

DM872

Math Optimization at Work

Dantzig-Wolfe Decomposition and Delayed Column Generation

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Outline

1. Solving the Linear Master Problem
2. Solving the Master Problem: Branch and Price

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Solving the Linear Master Problem

Integer Programming Problem with block structure:

$$\begin{aligned} z_{MP} = \max \quad & c^1 \sum_{t=1}^{T_1} \lambda_{1,t} x^{1,t} + \quad c^2 \sum_{t=1}^{T_2} \lambda_{2,t} x^{2,t} + \dots + \quad c^K \sum_{t=1}^{T_K} \lambda_{K,t} x^{K,t} \\ & A^1(\sum_{t=1}^{T_1} \lambda_{1,t} x^{1,t}) + \quad A^2(\sum_{t=1}^{T_2} \lambda_{2,t} x^{2,t}) + \dots + A^K(\sum_{t=1}^{T_K} \lambda_{K,t} x^{K,t}) = b \\ & \sum_{t=1}^{T_k} \lambda_{k,t} = 1 \quad k = 1, \dots, K \\ & \lambda_{k,t} \in \{0, 1\} \quad t \in T_k, k = 1, \dots, K \end{aligned}$$

Let's consider the case $K = 1$

$$\begin{aligned} z_{MP} = \max \quad & \sum_{t=1}^T (cx^t) \lambda_t \\ & \sum_{t=1}^T (Ax^t) \lambda_t = b \\ & \sum_{t=1}^T \lambda_t = 1 \\ & \lambda_t \in \{0, 1\} \quad t \in T \end{aligned}$$

$$\begin{aligned} z_{LMP} = \max \quad & \sum_{t=1}^T (cx^t) \lambda_t \\ & \sum_{t=1}^T (Ax^t) \lambda_t = b \\ & \sum_{t=1}^T \lambda_t = 1 \\ & \lambda_t \geq 0 \quad t \in T \end{aligned}$$

Restricted LMP and Dual

$$\begin{aligned} z_{LMP} = \max \quad & \sum_{t=1}^T (cx^t) \lambda_t \\ & \sum_{t=1}^T (Ax^t) \lambda_t = b \\ & \sum_{t=1}^T \lambda_t = 1 \\ & \lambda_t \geq 0 \quad t \in T \end{aligned}$$

$$\begin{aligned} z_{RLMP} = \max \quad & \sum_{t=1}^p (cx^t) \lambda_t \\ & \sum_{t=1}^p (Ax^t) \lambda_t = b \\ & \sum_{t=1}^p \lambda_t = 1 \\ & \lambda_t \geq 0 \quad t = 1, \dots, p \end{aligned}$$

$$\begin{aligned} z_{DLMP} = \min \quad & \pi b + \pi_0 \\ & \pi A^T x^t + \pi_0 \geq cx^t, \quad t = 1, \dots, T \\ & \pi \in \mathbb{R}^m \\ & \pi_0 \in \mathbb{R} \end{aligned}$$

$$\begin{aligned} z_{DRLMP} = \min \quad & \pi b + \pi_0 \\ & \pi A^T x^t + \pi_0 \geq cx^t, \quad t = 1, \dots, p \\ & \pi \in \mathbb{R}^m \\ & \pi_0 \in \mathbb{R} \end{aligned}$$

Column Generation Process and Dual Bound

- $z_{LMP} \geq z_{MP}$ because linear relaxation
- $z_{LMP} \geq z_{RLMP}$ because of simplex theory (some columns missing)
- subproblem (pricing or constraint violation)
 $\xi^P = \max\{cx^t - \pi A^T x^t - \pi_0 \mid x^t \in X\}$. Solution: $(x^*, (\pi^*, \pi_0^*))$
- $z_{MP} \leq z_{LMP} \leq z_{RLMP} + \xi^P$ hence, valid dual bound on z_{MP}
- if $\xi^P = 0$ then $z_{LMP} = z_{RLMP}$ and stop column generation process
- if $\xi^P > 0$ then
stop if $\pi^*(Ax^* - b) = 0$
else add column $(cx^*, Ax^*, 1)$

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Branching constraints

- branch on original variables or on column variables
- disadvantages of branching on column variables: B&B tree unbalanced and subproblem difficult to solve

Solving the LP master at a node

The constraints introduced for branching (and other cutting planes) change the master problem or the subproblem. Where they should be considered is a design choice.

Price and branch

Heuristic solution:

After solving the LMP, start the branch and bound with the existing columns.

Note, it can lead to infeasibility