Lagrangian Relaxation

in Integer Programming

Original Problem (OP)

$$z = \min \ c^T x$$
$$Ax \leq b$$
$$Dx \leq e$$
$$x \geq 0$$
$$x \text{ integer}$$

Lagrangian Relaxation Problem (LR)

$$z_{LR}(\lambda) = \min \ c^T x + \lambda(Ax - b)$$
$$Dx \leq e$$
$$x \geq 0$$
$$x \text{ integer}$$

z_{LP} : objective function value of linear relaxation of OP

$$z_{LD} = \max_{\lambda \geq 0} z_{LR}(\lambda)$$ Lagrangian dual problem

NB: In LR integrality constraint is not relaxed.

Facts:

- $z_{LP} \leqslant z$ because relaxation
- $z_{LR} \leqslant z$ because relaxation
- $z_{LR} \leqslant z_{LD}$ because of definition
- $z_{LP} \leqslant z_{LD}$ This is not trivial and important for motivating the use of Lagrangian Relaxation in Integer Programming:

motivation A: if $z_{LP} < z_{LD}$ then LR gives us a better bound to use in B&B

motivation B: if $z_{LP} = z_{LD}$ LR can still be worth because z_{LD} can be found more easily than with LP

under which conditions does it happen? See Corollary below

motivation C: in any case LR gives us an alternative way to solve the problem. It is an heuristic way with the rare chance of getting also a dual bound and eventually, a provable optimal solution

- **Proposition:**

$$Z_{LP} \leq Z_{LD}$$

Proof:

There are two ways of prooving this:

- via the convexification argument of the slides from the first class of this week, also presented in sec 16.4 of [AMO].
- via the duality argument also presented in sec 8 of [Fi]

I report here the duality arg:

$$Z_{LD} = \max_{\lambda \geq 0} \; Z_{LR}(\lambda) =$$

$$= \max_{\lambda \geq 0} \left\{ \min_x \left\{ c^T x + \lambda^T (Ax - b) \mid Dx \leq e,\ x \geq 0 \ \& \text{ integer} \right\} \right\}$$

because LP relaxation $\geq \max_{\lambda \geq 0} \left\{ \underbrace{\min_x \left\{ c^T x + \lambda^T (Ax - b) \mid Dx \leq e,\ x \geq 0 \right\}} \right\} =$

$$\begin{array}{llcl} \min & c^T x + \lambda^T (Ax - b) & & \max \ \lambda^T b + \mu^T e \\ \mu: & Dx \leq e & \xrightarrow[\text{Dual}]{} & \lambda^T A + \mu^T D \geq c \\ & x \geq 0 & & \mu \geq 0 \\ & & & \lambda \geq 0 \end{array}$$

Hence

$$= \underbrace{\max_{\lambda \geq 0} \left\{ \max_{\mu \geq 0} \left\{ \lambda^T b + \mu^T e \;\middle|\; \lambda^T A + \mu^T D \geq c,\ \mu \geq 0 \right\} \right\}}_{} =$$

$$\boxed{\begin{array}{lll} \max \ \lambda^T b + \mu^T e & & \min \ c^T x \\ x:\ \lambda^T A + \mu^T D \geq c & & Ax \leq b \\ \qquad \mu \geq 0 & \xrightarrow[\text{Dual}]{} & Dx \leq e \\ \qquad \lambda \geq 0 & & x \geq 0 \end{array}}$$

$$= \max_x \left\{ c^T x \mid Ax \leq b,\ Dx \leq e,\ x \geq 0 \right\} =$$

$$= Z_{LP} \qquad \blacksquare$$

- **Corollary**

  $Z_{LD} = Z_{LP}$ when the LR problem has the integrality property.

Proof:

The only inequality introduced in the derivations of the proof above becomes equality as well.