

DM872

Mathematical Optimization at Work

## Optimization under Uncertainty

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## 1. Structured LP models

## 2. Optimization under Uncertainty

Probabilistic Constraints

Recourse Problems

# Multiple Plant Models

	Factory A		Factory B	
	Standard	Deluxe	Standard	Deluxe
(Machine 1) Grinding	4	2	5	3
(Machine 2) Polishing	2	5	5	6

Maximize Profit  $10x_1 + 15x_2$   
 Subject to Raw A  $4x_1 + 4x_2 \leq 75$   
 Grinding A  $4x_1 + 2x_2 \leq 80$   
 Polishing A  $2x_1 + 5x_2 \leq 60$   
 $x_1, x_2 \geq 0$

Maximize Profit  $10x_3 + 15x_4$   
 Subject to Raw B  $4x_3 + 4x_4 \leq 45$   
 Grinding B  $5x_3 + 3x_4 \leq 60$   
 Polishing B  $5x_3 + 6x_4 \leq 75$   
 $x_3, x_4 \geq 0$

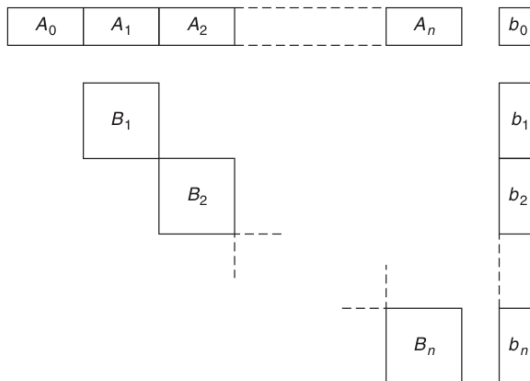
# Multiple Plant Models

$$\begin{array}{ll}
 \text{Maximize Profit} & 10x_1 + 15x_2 + 10x_3 + 15x_4 \\
 \text{Subject to Raw} & 4x_1 + 4x_2 + 4x_3 + 4x_4 \leq 120 \\
 \text{Grinding A} & 4x_1 + 2x_2 + \phantom{4x_3 + 4x_4} \leq 80 \\
 \text{Polishing A} & 2x_1 + 5x_2 + \phantom{4x_3 + 4x_4} \leq 60 \\
 \text{Grinding B} & \phantom{4x_1 + 2x_2 + } + 5x_3 + 3x_4 \leq 60 \\
 \text{Polishing B} & \phantom{4x_1 + 2x_2 + } + 5x_3 + 6x_4 \leq 75 \\
 & x_1, x_2, x_3, x_4 \geq 0
 \end{array}$$

10	15	10	15		
4	4	4	4	≤	120
4	2			≤	80
2	5			≤	60
		5	3	≤	60
		5	6	≤	75

allocation problems **between** plants +  
decision making **within** plants.

# Block Angular Structure

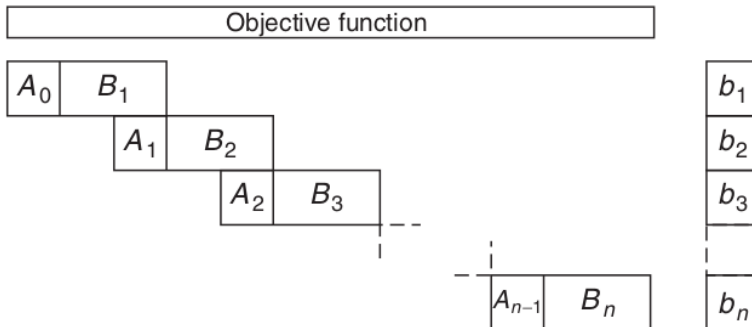


The rows  $A_0, \dots, A_n$  are known as **common rows**.  
The diagonally placed blocks are known as **submodels**.

# Staircase Structure

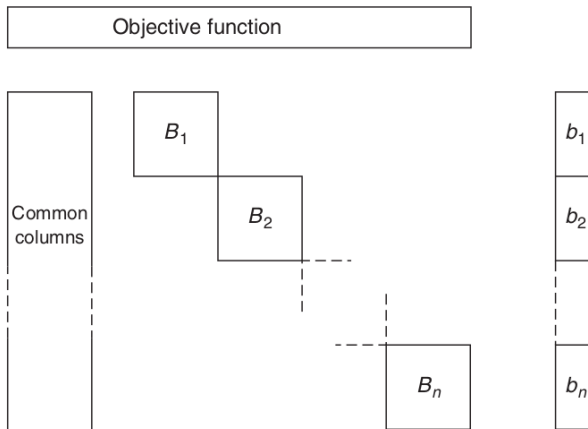
Multi-product and multi-period models lead also to staircase structures:

Amount in store at end of period  $(t - 1)$  + amount bought in period  $t$  =  
amount used in period  $t$  + amount in store at end of period  $t$



It can be converted into a block angular structure: alternate 'steps' such as  $(A_0, B_1), (A_2, B_3)$  can be treated as subproblem constraints and the intermediate 'steps', eg.  $(A_1, B_1)$ , as common rows.

# Block Angular Structure



It can be seen as the dual of the common row structure. However, this structure arises often in stochastic programming cases and it can be treated in its own way.

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Planning under uncertainty when data not known with certainty:

- inaccuracy of data (because of laziness of ignorance)
- multi-stage models where certain events, which need to be modelled, have not yet occurred.

Alternative approaches:

- sensitivity analysis, how solutions change with limited changes to data
- **robust optimization**, when we cannot quantify the uncertainty and the related risk. Stable solutions
- **risk-averse** (maximin, conditional value-at-risk): make the worst possible result as little bad as possible
- **stochastic optimization**, when uncertainty can be quantified.

Typical examples:

- News vendor problem
- Energy production
- Portfolio optimization
- Multi-period production planning

Stochastic programming (SP) is mathematical (i.e. linear, integer, mixed-integer, nonlinear) programming but with a stochastic element present in the data.

- in deterministic mathematical programming the data (coefficients) are known numbers
- in stochastic programming data are unknown, instead we may have a probability distribution present.

We consider two distinct stochastic programming problems:

- probabilistic constraints
- recourse problems.

The following slides are based on John E Beasley's OR-Notes on Stochastic Programming ([people.brunel.ac.uk/~mastjjb/jeb/or/sp.html](http://people.brunel.ac.uk/~mastjjb/jeb/or/sp.html))

Learn more about SP at <https://www.stoprog.org/>.

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Suppose that we have two six-sided dice.

Die one gives a result  $a_1$  when thrown and die two a result  $a_2$ .

Assuming the dice are fair we have discrete probability distributions for  $a_1$  and  $a_2$  as:

$$a_1 = i \quad (i = 1, \dots, 6) \quad \text{with probability } 1/6$$

$$a_2 = j \quad (j = 1, \dots, 6) \quad \text{with probability } 1/6$$

Consider a simple LP with two variables and one constraint:

$$\begin{aligned} &\text{minimise} && 5x + 6y \\ &\text{subject to:} && a_1x + a_2y \geq 3 \\ &&& x, y \geq 0 \end{aligned}$$

What does this LP mean?

One interpretation could be that we wish the constraint  $a_1x + a_2y \geq 3$  to hold for all possible values of  $a_1$  and  $a_2$ . Then we simply have a deterministic LP with two variables and 36 constraints:

$$\begin{array}{ll} \text{minimise} & 5x + 6y \\ \text{subject to:} & ix + jy \geq 3 \quad i = 1, \dots, 6 \quad j = 1, \dots, 6 \\ & x, y \geq 0 \end{array}$$

Suppose now that we insist that the constraint  $a_1x + a_2y \geq 3$  holds only with a specified probability  $1 - \alpha$  (where  $0 < \alpha < 1$ ).

For example  $\alpha = 0.05$  would mean that we want the constraint  $a_1x + a_2y \geq 3$  to hold with probability 0.95.

**Chance constraint:** A constraint needs not always be true now, rather it needs only be true, eg, 95% of the time.

$$\begin{array}{ll}\text{minimise} & 5x + 6y \\ \text{subject to:} & \text{Prob}(a_1x + a_2y \geq 3) \geq 1 - \alpha \\ & x, y \geq 0\end{array}$$

Here,  $a_1$  and  $a_2$  are unknown, we merely have probability distribution information for them. We are required to choose values for  $x$  and  $y$  such that the objective function is minimised and the probability that the constraint  $a_1x + a_2y \geq 3$  is satisfied is at least  $1 - \alpha$ .

# Is this problem well-defined?

For each pair of values  $(a_1, a_2)$  we have an associated joint probability ( $1/36$  in this simple case) then: given values for  $x \geq 0$  and  $y \geq 0$  we can easily check by enumeration whether the constraint is true with probability  $1 - \alpha$ .

Eg, for  $x = 0, y = 1$  and  $\alpha = 0.05$ :

$a_1$	$a_2$	Is $a_1 0 + a_2 1 \geq 3$ ?	Probability
1	1	No	$1/36$
2	1	No	$1/36$
...			

We already have a probability of  $2/36 = 0.0555$  that the constraint is infeasible. Hence, it is impossible for the constraint to be feasible with probability 0.95 (since  $1 - 0.0555 = 0.9445$ ). Hence,  $x = 0, y = 1$  is not a solution to the problem.

Conceptually, we could simply enumerate all possible values for  $x$  and  $y$  and choose those values that minimise  $5x + 6y$ .

Hence, the problem is well defined.



This problem is an example of a **stochastic (linear) program with probabilistic constraints**. Such problems are also sometimes called **chance-constrained linear programs**:

- mix of probabilistic and deterministic coefficients in the same problem
- mix of probabilistic and deterministic constraints in the same problem.

To solve SP's with probabilistic constraints we transform them into an **equivalent deterministic program**. Note here however that even if the original SP is linear the equivalent deterministic program may not be.

# Solving SP's with probabilistic constraints

Define zero-one variables  $z_{ij}$  using:

$$\begin{aligned} z_{ij} &= 1 && \text{if when } a_1 \text{ takes the value } i \ (i = 1, \dots, 6) \text{ and} \\ &&& a_2 \text{ takes the value } j \ (j = 1, \dots, 6) \text{ ix} + jy \geq 3 \\ &= 0 && \text{otherwise} \end{aligned}$$

Let  $p_{ij}$  be the probability that  $a_1$  takes the value  $i$  ( $i = 1, \dots, 6$ ) and  $a_2$  takes the value  $j$  ( $j = 1, \dots, 6$ ). That is,  $p_{ij} = 1/36$ .

The deterministic equivalent is:

$$\text{minimise } Mz_{ij} + (5x + 6y) \tag{1}$$

$$\text{subject to: } z_{ij} \geq [(ix + jy) - 3 + \delta]/M \quad i = 1, \dots, 6 \quad j = 1, \dots, 6 \tag{2}$$

$$\sum_{i=1}^6 \sum_{j=1}^6 p_{ij} z_{ij} \geq 1 - \alpha \tag{3}$$

$$z_{ij} \in \{0, 1\} \quad i = 1, \dots, 6 \quad j = 1, \dots, 6 \tag{4}$$

$$x, y \geq 0 \tag{5}$$

Suppose now that:

- $a_1$  has a normal distribution with mean  $A_1$  and standard deviation  $D_1$ , i.e.  $\mathcal{N}(A_1, (D_1)^2)$ ;
- $a_2$  has a normal distribution with mean  $A_2$  and standard deviation  $D_2$ , i.e.  $\mathcal{N}(A_2, (D_2)^2)$

and  $a_1$  and  $a_2$  independent.

$a_1x + a_2y \sim \mathcal{N}(A_1x + A_2y, [(D_1x)^2 + (D_2y)^2]^{1/2})$  because sum of normal distrs.

Hence,  $\text{Prob}(a_1x + a_2y \geq 3) \geq 1 - \alpha$  can be addressed in the standard way for normal distribution probability calculations.

Let  $K$  be the value of the standard normal distribution  $\mathcal{N}(0, 1)$  which has a probability of exactly  $\alpha$  of being exceeded (e.g. if  $\alpha = 0.025$  then  $K = 1.96$ ). Such values are easily obtained from statistical tables.

$$\frac{3 - (A_1x + A_2y)}{\sqrt{(D_1x)^2 + (D_2y)^2}} \geq K$$

So our SP becomes a non linear program:

$$\begin{aligned} &\text{minimise } 5x + 6y \\ &\text{subject to: } 3 - (A_1x + A_2y) \geq K \sqrt{(D_1x)^2 + (D_2y)^2} \\ &\quad x, y \geq 0 \end{aligned}$$

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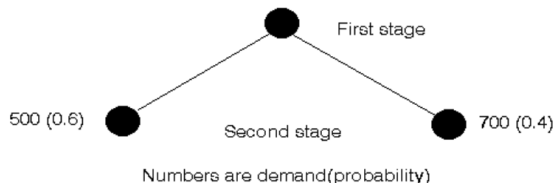
In the simplest model of this type we have two stages:

- in the first stage we make a decision
- in the second stage we see a realisation of the stochastic elements of the problem BUT are allowed to make further decisions to avoid the constraints of the problem becoming infeasible.

In the second stage the decisions that we make will be dependent upon the particular realisation of the stochastic elements observed.

- We produce product X
- each unit of X that we make costs us 20 kr.
- X is made to meet demand from customers in the next time period.
- demand is stochastic, with a discrete probability distribution: demand =  $D_s$  with probability  $p_s$  ( $s = 1, \dots, S$ ). Informally, we can think of having  $S$  scenarios for possible future demand.
- customer demand must be met.
- we have the flexibility to buy in the product from an external supplier to meet observed customer demand but this costs us 30 kr per unit (i.e. we have recourse to an additional source of supply if demand exceeds production).
- How much should we choose to make now before we know what the customer demand is?

$S = 2$  and  $D_1 = 500$ ,  $p_1 = 0.6$ ;  $D_2 = 700$ ,  $p_2 = 0.4$ .



If we were to produce 600 then if demand is 500 we are OK,  
if demand is 700 we need recourse to an extra 100 units to meet it.

### Two-stage model:

- action, make a decision (amount to produce)
- observation, observe a realisation of the stochastic elements (demand that occurs)
- reaction (recourse), further decisions, depending upon the realisation observed (extra production to meet demand if necessary)



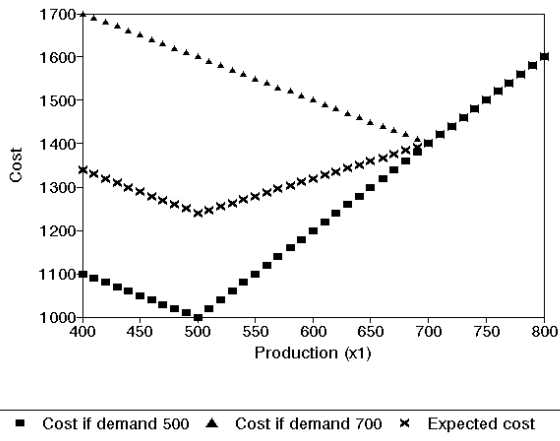
Let  $y_{2s} \geq 0$  be the number of units of X to buy from the external supplier at the second stage in scenario  $s$  when the stochastic realisation of the demand is  $D_s$  ( $s = 1, \dots, S$ ).

Goal: minimise total expected cost

$$\begin{aligned} & \text{minimise } 2x_1 + \sum_{s=1}^S p_s(3y_{2s}) \\ & \text{subject to } x_1 + y_{2s} \geq D_s \quad s = 1, \dots, S \\ & \quad x_1 \geq 0 \\ & \quad y_{2s} \geq 0 \quad s = 1, \dots, S \end{aligned}$$

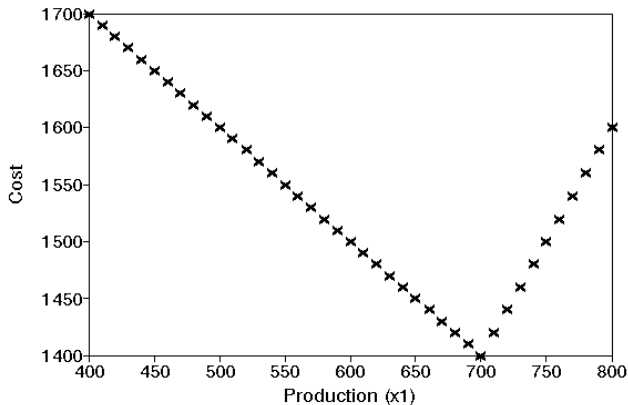
It is a deterministic program. We could require  $x_1$  and  $y_{2s}$  to be integer.

# Cost incurred



The production quantity that minimises expected cost is  $x_1 = 500$ .

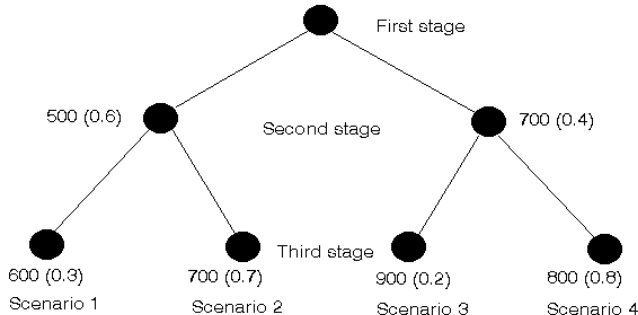
# Optimize the worst case



To minimise our worst case cost we should produce 700 now.

- the stochastic elements have a discrete distribution
- the realisations of the stochastic elements are represented as a number of future scenarios

We look forward two periods into the future in planning production.



Two level (three stage) binary scenario tree

- We initially make a decision about how much to produce.
- At the second stage have two possible realisations of the stochastic demand:
  - a demand of 500 with probability 0.6
  - a demand of 700 with probability 0.4
- After this realisation we make a decision as to how much to produce to meet demand in the next period (the third-stage).
- At the third stage we again have two possible realisations of the stochastic demand, but these are different depending upon the realisation at the second stage. If second stage was 500 then:
  - a demand of 600 with probability 0.3
  - a demand of 700 with probability 0.7
- (at each level in the scenario tree the appropriate probabilities must sum to one)

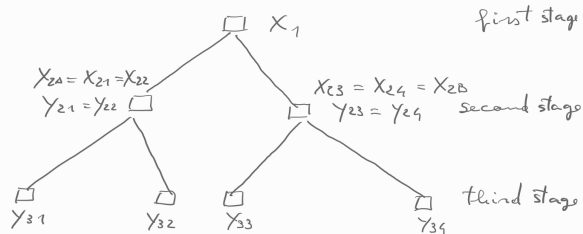
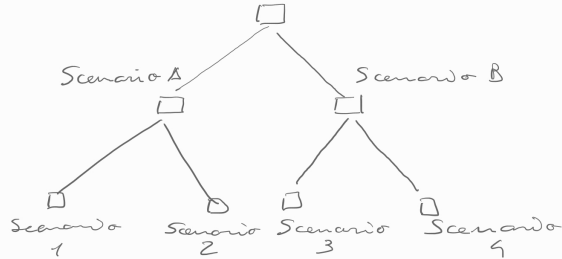
This two-level scenario tree actually represents  $2^2 = 4$  possible scenarios of the future:

Scenario	Second stage	Third stage	Probability
1	500	600	$0.6(0.3) = 0.18$
2	500	700	$0.6(0.7) = 0.42$
3	700	900	$0.4(0.2) = 0.08$
4	700	800	$0.4(0.8) = 0.32$

We have the following order of events:

- in the first stage a decision as to how much to produce; then
- in the second stage a realisation of the stochastic element (demand); then
- a decision as to the values of the recourse variables; then
- in the second stage a decision as to how much to produce; then
- in the third stage a realisation of the stochastic element (demand); and finally
- a decision as to the values of the recourse variables.

# Decisions and Decision Variables



- $x_1 \geq 0$  be the number of units of X to produce now (at the first stage)
- $y_{2s} \geq 0$  be the number of units of X to buy from the external supplier at the second stage in scenario  $s$  ( $s = 1, \dots, 4$ )
- $x_{2s} \geq 0$  be the number of units of X to produce at the second stage in scenario  $s$  ( $s=1, \dots, 4$ )
- $y_{3s} \geq 0$  be the number of units of X to buy from the external supplier at the third stage in scenario  $s$  ( $s=1, \dots, 4$ )



At the first stage, the constraints to ensure customer demand is satisfied are:

$$x_1 + y_{2s} \geq 500 \quad (s=1,2)$$

$$x_1 + y_{2s} \geq 700 \quad (s=3,4)$$

At the second stage we will have units left over (i.e. inventory) to help meet future demand. This inventory level will be:

$$x_1 + y_{2s} - 500 \quad (s=1,2)$$

$$x_1 + y_{2s} - 700 \quad (s=3,4)$$

To ensure that demand is met in the third stage we have:

inventory + amount produced + amount bought externally  $\geq$  demand

$$x_1 + y_{2s} - 500 + x_{2s} + y_{3s} \geq 600 \quad (s=1)$$

$$x_1 + y_{2s} - 500 + x_{2s} + y_{3s} \geq 700 \quad (s=2)$$

$$x_1 + y_{2s} - 700 + x_{2s} + y_{3s} \geq 900 \quad (s=3)$$

$$x_1 + y_{2s} - 700 + x_{2s} + y_{3s} \geq 800 \quad (s=4)$$

**non-anticipativity constraints**, scenarios with a common history must have the same set of decisions:

scenarios 1 and 2, second stage:

$$y_{21}=y_{22}$$

$$x_{21}=x_{22}$$

scenarios 3 and 4, second stage:

$$y_{23}=y_{24}$$

$$x_{23}=x_{24}$$

objective function: minimize expected costs

Scenario	Probability	Cost
1	0.18	$2x_{21} + 3y_{21} + 3y_{31}$
2	0.42	$2x_{22} + 3y_{22} + 3y_{32}$
3	0.08	$2x_{23} + 3y_{23} + 3y_{33}$
4	0.32	$2x_{24} + 3y_{24} + 3y_{34}$

Weighting each scenario cost by the associated scenario probability will give the expected cost.

minimise

$$2x_1 + 0.18(2x_{21} + 3y_{21} + 3y_{31}) + 0.42(2x_{22} + 3y_{22} + 3y_{32}) \\ + 0.08(2x_{23} + 3y_{23} + 3y_{33}) + 0.32(2x_{24} + 3y_{24} + 3y_{34})$$

# The full model

minimise

$$2x_1 + 0.18(2x_{21} + 3y_{21} + 3y_{31}) + 0.42(2x_{22} + 3y_{22} + 3y_{32}) \\ + 0.08(2x_{23} + 3y_{23} + 3y_{33}) + 0.32(2x_{24} + 3y_{24} + 3y_{34})$$

subject to

$$x_1 + y_{2s} \geq 500 \quad (s=1,2)$$

$$x_1 + y_{2s} \geq 700 \quad (s=3,4)$$

$$x_1 + y_{2s} - 500 + x_{2s} + y_{3s} \geq 600 \quad (s=1)$$

$$x_1 + y_{2s} - 500 + x_{2s} + y_{3s} \geq 700 \quad (s=2)$$

$$x_1 + y_{2s} - 700 + x_{2s} + y_{3s} \geq 900 \quad (s=3)$$

$$x_1 + y_{2s} - 700 + x_{2s} + y_{3s} \geq 800 \quad (s=4)$$

$$y_{21}=y_{22}$$

$$x_{21}=x_{22}$$

$$y_{23}=y_{24}$$

$$x_{23}=x_{24}$$

all variables  $\geq 0$

After taking a first stage decision, a random outcome (**scenario**) occurring with probability  $p_s$  involving one or more of the future data is observed. Then, an optimal second stage decision (**recourse action**) depending on the first stage and the scenario  $s$  is taken

Example:

- (Stage 1): decide production before the demand and future prices (uncertain) are known.
- (Stage 2): decide whether to sell any excess production at a lower price or extra produce to make up a shortfall at a higher cost.

(stage 1 variables) Production decisions:  $x_1, x_2, \dots, x_n$ .

(stage 2 variables) Excess production or shortfall:  $y_1, y_2, \dots, y_n, z_1, z_2, \dots, z_n$

stage 2 variables will be replicated  $m$  times according to each of the possible demand levels

$d_j^{(1)}, d_j^{(2)}, \dots, d_j^{(m)}$  with given probabilities  $p_s$  to occur.

$c_j$  production costs

$e_j$  excess costs (eg, storage)

$f_j$  shortfall costs (missed opportunity)

# Two-Stage Stochastic Program with Recursion

$$\text{Minimize } \sum_j c_j x_j + \sum_s p_s \left( \sum_j e_j y_j^{(s)} + \sum_j f_j z_j^{(s)} \right)$$

$$\text{subject to } \sum_j a_{ij} x_j \leq b_i$$

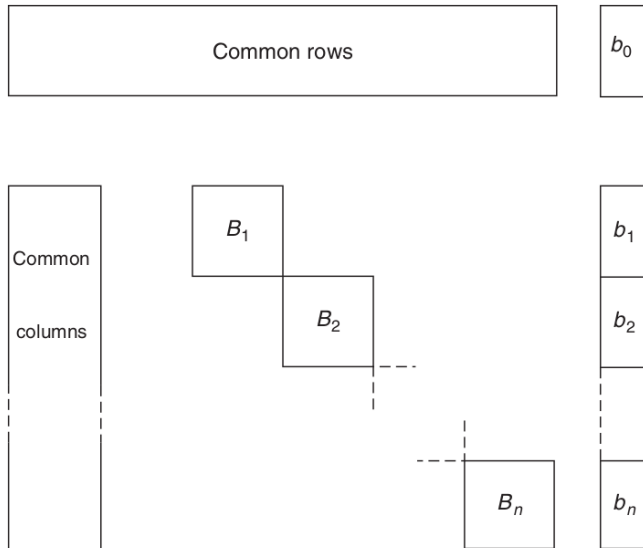
for all production constraints  $i$

$$x_j - y_j^{(s)} + z_j^{(s)} = d_j^{(s)}$$

for all  $j$  and  $s$

$$x_j, y_j^{(s)}, z_j^{(s)} \geq 0$$

for all  $j$  and  $s$



Minimise the maximum cost we would ever have to pay (minimise the maximum scenario cost).

$$Z \geq 2x_1 + (2x_{21} + 3y_{21} + 3y_{31}) \quad \text{scenario 1}$$

$$Z \geq 2x_1 + (2x_{22} + 3y_{22} + 3y_{32}) \quad \text{scenario 2}$$

$$Z \geq 2x_1 + (2x_{23} + 3y_{23} + 3y_{33}) \quad \text{scenario 3}$$

$$Z \geq 2x_1 + (2x_{24} + 3y_{24} + 3y_{34}) \quad \text{scenario 4}$$

The objective function would then become minimise  $Z$

After minimising  $Z$  with scenarios that cost less than this maximum cost we may have flexibility about variable values.

Hence if  $Z^*$  is the minimum value of  $Z$  from this formulation it is appropriate to then solve a further program:

minimise:     total scenario cost

subject to:    $Z \leq Z^*$  and the same constraints as above

i.e. minimise

$$4(2x_1) + (2x_{21} + 3y_{21} + 3y_{31}) + (2x_{22} + 3y_{22} + 3y_{32}) \\ + (2x_{23} + 3y_{23} + 3y_{33}) + (2x_{24} + 3y_{24} + 3y_{34})$$

subject to:    $Z \leq Z^*$  and the same constraints as above



- Tutorial by Giovanni Pantuso (more theoretical) <https://pantuso.sites.ku.dk/talks/>
- Example on VRP with stochastic demand
- Talk from gurobi, including robust, value-at-risk and conditional value-at-risk formulations  
<https://www.gurobi.com/events/solving-simple-stochastic-optimization-problems-with-gurobi/>
- Stochastic Programming community <https://www.stoprog.org/>