Vehicle Routing Problem

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Vehicle Routing Problem: Statement

There is a set of $\mathcal{C} = \{1, \ldots, n\}$ customers, a depot 0, and a matrix of travel distances (costs or travel times) c_{ij} that provides a cost of travelling directly from i to j. Every customer has demand $d_i > 0$ for delivery (pick-up) of a commodity. We need to determine:

- how many vehicles are needed to organize delivery,
- which customers to assign to every vehicle,
- in which specific order to visit those assigned customers,

such that

- every customer needs to be visited once,
- the capacity(s) of every vehicle is not violated,
- the total cost of the routes is minimized.

If the cost matrix is symmetric:

$$c_{ij} = c_{ji} \qquad\qquad i, j \in \mathcal{C}_0,\ i \neq j,$$

then the problem is called symmetric, otherwise it is asymmetric.

Vehicle Routing Problem: Variants

Vehicle capacity is the key complicating factor of the problem. It can be on

- the total quantity of the commodity a vehicle can carry (CVRP)
- the total distance a vehicle can travel (DVRP)

Side constraints may be on

- constraints on vehicle time visits (CVRP-TW)
- compatibility constraints between customers and vehicles.

In addition,

- vehicles may be identical (Homogeneous VRP)
- vehicles may have different capacities / speed (costs) of travelling over the network edges (Heterogeneous VRP)
- Vehicles may have fixed costs due when a vehicles is involved into serving customers (Fleet composition)

Capacitated Vehicle Routing Problem

We focus on the most fundamental version of the problem, the CVRP, in its symmetric and asymmetric versions:

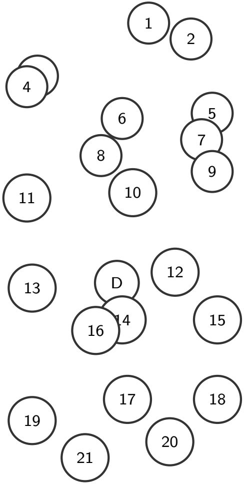
- A set of identical vehicles, located at the depot 0, that need to be routed over the set $\mathcal{C}$ ($|\mathcal{C}| = n$) of clients; $\mathcal{C}_0 = \mathcal{C} \cup \{0\}$
- Each client requires delivery of $d_i > 0$ units of a commodity
- Delivery of d_i units of the commodity can not be split among vehicles
- Each vehicle has the maximum capacity of Q units to carry the commodity, $Q \geq d_i$, $i \in \mathcal{C}$
- Distance / Cost matrix $C = \{c_{ij} : i, j \in \mathcal{C}_0, i \neq j\}$

All parameters (Q, c_{ij}, d_i) are assumed to take integer values.

CVRP Instance

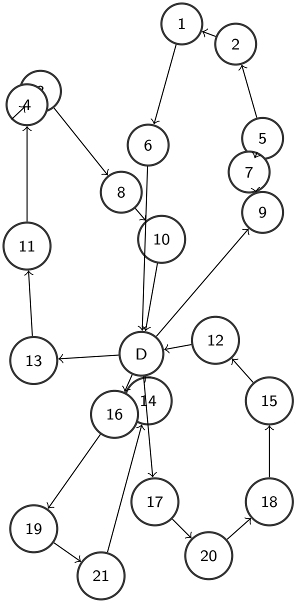
Location	Demand	Location	Demand
1	110	12	130
2	70	13	130
3	80	14	30
4	140	15	90
5	210	16	210
6	40	17	100
7	80	18	90
8	10	19	250
9	50	20	180
10	60	21	70
11	120		

$Q = 600,\ k = 4$ vehicles.



CVRP Solution

Route 1: Load 590
D - 17 - 20 - 18 - 15 - 12 - D
Route 2: Load 560
D - 16 - 19 - 21 - 14 - D
Route 3: Load 540
D - 13 - 11 - 4 - 3 - 8 - 10 - D
Route 4: Load 560
D - 9 - 7 - 5 - 2 - 1 - 6 - D
Total Cost: 375



Outline

- Compact, i.e., polynomial with $O(n^2)$ variables and constraints, extended formulations for the Asymmetric CVRP (ACVRP)
- Compact extended formulations for the Symmetric CVRP (simply referred to as the CVRP)
- Exponentially sized problem formulations
- Separation of the violated inequalities from the exponentially sized family

One-Commodity Network Flow CVRP Formulation (G−G)

The network flow idea initiated by Gavish and Graves (1978) can be used for the ACVRP. Suppose that the vehicle starting from the depot carries Q units of a commodity. Every client should get d_i units of that commodity.

A subtour not connected to the depot should not exist. Vehicle capacity is never violated.

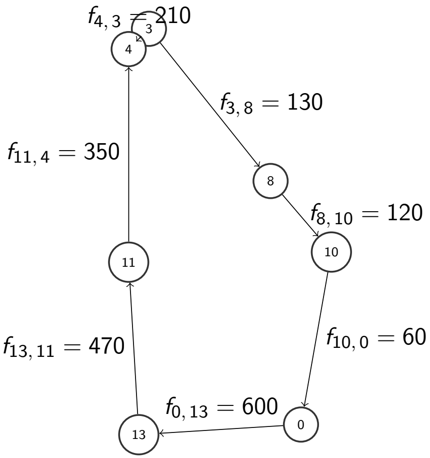
B. Gavish, S.C. Graves, "The Travelling Salesman Problem and Related Problems", Working Paper OR 078-78, Massachusetts Institute of Technology, Operations Research Center, Boston, 1978.

One-Commodity Network Flow ACVRP Illustration

The idea of Gavish and Graves (1978) (just for one route from the above example):

Route 3: Load 540
D - 13 - 11 - 4 - 3 - 8 - 10 - D

The one-commodity flow for the Route 3 will be organized as
$f_{D,13} = 600$, $f_{13,11} = 470$,
$f_{11,4} = 350$, $f_{4,3} = 210$, $f_{3,8} = 130$,
$f_{8,10} = 120$, $f_{10,D} = 60$



$d_{13} = 130$, $d_{11} = 120$, $d_4 = 140$, $d_3 = 80$, $d_8 = 10$, $d_{10} = 60$

ACVRP G−G Formulation

Decision variables: $x_{ij} \in \{0, 1\}$, $i, j \in \mathcal{C}_0$, $i \neq j$; $x_{ij} = 1 \iff$ a vehicle travels arc (i, j). f_{ij} = remaining # of units of the commodity when the vehicle leaves i and travels to j via (i, j) arc.

$$
\begin{aligned}
&\textbf{ACVRP G−G}: \\
&\min_{x_{ij},\, f_{ij}} \sum_{i, j \in \mathcal{C}_0,\, i \neq j} c_{ij} x_{ij} && && (1) \\
&\text{subject to} \\
&\sum_{j \in \mathcal{C}_0,\, i \neq j} x_{ij} = \sum_{j \in \mathcal{C}_0,\, j \neq i} x_{ji} = 1, && j \in \mathcal{C}, && (2) \\
&f_{ij} \leq Q x_{ij}, && i, j \in \mathcal{C},\, i \neq j, && (3) \\
&Q x_{0i} + \sum_{j \in \mathcal{C}} f_{ji} = \sum_{j \in \mathcal{C}_0} f_{ij} + d_i, && i \in \mathcal{C}, && (4) \\
&f_{ij} \geq 0, && i \in \mathcal{C},\, j \in \mathcal{C}_0,\, i \neq j, && (5) \\
&x_{ij} \in \{0, 1\}, && i, j \in \mathcal{C}_0,\, i \neq j. && (6)
\end{aligned}
$$

Note that the above formulation does not specify the number of vehicles involved. An optimal quantity of vehicles is determined as well, i.e., is part of decision-making. Alternatively, it can be specified as

$$\sum_{i \in \mathcal{C}} x_{0i} \leq m (= m).$$

Formulation validity proof consists of 2 parts:

1. Any integer feasible solution to **ACVRP G−G** does not contain a subtour disconnected from the depot
2. Any route, i.e., a subtour connected to the depot, is within the vehicle capacity.

Proof.

1. Home exercise. The same approach is employed as the one we used to prove the validity of the corresponding TSP formulation.
2. Suppose there is a route $0 \to i_1 \to i_2 \to \ldots \to i_k \to 0$: $\sum_{t=1}^{k} d_{i_t} > Q$. Consider the following flow balance constraints:

$$Qx_{0i_t} + \sum_{j \in \mathcal{C}} f_{ji_t} = \sum_{j \in \mathcal{C}_0} f_{i_t j} + d_{i_t}, \qquad t \in \{1, \ldots, k\}, \tag{7}$$

Aggregate them:

$$Q + \sum_{i,j \in \{i_1,\ldots,i_k\},\, i \neq j} f_{ij} = \sum_{i,j \in \{i_1,\ldots,i_k\},\, i \neq j} f_{ij} + \sum_{t \in \{1,\ldots,k\}} f_{i_t 0} + \sum_{t \in \{1,\ldots,k\}} d_{i_t}, \tag{8}$$

which implies that

$$Q \geq \sum_{t \in \{1,\ldots,k\}} d_{i_t}, \tag{9}$$

and contradicts the assumption.

□

ACVRP G−G Improvement

Gouveia (1995) observed that flow variables should be more accurately bounded as follows:

$$f_{ij} \geq d_j x_{ij}, \quad f_{ij} \leq (Q - d_i) x_{ij}, \qquad i, j \in \mathcal{C},\ i \neq j. \tag{10}$$

Employing the familiar transformation $f_{ij} = \tilde{f}_{ij} + d_j x_{ij}$, $\tilde{f}_{ij} \geq 0$ to satisfy the first inequality in (10), the modified and strengthened version of the **G−G** flow balance constraints, transform into

G−G m.

$$Q x_{0i} + \sum_{j \in \mathcal{C},\, i \neq j} \left(\widetilde{f}_{ji} + d_i x_{ji} \right) \geq$$

$$\sum_{j \in \mathcal{C},\, i \neq j} \left(\widetilde{f}_{ij} + d_j x_{ij} \right) + d_i, \qquad i \in \mathcal{C}, \tag{11}$$

$$\widetilde{f}_{ij} \leq (Q - d_i - d_j) x_{ij}, \qquad i, j \in \mathcal{C},\ i \neq j, \tag{12}$$

$$\widetilde{f}_{ij} \geq 0, \qquad i, j \in \mathcal{C},\ i \neq j. \tag{13}$$

Note that unnecessary variables f_{i0} have been eliminated.

The Label Setting M−T−Z Approach for the ACVRP

Suppose that vehicles do not deliver but pick up a commodity at clients $\mathcal{C}$ with supplies d_i. Vehicle capacity is Q. Let label $u_i > 0$ denote the cumulative load of a vehicle that visits client i **upon leaving** i.

ACVRP D−L :

$$\min_{x_{ij},\, f_{ij}} \sum_{i,j\in\mathcal{C}_0,\, i\neq j} c_{ij}x_{ij} \tag{14}$$

subject to

$$\sum_{j\in\mathcal{C}_0,\, i\neq j} x_{ij} = \sum_{j\in\mathcal{C}_0,\, j\neq i} x_{ji} = 1, \qquad j \in \mathcal{C}, \tag{15}$$

$$u_i - u_j + Qx_{ij} + (Q - d_i - d_j)x_{ji} \leq Q - d_j, \qquad i, j \in \mathcal{C},\ i \neq j, \tag{16}$$

$$u_i \leq Q - (Q - \max_{j\in\mathcal{C},\, j\neq i} d_j - d_i)x_{0i} - \sum_{j\in\mathcal{C},\, j\neq i} d_j x_{ij}, \qquad i \in \mathcal{C}, \tag{17}$$

$$u_i \geq d_i + \sum_{j\in\mathcal{C},\, j\neq i} d_j x_{ji}, \qquad i \in \mathcal{C}, \tag{18}$$

$$x_{ij} \in \{0, 1\}, \qquad i, j \in \mathcal{C}_0,\ i \neq j. \tag{19}$$

Desrochers, Martin, and Gilbert Laporte. "Improvements and extensions to the Miller-Tucker-Zemlin subtour elimination constraints." Operations Research Letters 10.1 (1991): 27–36.

Formulation validity proof consists of 2 parts:

1. Any integer feasible solution to **ACVRP D−L** does not contain a subtour disconnected from the depot
2. Any route, i.e., a subtour connected to the depot, is within the vehicle capacity

Proof.

1. Home exercise. The same approach is employed as the one we used to prove the validity of the corresponding TSP formulation.
2. Suppose there is a route $0 \rightarrow i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_k \rightarrow 0$: $\sum_{t=1}^{k} d_{i_t} > Q$. Then, prove that $u_{i_k} > Q$ as implied by constraints (16), which is a contradiction to (17).

□

Caveats

The paper of Desrochers and Laporte (1991) contains various typos:

- Typos in the formulation of the ACVRP, corrected later in Kara, Imdat, Gilbert Laporte, and Tolga Bektas. "A note on the lifted Miller–Tucker–Zemlin subtour elimination constraints for the capacitated vehicle routing problem." European Journal of Operational Research 158.3 (2004): 793-795.
- Typos in the formulation of the ACVRP with Time Windows constraints, corrected later in
  Yuan, Y., Cattaruzza, D., Ogier, M., & Semet, F. (2020). "A note on the lifted Miller–Tucker–Zemlin subtour elimination constraints for routing problems with time windows". Operations Research Letters, 48(2), 167-169.

Question

Does the ACVRP D$-$L formulation imply the

$$x_{ij} + x_{ji} \leq 1, \qquad i, j \in \mathcal{C},\ i < j \tag{20}$$

constraints?

Symmetric Capacitated Vehicle Routing (CVRP)

For a long time, there was no designated problem formulation for the symmetric CVRP. A specialized network flow formulation of the CVRP appeared in 2004:

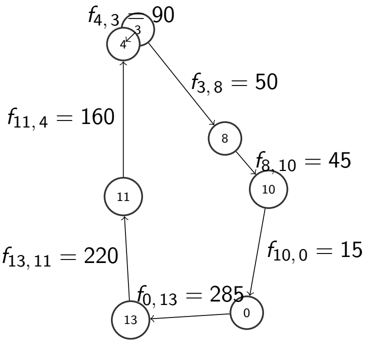
Baldacci, Roberto, Eleni Hadjiconstantinou, and Aristide Mingozzi. "An exact algorithm for the capacitated vehicle routing problem based on a two-commodity network flow formulation." Operations Research 52.5 (2004): 723-738.

We will study a very similar yet simpler approach by

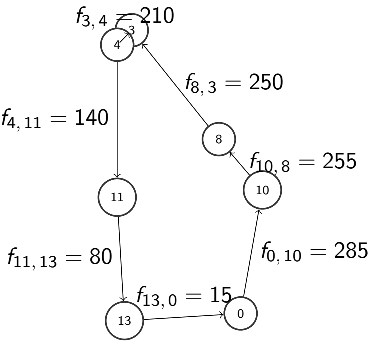
Pavlikov, K., Petersen, N. C. (2024). Two-Commodity Opposite Direction Network Flows for Vehicle Routing Problems.

Two-Commodity Opposite Direction Flow CVRP Illustration

One route only:



Flow of 285($< Q/2$) units of commodity in one direction



Flow of 285($< Q/2$) units of commodity in the opposite direction

Demand's splits is only an abstraction, the purpose of the splits is to prevent subtours not connected to the depot and find tours within vehicle capacities.

Two-Commodity Opposite Direction Network Flow Formulation for the CVRP

Decision variables: $x_{ij} \in \{0, 1\}$, $i, j \in \mathcal{C}$, $i < j$; $x_{ij} = 1 \iff$ a vehicle travels edge $\{i, j\}$.
$f_{ij} = \#$ of units of the commodity when the vehicle leaves i and travels to j via (i, j) arc.

CVRP−2CF :

$$\min \sum_{i,j\in\mathcal{C}_0,\, i<j} c_{ij} x_{ij} \tag{21}$$

subject to

$$\sum_{j\in\mathcal{C}_0,\, i<j} x_{ij} + \sum_{j\in\mathcal{C}_0,\, i>j} x_{ji} = 2, \qquad i \in \mathcal{C}, \tag{22}$$

$$\sum_{j\in\mathcal{C}_0,\, j\neq i} f_{ji} = \sum_{j\in\mathcal{C}_0,\, j\neq i} f_{ij} + d_i, \qquad i \in \mathcal{C}, \tag{23}$$

$$f_{ij} + f_{ji} = \frac{Q}{2} x_{ij}, \qquad i, j \in \mathcal{C}_0,\ i < j, \tag{24}$$

$$f_{ij} \geq 0, \qquad i, j \in \mathcal{C}_0,\ i \neq j, \tag{25}$$

$$\begin{cases} x_{0i} \geq 0, & \text{if one-city tour } 0 - i - 0 \text{ is allowed,} \\ x_{0i} \in [0, 1], & \text{otherwise} \end{cases} \qquad i \in \mathcal{C}, \tag{26}$$

$$x_{ij} \in \{0, 1\}, \qquad i, j \in \mathcal{C},\ i < j. \tag{27}$$

Observations

- Any integer feasible solution to **CVRP−2CF** implies integer values of x_{0i}
- CVRP is formulated using $n(n-1)/2$ binary variables while involves $n+1$ locations.
- The following inequality can be added if the number of vehicles is constrained: $\sum_{i\in\mathcal{C}} x_{0i} \leq 2m$.

Proof. We prove the proposition with the specified number m of vehicles to route first. Note that the CVRP with parameters $\mathcal{C}$, C, Q and m implies the corresponding set of feasible solutions, denoted by $FS_{cvrp}(\mathcal{C}, C, d, Q, m)$. Consider the set of integer feasible solutions (in terms of integer variables x only) to the **CVRP−2CF** problem. A feasible to **CVRP−2CF** integer solution x^* consists of m undirected routes that all pass through the depot, while all other subtours not involving the depot being eliminated due to constraints (23) and (24) (if $x^*_{0i} = 2$ for some i, this situation is interpreted as the simple one-city tour $0 - i - 0$). We denote the entire set of solutions feasible to **CVRP−2CF** by $FS(\textbf{CVRP}-\textbf{2CF})$ and will demonstrate that

$$FS(\textbf{CVRP}-\textbf{2CF}) = FS_{cvrp}(\mathcal{C}, C, d, Q, m). \qquad (28)$$

Consider a feasible integer solution to the **CVRP−2CF** program. Let $0 - i_1 - i_2 - \ldots - i_k - 0$ be an arbitrary undirected tour from the set of m tours and $S = \{i_1, \ldots, i_k\}$ be a set of customers assigned to, without loss of generality, the vehicle 1. What has to be proved is that $\sum_{i \in S} d_i \leq Q$. Consider the case of $|S| > 1$ first and suppose that $\sum_{i \in S} d_i > Q$. Due to (23), this assumption implies that the total amount of inflow of the commodity to set S is above Q: $\sum_{i \in S} f_{0i} > Q$. Due to (24), this inequality simplifies to $f_{0i_1} + f_{0i_k} > Q$. However, $y_{0i_1} + f_{i_1 0} = Q/2$ and $f_{0i_k} + f_{i_k 0} = Q/2$, which is why $f_{0i_1} + f_{0i_k} \leq Q$ and we obtain the contradiction to the assumption. Consider next the case of $|S| = 1$, i.e., $S = \{i\}$. Then $f_{0i} + f_{i0} = Q$ by (24), and $f_{0i} = f_{i0} + d_i$ due to (23). Therefore, $f_{0i} + f_{i0} = f_{i0} + d_i + f_{i0} = Q$, which is why $d_i \leq Q$ and $0 - i - 0$ is indeed a valid tour. Hence,

$$FS(\textbf{CVRP}-\textbf{2CF}) \subseteq FS_{cvrp}(\mathcal{C}, C, d, Q, m). \tag{29}$$

On the other hand, consider a feasible instance of the CVRP that consists of m undirected routes $R_t = \left(0 - i_1^t - \ldots - i_{k_t}^t - 0\right)$, $S_t = \{i_1^t, \ldots, i_{k_t}^t\}$, $t = 1, \ldots, m$. Take an arbitrary $t \in \{1, \ldots, m\}$, and there are again two cases:

1. $|S_t| > 1$. Let the flow of the commodity be initiated from the depot in both directions along the tour R_t as follows:

$$f_{0i_1^t} = f_{0i_{k_t}^t} = \frac{Q + \sum_{q \in S_t} d_q}{4}, \quad f_{i_1^t 0} = f_{i_{k_t}^t 0} = \frac{Q - \sum_{q \in S_t} d_q}{4}. \qquad (30)$$

The flows along the remaining arcs of R_t are defined as

$$f_{i_q^t i_{q+1}^t} = \frac{Q - \sum_{e \in S_t} d_e + \sum_{e \in S_t,\, e \geq q+1} d_e}{4}, \qquad q = 1, \ldots, k_t - 1, \qquad (31)$$

$$f_{i_{q+1}^t i_q^t} = \frac{Q - \sum_{e \in S_t} d_e + \sum_{e \in S_t,\, e \leq q} d_e}{4}, \qquad q = k_t - 1, \ldots, 1, \qquad (32)$$

with flow values along the other possible arcs between vertices in S_t to be equal to 0.

2. $|S_t| = 1$. Let

$$f_{0i_1^t} = \frac{Q + d_{i_1^t}}{4}, \quad f_{i_1^t 0} = \frac{Q - d_{i_1^t}}{4}. \qquad (33)$$

This way, all the flow balance constraints of the **CVRP−2CF** program for arcs associated with each tour R_t are satisfied, and we, therefore, have demonstrated that

$$FS(\mathbf{CVRP{-}2CF}) \supseteq FS_{cvrp}(\mathcal{C}, C, d, Q, m). \qquad (34)$$

Therefore, the statement (28) is proved. Note that the objective function (21) correctly represents the cost of any feasible integer solution, which, together with (28), proves the proposition for a fixed m.
Finally, if the number of vehicles is not defined in advance, then m is an unknown integer variable. The proposition is true for any fixed m, therefore the proposition is true for an unknown integer m as well. □

Improving CVRP−2CF

Consider the following constraint

$$f_{0i} + f_{i0} = \frac{Q}{2} x_{0i} \quad \text{as} \quad f_{0i} = \frac{Q}{2} x_{0i} - f_{i0}, \qquad i \in \mathcal{C}, \tag{35}$$

Using this representation in the flow balance constraints, we obtain:

$$\frac{Q}{2} x_{0i} - f_{i0} + \sum_{j \in \mathcal{C}, j \neq i} f_{ji} = \sum_{j \in \mathcal{C}, j \neq i} f_{ij} + d_i + f_{i0}, \qquad i \in \mathcal{C},$$

which can be equivalently presented as

$$\frac{Q}{2} x_{0i} + \sum_{j \in \mathcal{C}, j \neq i} f_{ji} \geq \sum_{j \in \mathcal{C}, j \neq i} f_{ij} + d_i, \qquad i \in \mathcal{C}.$$

Moreover, the following well-known set of valid inequalities Gouveira (1995) for the one-commodity network flow formulation

$$f_{ij} \geq d_j x_{ij}, \qquad i, j \in \mathcal{C}, \, i \neq j, \tag{36}$$

can be incorporated into the two-commodity network flow CVRP formulation as follows:

$$f_{ij} = \widetilde{f}_{ij} + \frac{d_j}{2} x_{ij}, \qquad f_{ji} = \widetilde{f}_{ji} + \frac{d_i}{2} x_{ij}, \qquad i, j \in \mathcal{C}, \, i < j, \tag{37}$$

$$\widetilde{f}_{ij}, \, \widetilde{f}_{ji} \geq 0, \qquad i, j \in \mathcal{C}, \, i < j. \tag{38}$$

Improved **CVRP−2CF**

CVRP−2CF m. :

$$\min \sum_{i,j\in\mathcal{C}_0,\, i<j} c_{ij}x_{ij} \tag{39}$$

subject to

$$\sum_{j\in\mathcal{C}_0,\, i<j} x_{ij} + \sum_{j\in\mathcal{C}_0,\, i>j} x_{ji} = 2, \qquad i \in \mathcal{C}, \tag{40}$$

$$\frac{Q}{2}x_{0i} + \sum_{j\in\mathcal{C},\, j\neq i}\left(\tilde{f}_{ji} + \frac{d_i}{2}x_{\min(i,j),\,\max(i,j)}\right) \geq \sum_{j\in\mathcal{C},\, j\neq i}\left(\tilde{f}_{ij} + \frac{d_j}{2}x_{\min(i,j),\,\max(i,j)}\right) + d_i, \qquad i \in \mathcal{C}, \tag{41}$$

$$\tilde{f}_{ij} + \tilde{f}_{ji} = \frac{Q - d_i - d_j}{2}x_{ij}, \qquad i,\, j \in \mathcal{C},\ i < j, \tag{42}$$

$$\tilde{f}_{ij} \geq 0, \qquad i,\, j \in \mathcal{C},\ i \neq j, \tag{43}$$

$$\begin{cases} x_{0i} \geq 0, & \text{if one-city tour } 0 - i - 0 \text{ is allowed}, \\ x_{0i} \in [0,\, 1], & \text{otherwise} \end{cases} \qquad i \in \mathcal{C}, \tag{44}$$

$$x_{ij} \in \{0,\, 1\}, \qquad i,\, j \in \mathcal{C},\ i < j. \tag{45}$$

The formulation implies Gouveira (1995) inequalities. Total # of constraints is $n(n+3)/2$.

Valid Inequalities ACVRP Case: Motivation

Even though the number of vehicles to serve $\mathcal{C}$ is not specified in a constraint, what if we try to find a lower bound on the min number of required vehicles and impose it as a constraint?

$$\sum_{i \in \mathcal{C}} x_{0i} \geq k(\mathcal{C}) \tag{46}$$

Possibilities for $k(\mathcal{C})$:

- $k(\mathcal{C}) = \sum_{i \in \mathcal{C}} d_i / Q$. this is a fractional number and is an absolute minimum number of the required vehicles
- $k(\mathcal{C}) = \lceil \sum_{i \in \mathcal{C}} d_i / Q \rceil$. this is an integer number and is slightly more accurate
- $k(\mathcal{C}) =$ min # of bins of size Q to pack all items $i \in \mathcal{C}$, i.e., obtained by solving the bin packing problem, a more accurate number.

Valid Inequalities ACVRP Case: Example

Suppose that

$$\vec{d} = (5, 7, 6, 6)$$

$$Q = 10$$

Possibilities for $k(\mathcal{C})$:

- $k(\mathcal{C}) = \sum_{i \in \mathcal{C}} d_i / Q = 2.4$
- $k(\mathcal{C}) = \lceil \sum_{i \in \mathcal{C}} d_i / Q \rceil = 3$
- $k(\mathcal{C}) =$ min # of bins of size Q to pack all items $i \in \mathcal{C} = 4$ (!!)

Inequalities of such type are called capacity-based inequalities.

Valid Inequalities ACVRP Case

An inequality for $S \subset \mathcal{C}$:

- $k(S) = \sum_{i \in S} d_i / Q$
- $k(S) = \lceil \sum_{i \in S} d_i / Q \rceil$
- $k(S) = \min \#$ of bins of size Q to pack all items $i \in S$

How to impose them?

- $x(\bar{S}, S) \geq k(S) = \sum_{i \in S} d_i / Q$, called fractional capacity inequality
- $x(\bar{S}, S) \geq k(S) = \lceil \sum_{i \in S} d_i / Q \rceil$, called rounded capacity inequality
- $x(\bar{S}, S) \geq k(S) = \min \#$ of bins of size Q to pack all items $i \in S$, called weak capacity inequality

where

$$\bar{S} = \mathcal{C}_0 \backslash S, \quad x(\bar{S}, S) = \sum_{j \in \bar{S}} \sum_{i \in S} x_{ji}.$$

Note that even weak capacity inequalities are not always tight!

Valid Inequalities CVRP Case

An inequality for $S \subset \mathcal{C}$:

- $k(S) = \sum_{i \in S} d_i / Q$
- $k(S) = \lceil \sum_{i \in S} d_i / Q \rceil$
- $k(S) = \min \#$ of bins of size Q to pack all items $i \in S$

How to impose them?

- $x(\bar{S}, S) \geq 2k(S) = 2 \sum_{i \in S} d_i / Q$, called fractional capacity inequality
- $x(\bar{S}, S) \geq 2k(S) = 2 \lceil \sum_{i \in S} d_i / Q \rceil$, called rounded capacity inequality
- $x(\bar{S}, S) \geq 2k(S) = 2 \min \#$ of bins of size Q to pack all items $i \in S$, called weak capacity inequality

where

$$\bar{S} = \mathcal{C}_0 \backslash S, \quad x(\bar{S}, S) = \sum_{j \in \bar{S}} \sum_{i \in S, j < j} x_{ji} + \sum_{j \in \bar{S}} \sum_{i \in S, i < j} x_{ij}.$$

Note that even weak capacity inequalities are not always tight!

CVRP: a Formulation

An early problem formulation is due to Laporte and Nobert (1983):

$$\min \sum_{i,j\in\mathcal{C}_0,\, i<j} c_{ij}x_{ij}, \tag{47}$$

subject to

$$\sum_{j\in\mathcal{C}_0,\, i<j} x_{ij} + \sum_{j\in\mathcal{C}_0,\, i>j} x_{ji} = 2, \qquad i \in \mathcal{C}, \tag{48}$$

$$x(\bar{S}, S) \geq 2\left\lceil \sum_{i\in S} d_i/Q \right\rceil, \qquad \forall S \subset \mathcal{C},\ |S| \geq 2, \tag{49}$$

$$\begin{cases} x_{0i} \geq 0, & \text{if one-city tour } 0 - i - 0 \text{ is allowed,} \\ x_{0i} \in [0, 1], & \text{otherwise} \end{cases} \qquad i \in \mathcal{C}, \tag{50}$$

$$x_{ij} \in \{0, 1\}, \qquad i, j \in \mathcal{C},\ i < j. \tag{51}$$

Laporte, Gilbert, and Yves Nobert. "A branch and bound algorithm for the capacitated vehicle routing problem." Operations-Research-Spektrum 5 (1983): 77-85.

ACVRP: a Formulation

An adaptation of the formulation due to Laporte and Nobert (1983) to the ACVRP case:

$$\min \sum_{i,j\in\mathcal{C}_0,\, i\neq j} c_{ij}x_{ij}, \tag{52}$$

subject to

$$\sum_{j\in\mathcal{C}_0,\, i\neq j} x_{ij} = 1, \qquad i\in\mathcal{C}, \tag{53}$$

$$\sum_{j\in\mathcal{C}_0,\, i\neq j} x_{ji} = 1, \qquad i\in\mathcal{C}, \tag{54}$$

$$x(\bar{S},\, S) \geq \left\lceil \sum_{i\in S} d_i/Q \right\rceil, \qquad \forall S\subset\mathcal{C},\, |S|\geq 2, \tag{55}$$

$$x_{ij}\in\{0,\, 1\}, \qquad i,\, j\in\mathcal{C}_0,\, i\neq j. \tag{56}$$

Remark 1: Binary constraint on x_{0i} can be safely relaxed. Hence, the problem involves $n+1$ locations and may need only $n(n-1)$ binary variables.
Remark 2: If tour $0\to i\to 0$ is not allowed, then impose $x_{0i}+x_{i0}\leq 1$.
Remark 3: The above problem formulation implies that the optimal number of vehicles needs to be obtained. If the number of vehicles to be used is known, impose $\sum_{i\in\mathcal{C}} x_{0i} = m$ $(\leq m)$.

Separation Problem, CVRP Case

Family of capacity inequalities

$$x(\bar{S}, S) \geq 2k(S), \qquad S \subset \mathcal{C}, \, |S| \geq 2, \tag{57}$$

- $k(S) = \sum_{i \in S} d_i / Q$, separation of fractional capacity inequalities is polynomially solvable but such inequalities are not efficient
- $k(S) = \lceil \sum_{i \in S} d_i / Q \rceil$, rounded capacity inequalities are relatively efficient, their separation problem complexity is NP-hard in general, but possible to do in practice using mixed integer programming!
- $k(S) = \min \#$ of bins of size Q to pack all items $i \in S$, algorithms for exact separation of weak capacity inequalities are not known in the literature (!)

Rounded Capacity Inequalities CVRP

Family of inequalities

$$x(\bar{S}, S) \geq 2\lceil \sum_{i \in S} d_i/Q \rceil, \qquad S \subset \mathcal{C}. \tag{58}$$

Let x^* be a feasible solution to a CVRP formulation. Does there exist S:

$$x^*(\bar{S}, S) < 2\lceil \sum_{i \in S} d_i/Q \rceil? \tag{59}$$

Why does the min cut based separation routine not work? We can easily find an optimal S^*:

$$x^*(\bar{S}^*, S^*) = \min_S \; x^*(\bar{S}, S) \tag{60}$$

and check whether

$$x^*(\bar{S}^*, S^*) < 2, \tag{61}$$

add violated S^* inequalities, re-optimize, until no violated constraint exists; but S^* is always limited to belong to the *arg* $\min_S \; x^*(\bar{S}, S)$!

Rounded Capacity Inequalities

In other words, we can make sure that

$$\min \text{cut}(x^*, 1, i) = \min_S \; x^*(\bar{S}, S) = x^*(\bar{S}^*, S^*) \geq 2, \tag{62}$$

for every $i \in \mathcal{C}$. However, what about $S \notin arg \min_S x^*(\bar{S}, S)$? It might be that set S exists, such that

$$min_S x^*(\bar{S}, S) = 2.1 \geq 2, \qquad \exists S : \; x^*(\bar{S}, S) = 2.2 \geq 2.1, \tag{63}$$

where

$$\lceil \sum_{i \in S} d_i / Q \rceil = 2,$$

which means that RC inequality

$$x^*(\bar{S}, S) \geq 4, \tag{64}$$

is violated. Such inequalities are not considered by the min cut-based separation routine! A complete answer to this question is here:
Diarrassouba, Ibrahima. "On the complexity of the separation problem for rounded capacity inequalities." Discrete Optimization 25 (2017): 86-104.

Exact Separation of Rounded Capacity Inequalities

For a given $x^* = \{x^*_{ij} \mid i, j \in \mathcal{C}_0, i < j\}$, feasible to

$$\sum_{j \in \mathcal{C}_0, i<j} x_{ij} + \sum_{j \in \mathcal{C}_0, j<i} x_{ji} = 2, \qquad i \in \mathcal{C}, \tag{65}$$

$$x_{0i} \in [0, 2], \qquad i \in \mathcal{C}, \tag{66}$$

$$x_{ij} \in [0, 1], \qquad i, j \in \mathcal{C}, i < j, \tag{67}$$

is there a RC inequality

$$x\left(\bar{S}, S\right) \geq 2 \left\lceil \sum_{i \in S} d_i / Q \right\rceil, \qquad S \subset \mathcal{C}, |S| \geq 2, \tag{68}$$

that is violated by x^*?

Model the set S: let $\delta_i \in \{0, 1\}$ define whether $i \in S$, hence the first constraint

$$\sum_{i \in \mathcal{C}} \delta_i \geq 2. \tag{69}$$

Then, an $\{i, j\} \in \left(\bar{S}, S\right)$ if and only if $i \in S$ and $j \in \bar{S}$ OR $i \in \bar{S}$ and $j \in S$, which is described by the condition:

$$\delta_i + \delta_j = 1 \quad \Longleftrightarrow \quad \delta_i + \delta_j - 2\delta_i\delta_j = 1.$$

Capacity of a Cut Set

Observed capacity of a cut set:

$$x^*\left(\bar{S},\, S\right) = \sum_{i \in \mathcal{C}} x^*_{0i}\delta_i + \sum_{i,j \in \mathcal{C},\, i<j} x^*_{ij}(\delta_i + \delta_j - 2\gamma_{ij}), \tag{70}$$

$$\sum_{i \in \mathcal{C}} \delta_i \geq 2, \tag{71}$$

$$\gamma_{ij} \geq \delta_i + \delta_j - 1, \quad \gamma_{ij} \geq 0, \qquad i, j \in \mathcal{C},\, i < j, \tag{72}$$

$$\gamma_{ij} \leq \delta_i, \qquad i, j \in \mathcal{C},\, i < j, \tag{73}$$

$$\gamma_{ij} \leq \delta_j, \qquad i, j \in \mathcal{C},\, i < j, \tag{74}$$

$$\delta_i \in \{0,\, 1\}, \qquad i \in \mathcal{C}. \tag{75}$$

Target capacity of cut set $\left(\bar{S},\, S\right)$:

$$2\lceil \sum_{i \in S} d_i / Q \rceil = \max_{\alpha}\; 2\left(\alpha + 1\right), \tag{76}$$

subject to

$$Q\alpha + 1 \leq \sum_{i \in S} d_i, \tag{77}$$

$$\alpha \in \mathbb{Z}. \tag{78}$$

Example: suppose that $S : \left\lceil \sum_{i \in S} d_i / Q \right\rceil = 1$, then $\alpha = 0$, and so the target capacity is equal to 2, which is correct.

Exact Separation of Rounded Capacity Inequalities

RCI−Sep:

$$\max_{\delta_i,\,\alpha,\,\gamma_{ij}} \; 2\,(\alpha+1) - \sum_{i\in\mathcal{C}} x^*_{0i}\delta_i - \sum_{i,j\in\mathcal{C},\,i<j} x^*_{ij}(\delta_i+\delta_j-2\gamma_{ij}), \tag{79}$$

subject to

$$Q\alpha + 1 \le \sum_{i\in\mathcal{C}} d_i\delta_i, \tag{80}$$

$$\sum_{i\in\mathcal{C}} \delta_i \ge 2, \tag{81}$$

$$\gamma_{ij} \le \delta_i, \qquad i,\,j\in\mathcal{C},\ i<j, \tag{82}$$

$$\gamma_{ij} \le \delta_j, \qquad i,\,j\in\mathcal{C},\ i<j, \tag{83}$$

$$\gamma_{ij} \ge 0, \qquad i,\,j\in\mathcal{C},\ i<j, \tag{84}$$

$$\alpha \in \mathbb{Z}, \tag{85}$$

$$\delta_i \in \{0,\,1\}, \qquad i\in\mathcal{C}. \tag{86}$$

Usage

- If the optimal objective of **RCI−Sep** is positive, then a violated RC inequality is identified
- If the optimal objective of **RCI−Sep** is 0 or negative, then no violated RC inequality exists
- Drawback: **RCI−Sep** returns at most one cut per run. How could we find multiple violated inequalities per iteration and hopefully reduce the number of iterations and thus the overall time?

A proposal to speed up separation:

Callback :

for every incumbent solution $\{\delta_i \in \{0, 1\}\}$ of **RCI−Sep**

$$\text{if } \sum_{i \in \mathcal{C}} x^*_{0i}\delta_i + \sum_{i,j \in \mathcal{C}, i<j} x^*_{ij}(\delta_i + \delta_j - 2\delta_i\delta_j) < 2\left\lceil \sum_{i \in \mathcal{C}} d_i\delta_i / Q \right\rceil : \tag{87}$$

$$\text{add } \left(\bar{S},\, S\right) \text{ to the set of violated RC inequalities.} \tag{88}$$

Notes

Presented theory was developed in Pavlikov, K., N. C. Petersen, and J. L. Sørensen. "Exact separation of the rounded capacity inequalities for the capacitated vehicle routing problem." Networks 83.1 (2024): 197-209.

This is not the first time an exact separation approach for RCI was introduced; the work of Fukasawa et al. (2006):

- described a very similar approach for exact separation of RCIs
- defined α as a fixed integer parameter instead of a variable, which required a larger number of calls for the separation routine to determine whether a violated RCI exists

For this reason, the heuristic approaches to RCI separation have been used; most widely known heuristic and code for that CVRPSEP:

Lysgaard, Jens, Adam N. Letchford, and Richard W. Eglese. "A new branch-and-cut algorithm for the capacitated vehicle routing problem." Mathematical programming 100 (2004): 423-445.

Summary

- Compact formulations of the CVRP / ACVRP are presented
- Several families of exponentially sized valid inequalities are introduced
- Separation problem of the Rounded Capacity Inequalities is considered
- Current state of development of the mixed integer linear programming solvers allows us to separate RCIs exactly and faster than when using a heuristic approach.