DM872
Mathematical Optimization at Work

Benders' Algorithm

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Outline

1. Structured LP models

2. Stochastic Programming

Multiple Plant Models

	Factory A		Factory B	
	Standard	Deluxe	Standard	Deluxe
(Machine 1) Grinding	4	2	5	3
(Machine 2) Polishing	2	5	5	6

$$
\begin{array}{llrcrcl}
\text{Maximize} & \text{Profit} & 10x_1 & + & 15x_2 & & \\
\text{Subject to} & \text{Raw A} & 4x_1 & + & 4x_2 & \leq & 75 \\
 & \text{Grinding A} & 4x_1 & + & 2x_2 & \leq & 80 \\
 & \text{Polishing A} & 2x_1 & + & 5x_2 & \leq & 60 \\
 & & & & x_1, x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{llrcrcl}
\text{Maximize} & \text{Profit} & 10x_3 & + & 15x_4 & & \\
\text{Subject to} & \text{Raw B} & 4x_3 & + & 4x_4 & \leq & 45 \\
 & \text{Grinding B} & 5x_3 & + & 3x_4 & \leq & 60 \\
 & \text{Polishing B} & 5x_3 & + & 6x_4 & \leq & 75 \\
 & & & & x_3, x_4 & \geq & 0
\end{array}
$$

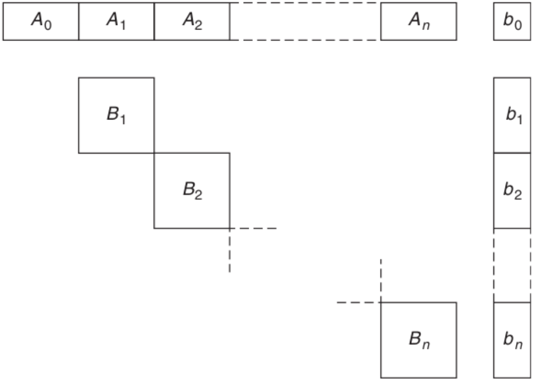
Multiple Plant Models

$$
\begin{array}{llcccccccccc}
\text{Maximize} & \text{Profit} & 10x_1 & + & 15x_2 & + & 10x_3 & + & 15x_4 & & \\
\text{Subject to} & \text{Raw} & 4x_1 & + & 4x_2 & + & 4x_3 & + & 4x_4 & \leq & 120 \\
 & \text{Grinding A} & 4x_1 & + & 2x_2 & + & & & & \leq & 80 \\
 & \text{Polishing A} & 2x_1 & + & 5x_2 & + & & & & \leq & 60 \\
 & & & & & + & 5x_3 & + & 3x_4 & \leq & 60 \\
 & & & & & + & 5x_3 & + & 6x_4 & \leq & 75 \\
 & & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

10	15	10	15		
4	4	4	4	≤	120
4	2			≤	80
2	5			≤	60
		5	3	≤	60
		5	6	≤	75

allocation problems between plants + decision making within plants.

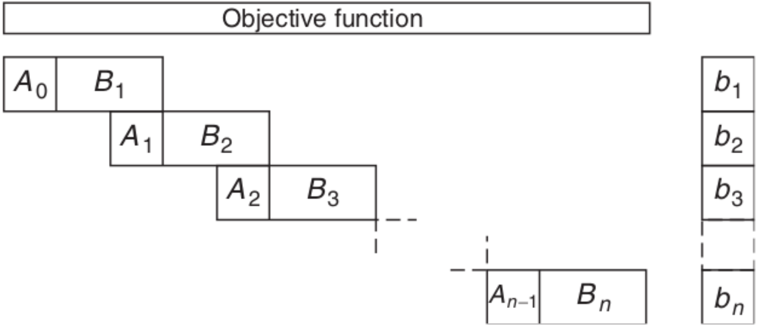
Block Angular Structure



The rows $A_0, \ldots, A_n$ are known as common rows.
The diagonally placed blocks are known as submodels.

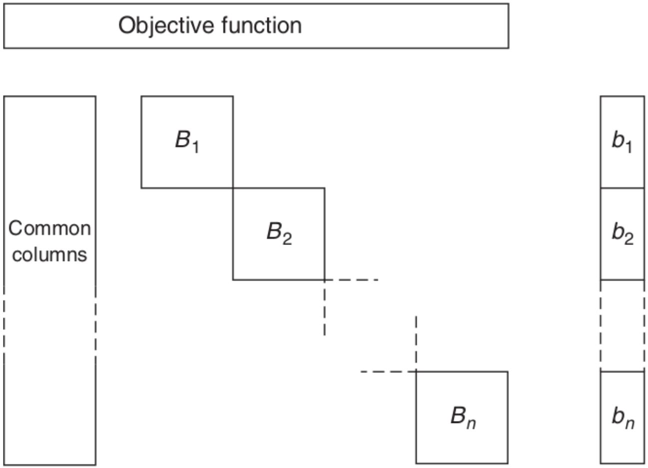
Staircase Structure

Multi-product and mulit-period models lead also to bloack angular structures. In case of this type:



It can be converted into a block angular structure: alternate 'steps' such as $(A_0, B_1), (A_2, B_3)$ can be treated as subproblem constraints and the intermediate 'steps' as common rows.

Block Angular Structure



It can be seen as the dual of the common row structure. However, this structure arises often in stochastic programming cases and it can be treated in its own way.

Outline

Stochastic Programming

Planning under uncertainty when data not known with certainty:

- inaccuracy of data
- multi-stage models where certain events, which need to be modelled, have not yet occurred.

Alternative approaches:

- robust optiomization, when we canot quantify the uncertainty and the related risk. Stable solutions
- sensitivity analysis, how solution change with limited changes to data
- risk-averse maximin approach: make the worst possible result as little bad as possible
- stochastic optimization, when uncertainty can be quantified.

A stochastic programme is a type of LP, which models uncertainty in a particular way.

Stochastic Programming: Multi-Stage Optimization

After taking a first stage decision, a random outcome (scenario) occurring with probability p_k involving one or more of the future data is observed. Then, an optimal second stage decision (recourse action) depending on the first stage and the scenario k is taken

Example:

- (Stage 1): decide production before the demand and future prices (uncertain) are known.
- (Stage 2): decide whether to sell any excess production at a lower price or extra produce to make up a shortfall at a higher cost.

(stage 1 variables) Production decisions: $x_1, x_2, \dots, x_n$.
(stage 2 variables) Excess production or shortfall: $y_1, y_2, \dots, y_n$, $z_1, z_2, \dots, z_n$
stage 2 variables will be replicated m times according to each of the possible demand levels $d_j^{(1)}, d_j^{(2)}, \dots, d_j^{(m)}$ with given probabilities p_r to occurr.
c_j production costs
e_j excess costs (eg, storage)
f_j shortfall costs (missed opportunity)

Two-Stage Stochastic Program with Recursion

$$
\begin{aligned}
\text{Minimize } & \sum_j c_j x_j + \sum_r p_r \left(\sum_j e_j y_j^{(r)} + \sum_j f_j z_j^{(r)} \right) & \\
\text{subject to } & \sum_j a_{ij} x_j \leq b_i & \text{for all production constraints } i \\
& x_j - y_j^{(r)} + z_j^{(r)} = d_j^{(r)} & \text{for all } j \text{ and } r \\
& x_j, y_j^{(r)}, z_j^{(r)} \geq 0 & \text{for all } j \text{ and } r
\end{aligned}
$$

Structure