

DM872
Math Optimization at Work

Preprocessing

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Outline

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Preprocessing rules

Consider $S = \{\mathbf{x} : a_0 x_0 + \sum_{j=1}^n a_j x_j \leq b, l_j \leq x_j \leq u_j, j = 0..n\}$

- Bounds on variables.

If $a_0 > 0$ then:

$$x_0 \leq \left(b - \sum_{j:a_j>0} a_j l_j - \sum_{j:a_j<0} a_j u_j \right) / a_0$$

and if $a_0 < 0$ then

$$x_0 \geq \left(b - \sum_{j:a_j>0} a_j l_j - \sum_{j:a_j<0} a_j u_j \right) / a_0$$

- Redundancy. The constraint $\sum_{j=0}^n a_j x_j \leq b$ is redundant if

$$\sum_{j:a_j>0} a_j u_j + \sum_{j:a_j<0} a_j l_j \leq b$$

- Infeasibility: $S = \emptyset$ if (swapping lower and upper bounds from previous case)

$$\sum_{j:a_j>0} a_j l_j + \sum_{j:a_j<0} a_j u_j > b$$

- Variable fixing. For a max problem in the form

$$\max\{\mathbf{c}^T \mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\}$$

if $\forall i = 1..m : a_{ij} \geq 0, c_j < 0$ then fix $x_j = l_j$

if $\forall i = 1..m : a_{ij} < 0, c_j > 0$ then fix $x_j = u_j$

- Integer variables:

$$\lceil l_j \rceil \leq x_j \leq \lfloor u_j \rfloor$$

- Binary variables. Probing: add a constraint, eg, $x_2 = 0$ and check what happens

Example

$$\begin{aligned} \max \quad & 2x_1 + x_2 - x_3 \\ \text{R1 : } & 5x_1 - 2x_2 + 8x_3 \leq 15 \\ \text{R2 : } & 8x_1 + 3x_2 - x_3 \geq 9 \\ \text{R3 : } & x_1 + x_2 + x_3 \leq 6 \\ & 0 \leq x_1 \leq 3 \\ & 0 \leq x_2 \leq 1 \\ & x_3 \geq 1 \end{aligned}$$

$$\begin{aligned} \text{R1 : } 5x_1 &\leq 15 + 2x_2 - 8x_3 \leq 15 + 2 \cdot \overbrace{1}^{u_2} - 8 \cdot \overbrace{1}^{l_3} = 9 && \rightsquigarrow x_1 \leq 9/5 \\ 8x_3 &\leq 15 + 2x_2 - 5x_1 \leq 15 + 2 \cdot 1 - 5 \cdot 0 = 17 && \rightsquigarrow x_3 \leq 17/8 \\ 2x_2 &\geq 5x_1 + 8x_3 - 15 \geq 5 \cdot 0 + 8 \cdot 1 = -7 && \rightsquigarrow x_2 \geq -7/2, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \text{R2 : } 8x_1 &\geq 9 - 3x_2 + x_3 \geq 9 - 3 + 1 = 7 && \rightsquigarrow x_1 \geq 7/8 \\ \text{R1 : } 8x_3 &\geq 15 + 2x_2 - 5x_1 \leq 15 + 2 - 5 \cdot 7/8 = 101/8 && \rightsquigarrow x_3 \leq 101/64 \end{aligned}$$

$$\text{R3 : } x_1 + x_2 + x_3 \leq 9/5 + 1 + 101/64 < 6 \quad \text{Hence R3 is redundant}$$

Example

$$\begin{aligned} \max \quad & 2x_1 + x_2 - x_3 \\ \text{R1 : } & 5x_1 - 2x_2 + 8x_3 \leq 15 \\ \text{R2 : } & 8x_1 + 3x_2 - x_3 \geq 9 \\ & 7/8 \leq x_1 \leq 9/5 \\ & 0 \leq x_2 \leq 1 \\ & 1 \leq x_3 \leq 101/64 \end{aligned}$$

Increasing x_2 makes constraints satisfied $\rightsquigarrow x_2 = 1$

Decreasing x_3 makes constraints satisfied $\rightsquigarrow x_3 = 1$

We are left with:

$$\max\{2x_1 : 7/8 \leq x_1 \leq 9/5\}$$

Preprocessing for Set Covering/Partitioning

1. if $e_i^T A = 0$ then the i th row can never be satisfied

$$[0 \ 0 \ \dots \ 1 \ \dots \ 0] \left[\begin{array}{c} \vdots \\ 0 \ \dots \ 0 \ \dots \ 0 \\ \vdots \end{array} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

2. if $e_i^T A = e_k$ then $x_k = 1$ in every feasible solution

$$[0 \ 0 \ \dots \ 1 \ \dots \ 0] \left[\begin{array}{c} \vdots \\ 0 \ \dots \ 1 \ \dots \ 0 \\ \vdots \end{array} \right] = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

In SPP can remove all rows t with $a_{tk} = 1$ and set $x_j = 0$ (ie, remove cols) for all cols that cover t

3. if $e_t^T A \geq e_p^T A$ then we can remove row t , row p dominates row t (by covering p we cover t)

$$\begin{array}{c} t \\ p \end{array} \left[\begin{array}{ccc} & 1 & 1 & 1 \\ & 1 & & 1 \end{array} \right]$$

In SPP we can remove all
cols j : $a_{tj} = 1, a_{pj} = 0$

4. if $\sum_{j \in S} A e_j = A e_k$ and $\sum_{j \in S} c_j \leq c_k$ then we can cover the rows by $A e_k$ more cheaply with S and set $x_k = 0$
(Note, we cannot remove S if $\sum_{j \in S} c_j \geq c_k$)

$$\left[\begin{array}{c|c|c|c|c} & 1 & & & 1 \\ & 1 & & & 1 \\ & & 1 & & 1 \\ 0 & 0 & 0 & & 0 \\ & 1 & & & 1 \\ 0 & 0 & 0 & & 0 \end{array} \right]$$

- MIP challenge 2024 on Presolving Reductions <https://github.com/dominiqs81/MIPcc24>
- Papilo, parallel presolving <https://github.com/dominiqs81/MIPcc24>

Summary

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