

DM872  
Mathematical Optimization at Work

**TSP practice**

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# Outline

1. Dynamic Programming
2. MILP Formulations
3. Cutting Plane Approaches to DFJ

# Traveling Salesman Problem

<https://www.math.uwaterloo.ca/tsp/>

# Outline

1. Dynamic Programming

2. MILP Formulations

3. Cutting Plane Approaches to DFJ

- Dynamic Programming (DP) is a technique to solve combinatorial optimization problems with applications, for example, in mathematical programming, optimal control, and economics
- DP is somehow related to branch-and-bound as it performs an intelligent enumeration of the feasible solutions of the problem considered
- Principle of Optimality (known as Bellman Optimality Conditions): Suppose that the solution of a problem is the result of a sequence of  $n$  decisions  $D_1, D_2, \dots, D_n$ ; if a given sequence is optimal, then the first  $k$  decisions must be optimal, but also the last  $n - k$  decisions must be optimal
- DP breaks down the problem into stages, at which decisions take place, and find a recurrence relation that relates each stage with the previous one

# Principle of Optimality

The TSP asks for the shortest tour that starts from 0, visits all cities of the set  $C = \{1, 2, \dots, n\}$  exactly once, and returns to 0, where the cost to travel from  $i$  to  $j$  is  $c_{ij}$  (with  $(i, j) \in A$ )

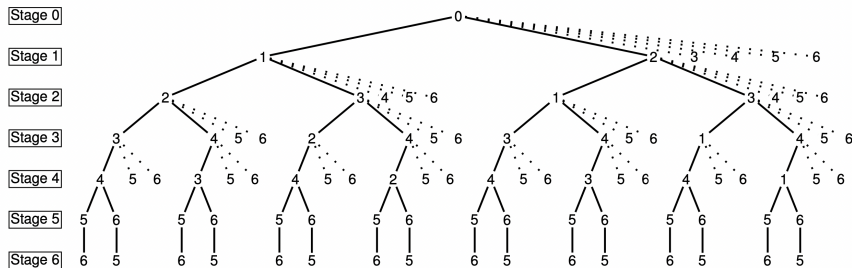
If the optimal solution of a TSP with six cities is  $(0, 1, 3, 2, 4, 6, 5, 0)$ , then...

- the optimal solution to visit  $\{1, 2, 3, 4, 5, 6\}$  starting from 0 and ending at 5 is  $(0, 1, 3, 2, 4, 6, 5)$
- the optimal solution to visit  $\{1, 2, 3, 4, 6\}$  starting from 0 and ending at 6 is  $(0, 1, 3, 2, 4, 6)$
- the optimal solution to visit  $\{1, 2, 3, 4\}$  starting from 0 and ending at 4 is  $(0, 1, 3, 2, 4)$
- the optimal solution to visit  $\{1, 2, 3\}$  starting from 0 and ending at 2 is  $(0, 1, 3, 2)$
- the optimal solution to visit  $\{1, 3\}$  starting from 0 and ending at 3 is  $(0, 1, 3)$
- the optimal solution to visit 1 starting from 0 is  $(0, 1)$

↪ The optimal solution is made up of a number of optimal solutions of smaller subproblems

# Enumerate All Solutions of the TSP

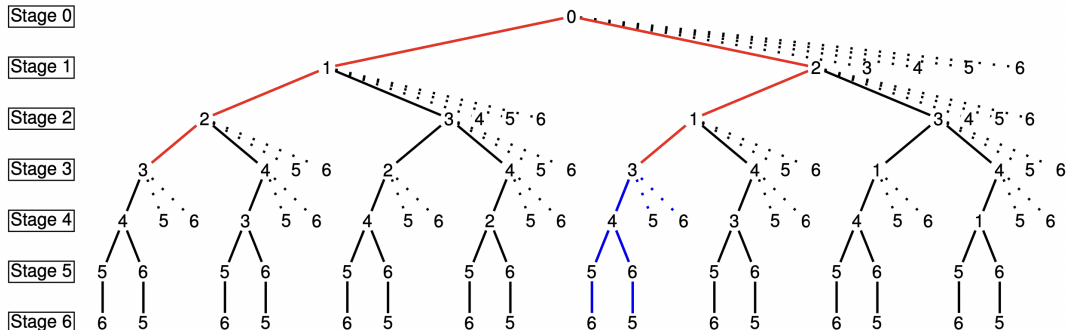
- A solution of a TSP with  $n$  cities derives from a sequence of  $n$  decisions, where the  $k$ th decision consists of choosing the  $k$ th city to visit in the tour



- The number of nodes (or states) grows exponentially with  $n$
- At stage  $k$ , the number of states is  $\binom{n}{k} k!$
- With  $n = 6$ , at stage  $k = 6$ , 720 states are necessary

↪ DP finds the optimal solution by implicitly enumerating all states but actually generating only some of them

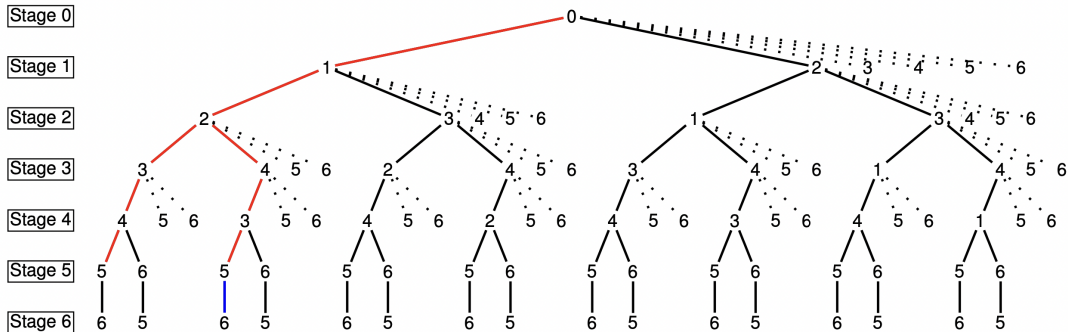
# Are All States Necessary?



If path  $(0, 1, 2, 3)$  costs less than  $(0, 2, 1, 3)$ , the optimal solution cannot be found in the blue part of the tree



# Are All States Necessary?



If path  $(0, 1, 2, 3, 4, 5)$  costs less than  $(0, 1, 2, 4, 3, 5)$ , the optimal solution cannot be found in the blue part of the tree

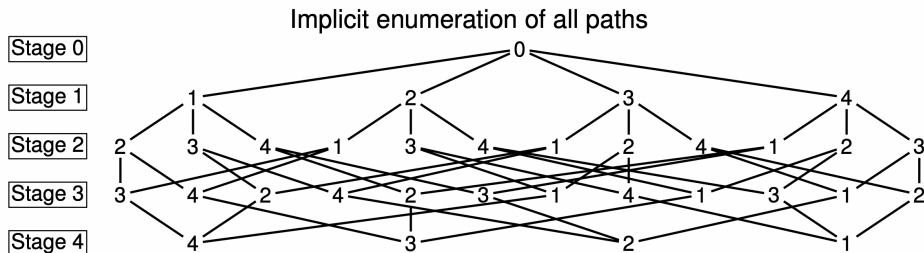
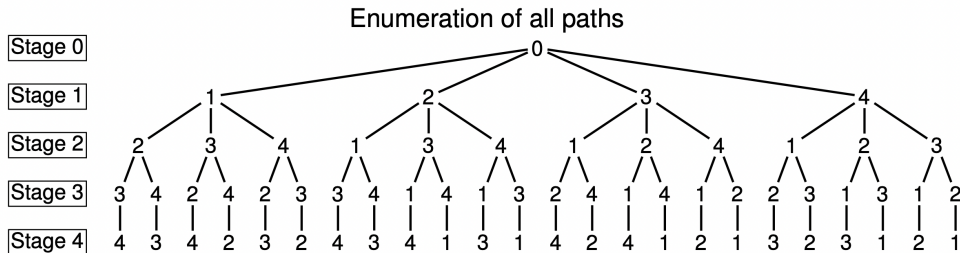
# Are All States Necessary?

- At stage  $k$  ( $1 \leq k \leq n$ ), for each subset of cities  $S \subseteq C$  of cardinality  $k$ , it is necessary to have only  $k$  states (one for each of the cities of the set  $S$ )
- At state  $k = 3$ , given the subset of cities  $S = \{1, 2, 3\}$ , three states are needed:
  - the shortest-path to visit  $S$  by starting from 0 and ending at 1
  - the shortest-path to visit  $S$  by starting from 0 and ending at 2
  - the shortest-path to visit  $S$  by starting from 0 and ending at 3
- At stage  $k$ ,  $\binom{n}{k} k$  states are required to compute the optimal solution (not  $\binom{n}{k} k!$ )

#States  $n = 6$

Stage	$\binom{n}{k} k!$	$\binom{n}{k} k$
1	6	6
2	30	30
3	120	60
4	360	60
5	720	30
6	720	6

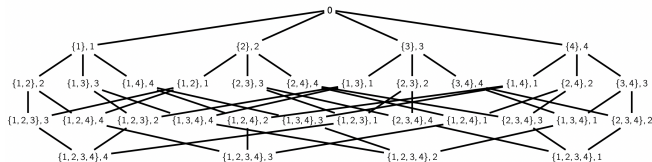
# Complete Trees with $n=4$



# Dynamic Programming Recursion for the TSP I

- Given a subset  $S \subseteq C$  of cities and  $k \in S$ , let  $f(S, k)$  be the optimal cost of starting from 0, visiting all cities in  $S$ , and ending at  $k$
- Begin by finding  $f(S, k)$  for  $|S| = 1$ , which is  $f(\{k\}, k) = c_{0k}, \forall k \in C$
- To compute  $f(S, k)$  for  $|S| > 1$ , the best way to visit all cities of  $S$  by starting from 0 and ending at  $k$  is to consider all  $j \in S \setminus \{k\}$  immediately before  $k$ , and look up  $f(S \setminus \{k\}, j)$ , namely

$$f(S, k) = \min_{j \in S \setminus \{k\}} \{f(S \setminus \{k\}, j) + c_{jk}\}$$



- The optimal solution cost  $z^*$  of the TSP is  $z^* = \min_{k \in C} \{f(C, k) + c_{k0}\}$

# Dynamic Programming Recursion for the TSP II

DP Recursion from [Held and Karp (1962)]

1. **Initialization.** Set  $f(\{k\}, k) = c_{0k}$  for each  $k \in C$

2. **RecursiveStep.** For each stage  $r = 2, 3, \dots, n$ , compute

$$f(S, k) = \min_{j \in S \setminus \{k\}} \{f(S \setminus \{k\}, j) + c_{jk}\} \forall S \subseteq C : |S| = r \text{ and } \forall k \in S$$

3. **Optimal Solution.** Find the optimal solution cost  $z^*$  as

$$z^* = \min_{k \in C} \{f(C, k) + c_{k0}\}$$

- With the DP recursion, TSP instances with up to 25 - 30 customers can be solved to optimality; other solution techniques (i.e., branch-and-cut) are able to solve TSP instances with up to... 85900 customers
- Nonetheless, DP recursions represents the state-of-the-art solution techniques to solve a wide variety of PDPs

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3. Cutting Plane Approaches to DFJ

# Dantzig, Fulkerson and Johnson (DFJ) Formulation

- Find the cheapest movement for a drilling, welding, drawing, soldering arm as, for example, in a printed circuit board manufacturing process or car manufacturing process
- $n$  locations,  $c_{ij}$  cost of travel

**Variables:**

$$x_{ij} = \begin{cases} 1 \\ 0 \end{cases}$$

**Objective:**

$$\sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}$$

## Constraints:

- 

$$\sum_{j:j \neq i} x_{ij} = 1$$

$$\forall i = 1, \dots, n$$

$$\sum_{i:i \neq j} x_{ij} = 1$$

$$\forall j = 1, \dots, n$$

- cut set constraints

$$\sum_{i \in S} \sum_{j \notin S} x_{ij} \geq 1$$

$$\forall S \subset N, S \neq \emptyset$$

- subtour elimination constraints

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1$$

$$\forall S \subset N, 2 \leq |S| \leq n - 1$$



# Miller, Tucker, Zemling (MTZ) Formulation

$$\min \sum_{(ij) \in A} c_{ij} x_{ij} \quad (1)$$

$$\sum_{i:i \neq j} x_{ij} = 1 \quad \forall j = 1, \dots, n \quad (2)$$

$$\sum_{j:i \neq j} x_{ij} = 1 \quad \forall i = 1, \dots, n \quad (3)$$

$$u_i - u_j + nx_{ij} \leq n - 1, \quad \forall i, j = 2, 3, \dots, n, i \neq j \quad (4)$$

$$x_{ij} \in \mathbb{B} \quad \forall ij \in A \quad (5)$$

$$u_i \in \mathbb{R} \quad \forall i = 1, \dots, n \quad (6)$$

# Gavish-Graves (GG) Formulation

$g_{ij} \in \mathbb{R}^+$  sequence variables (0 if  $x_{ij} = 0$  otherwise it indicates the position of node  $i$  and  $j$  in the tour).

$$\min \sum_{(ij) \in A} c_{ij} x_{ij} \quad (7)$$

$$\sum_{i: i \neq j} x_{ij} = 1 \quad \forall j = 1, \dots, n \quad (8)$$

$$\sum_{j: i \neq j} x_{ij} = 1 \quad \forall i = 1, \dots, n \quad (9)$$

$$\sum_{j=1}^n g_{ji} - \sum_{j=1}^n g_{ij} = 1 \quad \forall i = 2..n \quad (10)$$

$$g_{ij} \leq (n-1)x_{ij} \quad \forall ij \in A \quad (11)$$

$$x_{ij} \in \mathbb{B} \quad \forall ij \in A \quad (12)$$

$$g_{ij} \in \mathbb{R}^+ \quad \forall ij \in A \quad (13)$$

# Svestka (S) Formulation

- $f$ : gain in flow from city  $i$  to city  $j$
- $x_{ij} = 1$ : if we drive from city  $i$  to city  $j$ , else 0
- $y_{ij}$ : flow from city  $i$  to city  $j$

$$\min \sum_{ij \in A} c_{ij} x_{ij} \quad (14)$$

$$\sum_{j:ji \in A} y_{ji} \geq 1 \quad \forall i = 2, \dots, n \quad (15)$$

$$\sum_{j:ij \in A} y_{ij} - \sum_{j:ji \in A} y_{ji} = f \quad \forall i = 1, \dots, n \quad (16)$$

$$\sum_{ij \in A} x_{ij} \leq n \quad (17)$$

$$y_{ij} \leq (1 + n f) x_{ij} \quad \forall ij \in A \quad (18)$$

$$x_{ij} \in \mathbb{B} \quad \forall ij \in A \quad (19)$$

$$y_{ij} \in \mathbb{R}^+ \quad \forall ij \in A \quad (20)$$

# Dantzig (D) Formulation

- Indices:  $i, j, k$  for cities,  $t$  for step
- $x_{ijt} = 1$  if we drive from city  $i$  to city  $j$  at step  $t$ , else 0.

$$\min \sum_{ij \in A} \sum_t c_{ij} x_{ijt} \quad (21)$$

$$\sum_i x_{ijt} - \sum_k x_{j, k, t+1} = 0 \quad \forall j \text{ and } t = 1, \dots, n \quad (22)$$

$$\sum_j \sum_t x_{ijt} = 1 \quad \forall i = 1, \dots, n \quad (23)$$

$$x_{ijt} \in \mathbb{B} \quad \forall ij \in A, t \quad (24)$$

## Comparing LP relaxations

Source: Oncan, Altinel, Laporte, A comparative analysis of several asymmetric traveling salesman problem formulations (2009)

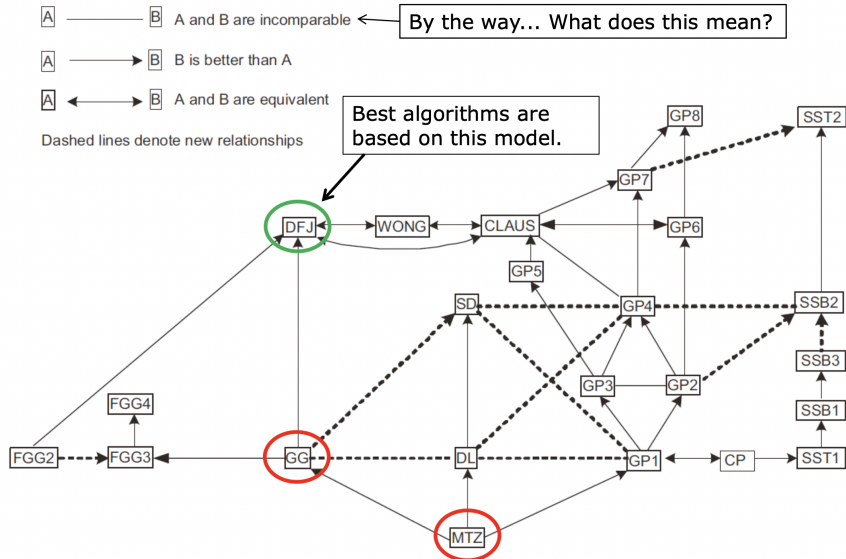


Fig. 2. Relative strength of the 24 ATSP formulations.

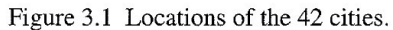
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- $E = \{i, j \mid i \in V, j \in V, i < j\}$

$$\begin{aligned} \text{(TSPIP)} \quad & \min \sum c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{ij \in \delta(i)} x_{ij} + \sum_{ji \in \delta(i)} x_{ji} = 2 \text{ for all } i \in V \\ & \sum_{ij \in E(S)} x_{ij} \leq |S| - 1 \text{ for all } \emptyset \subset S \subset V, 2 \leq |S| \leq n - 1 \\ & x_{ij} \in \{0, 1\} \text{ for all } ij \in E \end{aligned}$$

Dynamic Programming  
MILP Formulations  
**Cutting Plane Approaches to DFJ**





# Traveling Salesman Problem

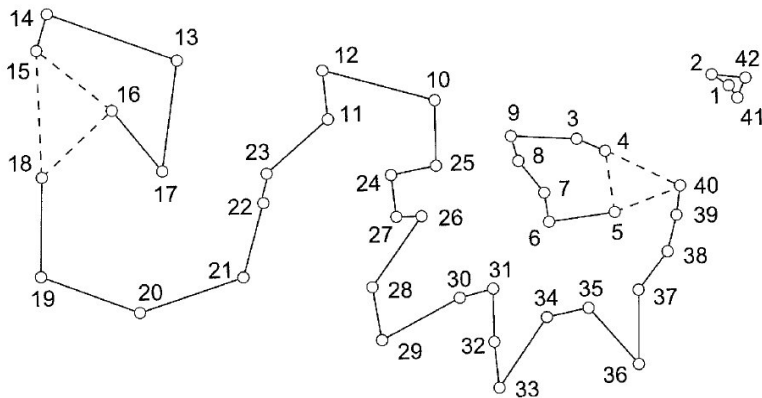


Figure 3.2 Solution of the initial LP relaxation.

# Traveling Salesman Problem

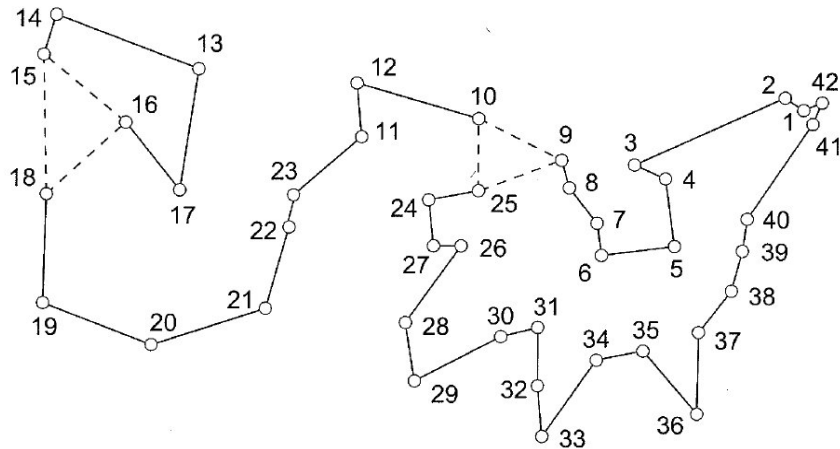


Figure 3.3 LP solution after three subtour constraints.

# Traveling Salesman Problem

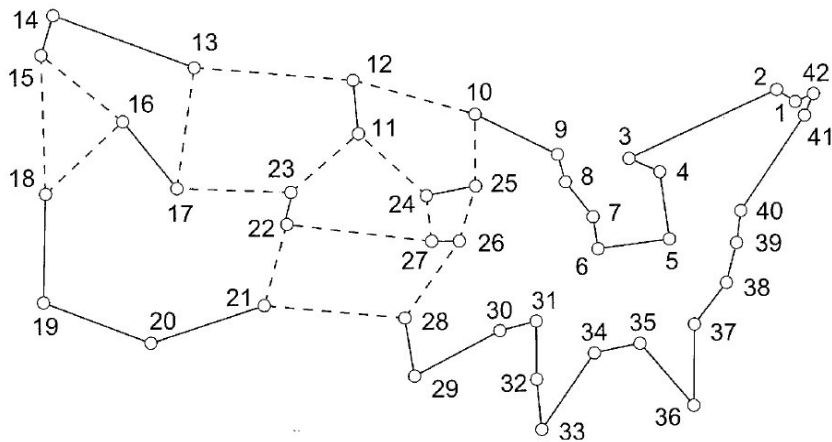


Figure 3.4 LP solution satisfying all subtour constraints.

# Traveling Salesman Problem

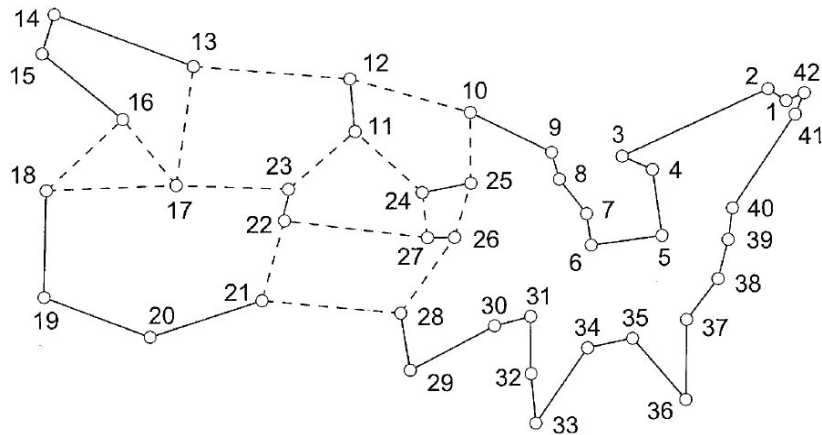


Figure 3.7 What is wrong with this vector?



# Traveling Salesman Problem

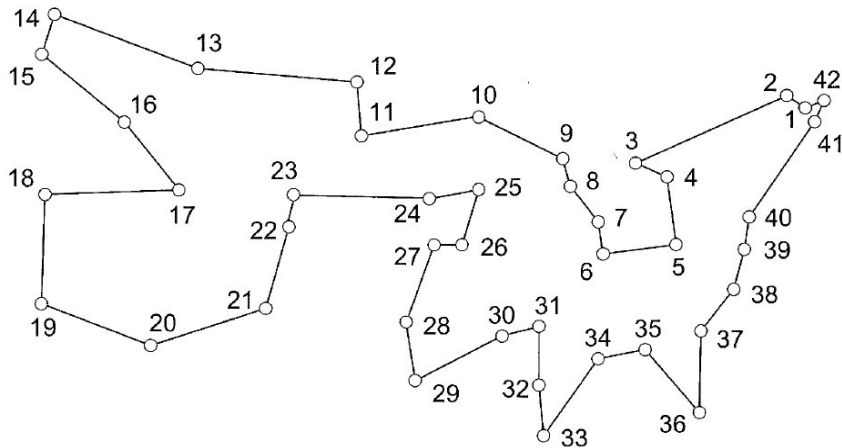


Figure 3.9 An optimal tour through 42 cities.

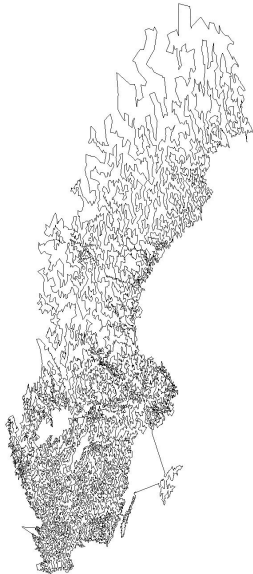
minimize  $c^T x$  subject to

$0 \leq x_e \leq 1$  for all edges  $e$ ,

$\sum(x_e : v \text{ is an end of } e) = 2$  for all cities  $v$ ,

$\sum(x_e : e \text{ has one end in } S \text{ and one end not in } S) \geq 2$   
for all nonempty proper subsets  $S$  of cities,

$\sum_{i=0}^{i=3} (\sum(x_e : e \text{ has one end in } S_i \text{ and one end not in } S_i) \geq 10,$   
for any comb



24,978 Cities

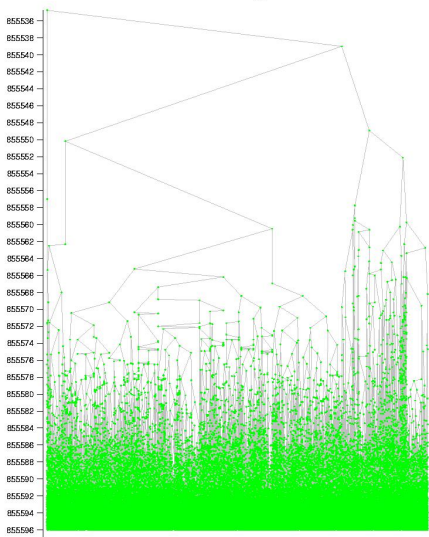
solved by LK-heuristic and proved  
optimal by branch and cut

10 months of computation on a cluster  
of 96 dual processor Intel Xeon 2.8  
GHz workstations

<http://www.tsp.gatech.edu>



### sw24978 Branching Tree - Run 5



24,978 Cities

solved by LK-heuristic and proved  
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# Relaxations of DFJ

- $\mathcal{S} = \{\emptyset \subset S \subset V\}$

- $\mathcal{S}' \subset \mathcal{S}$

$$\begin{aligned}
 (\text{RTSPIP}) \quad & \min \sum c_{ij} x_{ij} \\
 \text{s.t.} \quad & \sum_{ij \in \delta(i)} x_{ij} + \sum_{ji \in \delta(i)} x_{ji} = 2 \text{ for all } i \in V \\
 & \sum_{ij \in E(S)} x_{ij} \leq |S| - 1 \text{ for all } S \in \mathcal{S}' \\
 & x_{ij} \in \{0, 1\} \text{ for all } ij \in E
 \end{aligned}$$

$$\begin{aligned}
 (\text{RTSPLP}) \quad & \min \sum c_{ij} x_{ij} \\
 \text{s.t.} \quad & \sum_{ij \in \delta(i)} x_{ij} + \sum_{ji \in \delta(i)} x_{ji} = 2 \text{ for all } i \in V \\
 & \sum_{ij \in E(S)} x_{ij} \leq |S| - 1 \text{ for all } S \in \mathcal{S}' \\
 & x_{ij} \in \mathbb{R}^+ \text{ for all } ij \in E
 \end{aligned}$$

# Implementation V1

set  $\mathcal{S}' = \emptyset$

1.  $x^* \leftarrow \text{Solve RTSPIP}(\mathcal{S}')$
2.  $\mu_k, \mathcal{S} \leftarrow \text{Solve SEP}(x^*)$   
if  $\mu_k < 2$  then set  $\mathcal{S}' = \mathcal{S}' \cup \mathcal{S}$  and go to 1  
else return optimal solution  $x^*$

**SEP**: connected components or number of cycles

In gurobi and cplex implementation via Lazy constraints (`Model.cbLazy`) and call back function called when MIPSOL. See script: `tsp_gurobi_lazy`

# Implementation V2

set  $\mathcal{S} = \emptyset$

1.  $x^* \leftarrow \text{Solve RLP}(\mathcal{S}')$
2.  $\mu_k, \mathcal{S} \leftarrow \text{Solve SEPLP}(x^*)$   
if  $\mu_k < 2$  then set  $\mathcal{S}' = \mathcal{S}' \cup \mathcal{S}$  and go to 1  
else go to 3
3. branch and bound and repeat 1. and 2. at every node.

SEPLP: LP formulation or Max Flow

In gurobi and cplex implementation via Lazy constraints (`Model.cbLazy`) and call back functions when LP solution at node.