DM872
Mathematical Optimization at Work

Cut and solve

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Outline

Asymmetric Traveling Salesman Problem

$$
\begin{aligned}
\text{(TSPIP)} \quad \min \; & \sum_{ij \in A} c_{ij} x_{ij} \\
\text{s.t.} \; & \sum_{ij \in \delta^+(i)} x_{ij} = 1 \qquad \text{for all } i \in V \\
& \sum_{ji \in \delta^-(i)} x_{ji} = 1 \qquad \text{for all } i \in V \\
& \sum_{ij \in A(S)} x_{ij} \leq |S| - 1 \qquad \text{for all } \emptyset \subset S \subset V, 2 \leq |S| \leq n-1 \\
& x_{ij} \in \{0, 1\} \qquad \text{for all } ij \in E
\end{aligned}
$$

Relaxations:

- omit subtour elimination constraints $\rightsquigarrow$ Assignment Problem.
- relax itegrality requirement with $0 \leq x_{ij} \leq 1, \forall i, j \in V$
- relax

Tightening

- An IP can be tightened by ading additinal constraints or tightening the existing ones.
- the search space of the original problem is contained in the tightened problem.
- A solution to a tightened problem is an upper bound to the original problem.

Branching constraints are tightening constraints.

- Branch and Bound CDT
- Gomory cuts
- Branch and Cut

Cut and Solve

- Iteration $\equiv$ node in search path
- piercing cut a cut that removes at least one feasible solution from the original (unrelaxed) problem solution space.

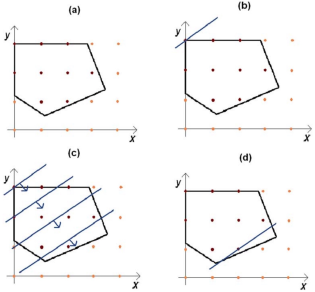
```
algorithm cut_and_solve (IP)
     select cut
     find optimal feasible solution in space removed by cut
        update best if necessary
     add cut to problem
     find lower bound
     if (lower bound >= best) return best
     otherwise, repeat
```

Example

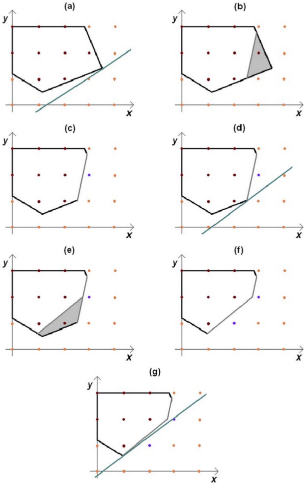
$$\min Z = y - \frac{4}{5}x$$

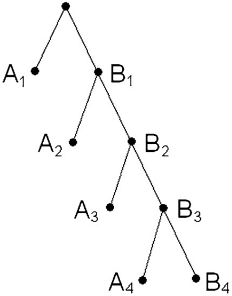
subject to:

$$x \geqslant 0$$
$$y \leqslant 3$$
$$y + \frac{3}{5}x \geqslant \frac{6}{5}$$
$$y + \frac{13}{6}x \leqslant 9$$
$$y - \frac{5}{13}x \geqslant \frac{1}{14}$$
$$x \in I$$
$$y \in I$$



Example





Generic piercing cut procedure

- Partition binary variables in a small set S and a large set L.

  - sparse problem solved on the set S while setting variables in L to zero.

$$\sum_{x_i \in L} x_i = 0$$

  - piercing cut

$$\sum_{x_i \in L} x_i \geq 1$$

- the assumption is that being sparse in feasible (integer) solutions, this problem should be easier to solve.
- general guidelines to select S:
  - Each piercing cut should remove the solution to the current relaxed problem so as to prevent this solution from being found in subsequent iterations.
  - The space that is removed by the piercing cut should be adequately sparse, so that the optimal solution can be found relatively easily.
  - The piercing cuts should attempt to capture an optimal solution for the original problem. The algorithm will not terminate until an optimal solution has been cut away and consequently made the incumbent.
  - In order to guarantee termination, each piercing cut should contain at least one feasible solution for the original, unrelaxed, problem.

Generic Cut and Solve

```
algorithm generic_cut_and_solve (BIP)
     relax integrality and solve LP
     if (LP solution >= best) return best
     let S = {variables with reduced costs <= alpha}
     find optimal feasible solution in S
        update best if necessary
        if (LP solution >= best) return best
     add (sum of variables not in S >= 1) to BIP
     repeat
```