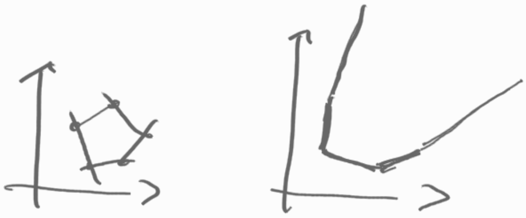
<u>Background to Benders' Algorithm</u>

- Extreme points and extreme rays [DJ, p 36]

$\min c^T x$

$$A_1 x \leq b_1$$
$$A_2 x \leq b_2 \longrightarrow$$ in general it is a polyhedron, which can be <u>bounded</u> (and hence a polytope) or <u>unbounded</u>

$$x \geq 0$$



In Dantzig Wolfe dec. we rewrite the problem as:

$\min c^T x$

$$A_1 x \leq b_1$$
$$x \in X$$

where $X = \{ x \mid A_2 x \leq b_2 \}$

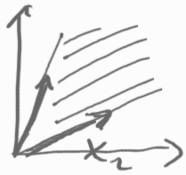
If X was bounded we have seen that it can be written as a convex combination of its extreme points P.

$$X = \{ x \mid x = \sum \lambda_p x_p, \lambda \geq 0, \sum \lambda_p = 1 \}$$

$$X = \{x \mid x = \sum_{p \in P} \lambda_p x_p,\ \lambda_p \geq 0,\ \sum_{p \in P} \lambda_p = 1\}$$

If we want to account also for the possibility that $A_2 x \leq b_2$ is unbounded then we need the following:

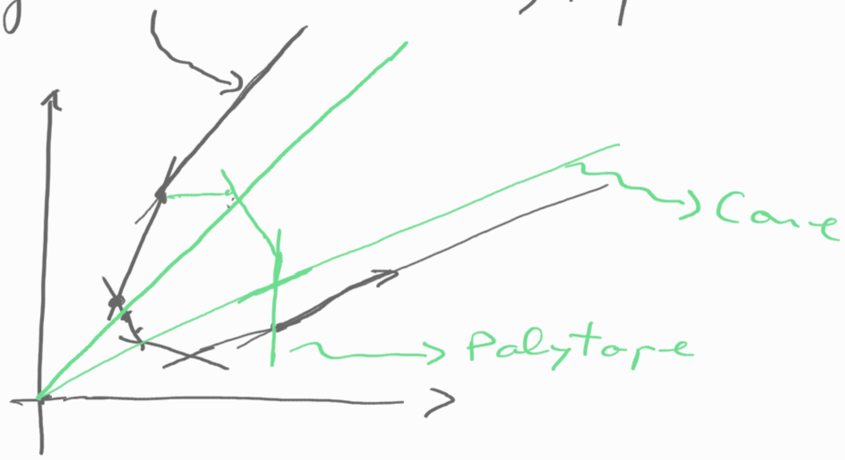
A <u>convex cone</u> is $\{x \mid Bx \leq 0\}$ that is the intersection of many halfspaces.
A convex cone is also described by the conic combination of its extreme rays R:



$$C = \{x \mid x = \sum_{r \in R} \delta_r x_r,\ \delta_r \geq 0\}.$$

<u>Minkowski-Weyl theorem</u> from polyhedral analysis:

$$\text{Polyhedron} = \text{Polytope} + \text{Cone}$$



Hence, a point of a polyhedron can be described as:

$$X = \{ x \mid x = \sum_{p \in P} \lambda_p x_p + \sum_{r \in R} \delta_r x_r ,$$
$$\lambda_p \geq 0 ,$$
$$\sum_{p \in P} \lambda_p = 1 ,$$
$$\delta_r \geq 0 \}$$

Hence in Dantzig Wolfe decomposition the substitution would be

$$\min c^T x \qquad\qquad \min c^T \sum_{p \in P} \lambda_p x_p + c^T \sum_{r \in R} \delta_r x_r$$
$$A_1 x \leq b_1 \qquad\qquad A_1 \Big(\sum_{p \in P} \lambda_p x_p + \sum_{r \in R} \delta_r x_r \Big) \leq b_1$$
$$x \in X \qquad\qquad \sum_{p \in P} \lambda_p = 1$$
$$\lambda_p \geq 0 \qquad \forall p \in P$$
$$\delta_r \geq 0 \qquad \forall r \in R$$

We previously ignored the rays and can continue to do so if the pricing problem is bounded and feasible.

- How do we find the extreme rays?
  From the simplex:

$$\max \quad x_1 + x_2$$
$$-2x_1 + x_2 + x_3 = 1$$

$$x_1 - x_2 + x_4 = 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

x_1	x_2	x_3	x_4		
-2	1	1	0	0	1
1	-1	0	1	0	1
1	1	0	0	1	0

$\Rightarrow$

x_1	x_2	x_3	x_4		
-2	1	1	0	0	1
-1	0	1	1	0	2
$+3$	0	-1	0	1	1

basic sol. $[0, 1, 0, 2]$. Trying to bring x_1 in basis unveals that we can increase x_1 arbitrarily

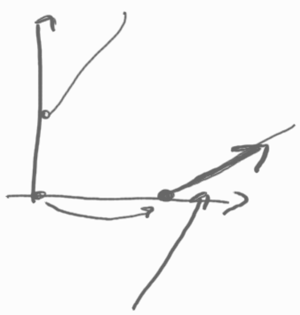
Hence:

$$x_2 = 1 + 2x_1 (- x_3)$$

$$x_4 = 2 + x_1 (- x_3)$$

Not in basis stays zero

$$x = \begin{bmatrix} k \\ 1 + 2k \\ 0 \\ 2 + k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} + k \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$



extreme ray

Hence, we find it with the simplex and it is returned by solvers when the problem is unbounded.

Note: if a problem is infeasible its dual is unbounded and the extreme ray is a proof of infeasibility for the primal (Farkas Lemma)

- <u>Farkas Lemma</u>

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m}$. Then:

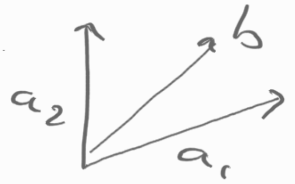
either : $\exists x \in \mathbb{R}^n : Ax \leq b$ and $x \geq 0$

or : $\exists u \in \mathbb{R}^m : u^T A \geq 0$ and $u^T b < 0, u \geq 0$

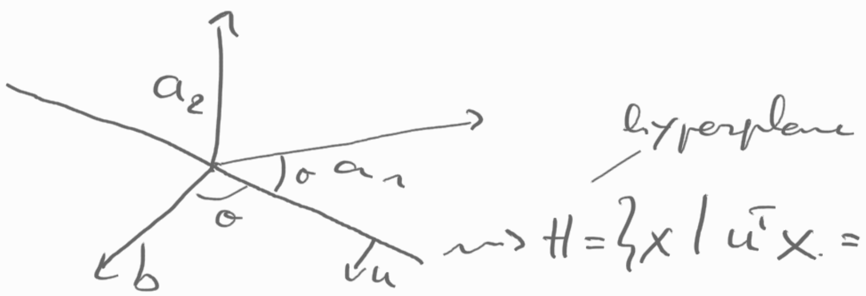
(Consult in case wikipedia for variants)

Meaning:

either the system $Ax \geq b$ admits a solution $x \geq 0$, that is, b can be attained as a linear combination of the column vectors of A:



or the system $Ax \geq b$ does not admit a solution and there is a certificate of this fact: an hyperplane described by $u \geq 0$ that separates b from A:



$\rightsquigarrow H = \{x \mid u^T x = 0\}$

$u^T b = \|u\| \|b\| \cos\theta$

$u^T A = [\|u\| \|a\| \cos\theta]$

$\Leftarrow$ if different sign then vector b stays on one side and a_i on the other of H

- Benders' reformulation

(OP)
original problem

$$\max \; cx + hy$$
$$Fx + Gy \le d$$
$$x \in X \subseteq \mathbb{R}^n_+$$
$$y \in \mathbb{R}^p_+$$

(EF)
extended formulation

$$\max \; cx + \eta$$
$$v^r(d - Fx) \ge 0 \qquad r \in R$$
$$u^p(d - Fx) \ge \eta \qquad p \in P$$
$$x \in X$$
$$\eta \in \mathbb{R}^1$$

extreme rays of DSP

extreme points of DSP

Derivation of the reformulation.
Let's rewrite (OP) as:

$$z = \max_x \{ cx + \phi(x) : x \in X \}$$

where $\phi(x) = \underbrace{\max \{ hy : Gy \le d - Fx, \; y \in \mathbb{R}^p_+ \}}_{\text{Subproblem (SP)}}$

SP is feasible $\iff$ by Farkas:

$$\forall u \in \mathbb{R}^m : u^T G \ge 0, \; u \ge 0$$

$$u(d - Fx) \geq 0$$

u are the extreme rays of $u^T G \geq 0$ (cone)

Let's call them $v^r,\ r \in R$

if $\phi(x)$ exists and is finite:
by strong duality theorem of LP:

$$\phi(x) = \min_u \left\{ u(d - Fx) \,\middle|\, uG \geq h,\ u \in \mathbb{R}^m_+ \right\}$$

Dual Subproblem DSP

If SP bounded and DSP bounded, then the solution of DSP will be in one of the extreme points by the fundamental th. of lin. programming. Hence:

$$\phi(x) = \min_{p \in P} \left\{ w^p (d - Fx) \right\}$$

where $w^p,\ p \in P$ are the extreme points of $\{uG \geq h, u \geq 0\}$

similar to $\min\{2x, 3x, 5x, 6x\}$

that can be linearized as
$$\eta \le 2x$$
$$\eta \le 3x$$
$$\eta \le 5x$$
$$\eta \le 6x$$

SP		DSP
bounded	----	bounded
unbounded ($+\infty$)	----	infeasible

After linearization this is equivalent to:

$$
(EF) \quad \begin{aligned}
\max_x \quad & cx + \eta \\
& w^p(d - Fx) \geq \eta \qquad \forall p \in P \\
& v^r(d - Fx) \geq 0 \qquad \forall r \in R \\
& x \in X
\end{aligned}
$$

$\rightsquigarrow$ from def. of $\bar{X}$, ie, SP is feasible

which is the reformulation we wanted to achieve. Hence

OP has feasible sol $\iff$ EF has feasible sol

OP has unbounded sol $\iff$ EF has unbounded sol

OP is infeasible $\iff$ EF is infeasible

EF has too many constraints to list

them all hence, we solve it by defin-
ing it incrementally:

$$
(\mathrm{REF}) \quad
\begin{aligned}
z^* = \max_{x} \quad & cx + \eta \\
& w^p(d - Fx) \geq \eta \quad \forall p \in \bar{P} \subseteq P \\
& v^r(d - Fx) \geq 0 \quad \forall r \in \bar{R} \subseteq R \\
& x \in X
\end{aligned}
$$

- Benders' algorithm

Solve (REF) and find (η^*, x^*)

Solve (DSP) $\phi(x^*) = \min \; u(d - Fx^*)$

$$
\begin{aligned}
& uG \geq h \\
& u \in \mathbb{R}^m_+
\end{aligned}
$$

if unbounded then

$\exists\, v^r : v^r(d - Fx^*) < 0 \Rightarrow v^r(d - Fx) \geq 0$

must be added to make

the primal feasible (from

formulation of $\bar{X}$)

if bounded and $\phi(x^*) < \eta^*$

then the solution to DSP u^p

then the solution of DSP, w,

gives a new extreme point;

whose relative constr. in EF is violated

$$\phi(x^*) = w^p(d - Fx^*) < \eta^*$$

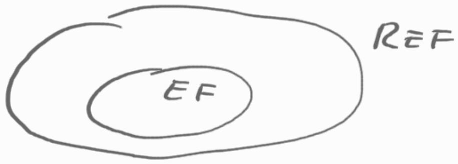
hence we add

$$w^p(d - Fx) \geq \eta$$

if bounded and $\phi(x^*) = \eta^*$

then all constr. are satisfied

so the lin. prog. EF is solved



REF is a relaxation of EF, hence

if the sol is feasible for EF it

is opt. for EF

Resuming:

if (REF) has no feasible sol $\Rightarrow$ STOP and return infeasible, adding constraints will not remove infeas.

if (REF) has unbounded sol $\Rightarrow$ $\eta^* = +\infty$

guess x^* and solve (DSP)

if (REF) has a bounded sol $\Rightarrow$ (η^*, x^*) and solve (DSP)

if (DSP) is infeasible $\Rightarrow$ STOP the (EF) is unbound.
if (DSP) is unbounded $\Rightarrow$ add extreme ray constr.
if (DSP) is bounded $\Rightarrow$ add extreme point constr.
or STOP because opt. found.

- if $X \subseteq \mathbb{Z}^n$ instead of $\mathbb{R}^n$

Then Branch and Cut [Wo, p. 237]

Solve LP relaxation by Benders' alg at each node of the enumeration Tree. Early pruning if DSP is infeasible.

- if $y \in \mathbb{Z}^p$ instead of $\mathbb{R}^p$ $(+ X \subseteq \mathbb{Z}^n)$

Then integer subproblems

1) branch on (x, y, η) space
(extension of the branch and cut alg seen above)

branch on x var except when they are integer.

$\rightarrow$ DSP branching constraints
$\ell \le y \le u$

yield the following changes

$$\min\{u(d-Fx^*)-u^1 \ell+u^2 k : (u,u^1,u^2)\in U^*\}$$

where $U^* = \{u, u^1, u^2 \in \mathbb{R}_+^m \times \mathbb{R}_+^p \times \mathbb{R}_+^p :$

$uG - u^1 I_m + u^2 I_m = h\}$

2) branch on (x, η)
subproblems are integer programs ⇝ cuts are weak

i) $SP^I(x^*)$ infeasible

add no-good cuts

ii) $SP^I(x^*)$ has UB ϕ'

add no-good optimality cuts

iii) $\phi' = \phi^I(x^*) = \eta^*$

(x^*, y^*, η^*) is a feasible sol to (OP)

update incumbent and

continue the search.