DM872

Math Optimization at Work

Lagrangian Relaxation

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[*Partly based on slides by David Pisinger, DIKU (now DTU)*]

Outline

1. Relaxations and Bounds
2. Subgradient Optimization
3. LR in IP

Outline

Relaxation

In branch and bound we find upper bounds by relaxing the problem

Relaxation

$$\max_{s \in P} g(s) \geq \left\{ \begin{array}{l} \max_{s \in P} f(s) \\ \max_{s \in S} g(s) \end{array} \right\} \geq \max_{s \in S} f(s)$$

- P: candidate solutions;
- $S \subseteq P$ feasible solutions;
- $g(x) \geq f(x)$

Which constraints should be relaxed?

- Quality of bound (tightness of relaxation)
- Remaining problem can be solved efficiently
- Proper multipliers can be found efficiently
- Constraints difficult to formulate mathematically
- Constraints which are too expensive to write up

Relevant Relaxations

Different relaxations

- LP-relaxation
- Deleting constraint
- Lagrange relaxation
- Surrogate relaxation
- Semidefinite relaxation

Relaxations are often used in combination.

Tighter

Best surrogate relaxation

Best Lagrangian relaxation

LP relaxation

Surrogate Relaxation

Integer Programming Problem: $\max\{cx \mid Ax \leq b, Dx \leq d, x \in \mathbb{Z}^n_+\}$[1]
Relax complicating constraints $Dx \leq d$.
Surrogate Relax $Dx \leq d$ using multipliers $\lambda \geq 0$, i.e., add together constraints using weights λ

$$\begin{aligned} z_{SR}(\lambda) = \max\ & cx \\ \text{s.t. } & Ax \leq b \\ & \lambda Dx \leq \lambda d \\ & x \in \mathbb{Z}^n_+ \end{aligned}$$

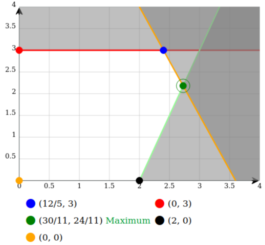
Proposition: Optimal Solution to relaxed problem gives an upper bound on original problem
Proof: show that it is a relaxation

Each multiplier λ_i is a weighting of the corresponding constraint
If λ_i large $\Longrightarrow$ constraint satisfied (at expenses of other constraints)
If $\lambda_i = 0 \Longrightarrow$ drop the constraint

[1] Notation: in this set of slides vectors are not in bold

Surrogate Relaxation: Example

$$\begin{array}{rrl} \text{maximize} & 4x_1 + x_2 & \\ \text{subject to} & 3x_1 - x_2 & \leq 6 \\ & x_2 & \leq 3 \\ & 5x_1 + 2x_2 & \leq 18 \\ & x_1, \quad x_2 & \geq 0, \text{integer} \end{array}$$



IP solution $(x_1, x_2) = (2, 3)$ with $z_{IP} = 11$
LP solution $(x_1, x_2) = (\frac{30}{11}, \frac{24}{11})$ with $z_{LP} = \frac{144}{11} = 13.1$
First and third constraints complicating, surrogate relax using multipliers $\lambda_1 = 2$ and $\lambda_3 = 1$:

$$\begin{array}{rrl} \text{maximize} & 4x_1 + x_2 & \\ \text{subject to} & x_2 & \leq 3 \\ & 11x_1 & \leq 30 \\ & x_1, \quad x_2 & \geq 0, \text{integer} \end{array}$$

Solution $(x_1, x_2) = (2, 3)$ with $z_{SR} = 4 \cdot 2 + 3 = 11$. Upper bound.

Lagrangian Relaxation

Integer Linear Programming problem

$$\begin{aligned} z = \max\ & cx \\ \text{s.t. } & Ax \leq b \\ & Dx \leq e \\ & x \in \mathbb{Z}_+^n \end{aligned}$$

We relax the $Dx \leq e$ constraints:

Lagrangian Relaxation, $\lambda \geq 0$:

$$\begin{aligned} z_{LR}(\lambda) = \max\ & cx - \lambda(Dx - e) \\ \text{s.t. } & Ax \leq b \\ & x \in \mathbb{Z}_+^n \end{aligned}$$

optimizes over the x variables with λ fixed

Lagrange Dual Problem

$$z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)$$

optimizes over the λ variables with x fixed

Tightness of Relaxations (1/2)

Integer Linear Programming problem

$$\begin{aligned} z = \max\ & cx \\ \text{s.t. } & Ax \leq b \\ & Dx \leq e \\ & x \in \mathbb{Z}^n_+ \end{aligned}$$

It corresponds to:

$$\boxed{z = \max\left\{cx \ : \ x \in \operatorname{conv}(Ax \leq b, Dx \leq e, x \in \mathbb{Z}^n_+)\right\}}$$

LP-relaxation:

$$\boxed{z_{LP} = \max\left\{cx \ : \ x \in Ax \leq b, Dx \leq e, x \in \mathbb{R}^n_+\right\}}$$

Lagrangian Relaxation, $\lambda \geq 0$:

$$\begin{aligned} z_{LR}(\lambda) = \max\ & cx - \lambda(Dx - e) \\ \text{s.t. } & Ax \leq b \\ & x \in \mathbb{Z}^n_+ \end{aligned}$$

Lagrange Dual Problem

$$z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)$$

with best multipliers λ it corresponds to:

$$\boxed{z_{LD} = \max\left\{cx \ : \ Dx \leq e, x \in \operatorname{conv}(Ax \leq b, x \in \mathbb{Z}^n_+)\right\}}$$

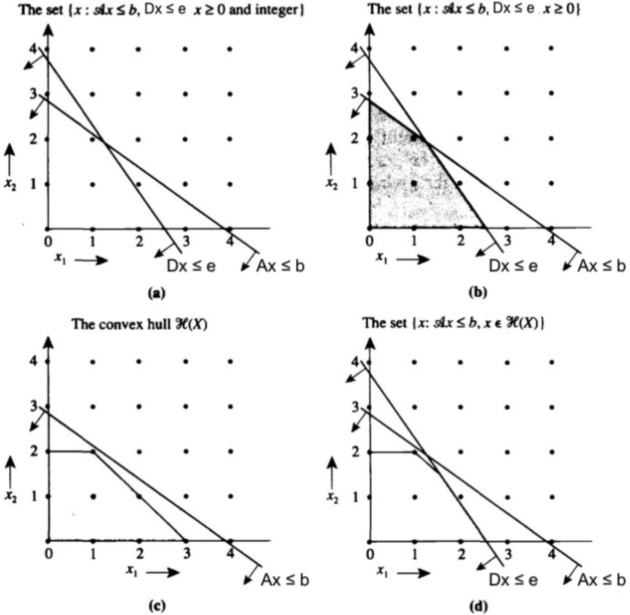


Fig 16.6 from [AMO]

Tightness of Relaxations (2/2)

Surrogate Relaxation, $\lambda \geq 0$

$$
\begin{aligned}
z_{SR}(\lambda) = \max\ & cx \\
\text{s.t. } & Ax \leq b \\
& \lambda Dx \leq \lambda e \\
& x \in \mathbb{Z}_+^n
\end{aligned}
$$

Surrogate Dual Problem

$$z_{SD} = \min_{\lambda \geq 0} z_{SR}(\lambda)$$

with best multipliers λ:

$$\boxed{z_{SD} = \max \left\{ cx \ : \ x \in \text{conv}\left(Ax \leq b, \lambda Dx \leq \lambda e, x \in \mathbb{Z}_+^n\right) \right\}}$$

$\rightsquigarrow$ Best surrogate relaxation (i.e., best λ multipliers) is tighter than best Lagrangian relaxation.

Relaxation strategies

Which constraints should be relaxed

- "the complicating ones"
- remaining problem is polynomially solvable
  (e.g. min spanning tree, assignment problem, linear programming)
- remaining problem is totally unimodular
  (e.g. network problems)
- remaining problem is NP-hard but good techniques exist
  (e.g. knapsack)
- constraints which cannot be expressed in MIP terms
  (e.g. cutting)
- constraints which are too extensive to express
  (e.g. subtour elimination in TSP)

Outline

Subgradient Optimization of Lagrangian Multipliers

$$
\begin{aligned}
z = \max \ & cx \\
\text{s. t.} \ & Ax \leq b \\
& Dx \leq e \\
& x \in \mathbb{Z}^n_+
\end{aligned}
$$

Lagrange Relaxation, multipliers $\lambda \geq 0$

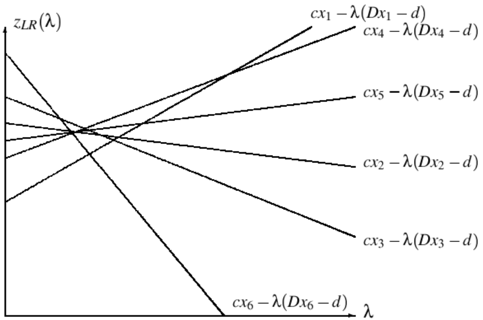
$$
\begin{aligned}
z_{LR}(\lambda) = \max \ & cx - \lambda(Dx - e) \\
\text{s. t.} \ & Ax \leq b \\
& x \in \mathbb{Z}^n_+
\end{aligned}
$$

Lagrange Dual Problem

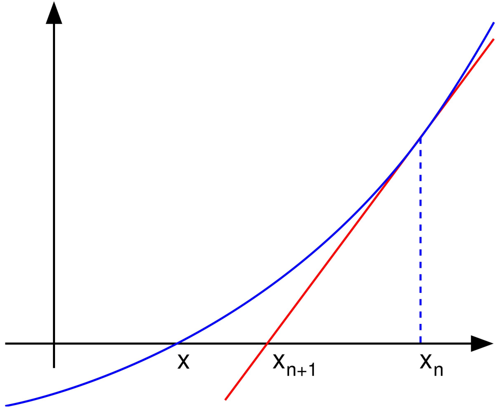
$$
z_{LD} = \min_{\lambda \geq 0} z_{LR}(\lambda)
$$

- We do not need best multipliers in B&B algorithm
- Subgradient optimization fast method
- Works well due to convexity
- Roots in nonlinear programming, Held and Karp (1971)

Subgradient optimization, motivation



Lagrange function $z_{LR}(\lambda)$ is piecewise linear and convex



Netwon-like method to minimize a function in one variable

Digression: Gradient methods

Gradient methods are iterative methods:

- find a descent direction with respect to the objective function f
- move x in that direction by a step size

The descent direction can be computed by various methods, such as gradient descent, Newton-Raphson method and others. The step size can be computed either exactly or loosely by solving a line search problem.

Gradient descent algorithm:

Set iteration counter $t = 0$, and make an initial guess x_0 for the minimum
Repeat:
  Compute a descent direction $\boldsymbol{d}_t = -\nabla(f(x_t))$
  Choose α_t to minimize $f(x_t + \alpha \boldsymbol{d}_t)$ over $\alpha \in \mathbb{R}_+$
  Update $x_{t+1} = x_t + \alpha_t \boldsymbol{d}_t$, and $t = t + 1$
Until $\|\nabla f(x_k)\| < \mathit{tolerance}$

We will set α_t 'loosely' by taking small enough values $\alpha_t > 0$

Newton-Raphson method

Example of gradient algorithm:

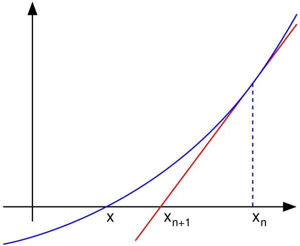
Find zeros of a real-valued, derivable function

$$x : f(x) = 0\,.$$

- Start with a guess x_0
- Repeat:
  Move to a better approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

  until a sufficiently accurate value is reached.



Geometrically, $(x_{n+1}, 0)$ is the intersection with the x-axis of a line tangent to f at $(x_n, f(x_n))$.

$$f'(x_n) = \frac{\Delta y}{\Delta x} = \frac{f(x_n) - 0}{x_n - x_{n+1}}.$$

Subgradient

Subgradient: Generalization of gradients to non-differentiable functions.

Definition

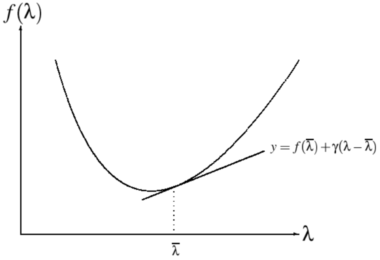
An m-vector γ is subgradient of $f(\lambda)$ at $\bar{\lambda}$ if

$$f(\lambda) \geq f(\bar{\lambda}) + \gamma(\lambda - \bar{\lambda})$$

The inequality says that the hyperplane

$$y = f(\bar{\lambda}) + \gamma(\lambda - \bar{\lambda})$$

is tangent to $y = f(\lambda)$ at $\lambda = \bar{\lambda}$ and supports $f(\lambda)$ from below



Proposition Given a choice of nonnegative multipliers $\bar{\lambda}$, if x' is an optimal solution to $z_{LR}(\bar{\lambda})$ then

$$\gamma = e - Dx'$$

is a subgradient of $z_{LR}(\lambda)$ at $\lambda = \bar{\lambda}$.

Proof Note that for us in the LD problem: $f(\lambda) = \max_{Ax \leq b} (cx - \lambda(Dx - e))$.
We wish to prove that the inequality from the subgradient definition holds:

$$\max_{Ax \leq b} (cx - \lambda(Dx - e)) \geq \max_{Ax \leq b} \left(cx - \bar{\lambda}(Dx - e)\right) + \gamma(\lambda - \bar{\lambda})$$

Indeed:

- We note that in the RHS: $\max_{Ax \leq b} \left(cx - \bar{\lambda}(Dx - e)\right) = (cx' - \bar{\lambda}(Dx' - e))$ because x' is by hypothesis the optimal solution of $f(\bar{\lambda})$.
- Rewriting the inequality using the hypothesis on γ we have:

  $$\max_{Ax \leq b} (cx - \lambda(Dx - e)) \;\geq\; (cx' - \bar{\lambda}(Dx' - e)) + (e - Dx')(\lambda - \bar{\lambda}) \;=\; cx' - \lambda(Dx' - e)$$

  The right most part is the evaluation of the left most problem at a single feasible solution. Hence, it can be at most $\leq$ of the right most part, as we wanted to prove.

Intuition

Lagrange dual:

$$\begin{aligned} \min\ & z_{LR}(\lambda) = cx - \lambda(Dx - e) \\ \text{s.t. } & Ax \leq b \\ & x \in \mathbb{Z}_+^n \end{aligned}$$

Gradient in x' is

$$\gamma = e - Dx'$$

Subgradient Iteration

Recursion

$$\lambda_{k+1} = \max\{\lambda_k - \theta\gamma_k, 0\}$$

where $\theta_k > 0$ is step-size

If $\gamma_k > 0$ and θ_k is sufficiently small $z_{LR}(\lambda)$ will decrease.

- Small θ_k slow convergence
- Large θ_k unstable

Held and Karp procedure (gradient descent)

Initially

$$\lambda^0 = [0, \dots, 0]$$

compute the new multipliers by recursion

$$\lambda_{i,k+1} := \begin{cases} \lambda_{i,k} & \text{if } |\gamma_i| \leq \epsilon \\ \max(\lambda_{i,k} - \theta_k \gamma_i, 0) & \text{if } |\gamma_i| > \epsilon \end{cases}$$

where γ is subgradient.
The step size θ_k is defined by

$$\theta_k = \mu \frac{z_{LR}(\lambda_k) - \underline{z}}{\sum_i \gamma_i^2}$$

where μ is an appropriate constant and $\underline{z}$ a heuristic lower bound for the orginal ILP problem.
E.g. $\mu = 1$ and halved if upper bound not decreased in 20 iterations.

Lagrangian relaxation and LP

For an LP-problem where we Lagrange relax all constraints

- Dual variables are best choice of Lagrange multipliers
- Lagrange relaxation and LP “relaxation” give same bound

Gives a clue to solve LP-problems without Simplex

- Iterative algorithms
- Polynomial algorithms

Outline

Lagrangian Relaxation in Integer Programming

Original Problem (OP)

$$
\begin{aligned}
z = \min\ & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t. } & A\boldsymbol{x} \leq \boldsymbol{b} \\
& D\boldsymbol{x} \leq \boldsymbol{e} \\
& \boldsymbol{x} \geq 0 \\
& \boldsymbol{x} \text{ integer}
\end{aligned}
$$

Lagrangian Relaxation Problem (LR) $\lambda \geq 0$:

$$
\begin{aligned}
z_{LR}(\lambda) = \min\ & \boldsymbol{c}^T \boldsymbol{x} + \lambda(D\boldsymbol{x} - \boldsymbol{e}) \\
\text{s.t. } & A\boldsymbol{x} \leq \boldsymbol{b} \\
& \boldsymbol{x} \geq \boldsymbol{0} \\
& \boldsymbol{x} \text{ integer}
\end{aligned}
$$

- Note that in Lagrangian Relaxation the intgrality constraint is not relaxed
- z_{LP} objective function value of linear relaxation of OP
- $z_{LD} = \max_{\lambda \geq 0} z_{LR}(\lambda)$ Lagrangian dual problem.

Facts

$z_{LP} \leq z$ because relaxation

$z_{LR} \leq z$ because relaxation

$z_{LR} \leq z_{LD}$ because of definition

$z_{LP} \leq z_{LD}$ this is not trivial but important for motivating the use of Lagrangian Relaxation in Integer Programming

- Motivation A: if $z_{LP} < z_{LD}$ then LR gives us a better bound to in B&B.
- Motivation B: if $z_{LP} = z_{LD}$ LR can still be worth because Z_{LD} can be found more easily than with LP
- Motivation C: in any case LR gives us an alternative way to solve the problem. It is an heuristics way with the rare chance of getting also a dual bound and eventualy a provable optimal solution.

For a minimization problem: $z_{LR} \lesseqgtr z_{LP} \leq z_{LD} \leq z$

Proposition

$z_{LD} \geq z_{LP}$

Proof: There are two ways of prooving this:

1. via the convexification argument as in the previous slides (see also sec 16.4 of [AMO])
2. via the duality argument also presented in sec 8 of [Fi]

Let's use the second.

$$
\begin{aligned}
z_{LD} &= \max_{\lambda \geq 0} z_{LR}(\lambda) = \\
&= \max_{\lambda \geq 0} \left\{ \min_{x} \left\{ c^T x + \lambda^T (Dx - e) \mid Ax \leq b, x \geq 0, x \text{ integer} \right\} \right\} \geq \\
&\geq \max_{\lambda \geq 0} \left\{ \min_{x} \left\{ c^T x + \lambda^T (Dx - e) \mid Ax \leq b, x \geq 0 \right\} \right\} =
\end{aligned}
$$

$$= \max_{\lambda \geq 0} \left\{ \underbrace{\min_{x} \left\{ c^T x + \lambda^T (Dx - e) \mid Ax \leq b, x \geq 0 \right\}} \right\} =$$

$$\begin{aligned} &\min cx + \lambda(Dx - d) \\ &\mu : \; Ax \leq b \\ &\quad\;\; \boldsymbol{x} \geq 0 \end{aligned} \qquad \xRightarrow{Dual} \qquad \begin{aligned} &\max \lambda^T b + \mu^T e \\ &\quad \lambda^T A + \mu^T D \geq c \\ &\quad \mu \geq 0 \\ &\quad \lambda \geq 0 \end{aligned}$$

$$= \max_{\lambda \geq 0} \left\{ \max_{\mu \geq 0} \left\{ \lambda^T b + \mu^T e \mid \lambda^T A + \mu^T D \geq c, \mu \geq 0 \right\} \right\} =$$

$$= \underbrace{\max_{\lambda \geq 0} \left\{ \max_{\mu \geq 0} \{ \lambda^T b + \mu^T e \mid \lambda^T A + \mu^T D \geq c, \mu \geq 0 \} \right\}} =$$

$$\begin{aligned} \max\ & \lambda^T b + \mu^T e \\ x :\ & \lambda^T A + \mu^T D \geq c \\ & \mu \geq 0 \\ & \lambda \geq 0 \end{aligned} \quad \overset{Dual}{\Longrightarrow} \quad \begin{aligned} \min\ & c^T x \\ & Ax \leq b \\ & Dx \leq e \\ & x \geq 0 \end{aligned}$$

$$= z_{LP} \quad \square$$

Corollary

$z_{LD} = z_{LP}$ *when the LR problem has the integrality property*

Proof: The only inequality introduced in the derivations of the previous proof becomes equality as well. $\square$