DM872

Math Optimization at Work

Dantzig-Wolfe Decomposition and Delayed Column Generation

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[*Partly based on slides by David Pisinger, DIKU (now DTU)*]

Outline

1. Dantzig-Wolfe Decomposition
2. Solving the LP Master Problem

Outline

Dantzig-Wolfe Decomposition

Motivation: Large difficult IP models

$\Longrightarrow$ split them up into smaller pieces

Applications

- Cutting Stock problems
- Multicommodity Flow problems
- Facility Location problems
- Capacitated Multi-item Lot-sizing problem
- Air-crew and Manpower Scheduling
- Vehicle Routing Problems
- Scheduling

Leads to methods also known as:

- Branch-and-price (column generation + branch and bound)
- Branch-and-cut-and-price (column generation + branch and bound + cutting planes)

Dantzig-Wolfe Decomposition

From an original or compact formulation to an extensive formulation made of a master problem and a subproblem

+ Tighter bounds
+ Better control of subproblem
– Model may become (very) large

Delayed column generation

Write up the decomposed model gradually as needed

- Generate a few solutions to the subproblems
- Solve the master problem to LP-optimality
- Use the dual information to find most promising solutions to the subproblem
- Extend the master problem with the new subproblem solutions.

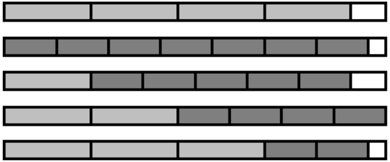
Motivation: Cutting stock problem

- Infinite number of raw stocks, having length L.
- Cut m piece types i, each having width w_i and demand b_i.
- Satisfy demands using least possible raw stocks.

Example:

- $w_1 = 5$, $b_1 = 7$ 
- $w_2 = 3$, $b_2 = 3$ 
- Raw length $L = 22$

Some possible cuts



Formulation 1

$$
\begin{array}{ll}
\text{minimize} & u_1 + u_2 + u_3 + u_4 + u_5 \\
\text{subject to} & 5x_{11} + 3x_{12} \le 22u_1 \\
& 5x_{21} + 3x_{22} \le 22u_2 \\
& 5x_{31} + 3x_{32} \le 22u_3 \\
& 5x_{41} + 3x_{42} \le 22u_4 \\
& 5x_{51} + 3x_{52} \le 22u_5 \\
& x_{11} + x_{21} + x_{31} + x_{41} + x_{51} \ge 7 \\
& x_{12} + x_{22} + x_{32} + x_{42} + x_{52} \ge 3 \\
& u_j \in \{0, 1\} \\
& x_{ij} \in \mathbb{Z}_+
\end{array}
$$

LP-relaxation gives solution value $z = 2$ with

$$u_1 = u_2 = 1, x_{11} = 2.6, x_{12} = 3, x_{21} = 4.4$$

Block structure:

	$x[0,0]$	$x[0,1]$	$u[0]$	$x[1,0]$	$x[1,1]$	$u[1]$	$x[2,0]$	$x[2,1]$	$u[2]$	$x[3,0]$	$x[3,1]$	$u[3]$	$x[4,0]$	$x[4,1]$	$u[4]$		
Minimize			$u[0]$			$+u[1]$			$+u[2]$			$+u[3]$			$+u[4]$		
stock[0]:	$5x[0,0]$	$+3x[0,1]$	$+22u[0]$													$\le$	0
stock[1]:				$5x[1,0]$	$+3x[1,1]$	$-22u[1]$										$\le$	0
stock[2]:							$5x[2,0]$	$+3x[2,1]$	$-22u[2]$							$\le$	0
stock[3]:										$5x[3,0]$	$+3x[3,1]$	$-22u[3]$				$\le$	0
stock[4]:													$5x[4,0]$	$+3x[4,1]$	$-22u[4]$	$\le$	0
type[0]:	$x[0,0]$			$+x[1,0]$			$+x[2,0]$			$+x[3,0]$			$+x[4,0]$			$\ge$	7
type[1]:		$x[0,1]$			$+x[1,1]$			$+x[2,1]$			$+x[3,1]$			$+x[4,1]$		$\ge$	3

Formulation 2

The matrix A contains all different cutting patterns
All (undominated) patterns:

$$A = \begin{pmatrix} 4 & 0 & 1 & 2 & 3 \\ 0 & 7 & 5 & 4 & 2 \end{pmatrix}$$

Problem

$$\begin{aligned} \text{minimize } & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ \text{subject to } & 4\lambda_1 + 0\lambda_2 + 1\lambda_3 + 2\lambda_4 + 3\lambda_5 \geq 7 \\ & 0\lambda_1 + 7\lambda_2 + 5\lambda_3 + 4\lambda_4 + 2\lambda_5 \geq 3 \\ & \lambda_j \in \mathbb{Z}_+ \end{aligned}$$

LP-relaxation gives solution value $z = 2.125$ with

$$\lambda_1 = 1.375, \lambda_4 = 0.75$$

Due to integer property a lower bound is $\lceil 2.125 \rceil = 3$.
Optimal solution value is $z^* = 3$.

Round up LP-solution getting heuristic solution $z_H = 3$.

Decomposition Approach: Lagrangian Approach

Integer Programming Problem with block structure:

$$
\begin{array}{rlllllll}
z_{IP} = \max & c^1x^1 & + & c^2x^2 & + & \ldots + & c^Kx^K & \\
& A^1x^1 & + & A^2x^2 & + & \ldots + & A^Kx^K & = \; b \\
& D^1x^1 & & & & & & \leq \; d_1 \\
& & & D^2x^2 & & & & \leq \; d_2 \\
& & & & & \ldots & & \leq \; \vdots \\
& & & & & & D^Kx^K & \leq \; d_K \\
& \multicolumn{6}{l}{x^1 \in \mathbb{Z}_+^{n_1}, x^2 \in \mathbb{Z}_+^{n_2}, \ldots, x^K \in \mathbb{Z}_+^{n_K}} &
\end{array}
$$

Lagrangian relaxation, multipliers $\lambda \in \mathbb{R}^K$
Objective becomes: $\max c^1x^1 + c^2x^2 + \ldots + c^Kx^K - \lambda(A^1x^1 + A^2x^2 + \ldots + A^Kx^K - b)$

$$
\begin{array}{rllllll}
z_{LR}(\lambda) = \max & c^1x^1 - \lambda A^1x^1 & + & c^2x^2 - \lambda A^2x^2 & + \; \ldots + & c^Kx^K - \lambda A^Kx^K & +b \\
& D^1x^1 & & & & & \leq \; d_1 \\
& & & D^2x^2 & & & \leq \; d_2 \\
& & & & \ldots & & \leq \; \vdots \\
& & & & & D^Kx^K & \leq \; d_K \\
& x^1 \in \mathbb{Z}_+^{n_1}, & & x^2 \in \mathbb{Z}_+^{n_2}, & \ldots, & x^K \in \mathbb{Z}_+^{n_K} &
\end{array}
$$

model is separable

Strength of the Lagrangian Relaxation

General result

Integer Programming Problem:

$$
\begin{aligned}
z_{IP} = \max \ & cx \\
& Ax \leq b \\
& Dx \leq d \\
& x_j \in \mathbb{Z}_+ \ i = 1, ..., n
\end{aligned}
$$

Lagrangian relaxation, multipliers $\lambda \geq 0$

$$
\begin{aligned}
z_{LR}(\lambda) = \max \ & cx - \lambda(Ax - b) \\
& Dx \leq d \\
& x_j \in \mathbb{Z}_+ \ i = 1, ..., n
\end{aligned}
$$

for the best multiplier λ (from the Lagrangian Dual problem)

$$\boxed{z_{LD} = \max \{cx \mid Ax \leq b, x \in \text{conv}(Dx \leq d, x \in \mathbb{Z}_+)\}}$$

$z_{IP} \leq z_{LD} \leq z_{LP}$ hence z_{LD} is a better bound than z_{LP} from the linear relaxation of IP.

Dantzig-Wolfe decomposition

If model has "block" structure

$$
\begin{array}{llllllll}
\max & c^1x^1 & + & c^2x^2 & + \ldots + & c^Kx^K & & \\
\text{s.t.} & A^1x^1 & + & A^2x^2 & + \ldots + & A^Kx^K & = & b \\
 & D^1x^1 & + & & & & \leq & d_1 \\
 & & + & D^2x^2 & & & \leq & d_2 \\
 & & & & \ldots & & \leq & \vdots \\
 & & & & & D^Kx^K & \leq & d_K \\
 & x^1 \in \mathbb{Z}_+^{n_1} & & x^2 \in \mathbb{Z}_+^{n_2} & \ldots & x^K \in \mathbb{Z}_+^{n_K} & &
\end{array}
$$

Describe each set X^k, $k = 1, \ldots, K$

$$
\begin{array}{lllllll}
\max & c^1x^1 & + & c^2x^2 & + \ldots + & c^Kx^K & \\
\text{s.t.} & A^1x^1 & + & A^2x^2 & + \ldots + & A^Kx^K & = b \\
 & x^1 \in X^1 & & x^2 \in X^2 & \ldots & x^K \in X^K &
\end{array}
$$

where $X^k = \{x^k \in \mathbb{Z}_+^{n_k} : D^kx^k \leq d_k\}$

Assuming that X^k has finite number of points $\{x^{k,t}\}$ $t \in T_k$

$$
X^k = \left\{ \begin{array}{ll} x^k \in \mathbb{R}^{n_k} : & x^k = \sum_{t \in T_k} \lambda_{k,t} x^{k,t}, \\ & \sum_{t \in T_k} \lambda_{k,t} = 1, \\ & \lambda_{k,t} \in \{0,1\}, t \in T_k \end{array} \right\}
$$

Dantzig-Wolfe decomposition

Substituting X^k in original model getting *Master Problem*

$$\max c^1(\sum_{t\in T_1}\lambda_{1,t}x^{1,t}) + c^2(\sum_{t\in T_2}\lambda_{2,t}x^{2,t}) + \ldots + c^K(\sum_{t\in T_K}\lambda_{K,t}x^{K,t})$$

$$\text{s.t.}\quad A^1(\sum_{t\in T_1}\lambda_{1,t}x^{1,t}) + A^2(\sum_{t\in T_2}\lambda_{2,t}x^{2,t}) + \ldots + A^K(\sum_{t\in T_K}\lambda_{K,t}x^{K,t}) = b$$

$$\sum_{t\in T_k}\lambda_{k,t} = 1 \qquad\qquad k = 1,\ldots,K$$

$$\lambda_{k,t} \in \{0,1\}, \qquad t\in T_k \quad k = 1,\ldots,K$$

Strength of linear master model

Solving LP-relaxation of master problem, is equivalent to (Wolsey Prop 11.1)

$$\begin{array}{llllllll}
\max & c^1x^1 & + & c^2x^2 & +\ldots+ & c^kx^k & \\
\text{s.t.} & A^1x^1 & + & A^2x^2 & +\ldots+ & A^kx^k & =b \\
& x^1 \in \operatorname{conv}(X^1) & & x^2 \in \operatorname{conv}(X^2) & \ldots & x^k \in \operatorname{conv}(X^k) &
\end{array}$$

Proof: Consider LP-relaxation

$$\max c^1(\sum_{t\in T_1} \lambda_{1,t}x^{1,t}) + c^2(\sum_{t\in T_2} \lambda_{2,t}x^{2,t}) + \ldots + c^K(\sum_{t\in T_K} \lambda_{K,t}x^{K,t})$$

$$\text{s.t.} \quad A^1(\sum_{t\in T_1} \lambda_{1,t}x^{1,t}) + A^2(\sum_{t\in T_2} \lambda_{2,t}x^{2,t}) + \ldots + A^K(\sum_{t\in T_K} \lambda_{K,t}x^{K,t}) = b$$

$$\sum_{t\in T_k} \lambda_{k,t} = 1 \qquad k = 1,\ldots,K$$

$$\lambda_{k,t} \geq 0, \qquad t \in T_k \qquad k = 1,\ldots,K$$

Informally speaking we have

- joint constraint is solved to LP-optimality
- block constraints are solved to IP-optimality

Theorem

- z_{LMP} *be the LP-solution value of the master problem*
- z_{LD} *be solution value of Lagrangian dual problem*

$$z_{LMP} = z_{LD}$$

Proof: as a consequence of the previous five slides the linear relaxation of the master problem and the Lagrangian dual correspond to solving the following problem:

$$\begin{array}{llllllll}
\max\ c^1x^1 & + & c^2x^2 & + & \dots + & c^Kx^K & & \\
\quad A^1x^1 & + & A^2x^2 & + & \dots + & A^Kx^K & = b & \\
\multicolumn{8}{l}{\quad x^1 \in \text{conv}(X^1),\ x^2 \in \text{conv}(X^2),\ \dots,\ x^K \in \text{conv}(X^K)}
\end{array}$$

Hence, also the DW decomposition leads to a better dual bound than the linear relaxation of the original problem

$$z_{IP} \le z_{LMP} = z_{LD} \le z_{LP} \qquad \text{(for a maximization problem)}$$

□

Outline

Delayed Column Generation

- Master problem can (and will) contain many columns
- To find bound, solve LP-relaxation of master
- Delayed column generation gradually writes up master

Solve the linear relaxation of the master problem by delayed column generation

Consider the general linear program

$$\begin{array}{rrcl} \text{minimize} & c^T x & & \\ \text{subject to} & Ax & = & b, \\ & x & \geq & 0, \end{array} \tag{3}$$

with $A \in \Re^{m\times n}$, $c \in \Re^n$, $b \in \Re^m$. The dual of (3) is

$$\begin{array}{rrcl} \text{maximize} & b^T y & & \\ \text{subject to} & A^T y & \leq & c. \end{array} \tag{4}$$

The sifting procedure begins by taking a "working set" of columns $\mathcal{W} \subset \{1, \ldots, n\}$ such that

$$\begin{array}{rrcl} \text{minimize} & c_{\mathcal{W}}^T x_{\mathcal{W}} & & \\ \text{subject to} & A_{\mathcal{W}} x_{\mathcal{W}} & = & b, \\ & x_{\mathcal{W}} & \geq & 0, \end{array} \tag{5}$$

is feasible. (This assumption is not essential.) Let π^* be an optimal solution to

$$\begin{array}{rrcl} \text{maximize} & b^T \pi & & \\ \text{subject to} & A_{\mathcal{W}}^T \pi & \leq & c_{\mathcal{W}}, \end{array} \tag{6}$$

the dual of (5), and let $x_{\mathcal{W}}^*$ be an optimal solution of (5). Then the vector $x^T = ((x_{\mathcal{W}}^*)^T, 0) \in \Re^n$ is optimal for (3) if

$$c - A^T \pi^* \geq 0. \tag{7}$$

Given the linear program (3) and a set $\mathcal{W}$ such that (5) is feasible:
Solve (5) obtaining x^* and π^*.

while $(c - A^T\pi^* \ngeq 0)$ **do** (major iteration)

  Choose $\mathcal{P} \subset \{1, \ldots, n\} \setminus \mathcal{W}$. (price)

  Set $\mathcal{W} \leftarrow \mathcal{W} \cup \mathcal{P}$. (augment problem)

  (Optionally) If $\mathcal{W}$ is too big,

    reduce the size of $\mathcal{W}$. (purge)

  Solve (5) obtaining x^* and π^*. (solve)

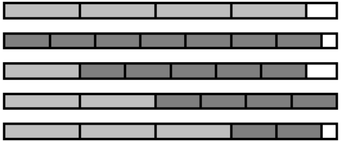
end while

Delayed column generation, linear master

- $w_1 = 5, b_1 = 7$
- $w_2 = 3, b_2 = 3$

- Raw length $L = 22$

Some possible cuts



In matrix form

$$A = \begin{pmatrix} 4 & 0 & 1 & 2 & 3 & \cdots \\ 0 & 7 & 5 & 4 & 2 & \cdots \end{pmatrix}$$

LP-problem

$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

where

- $b = (7, 3)$,
- $x = (x_1, x_2, x_3, x_4, x_5, \cdots)$
- $c = (1, 1, 1, 1, 1, \cdots)$.

Revised Simplex Method

- $\max\{cx \mid Ax \leq b, x \geq 0\}$
- $B = \{1 \ldots m\}$ basic variables
- $N = \{m+1 \ldots m+n\}$ non-basic variables (will be set to lower bound 0)
- $A_B = [A_1 \ldots A_m]$
- $A_N = [A_{m+1} \ldots A_{m+n}]$

Standard form

$$\left[\begin{array}{c:c:c:c} A_B & A_N & 0 & b \\ \hdashline c_B & c_N & 1 & 0 \end{array}\right]$$

$$
\begin{aligned}
Ax &= A_N x_N + A_B x_B = b \\
A_B x_B &= b - A_N x_N \\
x_B &= A_B^{-1} b - A_B^{-1} A_N x_N
\end{aligned}
$$

basic feasible solution:

- $X_N = 0$
- A_B lin. indep.
- $X_B \geq 0$

$$
\begin{aligned}
z = c^T x = c_B^T (A_B^{-1} b - A_B^{-1} A_N x_N) + c_N^T x_N &= \\
= c_B^T A_B^{-1} b + (c_N^T - c_B^T A_B^{-1} A_N) x_N &
\end{aligned}
$$

Canonical form

$$
\left[\begin{array}{c:c:c:c}
I & A_B^{-1} A_N & 0 & A_B^{-1} b \\
\hdashline
0 & c_N^T - c_B^T A_B^{-1} A_N & 1 & -c_B^T A_B^{-1} b
\end{array}\right]
$$

$c_N^T - c_B^T A_B^{-1} A_N = c_N^T - \pi A_N$ (π dual variables) are the reduced costs of the non-basis variables

In scalar form: the objective function is obtained by multiplying and subtracting constraints by means of multipliers $\mu = -\pi$: $\pi = c_B^T A_B^{-1}$ (the dual variables)

Note! (multipliers) $\mu_i = -\pi_i$ (dual variables)

$$z = \sum_{j=1}^{m} \left[c_j + \sum_{i=1}^{m} \mu_i a_{ij} \right] x_j + \sum_{j=m+1}^{m+n} \left[c_j + \sum_{i=1}^{m} \mu_i a_{ij} \right] x_j + \sum_{i=1}^{m} \mu_i b_i$$

Each basic variable has cost null in the objective function

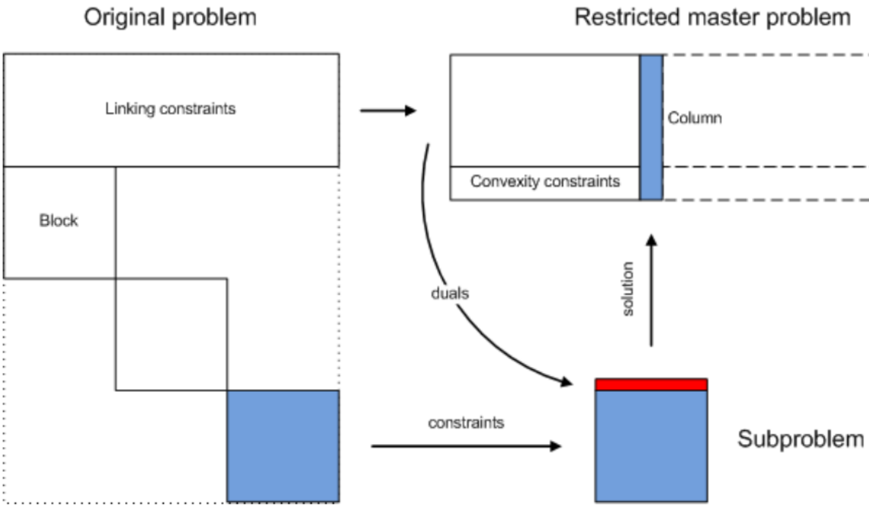
$$c_j + \sum_{i=1}^{m} \mu_i a_{ij} = 0 \qquad j = 1, \ldots, m$$

Reduced costs of non-basic variables:

$$\bar{c}_j = c_j + \sum_{i=1}^{m} \mu_i a_{ij} = c_j - \sum_{i=1}^{m} \pi_i a_{ij} \qquad j = m+1, \ldots, m+n$$

If basis is optimal then $\bar{c}_j \leq 0$ for all $j = m+1, \ldots, m+n$.

Dantzig Wolfe Decomposition with Delayed Column Generation



Delayed column generation (example)

- $w_1 = 5, b_1 = 7$ 
- $w_2 = 3, b_2 = 3$ 
- Raw length $L = 22$

Initially we choose only the trivial cutting patterns

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix}$$

Solve LP-problem

$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

i.e.

$$\begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

with solution $x_1 = \frac{7}{4}$ and $x_2 = \frac{3}{7}$.
The dual variables are $y = c_B A_B^{-1}$ i.e.

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{7} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{7} \end{pmatrix}$$

Small example (continued)

Find entering variable

$$\begin{array}{rcl} A & = & \begin{pmatrix} 1 & 2 & 3 & \cdots \\ 5 & 4 & 2 & \cdots \end{pmatrix} \qquad \begin{array}{l} \frac{1}{4} \leftarrow y_1 \\ \frac{1}{7} \leftarrow y_2 \end{array} \\ c_N - yA_N & = & \left(1 - \frac{27}{28} \;\; 1 - \frac{30}{28} \;\; 1 - \frac{29}{28} \;\; \cdots \right) \end{array}$$

We could also solve optimization problem

$$\begin{array}{ll} \min & 1 - \frac{1}{4}x_1 - \frac{1}{7}x_2 \\ \text{s.t.} & 5x_1 + 3x_2 \leq 22 \\ & x \geq 0, \text{integer} \end{array}$$

which is equivalent to knapsack problem

$$\begin{array}{ll} \max & \frac{1}{4}x_1 + \frac{1}{7}x_2 \\ \text{s.t.} & 5x_1 + 3x_2 \leq 22 \\ & x \geq 0, \text{integer} \end{array}$$

This problem has optimal solution $x_1 = 2, x_2 = 4$.
Reduced cost of entering variable

$$1 - 2\frac{1}{4} - 4\frac{1}{7} = 1 - \frac{30}{28} = -\frac{1}{14} < 0$$

Small example (continued)

Add new cutting pattern to A getting

$$A = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 7 & 2 \end{pmatrix}$$

Solve problem to LP-optimality, getting primal solution

$$x_1 = \frac{5}{8}, x_3 = \frac{3}{2}$$

and dual variables

$$y_1 = \frac{1}{4}, y_2 = \frac{1}{8}$$

Note, we do not need to care about "leaving variable"
To find entering variable, solve

$$\begin{array}{ll} \max & \frac{1}{4}x_1 + \frac{1}{8}x_2 \\ \text{s.t.} & 5x_1 + 3x_2 \leq 22 \\ & x \geq 0, \text{integer} \end{array}$$

This problem has optimal solution $x_1 = 4$, $x_2 = 0$.
Reduced cost of entering variable

$$1 - 4\frac{1}{4} - 0\frac{1}{7} = 0$$

Terminate with $x_1 = \frac{5}{8}$, $x_3 = \frac{3}{2}$, and $z_{LP} = \frac{17}{8} = 2.125$.