Computational Optimization and Applications, 20, 171–183, 2001

Certificates of Primal or Dual Infeasibility in Linear Programming

ERLING D. ANDERSEN e.d.andersen@mosek.com
EKA Consulting Aps, Fruebjervej 3, Box 16, Copenhagen 2100, Denmark

Received November 18, 1998; Revised May 2, 2000

Abstract. In general if a linear program has an optimal solution, then a primal and dual optimal solution is a certificate of the solvable status. Furthermore, it is well known that in the solvable case, then the linear program always has an optimal basic solution. Similarly, when a linear program is primal or dual infeasible then by Farkas's Lemma a certificate of the infeasible status exists. However, in the primal or dual infeasible case then there is not an uniform definition of what a suitable basis certificate of the infeasible status is.

In this work we present a definition of a basis certificate and develop a strongly polynomial algorithm which given a Farkas type certificate of infeasibility computes a basis certificate of infeasibility. This result is relevant for the recently developed interior-point methods because they do not compute a basis certificate of infeasibility in general. However, our result demonstrates that a basis certificate can be obtained at a moderate computational cost.

Keywords: linear programming, infeasibility, interior-point methods, basis identification

1. Introduction

In general a linear program (LP) has an optimal solution if and only if the primal problem and the corresponding dual problem is feasible and in this case the optimal value of the two problems coincide. This implies if a primal and dual solution is computed and reported, then optimality of the primal solution can easily be verified by checking whether the primal and dual solution is feasible and the duality gap is zero. Hence, the optimal dual solution is a *certificate* of the optimality of the primal solution. In practice this also implies that it is easy for an user of an LP software to verify the correctness of the solution produced by the software.

Similarly, in the case an LP is primal or dual infeasible, then a certificate should be reported which allows easy verification of the infeasible status of the LP.

The reader may question why optimization software should return a certificate of the infeasible status and not just a status indicator which indicates that the problem is infeasible. One reason is optimization software may produce incorrect results, so it advantageous to be able to check the software. Moreover, if for example the LP is a subproblem arising within a column generation based solution method for large-scale LPs, then a certificate is required to generate more columns or to prove that the "real" problem is infeasible.

Hence, a certificate of the infeasible status may be important for both theoretical and practical reasons.

Recently interior-point methods have emerged as an efficient alternative to simplex based solution methods for LPs. Unfortunately some of these methods such as the primal-dual algorithm discussed in [6] do not handle primal or dual infeasible LPs very well [6, p. 177]. However, interior-point methods based on the homogeneous model (or equivalently the self-dual embedding) detects a possible infeasible status both in theory and practice [1, 5]. These interior-point methods generates an infeasibility certificate using Farkas's lemma. This leads to the question about the relation between an "interior-point" certificate of infeasibility and the certificate of infeasibility generated by the simplex method. Moreover, in the case an optimal interior-point solution is known, then it is possible to compute an optimal basic solution in strongly polynomial time [3]. However, is it also possible given an "interior-point" certificate of infeasibility to compute a basis certificate of infeasibility?

The computation of a basis certificate of infeasibility starting from an arbitrary certificate of infeasibility may not only be relevant for interior-point methods. For instance when an LP is solved using the primal simplex method, then in the case the LP is primal infeasible an optimal basis to a phase 1 problem is computed and reported. The optimal basic solution is a certificate of the primal infeasibility, but unfortunately a phase 1 problem is not uniquely defined [4, ch. 8]. Hence, it may be difficult to verify the conclusion of the certificate. However, if such a certificate could easily be converted to a basis certificate of infeasibility (to be defined precisely later), then a well defined certificate could always be reported.

The outline of the paper is as follows. First in Section 2 we present our notation. In Section 3 we discuss primal infeasible LPs and give a definition of an optimal basis to an infeasible LP. Moreover, by generalizing previous work in [2, 3] we show that such a basis can be computed in strongly polynomial time given a Farkas type certificate of infeasibility. In Section 4 we analyze dual infeasible problems. In Section 5 we discuss the usefulness of computing a basis certificate of infeasibility and finally in Section 6 we present our conclusions.

2. Notation and theory

The problem of study is an LP in standard form

$$
\begin{array}{lll}
(P) & \text{minimize} & c^T x \\
 & \text{subject to} & Ax = b, \\
 & & x \geq 0,
\end{array}
$$

where $b \in R^m$, $A \in R^{m \times n}$, and $c,\ x \in R^n$. For convenience and without loss of generality we assume that $rank\,(A) = m$. The dual problem corresponding to (P) is

$$
\begin{array}{lll}
(D) & \text{maximize} & b^T y \\
 & \text{subject to} & A^T y + s = c, \\
 & & s \geq 0,
\end{array}
$$

where $y \in R^m$ and $s \in R^n$. (P) is said to be feasible if a solution satisfying the constraints of (P) exists. Similarly (D) is said to be feasible if (D) has at least one solution satisfying the constraints of (D). The following lemma is a well-known fact of linear programming.

Lemma 2.1.
a. (P) *has an optimal solution if and only if* (x^*, y^*, s^*) *exists such that*
$$Ax^* = b, \quad A^T y^* + s^* = c, \quad c^T x^* = b^T y^*, \quad x^*, s^* \geq 0.$$
b. (P) *is infeasible if and only if an* y^* *exists such that*

$$A^T y^* \leq 0, \quad b^T y^* > 0. \tag{1}$$

c. (D) *is infeasible if and only if an* x^* *exists such that*

$$Ax^* = 0, \quad c^T x^* < 0, \quad x^* \geq 0. \tag{2}$$

Proof: See [5]. □

Hence, (P) has an optimal solution if and only if (P) and (D) are both feasible. Moreover, a primal and dual optimal solution is a certificate of that the problem has an optimal solution. In the case the problem is primal or dual infeasible, then any y^* satisfying (1) and any x^* satisfying (2) is a certificate of the primal and dual infeasible status respectively.

An LP may be both primal and dual infeasible and in that case both a certificate for the primal and dual infeasibility exists.

Note that the statements b and c in Lemma 2.1 are nothing but Farkas's lemma. Also note that in the case an LP is solved using a column generation method and for example the first subproblem is infeasible, then any column having a positive inner product with a certificate y^* of primal infeasibility is a potential candidate to be included in the next subproblem. If no such column exists, then the complete problem can be concluded to be infeasible.

Furthermore, by using the so-called homogeneous model and an interior-point algorithm combined with a finite termination method then either an optimal solution or an infeasibility certificate can be computed in polynomial time, see [7, pp. 159–167, 5]. However, in the case the problem is both primal and dual infeasible then this method is only guaranteed to compute one of the primal and dual infeasibility certificates.

Given the assumption A is of full row rank, then a basic partition of the indices of the variables denoted $(\mathcal{B}, \mathcal{N})$ satisfying

$$|\mathcal{B}| = m \quad \text{and} \quad \operatorname{rank}(B) = m$$

always exists where we use the definition that

$$B := A_{:\mathcal{B}} \quad \text{and} \quad N := A_{:\mathcal{N}}.$$

Also $\mathcal{B}$ is denoted the basis.

An optimal basic partition of the indices of the variables is defined by

$$x_{\mathcal{B}} = B^{-1} b \geq 0, \quad x_{\mathcal{N}} = 0, \tag{3}$$

and

$$y = B^{-T}(c_{\mathcal{B}} - s_{\mathcal{B}}), \quad s_{\mathcal{B}} = 0, \quad s = c - A^T y \geq 0. \tag{4}$$

Subsequently we say a basic partition of the indices of the variables is primal feasible if it satisfies (3), where the requirement $x_{\mathcal{N}} = 0$ may be relaxed to $x_{\mathcal{N}} \geq 0$. Similarly, a basic partition of the indices of the variables is dual feasible if it satisfies (4) possible with $s_{\mathcal{B}} = 0$ relaxed to $s_{\mathcal{B}} \geq 0$.

It has been proved in [3] that given any primal and dual optimal solution, then an optimal basic partition of the indices of the variables can be computed in strongly polynomial time. However, in the case (P) is primal or dual infeasible, then it is undefined what constitutes an "optimal" basic partition of the indices of the variables. Furthermore, given a definition of an optimal basic partition of the indices of the variables to an infeasible LP, then it is unclear whether such a basis can be computed in strongly polynomial time starting from an arbitrary certificate of the infeasible status.

In the subsequent two sections we will give a definition of an optimal basic partition of the indices of the variables for a primal and dual infeasible LP. Moreover, we will present a strongly polynomial procedure to compute such a basic partition starting from any infeasibility certificate.

3. Primal infeasible linear programs

The primal simplex method as commonly presented must be initiated with a basic feasible solution. However, in general such a solution is of course not known and therefore a phase 1 method is used to initialize the simplex method. The principle of the phase 1 method is to construct an LP having a known feasible basic solution. In addition the constructed LP always has an optimal basic solution which either demonstrates the original problem is infeasible or is a feasible basic solution to (P).

A "textbook" phase 1 problem corresponding to (P) is

$$\begin{array}{ll} \text{minimize} & z_{1p} = e^T t^+ + e^T t^- \\ \text{subject to} & Ax + It^+ - It^- = b, \\ & x, t^+, t^- \geq 0. \end{array} \tag{5}$$

e is a vector of appropriate dimension containing all ones. The problem (5) clearly has a feasible solution and the purpose of the objective function is to minimize the sum of infeasibility. It follows that (P) has a feasible solution if and only if

$$z_{1p}^* = 0.$$

The dual problem corresponding to (5) is

$$\begin{array}{ll} \text{maximize} & z_{1d} = b^T y \\ \text{subject to} & A^T y + s = 0, \\ & e \geq y \geq -e, \quad s \geq 0. \end{array} \tag{6}$$

The constraints of the problem (6) contain a homogeneous component and a normalization component which implies (6) is trivially feasible. Furthermore, from weak duality we have that $z_{1p}^* \geq b^T y$ for any dual feasible y. This implies that any feasible solution y to (6) having a positive objective value is a certificate of primal infeasibility. Note such an y also satisfies (1). On the other hand any certificate y^* of primal infeasibility satisfying (1) is almost a feasible solution to (6), because

$$(y; s) = \frac{(y^*; -A^T y^*)}{\|y^*\|_\infty}$$

is a feasible solution to (6) having a positive objective value. Perhaps not that surprising we see there is a close relationship between certificates of primal infeasibility and feasible solutions to the dual problem of the phase 1 problem (5).

If an infeasible problem (P) is solved using the dual simplex algorithm and assuming (P) is dual feasible, then the dual simplex method will terminate with a basis B and an index i such that

$$e_i^T B^{-1} A \geq 0 \quad \text{and} \quad e_i^T B^{-1} b < 0$$

indicating the dual problem is unbounded, where e_i is the ith unit vector of appropriate dimension. Observe that

$$y = \frac{-B^{-T} e_i}{\|B^{-T} e_i\|_\infty} \quad \text{and} \quad s = -A^T y$$

is a feasible solution to (6) having a positive objective value. Therefore, we define a basic partition $(\mathcal{B}, \mathcal{N})$ of the indices of the variables to be a certificate of primal infeasibility if it satisfies Definition 3.1.

Definition 3.1. A basic partition $(\mathcal{B}, \mathcal{N})$ of the indices of the variables to (P) is a certificate of primal infeasibility if

$$\exists i : e_i^T B^{-1} A \geq 0, \quad e_i^T B^{-1} b < 0. \tag{7}$$

Any primal infeasible LP has a basic partition of the indices of the variables which satisfies Definition 3.1, because the problem is primal infeasible independent of the objective vector c. Clearly, the problem is trivially dual feasible for $c = 0$ and if the dual simplex method is applied to (P) with zero objective, then it will ultimately determine the required basic partition.

3.1. Basis identification in the primal infeasible case

Given a certificate (y^*, s^*) of primal infeasibility satisfying

$$A^T y^* + s^* = 0, \quad s^* \geq 0, \quad \text{and} \quad b^T y^* > 0 \tag{8}$$

we will show that a basis certificate satisfying Definition 3.1 can be computed in strongly polynomial time.

In practice an exact solution (y^*, s^*) to (8) may not be known. For example (y^*, s^*) may be the limit point of an infinite sequence generated by an algorithm. Hence, it is advantageous to be able to work with only an approximate solution to (8). Although only in the case an exact solution to (8) is known, then the proposed procedure is guaranteed to compute a basis satisfying Definition 3.1.

Therefore, assume an approximate solution (y^*, s^*) to (8) is known such that $s^* \geq 0$. Moreover, assume that an estimate $(\mathcal{P}^*, \bar{\mathcal{P}}^*)$ for the optimal partition is known. The optimal partition is in this case defined by

$$\mathcal{P}^* = \left\{ j : s_j^* = -e_j^T A^T y^* = 0, \quad \forall y^* \in \{y : A^T y \leq 0, b^T y > 0\} \right\}.$$

One simple way of generating this estimate is to use

$$\mathcal{P}^* = \{j : s_j^* = 0\} \quad \text{and} \quad \bar{\mathcal{P}}^* = \{1, \ldots, n\} \backslash \mathcal{P}^*.$$

Given these assumptions we can employ an idea originally suggested in [2] to define a perturbed objective vector as follows

$$\hat{c}_j := \begin{cases} A_{:j}^T y^*, & j \in \mathcal{P}^*, \\ s_j^* + A_{:j}^T y^*, & j \in \bar{\mathcal{P}}^*. \end{cases}$$

In the case (y^*, s^*) is an exact optimal solution to (8) then $\hat{c} = 0$ is the case. Next define the perturbed LP

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = \hat{c}, \\ & s \geq 0 \end{array} \tag{9}$$

which by construction has the feasible solution $(y, s) = (y^*, \hat{c} - A^T y^*)$. Therefore, given (P) is infeasible then (9) is unbounded. Now if we apply a specialized version of the dual simplex method to (9) starting from the known solution, then the dual simplex algorithm must determine a basis which proves that (9) is unbounded. Such a basis satisfies Definition 3.1.

The specialized dual simplex algorithm can be derived as follows. Given an initial feasible solution (y^0, s^0) to (9) and any basic partition $(\mathcal{B}, \mathcal{N})$ of the indices of the variables, then the constraints of the dual problem can be written as follows

$$\begin{aligned} B^T y + s_{\mathcal{B}} &= \hat{c}_{\mathcal{B}}, \\ N^T y + s_{\mathcal{N}} &= \hat{c}_{\mathcal{N}}. \end{aligned}$$

It follows from B is nonsingular and feasibility that

$$y = B^{-T}\left(\hat{c}_{\mathcal{B}} - s_{\mathcal{B}}^0\right) = y^0.$$

The basic partition of the indices of the variables is only a super-basic solution because some of the components in $s^0_{\mathcal{B}}$ may be positive. The positive variables in $s^0_{\mathcal{B}}$ are denoted super-basic dual variables. In order to obtain a basic solution then the super-basic variables should be moved to zero while not decreasing the dual objective value and maintaining dual feasibility.

If $s^0_{\mathcal{B}_i} > 0$, then we can use the search direction

$$d_y := -B^{-T} e_i \quad \text{and} \quad d_s := -A^T d_y$$

to achieve this aim. The new dual solution is defined by

$$y^+ := y^0 + \alpha d_y \quad \text{and} \quad s^+ := s^0 + \alpha d_s$$

which maintains dual feasibility for a sufficiently small value of the step size α in absolute terms. Moreover, the search direction is constructed such that

$$s^+_{\mathcal{B}} = s^0_{\mathcal{B}} + \alpha e_i. \tag{10}$$

The new objective value is given by

$$b^T y^+ = b^T y^0 + \alpha b^T d_y$$

showing that if $b^T d_y$ is positive, then α should be increased and otherwise it should be decreased. Hence, the optimal step size α^* is given by

$$\begin{aligned} \alpha^* = \arg\max_{\alpha} \quad & (b^T d_y)\alpha \\ \text{subject to} \quad & s^0 + \alpha d_s \geq 0. \end{aligned} \tag{11}$$

Note this problem is unbounded if and only if $(b^T d_y)\alpha^* = \infty$ and in that case we have

$$e_i^T B^{-T} A = d_s^T \geq 0 \quad \text{and} \quad -e_i^T B^{-1} b = b^T d_y > 0$$

which demonstrates that a basis satisfying Definition 3.1 has been computed. If on the other hand the problem (11) is not unbounded, then two cases occur. In the first case we have that

$$e_i^T s^+_{\mathcal{B}} = 0$$

leading to a reduction in the number of super-basic variables by one. Note no basic exchange occurs. In the second case we have that

$$\exists j \in \mathcal{N} : d_{s_j} < 0 \quad \text{and} \quad s^+_j = 0$$

because otherwise the problem (11) would be unbounded. Moreover, if we exchange such a binding nonbasic variable with the super-basic variable $(s_{\mathcal{B}_i})$, then the number of the super-basic variables is reduced by one. In both cases the dual objective value is nondecreasing and the number of super-basics is reduced by one. Hence, after at most m pivots all the super-basic dual variables are removed. Observe degeneration ($\alpha^* = 0$) does not affect this conclusion.

Using these ideas Algorithm 3.1 can be stated.

Algorithm 3.1.

1. *procedure dual-improve* $(c, A, b, y^0, s^0, \mathcal{B}, \mathcal{N})$
2. *Assumption:* $s^0 = c - A^T y^0 \geq 0.$
3. $k := 0.$
4. *while* $\exists i^k : s^k_{\mathcal{B}_{i^k}} > 0$ *do*
5. $\quad d_y := -B^{-T} e_{i^k},\ d_s := -A^T d_y$
6. $\quad \alpha^k := arg \max_{\alpha} \quad (b^T d_y)\alpha$
 $\qquad\qquad$ *subject to* $s^k + \alpha d_s \geq 0.$
7. $\quad$ *if* $(b^T d_y)\alpha^k = \infty$, *then terminate.*
8. $\quad (s^{k+1}; y^{k+1}) := (s^k; y^k) + \alpha^k (d_s; d_y)$
9. $\quad$ *Choose* $j^k \in \{j : s^{k+1}_j = 0, d_{s_j} \neq 0\}$
10. $\quad \mathcal{B} := (\mathcal{B} \backslash \{\mathcal{B}_{i^k}\}) \cup \{j^k\}, \mathcal{N} := \{1, \ldots, n\} \backslash \mathcal{B}$
11. $\quad k := k + 1$
12. *end while*.

Let

$$I^k := \left\{ i : s^k_i > 0,\ i \in \mathcal{B}^k \right\},$$

where $\mathcal{B}^k$ is the basis in the kth iteration, then by construction

$$|I^0| \leq m \quad \text{and} \quad |I^{k+1}| \leq |I^k| - 1$$

is true for all k. Therefore, Algorithm 3.1 terminates after at most m iterations which implies that the algorithm is strongly polynomial. Moreover, if we assume that the initial solution (y^*, s^*) is an exact solution to (8) and use the choice $\mathcal{P}^* = \{j : s^*_j = 0\}$, then it follows that $c = \hat{c} = 0$. Given these assumptions and that Algorithm 3.1 is not terminated in step 7, then the algorithm computes a basis $\mathcal{B}$ and a final dual solution $(\hat{y}, \hat{s})$ satisfying $\hat{s}_{\mathcal{B}} = 0$. This implies

$$\begin{aligned} 0 &< b^T y^* \\ &= b^T y^0 \\ &\leq b^T \hat{y} \\ &= b^T B^{-T} (c_{\mathcal{B}} - \hat{s}_{\mathcal{B}}) \\ &= b^T B^{-T} (0 - 0) \\ &= 0 \end{aligned}$$

which is a contradiction. The conclusion is if Algorithm 3.1 is initiated with an exact solution to (8), then in less than m iterations a basic partition of the indices of the variables satisfying Definition 3.1 is determined.

In the case Algorithm 3.1 is not initialized with an exact feasible solution and the algorithm is not terminated in step 7, then the problem (9) can be optimized using the dual simplex method starting from the basis generated by Algorithm 3.1. The dual simplex method may either conclude (9) is unbounded in which case (P) is infeasible. Otherwise the optimal basis to (9) is a feasible starting basis for phase 2 of the primal simplex method when applied to (P).

To summarize we have the following theorem.

Theorem 3.1. *Given any certificate* (y^*, s^*) *of primal infeasibility* (*see* (1)), *then a basis certificate satisfying Definition* 3.1 *can be computed in strongly polynomial time.*

4. Dual infeasible linear programs

In this section dual infeasible LPs are discussed. The discussion is kept brief because it is fairly similar to the discussion of the primal infeasible case.

If the dual simplex method is applied to an LP, then first a phase 1 problem is created which is used to compute an initial dual basic feasible solution. One possible phase 1 problem to (D) is

$$\begin{array}{ll} \text{maximize} & \hat{z}_{d1} = -e^T s^- \\ \text{subject to} & A^T y + s - s^- = c, \\ & s, s^- \geq 0. \end{array} \tag{12}$$

and the corresponding dual problem is

$$\begin{array}{ll} \text{minimize} & \hat{z}_{p1} = c^T x \\ \text{subject to} & Ax = 0, \\ & 0 \leq x \leq e. \end{array} \tag{13}$$

(D) has a feasible solution if and only if

$$\hat{z}_{d1}^* = \hat{z}_{p1}^* = 0,$$

where $\hat{z}_{d1}^*$ and $\hat{z}_{p1}^*$ denote optimal values.

Now assume (D) is infeasible then a certificate x^* satisfying (2) exists. This implies that

$$x = \frac{x^*}{\|x^*\|_\infty}$$

is a feasible solution to (13) having a strictly negative objective value. Similar, if x is a feasible solution to (13) having a negative objective value, then x is certificate of dual infeasibility satisfying (2).

If a primal feasible but dual infeasible LP is solved using the primal simplex method, then ultimately the problem is detected primal unbounded in the ratio test which implies the problem is dual infeasible. Hence, a basis satisfying Definition 4.1 is computed.

Definition 4.1. A basic partition $(\mathcal{B}, \mathcal{N})$ of the indices of the variables to (P) is a certificate of dual infeasibility if

$$\exists j : B^{-1}Ae_j \leq 0, \quad (c - A^T B^{-T} c_{\mathcal{B}})^T e_j < 0. \tag{14}$$

Note independent of whether the primal problem is feasible then a basis satisfying Definition 4.1 exists. Furthermore, we have that

$$x_{\mathcal{B}} = -\frac{B^{-1}Ae_j}{1 + \|B^{-1}Ae_j\|_\infty} \quad \text{and} \quad x_k = \begin{cases} \dfrac{1}{1 + \|B^{-1}Ae_j\|_\infty}, & k = j \\ 0, & k \in \mathcal{N} \setminus \{j\} \end{cases}$$

which satisfies (2). Hence, given a basis certificate satisfying Definition 4.1 it is trivial to compute a certificate of dual infeasibility satisfying (2).

4.1. Basis identification in the dual infeasible case

Given a certificate x^* of dual infeasibility is known such that

$$Ax^* = 0, \quad x^* \geq 0, \quad \text{and} \quad c^T x^* < 0, \tag{15}$$

then we show a basis certificate satisfying Definition 4.1 can be computed in strongly polynomial time.

Similar to the discussion in Section 3.1 it is only assumed that an approximate solution to (15) is known such that $x^* \geq 0$. Moreover, it is assumed that an estimate $(\mathcal{P}^*, \bar{\mathcal{P}}^*)$ for the optimal partition is known where the optimal partition is defined by

$$\mathcal{P}^* := \{j : Ax^* = 0, c^T x^* < 0, x^* \geq 0, x_j^* > 0, \text{ for some } x^*\}.$$

One possible estimate is

$$\mathcal{P}^* = \{j : x_j^* > 0\}. \tag{16}$$

Next define

$$\hat{b} := \sum_{j \in \mathcal{P}^*} A_{:j} x_j^*$$

and the perturbed problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = \hat{b}, \\ & x \geq 0. \end{array} \tag{17}$$

Clearly, the problem (17) is feasible due to the assumptions and the way $\hat{b}$ is constructed. Furthermore,

$$\hat{b} = 0$$

if x^* satisfies (15). Furthermore, if (D) is infeasible, then (17) is unbounded.

The idea is now to apply a specialized version of the primal simplex method to the problem (17) starting from the known feasible solution. In the case x^* is an exact solution to (15), then $\hat{b} = 0$ and the specialized algorithm is guaranteed to compute a basis satisfying Definition 4.1 in at most $(n - m)$ simplex type pivots.

Now let x^0 be an initial feasible solution to (17) such that $c^T x^0 < 0$ and let $(\mathcal{B}, \mathcal{N})$ be any basic partition of the indices of the variables to (17). This implies

$$x_{\mathcal{B}} = B^{-1}\big(\hat{b} - N x^0_{\mathcal{N}}\big) = x^0_{\mathcal{B}} \geq 0.$$

Moreover, not all the components in $x^0_{\mathcal{N}}$ are zero because

$$0 > c^T x^0 = (c_{\mathcal{N}} - N^T B^{-T} c_{\mathcal{B}})^T x^0_{\mathcal{N}}$$

holds. Positive nonbasic variables are denoted super-basic variables and they should be moved to zero or introduced into the basis in order to create a basic solution. Given $\hat{b} = 0$ then such a purified basic solution will be a basis certificate of infeasibility.

This idea can be formalized as follows. First define the search direction

$$d_{x_{\mathcal{N}}} := e_j \quad \text{and} \quad d_{x_{\mathcal{B}}} := -B^{-1} A_{:j}$$

which has the property

$$A d_x \; = \; 0.$$

Next a new point x^+ is obtained using

$$x^+ := x^0 + \alpha d_x$$

which for a sufficiently small value of α in absolute terms implies x^+ is primal feasible. If $c^T d_x < 0$, then α should be increased or otherwise it should be decreased because it leads to a decrease in the objective value. In the first case it might be possible to increase α indefinitely and in this case a basic partition of the indices of the variables satisfying Definition 4.1 has been computed. Otherwise either x^+_j is reduced to zero or a basis exchange occurs in the usual way. Hence, the number of super-basic variables is reduced by one. Summarizing these observations leads to Algorithm 4.1.

Algorithm 4.1.

1. *procedure primal-improve* $(c, A, b, x^0, \mathcal{B}, \mathcal{N})$
2. *Assumption:* $Ax^0 = b, \quad x^0 \geq 0.$
3. $k = 0.$
4. *while* $\exists j^k \in \mathcal{N} : x^k_j > 0$ *do*

5. $d_{x_{\mathcal{B}}} := -B^{-1}A_{:j^k}, d_{x_{\mathcal{N}}} := e_j^k,$
6. $\alpha^k = arg \min_{\alpha} \quad (c^T d_x)\alpha$
   $\quad\quad subject\ to\ x^k + \alpha d_x \geq 0$
7. *if* $\alpha^k = -(c^T d_x)\infty,$ *then terminate.*
8. $x^{k+1} := x^k + \alpha^k d_x$
9. *Choose* $i^k \in \{j : x_j^{k+1} = 0, d_{x_j} \neq 0\}$
10. $\mathcal{B} := (\mathcal{B} \cup \{j^k\}) \backslash \{i^k\}, \mathcal{N} := \{1, \ldots, n\} \backslash \mathcal{B}$
11. $k := k + 1$
12. *end while*

Algorithm 3.1 will terminate after at most $(n - m)$ iterations, because initially there are at most $(n - m)$ super-basic variables and one super-basic variable is removed in each iteration. Moreover, in the case x^* is an exact solution to (15) and the estimate (16) is used then Algorithm 4.1 will terminate in step 7 with a basic partition of the indices of the variables satisfying Definition 4.1.

Finally, we have the following theorem.

Theorem 4.1. *Given a certificate x^* of primal infeasibility* (*see* (2)) *then a basis certificate of dual infeasibility satisfying Definition* 4.1 *can be computed in strongly polynomial time.*

5. Discussion

The main result in the previous sections is the development of two procedures which can convert a Farkas type certificate of primal (dual) infeasibility into a basis certificate of primal (dual) infeasibility in strongly polynomial time. This may be of theoretical interest only. However, it may also be of practical significance.

One practical use may arise from that a basis certificate can have quite different properties than an arbitrary infeasibility certificate.

Indeed if an LP is for example primal infeasible, then many alternative basis certificates of the primal infeasibility may exists. Obviously any convex combination of those basis certificates is also a certificate of the primal infeasibility. However, one particular basis certificate of the infeasibility and a convex combination of all possible certificates is likely to be very different. For example a basis certificate is likely to be sparser than an arbitrary convex combination of all infeasibility certificates. This may be a very advantageous feature in some practical applications. However, if the certificate is used to generate new columns within a column generation scheme, then a certificate reflecting all the infeasibilities in the problem might be better.

6. Conclusion

Independent of whether an LP has an optimal solution, is primal infeasible, or is dual infeasible then there exists a certificate which allows easy verification of the status of the problem. In this paper we argue that optimization software should return such a certificate and not just a status flag indicating the status of the problem.

Finally, we show that basis certificates of primal or dual infeasibility can be computed in strongly polynomial time starting from a more general infeasibility certificate.

Acknowledgment

The author is thankful for comments made by Tamas Terlaky and the referees regarding the present manuscript.

References

1. E.D. Andersen and K.D. Andersen, "The MOSEK interior point optimizer for linear programming: An implementation of the homogeneous algorithm," in High Performance Optimization, H. Frenk, K. Roos, T. Terlaky, and S. Zhang (Eds.), Kluwer Academic Publishers: Dordrecht/Boston/New York, 2000.
2. E.D. Andersen and Y.Ye, "Combining interior-point and pivoting algorithms," Management Sci., vol. 42, no. 12, pp. 1719–1731, 1996.
3. N. Megiddo, "On finding primal- and dual-optimal bases," *ORSA J. on Comput.*, vol. 3, no. 1, pp. 63–65, 1991.
4. J.L. Nazareth, Computer Solution of Linear Programs, Oxford University Press: New York, 1987.
5. C. Roos, T. Terlaky, and J.-Ph. Vial, Theory and Algorithms for Linear Optimization: An Interior Point Approach, John Wiley and Sons: New York, 1997.
6. S.J. Wright, Primal-Dual Interior-Point Methods, SIAM: Philadelphia, 1997.
7. Y. Ye, Interior Point Algorithms: Theory and Analysis, John Wiley and Sons: New York, 1997.