DM545/DM871

Linear and Integer Programming

Lecture 11

Network Flows

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Outline

$$
\begin{array}{r|cccccccc}
 & x_{e_1} & x_{e_2} & \cdots & x_{ij} & \cdots & x_{e_m} & & \\
 & c_{e_1} & c_{e_2} & \cdots & c_{ij} & \cdots & c_{e_m} & & \\
\hline
1 & -1 & . & \cdots & . & \cdots & . & = & b_1 \\
2 & . & . & \cdots & . & \cdots & . & = & b_2 \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
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j & . & . & \cdots & 1 & \cdots & . & = & b_j \\
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n & . & . & \cdots & . & \cdots & . & = & b_n \\
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e_1 & 1 & & & & & & \leq & u_1 \\
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(i,j) & & & & 1 & & & \leq & u_{ij} \\
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e_m & & & & & & 1 & \leq & u_m
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Outline

Shortest Path - Dual LP

$$z = \min \sum_{ij \in A} c_{ij} x_{ij}$$

$$\sum_{j:ji \in A} x_{ji} - \sum_{j:ij \in A} x_{ij} = 1 \qquad \textit{for } i = s \qquad (\pi_s)$$

$$\sum_{j:ji \in A} x_{ij} - \sum_{j:ij \in A} x_{ji} = 0 \qquad \forall i \in V \setminus \{s, t\} \qquad (\pi_i)$$

$$\sum_{j:ji \in A} x_{ji} - \sum_{j:ij \in A} x_{ij} = -1 \qquad \textit{for } i = t \qquad (\pi_t)$$

$$x_{ij} \geq 0 \qquad \forall ij \in A$$

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& x_{ij} \geq 0 & \forall ij \in A &
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Dual problem:

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\begin{aligned}
g^{LP} &= \max \pi_s - \pi_t & & \\
& \pi_j - \pi_i \leq c_{ij} & \forall ij \in A &
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Hence, the shortest path can be found by potential values π_i on nodes such that $\pi_s = z, \pi_t = 0$ and $\pi_j - \pi_i \leq c_{ij}$ for $ij \in A$

Maximum (s, t)-Flow

Adding a backward arc from t to s:

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z = \max x_{ts} & & \\
\sum_{j:ji \in A} x_{ij} - \sum_{j:ij \in A} x_{ji} = 0 & \quad \forall i \in V & \quad (\pi_i) \\
x_{ij} \leq u_{ij} & \quad \forall ij \in A & \quad (w_{ij}) \\
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g^{LP} = \min \sum_{ij\in A} u_{ij} w_{ij} & & \\
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for those arcs that minimize the cut capacity $\sum_{ij \in A} u_{ij} w_{ij}$

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  for those arcs that minimize the cut capacity $\sum_{ij \in A} u_{ij} w_{ij}$
- Complementary slackness: $w_{ij} = 1 \implies x_{ij} = u_{ij}$

Theorem

A strong dual to the max (st)-flow is the minimum (st)-cut problem:

$$\min_X \left\{ \sum_{ij \in A : i \in X, j \notin X} u_{ij} : s \in X \subset V \setminus \{t\} \right\}$$

Max Flow Algorithms

Optimality Condition

- Ford Fulkerson augmenting path algorithm $O(m|x^*|)$
- Edmonds-Karp algorithm (augment by shortest path) in $O(nm^2)$
- Dinic algorithm in layered networks $O(n^2m)$
- Karzanov's push relabel $O(n^2m)$

Min Cost Flow - Dual LP

$$\min \sum_{ij \in A} c_{ij} x_{ij}$$

$$\sum_{j:ji \in A} x_{ij} - \sum_{j:ij \in A} x_{ji} = b_i \qquad \forall i \in V \qquad (\pi_i)$$

$$x_{ij} \leq u_{ij} \qquad \forall ij \in A \qquad (w_{ij})$$

$$x_{ij} \geq 0 \qquad \forall ij \in A$$

Dual problem:

$$\max \sum_{i \in V} b_i \pi_i - \sum_{ij \in E} u_{ij} w_{ij} \tag{1}$$

$$-c_{ij} - \pi_i + \pi_j \leq w_{ij} \qquad \forall ij \in E \tag{2}$$

$$w_{ij} \geq 0 \qquad \forall ij \in A \tag{3}$$

- define reduced costs $\bar{c}_{ij} = c_{ij} + \pi_j - \pi_i$, hence (2) becomes $-\bar{c}_{ij} \leq w_{ij}$

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- Complementary slackness th. for optimal solutions:
  each primal variable $\times$ the corresponding dual slack must be equal 0, ie, $x_e(\bar{c}_e + w_e) = 0$;

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    $-\bar{c} > 0 \implies x_e = u_e$ or equivalently $\bar{c} < 0 \implies x_e = u_e$

  Hence:
    $\bar{c}_e > 0$ then $x_e = 0$
    $\bar{c}_e < 0$ then $x_e = u_e \neq \infty$

Min Cost Flow Algorithms

Theorem (Optimality conditions)

Let $\mathbf{x}$ be feasible flow in $N(V, A, \mathbf{l}, \mathbf{u}, \mathbf{b})$ then $\mathbf{x}$ is min cost flow in N iff $N(\mathbf{x})$ contains no directed cycle of negative cost.

- Cycle canceling algorithm with Bellman Ford Moore for negative cycles $O(nm^2UC)$, $U = \max|u_e|$, $C = \max|c_e|$
- Build up algorithms $O(n^2mM)$, $M = \max|b(v)|$