DM872

Math Opt @ Work

More on Polyhedra and Farkas' Lemma

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

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We look at Farkas' Lemma with two objectives:

- (giving another proof of strong duality)
- understanding a certificate of infeasibility

Farkas' Lemma

Theorem (Farkas' Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Then,

either	*I.*	$\exists \boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b}$ *and* $\boldsymbol{x} \geq \boldsymbol{0}$
or	*II.*	$\exists \boldsymbol{y} \in \mathbb{R}^m : \boldsymbol{y}^T A \geq 0^T$ *and* $\boldsymbol{y}^T \boldsymbol{b} < \boldsymbol{0}$

Easy to see that both I and II cannot occur together:

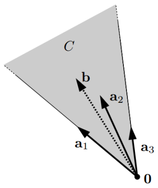
$$(0 \leq) \qquad \boldsymbol{y}^T A\boldsymbol{x} = \boldsymbol{y}^T \boldsymbol{b} \qquad (< 0)$$

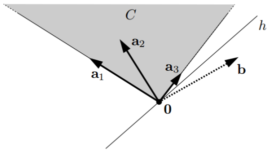
Geometric interpretation of Farkas' Lemma

Linear combination of $\boldsymbol{a}_i$ with nonnegative terms generates a convex cone:

$$\{\lambda_1 \boldsymbol{a}_1 + \ldots + \lambda_n \boldsymbol{a}_n, \mid \lambda_1, \ldots, \lambda_n \geq \mathbf{0}\}$$

Polyhedral cone: $C = \{\boldsymbol{x} \mid A\boldsymbol{x} \leq \mathbf{0}\}$, intersection of many $\boldsymbol{a}\boldsymbol{x} \leq 0$
Conic hull of rays $\boldsymbol{p}_i = \{\lambda_i \boldsymbol{a}_i, \lambda_i \geq 0\}$





Either	point $\boldsymbol{b}$ lies in convex cone C
or	$\exists$ hyperplane h passing through point 0 $h = \{\boldsymbol{x} \in \mathbb{R}^m : \boldsymbol{y}^T \boldsymbol{x} = 0\}$ for $\boldsymbol{y} \in \mathbb{R}^m$ such that all vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ (and thus C) lie on one side and $\boldsymbol{b}$ lies (strictly) on the other side (ie, $\boldsymbol{y}^T \boldsymbol{a}_i \geq 0, \forall i = 1 \ldots n$ and $\boldsymbol{y}^T \boldsymbol{b} < 0$).

Alternative Formulation

Theorem (Farkas' Lemma)

The inequality $\boldsymbol{c}^T\boldsymbol{x} \geq c_0$ is valid for the non-empty polyhedron $P := \{\boldsymbol{x} \geq 0 \mid A\boldsymbol{x} = \boldsymbol{b}\}$ ***if and only if*** *$\boldsymbol{y} \in \mathbb{R}^m$ exists such that:*

$$\boldsymbol{c}^T \geq \boldsymbol{y}^T A$$
$$c_0 \leq \boldsymbol{y}^T \boldsymbol{b}$$

$\Longleftarrow$ (sufficiency) (used in Gomory cuts)

$$\boldsymbol{c}^T\boldsymbol{x} \geq \quad \boldsymbol{y}^T A\boldsymbol{x} = \boldsymbol{y}^T\boldsymbol{b} \quad \geq c_0$$

$\Longrightarrow$ (necessity)
by simplex algorithm similar to our proof of the strong duality theorem

Other Variants of Farkas' Lemma

Corollary

(i) $A\boldsymbol{x} = \boldsymbol{b}$ *has sol* $\boldsymbol{x} \geq \boldsymbol{0} \iff \forall \boldsymbol{y} \in \mathbb{R}^m$ *with* $\boldsymbol{y}^T A \geq \boldsymbol{0}^T, \boldsymbol{y}^T \boldsymbol{b} \geq 0$

(ii) $A\boldsymbol{x} \leq \boldsymbol{b}$ *has sol* $\boldsymbol{x} \geq \boldsymbol{0} \iff \forall \boldsymbol{y} \geq \boldsymbol{0}$ *with* $\boldsymbol{y}^T A \geq \boldsymbol{0}^T, \boldsymbol{y}^T \boldsymbol{b} \geq 0$

(iii) $A\boldsymbol{x} \leq \boldsymbol{0}$ *has sol* $\boldsymbol{x} \in \mathbb{R}^n \iff \forall \boldsymbol{y} \geq \boldsymbol{0}$ *with* $\boldsymbol{y}^T A = \boldsymbol{0}^T, \boldsymbol{y}^T \boldsymbol{b} \geq 0$

Certificate of Infeasibility

Farkas' Lemma provides a way to certificate infeasibility.

Theorem

Let $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$.
Given a certificate $\boldsymbol{y}^*$ *it is easy to check the conditions (by linear algebra):*

$$\begin{aligned} A^T\boldsymbol{y}^* &\geq \boldsymbol{0} \\ \boldsymbol{b}\boldsymbol{y}^* &< 0 \end{aligned}$$

Why would $\boldsymbol{y}^*$ be a certificate of infeasibility?
Proof (by contradiction)
Assume, $A^T\boldsymbol{y}^* \geq \boldsymbol{0}$ and $\boldsymbol{b}\boldsymbol{y}^* < 0$.
Moreover assume $\exists \boldsymbol{x}^*$: $A\boldsymbol{x}^* = \boldsymbol{b}$, $\boldsymbol{x}^* \geq \boldsymbol{0}$,then:

$$(\geq 0) \qquad (\boldsymbol{y}^*)^T A\boldsymbol{x}^* = (\boldsymbol{y}^*)^T\boldsymbol{b} \qquad (< 0)$$

Contradiction

General form:

$$\begin{aligned} \max\ & c^T x \\ A_1 x &= b_1 \\ A_2 x &\leq b_2 \\ A_3 x &\geq b_3 \\ x &\geq 0 \end{aligned}$$

infeasible $\Leftrightarrow \exists y^*$

$$\begin{aligned} b_1^T y_1 + b_2^T y_2 + b_3^T y_3 &> 0 \\ A_1^T y_1 + A_2^T y_2 + A_3^T y_3 &\leq 0 \\ y_2 &\leq 0 \\ y_3 &\geq 0 \end{aligned}$$

Example

$$\begin{aligned} \max\ & c^T x \\ x_1 &\leq 1 \\ x_1 &\geq 2 \end{aligned}$$

$$\begin{aligned} b_1^T y_1 + b_2^T y_2 &> 0 \\ A_1^T y_1 + A_2^T y_2 &\leq 0 \\ y_1 &\leq 0 \\ y_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} y_1 + 2y_2 &> 0 \\ y_1 + y_2 &\leq 0 \\ y_1 &\leq 0 \\ y_2 &\geq 0 \end{aligned}$$

$y_1 = -1, y_2 = 1$ is a valid certificate.

- Observe that it is not unique!
- It can be reported in place of the dual solution because same dimension.
- To repair infeasibility we should change the primal at least so much as that the certificate of infeasibility is no longer valid.
- Only constraints with $y_i \neq 0$ in the certificate of infeasibility cause infeasibility