$

Two-stage SP formulation: Second stage problem

$$Q(\boldsymbol{y}, \boldsymbol{u}, \xi) = \min_{\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{w}}} \underbrace{\sum_{i=1}^{5} \sum_{[e,e'] \in C} 2^{(5-i)} n_{e,e'} \tilde{u}^{i}_{e,e'}}_{\text{second stage distances}} +$$

$$+ \underbrace{\alpha \sum_{e \in E} \sum_{t=1}^{T} \frac{n_e}{2} \left(|\tilde{y}_{e,t} - y_{e,t}| - \tilde{w}_{e,t} \right)}_{\text{rescheduled exams}} + \underbrace{\beta \sum_{e \in E} \sum_{t=1}^{T} n_e \tilde{w}_{e,t}}_{\text{spot-market room}}$$

Recourse actions' constraints:

$$y_{e,t} \leq \sum_{t'=t}^{T} \tilde{y}_{e,t'} + \tilde{w}_{e,t} \qquad e \in E, t = 1, \ldots, T$$

Deterministic equivalent formulation

Approximate the behavior of the random variables through a finite set $\boldsymbol{\Omega}$ of **future scenarios**, each occurring with a **probability** $\boldsymbol{p^{\omega}}$.

$$\min_{\boldsymbol{y},\boldsymbol{u},\tilde{\boldsymbol{y}},\tilde{\boldsymbol{u}},\tilde{\boldsymbol{w}}} \quad \underbrace{\sum_{i=1}^{5} \sum_{[e,e']\in C} 2^{(5-i)} n_{e,e'} u^{i}_{e,e'}}_{\text{first stage distances}} + \sum_{\omega\in\Omega} p^{\omega} \Bigg[\underbrace{\sum_{i=1}^{5} \sum_{[e,e']\in C} 2^{(5-i)} n_{e,e'} \tilde{u}^{i,\omega}_{e,e'}}_{\text{second stage distances}} +$$

$$+ \alpha \underbrace{\sum_{e\in E} \sum_{t=1}^{T} \frac{n_e}{2} \left(|\tilde{y}^{\omega}_{e,t} - y_{e,t}| - \tilde{w}^{\omega}_{e,t} \right)}_{\text{rescheduled exams}} + \beta \underbrace{\sum_{e\in E} \sum_{t=1}^{T} n_e \tilde{w}^{\omega}_{e,t}}_{\text{spot-market room}} \Bigg]$$

subject to

...

second stage constraints replicated for **each scenario** $\omega \in \boldsymbol{\Omega}$

Impact of uncertainty: settings

Instances:

- 20 instances
- $E = 10$, $T = 7$, $S = 38$–77, $B = 2$
- $\tilde{B}_t \sim \mathcal{N}(\mu = 0, \sigma = 2)$
- $|\Omega| = 20$ scenarios

SP indicators:

- **Stochastic Loss (SL)**: penalty loss in the provisional schedule (first stage) by using SP instead of a deterministic model
- **Value of the Stochastic Solution (VSS)**: penalty saving by using an SP approach instead of a deterministic model
- **Expected Value of the Perfect Information (EVPI)**: how much we would be willing to pay for not having uncertain data

Tool: `Gurobi v.11.0.3`

Value of the Stochastic Solution (VSS)

- The *VSS* (Birge, 1982) represents the potential benefit from solving the stochastic program over solving a deterministic program in which expected values have replaced random parameters.

- The higher the VSS, the more critical the use of the two-stage SP model for examination timetabling is.

- $$VSS[\%] = 100 \cdot \frac{EEV - z^*_{DEP}}{z^*_{DEP}}$$

  - z^*_{DEP} is the optimal solution of the DEP formulation;
  - EEV is the expectation over all the scenarios of implementing at the first stage the optimal solution of the so-called *expected value* problem (EV), i.e., the SCETT_B formulation in which all the random variables are substituted by their mean values.

Expected Value of the Perfect Information (EVPI)

- The *EVPI* (Dempster, 1982) represents the value, averaged over all the future scenarios, obtainable if one could completely forecast the realization of the uncertainties.

- It can be interpreted as the value we would be willing to pay for perfect information on future events.

- $$EVPI[\%] = 100 \cdot \frac{z^*_{DEP} - WS}{z^*_{DEP}}$$

    - z^*_{DEP} is the optimal solution of the DEP formulation;
    - WS is the value of the so-called *wait-and-see* solution, i.e., the mean of the optimal solution value of the DEP solved separately for each scenario;

Impact of uncertainty: validation results

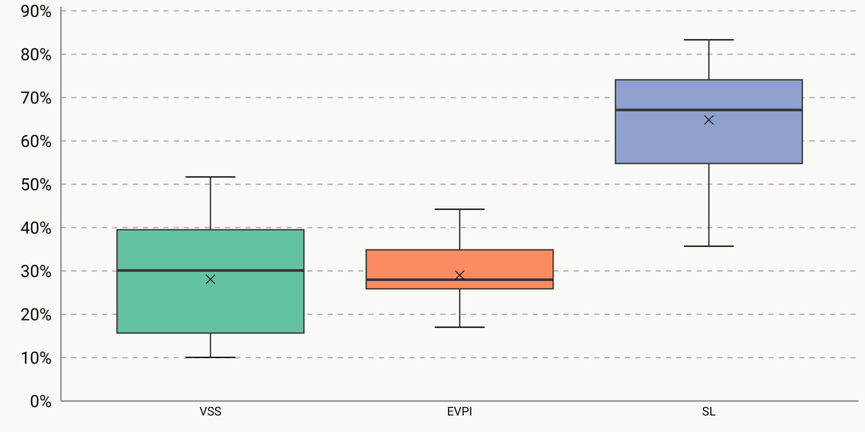

Figure: Percentage values of VSS, EVPI, and SL over all the benchmark instances.

A Progressive Hedging approach (Rockafellar and Wets, 1991)

1. Relax non-anticipativity constraints in a Lagrangian fashion
2. Decompose the Lagrangian problem by scenarios
3. Solve each mono-scenario problem independently
4. Compare first-stage schedules and calculate a *Temporary Global Solution* (TGS)
5. According to the differences between first-stage schedules over the scenarios, update Lagrangian penalties and re-iterate from 3 until a *global consensus* is met

Convergence proved only for convex programs, but

$$\Downarrow$$

PH can be used as a heuristic convergence framework

Problem reformulation

Let:

- $y^{\omega}_{e,t}$ as a copy of the first stage variable $y_{e,t}$ relative to scenario $\omega \in \Omega$
- $u^{i,\omega}_{e,e'}$ as a copy of the first stage variable $u^{i}_{e,e'}$ relative to scenario $\omega \in \Omega$
- $\bar{y}_{e,t}$ as the *Reference Solution* (RS) decisions.

$$\min_{\boldsymbol{y}^{\omega},\boldsymbol{u}^{\omega},\tilde{\boldsymbol{y}},\tilde{\boldsymbol{u}},\tilde{\boldsymbol{w}},\bar{\boldsymbol{y}}} \quad \sum_{\omega\in\Omega} p^{\omega} \left[\underbrace{\sum_{i=1}^{5} \sum_{[e,e']\in C} 2^{(5-i)} n_{e,e'} \left(u^{i,\omega}_{e,e'} + \tilde{u}^{i,\omega}_{e,e'} \right)}_{\text{distances}} + \right.$$

$$\left. + \underbrace{\alpha \sum_{e\in E} \sum_{t=1}^{T} \frac{n_e}{2} \left(|\tilde{y}^{\omega}_{e,t} - y^{\omega}_{e,t}| - \tilde{w}^{\omega}_{e,t} \right)}_{\text{rescheduled exams}} + \underbrace{\beta \sum_{e\in E} \sum_{t=1}^{T} n_e \tilde{w}^{\omega}_{e,t}}_{\text{spot-market room}} \right] \tag{8}$$

Non-anticipativity constraints:

$$y^{\omega}_{e,t} - \bar{y}_{e,t} = 0 \qquad e \in E, t = 1, \dots, T, \omega \in \Omega$$

Augmented Lagrangian relaxation

- Relaxing non-anticipativity constraints the model is **separable per scenario**
- Let $\lambda^{\omega}_{e,t}$ be the Lagrangian multiplier for each non-anticipativity constraint,
- ρ be a penalty factor for the associated quadratic term added to speed up convergence

$$\mathcal{L}(\bar{\boldsymbol{y}}, \boldsymbol{\lambda}, \rho) := \min_{\boldsymbol{y}^\omega, \boldsymbol{u}^\omega, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{w}}} \sum_{\omega \in \Omega} p^\omega \Bigg[\sum_{i=1}^{5} \sum_{[e,e'] \in C} 2^{(5-i)} n_{e,e'} (u^{i,\omega}_{e,e'} + \tilde{u}^{i,\omega}_{e,e'}) + \\ + \alpha \sum_{e \in E} \sum_{t=1}^{T} \frac{n_e}{2} \left(|\tilde{y}^\omega_{e,t} - y^\omega_{e,t}| - \tilde{w}^\omega_{e,t} \right) + \beta \sum_{e \in E} \sum_{t=1}^{T} n_e \tilde{w}^\omega_{e,t} + \\ + \underbrace{\sum_{e \in E} \sum_{t=1}^{T} \lambda^\omega_{e,t} (y^\omega_{e,t} - \bar{y}_{e,t}) + \frac{\rho}{2} \sum_{e \in E} \sum_{t=1}^{T} (y^\omega_{e,t} - \bar{y}_{e,t})^2}_{\text{Lagrangian terms}} \Bigg]$$

The overall Lagrangian problem can be decomposed per scenario in $|\Omega|$ sub-problems.

Problem decomposition

For each scenario $\omega \in \Omega$, the single-scenario Lagrangian sub-problem is:

$$\mathcal{L}^{\omega}(\bar{\boldsymbol{y}}, \boldsymbol{\lambda}, \rho) := \min_{\boldsymbol{y}^{\omega}, \boldsymbol{u}^{\omega}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{w}}} \quad \underbrace{\sum_{i=1}^{5} \sum_{[e,e'] \in C} 2^{(5-i)} n_{e,e'} (u^{i,\omega}_{e,e'} + \tilde{u}^{i,\omega}_{e,e'})}_{\text{distances}}$$

$$+ \underbrace{\alpha \sum_{e \in E} \sum_{t=1}^{T} \frac{n_e}{2} \left(|\tilde{y}^{\omega}_{e,t} - y^{\omega}_{e,t}| - \tilde{w}^{\omega}_{e,t} \right)}_{\text{rescheduled exams}} + \underbrace{\beta \sum_{e \in E} \sum_{t=1}^{T} n_e \tilde{w}^{\omega}_{e,t}}_{\text{spot-market}}$$

$$+ \underbrace{\sum_{e \in E} \sum_{t=1}^{T} \left(\lambda^{\omega}_{e,t} + \frac{\rho}{2} - \rho \bar{y}_{e,t} \right) y^{\omega}_{e,t}}_{\text{Lagrangian term}} \cancel{- \lambda^{\omega}_{e,t} \bar{y}_{e,t} + \frac{\rho}{2} (\bar{y}_{e,t})^2}$$

PH algorithm

Initialize $\bar{\boldsymbol{y}}$, $\boldsymbol{\lambda}^{(0)} = \boldsymbol{0}$, $\rho^{(0)} > 0$, $\rho_{step} > 1$, $i \leftarrow 0$
while i $\leq$ max_iterations **do**
    for each scenario $\omega \in \Omega$ **do**
        Solve the corresponding subproblem $\mathcal{L}^{\omega}(\bar{\boldsymbol{y}}, \boldsymbol{\lambda}^{(i)}, \rho^{(i)})$.
        Let $\boldsymbol{y}^{\omega(i)}$ be the resulting first stage solution.
    end for
    $y_{e,t}^{TGS} \leftarrow \sum_{\omega \in \Omega} p^{\omega} y_{e,t}^{\omega(i)}, \quad e \in E, t = 1, \ldots, T$ //Calculate a temporary global solution (TGS)
    if the TGS is integer-valued **then**
        break //Consensus met and optimal solution found
    else
        $\rho^{(i+1)} \leftarrow \rho_{step} \cdot \rho^{(i)}$ //Update the penalty factor
        $\lambda_{e,t}^{\omega(i+1)} \leftarrow \lambda_{e,t}^{\omega(i)} + \rho^{(i)}(y_{e,t}^{\omega(i)} - \bar{y}_{e,t}), \quad \forall e \in E, t = 1, \ldots, T, \omega \in \Omega$ //Update the Lagrangian multipliers
        $\bar{\boldsymbol{y}} \leftarrow \boldsymbol{y}^{TGS}$ //Update the reference solution
    end if
    $i \leftarrow i + 1$
end while

Local search-based metaheuristic

- Search space:
  vector: period for each exam
- Cost function:
  objective function of the subproblem $\mathcal{L}^{\omega}(\bar{\boldsymbol{y}}, \boldsymbol{\lambda}, \rho)$
- Initial solution:
  reference solution $\bar{\boldsymbol{y}}$ or the solution found in the previous iteration
- Neighborhoods:
  - **MoveExam**: change the period of an exam in the first stage
  - **MoveRescheduledExam**: change the period of an exam in the second stage
  - **MoveExamAndRescheduledExam**: change the period of an exam both in the first and the second stage
- Simulated Annealing with cutoff [Bellio et al, 2021]

Conclusions

- Stochastic CETT with Uncertain Timeslot Capacity
- Two-stage Stochastic Programming method with rescheduling and spot-market
- Validation results confirm the impact of uncertainty and the value of using SP
- Progressive Hedging framework that embeds a SA metaheuristics

Current and future work

- Experimental validation and tuning of the PH with different variants
- Sensitivity of analysis of the different weights of the obj
- Extensive experimental campaign on real-size instances

Thanks for your attention!