# Crew Scheduling: Models and Algorithms

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2 Urban Crew Scheduling

3 Regional Crew Scheduling

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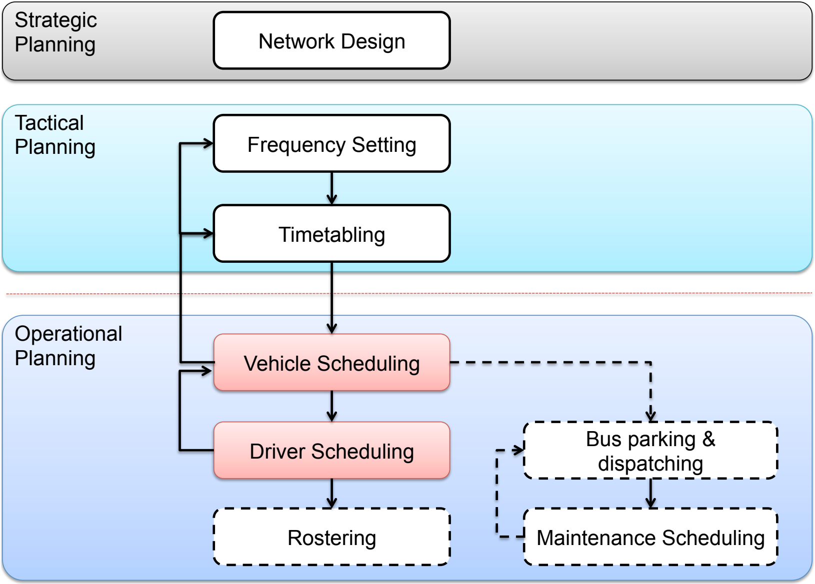
# Overview of Planning Activities (Desaulniers&Hickman2007)



# Crew Scheduling

**Definition (Relief times)**

Each **vehicle duty** (herein called **block**) has a set of **relief times** where a driver substitution may occur.

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# Crew Scheduling

**Definition (Piece of Work (PoW))**

A piece of work $p$ is a continuous driving period from $s(p)$ to $e(p)$.
A piece of work is feasible for a block $k$ if both $s(p)$ and $e(p)$ are relief times of $k$.

**Example:** Given

- a block that starts at $8:30$ and ends $12:30$
- relief times at $\{8:30, 9:30, 10:20, 11:20, 12:30\}$
- constraint: a PoW lasts at least 01:00 and at most 02:00

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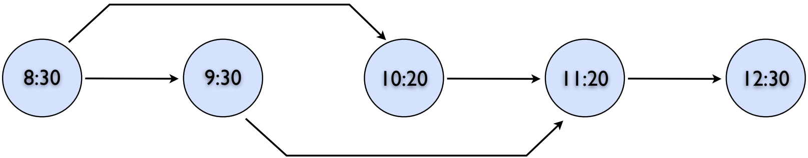

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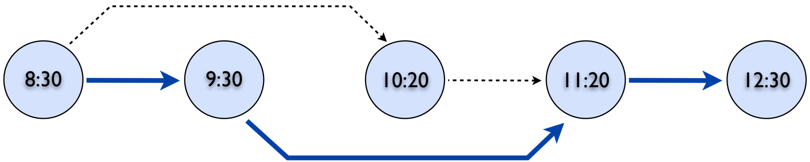

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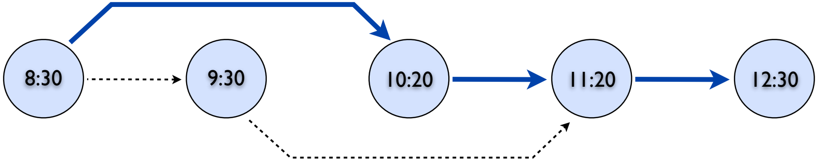

# Crew Scheduling

**Definition (Crew duty)**

A crew duty consists of a set of pairs $(p, k)$ where $p$ is a piece of work associated to block $k$.

**Definition (Crew Scheduling)**

Given a Vehicle Schedule (i.e. a collection of vehicle duties), the **Crew Scheduling** problem consists of finding a set of crew duties to be assigned to drivers in order to guarantee the daily service.

# Crew Scheduling: Urban and Regional


# Crew Scheduling

- $\{1, \ldots, r\}$ vehicle duties (blocks) indexed by $k$
- $T_k = \{t_1^k, \ldots, t_{u_k}^k\}$ is the set of relief times for block $k$
- $t_1^k$ and $t_{u_k}^k$ are the starting and ending time of the block $k$
- $P_k$ set of pieces of work feasible for block $k$
- $\mathcal{D} = \{d_1, \ldots, d_{|\mathcal{D}|}\}$ set of all feasible crew duties

# Partition of blocks into pieces of work

For each block, we define the network $G_k = (N_k, A_k)$ where

- $N_k = T_k$ one node for each relief time
- $A_k = \{(s(p), e(p)) \mid p \in P_k\}$ an arc for each piece of work

The problem of finding a partition of a block into pieces of work is:

$$
-\sum_{p \in P_K | e(p)=i} y_p^k + \sum_{p \in P_k | s(p)=i} y_p^k = \begin{cases} 1 & \text{if } i = t_1^k \\ 0 & \text{if } i = t_j^k, j = 2, \dots, u_k - 1 \\ -1 & \text{if } i = t_{u_k}^k \end{cases}
$$

$$
y_p^k \in \{0, 1\} \qquad \forall p \in P_k
$$

We can write in compact form:

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E^k y^k = b^k, \qquad y^k \in \{0, 1\}
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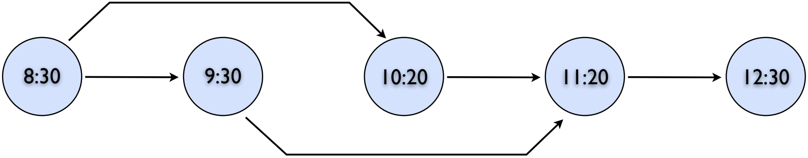
# Partition of blocks into pieces of work

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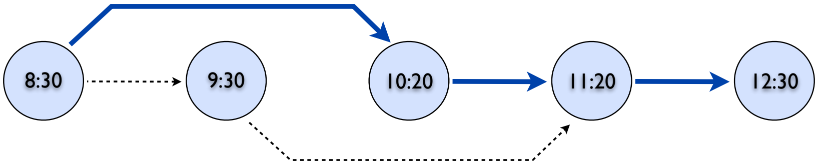


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We can write in compact form:

$$E^k y^k = b^k, \qquad y^k \in \{0,1\}$$

# Crew Scheduling: Basic Model

- Let $x$ be a $|\mathcal{D}|$-vector of binary variables corresponding to the set of all feasible crew duties
- Let $I_{pk}$ be the subset of all the crew duty indices corresponding in $G$ to arcs incident to $(p, k)$

$$\begin{aligned}
\min \quad & \sum_{d \in \mathcal{D}} c_d x_d && && (1)\\
\text{s.t.} \quad & E^k y^k = b^k && \forall && (2)\\
& \sum_{d \in I_{pk}} x_d = y_p^k && \forall p \in P_k, k = 1, \ldots, r && (3)\\
& y^k \in \{0,1\}^{m_k} && \forall k = 1, \ldots, r && (4)\\
& x \in \{0,1\}^{|\mathcal{D}|} && && (5)\\
& x \in X. && && (6)
\end{aligned}$$

# Crew Scheduling and Regional Transit

In Regional Transit, Crew Scheduling is performed before of Vehicle Scheduling, and in practice the set of pieces of work is given (there are very few relief times).

- Let $P$ be the set of piece of work
- Let $\mathcal{D}$ be the set of every possible crew duty
- The cost of a duty $j$ is denoted by $c_j$
- $b_{ij} = \begin{cases} 1 & \text{if the piece of work } i \text{ appears in duty } j \\ 0 & \text{otherwise} \end{cases}$

# Crew Scheduling and Regional Transit

$$\min \quad \sum_{j \in \mathcal{D}} c_j \lambda_j \tag{7}$$

$$\text{s.t.} \quad \sum_{j \in D} b_{ij} \lambda_j = 1 \qquad \forall i \in P \quad \rightarrow \quad \text{partition of PoW} \tag{8}$$

$$\lambda_j \in \{0, 1\} \qquad \forall j \in \mathcal{D} \quad \rightarrow \quad \text{every possible duty} \tag{9}$$

"The set partitioning problem is arguably the easiest optimization model in the world to represent on paper"

"In contrast, the real-life computer code used to manage this simple model can easily run in the order of many hundred thousand lines"

# Crew Scheduling: Set Partitioning Formulation

$$\min \quad \sum_{j \in D} c_j \lambda_j \tag{10}$$

$$\text{s.t.} \quad \sum_{j \in D} b_{ij} \lambda_j = 1 \qquad \forall i \in P \quad \rightarrow \quad \text{partition of PoW} \tag{11}$$

$$\lambda_j \geq 0 \qquad \forall j \in \mathcal{D} \quad \rightarrow \quad \text{every possible duty} \tag{12}$$

First step: to solve the continuous relaxation

QUESTION: Is it easy to solve the LP?

ISSUE: the size of $\mathcal{D}$ is exponential in $|P|$!

# Column Generation

$$(LP) \quad \min \{cx \mid Ax \geq b, x \in \mathbb{R}^n\}$$

- Column Generation is an efficient algorithm for solving **very large linear programs as (LP-MP)**
- Since most of the variables will be **non-basic** and assume a value of zero in the optimal solution, only a subset of variables need to be considered
- Column generation leverages this idea to generate only the variables which have the potential to improve the objective function, that is, to find variables with negative reduced cost

# Dealing with Finitely Many Columns

The main idea is to start with a subset of columns $\bar{\mathcal{D}} \subset \mathcal{D}$ such that a feasible solution to the following problem exists:

$$z_{RMP} = \min \sum_{j \in \bar{\mathcal{D}}} c_j \lambda_j \tag{13}$$

$$\text{s.t.} \sum_{j \in \bar{\mathcal{D}}} b_{ij} \lambda_j \geq 1 \qquad \forall i \in P \tag{14}$$

$$\lambda_j \geq 0 \qquad \forall j \in \bar{\mathcal{D}} \tag{15}$$

Using the Duality Theory of Linear Programming we can generate as set of improving columns. . .

# Column Generation: A Dual Persepective

Consider the LP relaxation of the "master" problem and its dual:

$$(P)\ \min \sum_{j \in \bar{\mathcal{D}}} c_j \lambda_j$$
$$\text{s.t.} \sum_{j \in \bar{\mathcal{D}}} b_{ij} \lambda_j \geq 1, \quad \forall i \in P,$$
$$\lambda_j \geq 0, \qquad \forall j \in \bar{\mathcal{D}}.$$

$$(D)\ \max \sum_{i \in P} \pi_i$$
$$\text{s.t.} \sum_{i \in P} b_{ij} \pi_i \leq c_j, \quad \forall j \in \bar{\mathcal{D}},$$
$$\pi_i \geq 0, \qquad \forall i \in P.$$

Using the Duality Theory of Linear Programming with can generate as set of improving columns... by separating inequalities on the dual of the master problem!

# Pricing Subproblem (Separation on the Master Dual)

The question is:

Does a column (duty) in $\mathcal{D} \setminus \bar{\mathcal{D}}$ that could improve the current optimal solution of the linear relaxation exist?

**Does a column (row of the dual) exist such that . . . ?**

$$\exists j \in \mathcal{D} \setminus \bar{\mathcal{D}} : \quad \sum_{i \in P} b_{ij} \pi_i > c_j$$

# Pricing Subproblem (Separation on the Master Dual)

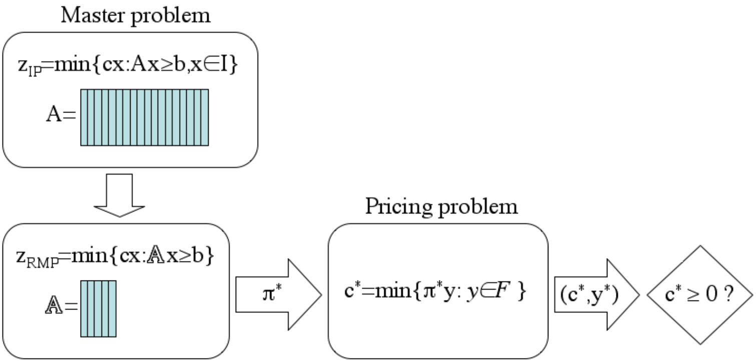
Given the vector of optimal dual multipliers $\bar{\pi}$ for (RMP), we look for a column (duty) such that:

$$\begin{aligned} c^* = \min \quad & c_j - \sum_{i \in P} \bar{\pi}_i y_i \\ \text{s.t.} \quad & y \in F \\ & y_i \in \{0, 1\}. \end{aligned}$$

If $c^* < 0$, the vector of variables $y$ is the incidence vector of an *"improving"* column. It corresponds to a variable with **negative reduced cost** in the (restricted) master problem.

What is $F$ in Crew Scheduling problems?

# Column Generation: Algorithmic Persepective



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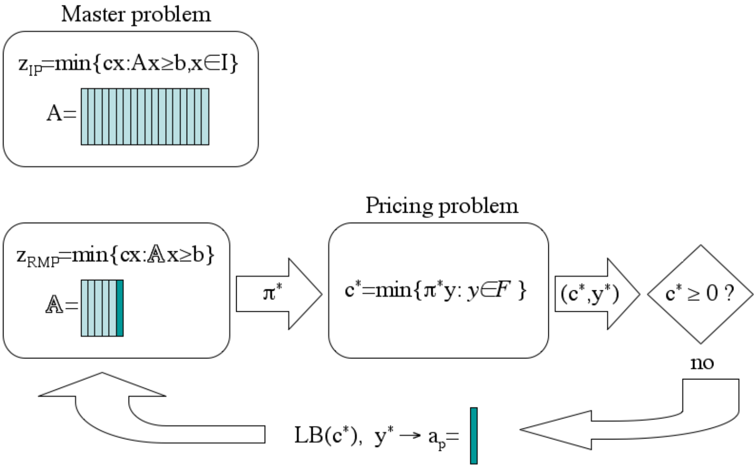

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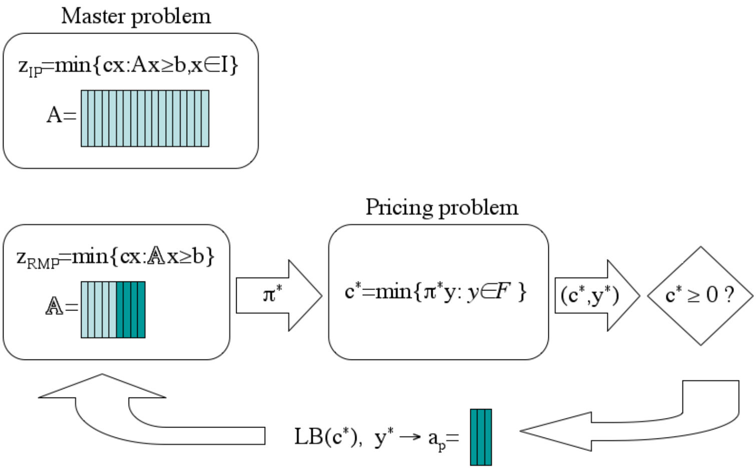

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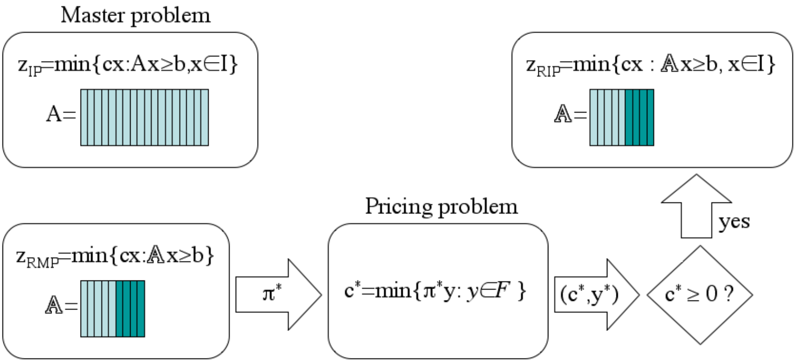

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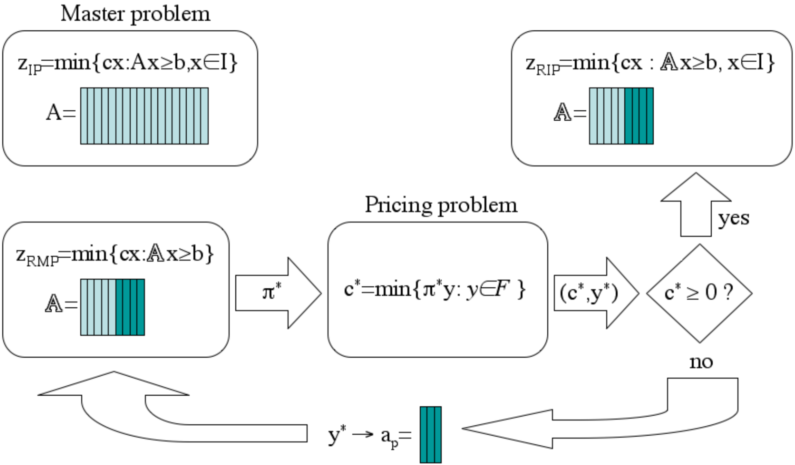

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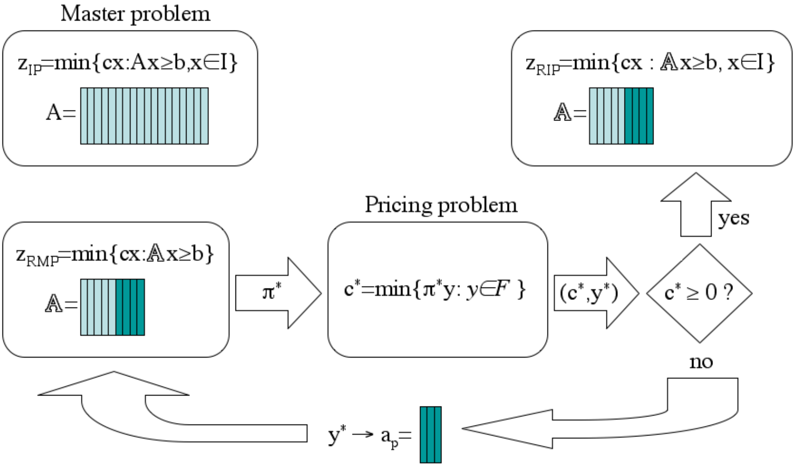



What is $F$ in Crew Scheduling problems?

# Column or Variable Generation

The problem of putting together a set of pieces of work into a single duty, that is a column or variable of problem (LP-MP), is formalized as a

**Resource Constrained Shortest Path Problem**

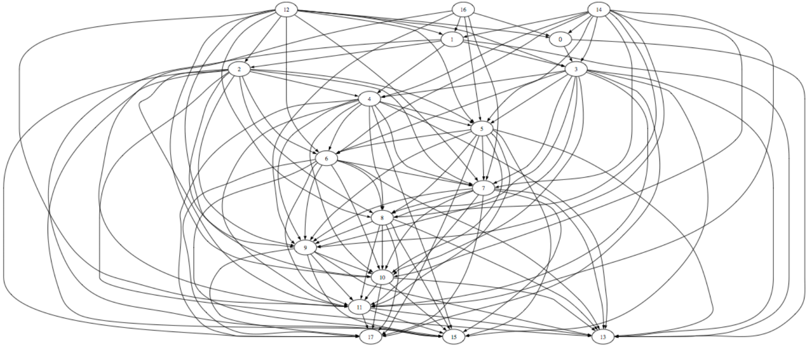
**Example** 12 pieces of work, 3 depots

| ID | Da | A | Inizio | Fine |
|---|---|---|---|---|
| 0 | NETTPO | RMANAG | 04:30 | 06:20 |
| 1 | NETTPO | RMLAUREN | 04:40 | 06:20 |
| 2 | RMLAUREN | NETTPO | 06:20 | 08:15 |
| 3 | APRILI | LATINA | 07:25 | 08:05 |
| 4 | ANZICO | NETTPO | 13:00 | 13:40 |
| 5 | NETTPO | ANZIO | 14:00 | 14:25 |
| 6 | ANZIO | NETTPO | 14:30 | 14:50 |
| 7 | NETTPO | ANZIO | 14:50 | 15:20 |
| 8 | ANZIO | NETTPO | 15:30 | 16:00 |
| 9 | NETTPO | ANZIO | 16:00 | 16:20 |
| 10 | ANZIO | NETTPO | 16:30 | 16:55 |
| 11 | NETTPO | ANZIO | 17:30 | 18:00 |

# Resource Constraint Shortest Path

Let $G = (N, A)$ be the compatibility graph, weighted, directed, and acyclic:

- $N = P \cup \{\{s^h, t^h\} | h \in D\}$ a node for each PoW, and a pair of nodes for each depot
- $A$ has an arc for each pair $(i, j)$ of compatible PoW, and $(s^h, i)$ (pull-out) and $(i, t^h)$ (pull-in) $\forall h \in D$ and $i \in P$



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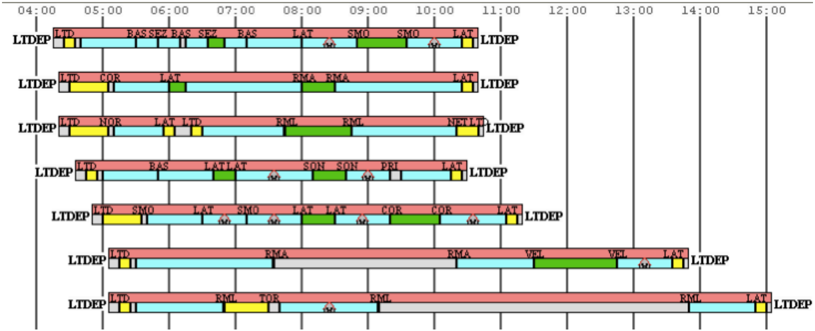
- $A$ has an arc for each pair $(i, j)$ of compatible PoW, and $(s^h, i)$ (pull-out) and $(i, t^h)$ (pull-in) $\forall h \in D$ and $i \in P$
- each arc $(i, j)$ has associated a set of resources $r_{ij}^k$, for each $k \in K$, e.g. **working time**, driving time, and break time (other resources may be used to model working regulation)

| | | | | | |
|---|---|---|---|---|---|
| | NEDEP | ANZICO | 12:35 | 12:55 | VAV |
| 4 | ANZICO | NETTPO | 13:00 | 13:40 | PG |
| 5 | NETTPO | ANZIO | 14:00 | 14:25 | PG |
| 6 | ANZIO | NETTPO | 14:30 | 14:50 | PG |
| 7 | NETTPO | ANZIO | 14:50 | 15:20 | PG |
| 8 | ANZIO | NETTPO | 15:30 | 16:00 | PG |
| 9 | NETTPO | ANZIO | 16:00 | 16:20 | PG |
| 10 | ANZIO | NETTPO | 16:30 | 16:55 | PG |
| 11 | NETTPO | ANZIO | 17:30 | 18:00 | PG |
| | ANZIO | NEDEP | 18:00 | 18:10 | VAV |
| | | | durata: | 5:35 | |

# Example of Crew Schedule (Resources)




Resources:

1. spread time (red)
2. driving time (light blue), corresponds to PoW
3. *out-of-service* time (yellow)
4. long break (grey)
5. breaks (green), very important how they are located

# Duty Generation: Pricing Problem

- Duties (or shifts) with max duration between 4h30 (270m) and 6h30 (390m), with a maximum driving time of 5h30 (330m).
- For each interval of 4h30m (270 minutes), inside a duty, there must be at least a break of 15 minutes and at least a break of 30 minutes.
- The cost of each duty is determined by the minutes out of service.

We lay on every arc $(i, j) \in A$ the values:

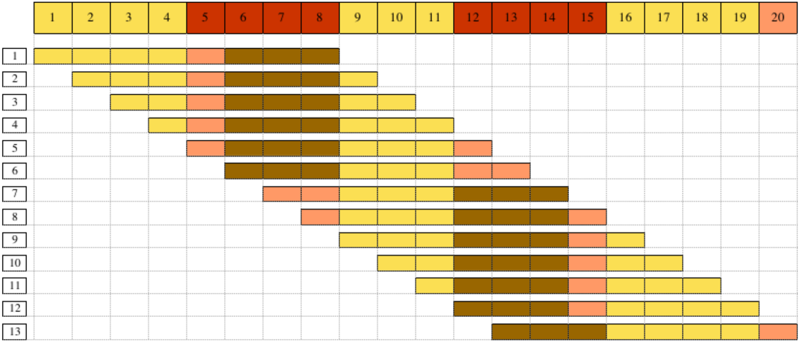
- PG : driving minutes
- FS : minutes of out of service
- PD : minutes of break at the depot
- T1 : number of breaks of type 1 (30 minutes)
- T2 : number of breaks of type 2 (15 minutes)

# Pricing Problem MIP Model

$$
\begin{aligned}
\min \quad & \left(1 + \frac{1}{500} \sum_{ij \in A} t_{ij}^{FS} x_{ij}\right) - \sum_{i \in P} \bar{\pi}_i y_i \\
\text{s.t.} \quad & \sum_{ij \in A} x_{ij} = y_i, \sum_{ji \in A} x_{ij} = y_i, \quad \forall i \in N \setminus \{s, t\}, \\
& \sum_{ij \in A} x_{ij} + \sum_{ji \in A} x_{ij} = b_i, \quad \forall i \in \{s, t\}, \\
& \sum_{ij \in A} t_{ij}^{PG} x_{ij} + \sum_{i \in P} t_i^{PG} y_i \leq t^{MAX-PG}, \\
& \sum_{ij \in A} (t_{ij}^{PG} + t_{ij}^{FS} + t_{ij}^{PD}) x_{ij} + \sum_{i \in P} t_i^{PG} y_i \geq t^{MIN}, \\
& \sum_{ij \in A} (t_{ij}^{PG} + t_{ij}^{FS} + t_{ij}^{PD}) x_{ij} + \sum_{i \in P} t_i^{PG} y_i \leq t^{MAX}, \\
& + \text{Vincolo delle Sequenze}, \\
& x_{ij} \in \{0, 1\}, \ \forall (i, j) \in A, \quad y_i \in \{0, 1\}, \forall i \in P.
\end{aligned}
$$

# Sequence Constraint: Example

Let's assume to have a duty with 20 units of time, and two types of breaks, one that lasts one unit and one 3 units of time. Every 8 units we want at least one break of each type.



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