Multi-Asset Jump-Diffusion Heston Variational Bayesian SDE Framework for Decentralized Financial Markets Analysis: Mathematical Formulation

Mathematical Finance

1 Fundamental Mathematical Setup

1.1 State Space and Coordinate System

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Consider $n = 6$ cryptocurrency assets with prices $S_i(t)$ for $i \in \{1, 2, \ldots, 6\}$.

Define the log-price coordinates:

$$q_i(t) = \log\left(\frac{S_i(t)}{S_i(0)}\right), \quad i = 1, \ldots, n \tag{1}$$

The state vector is $\mathbf{q}(t) = [q_1(t), q_2(t), \ldots, q_n(t)]^T \in \mathbb{R}^n$.

1.2 Stochastic Basis

Let $\{\mathbf{W}(t)\}_{t\geq 0} = \{W_1(t), W_2(t), \ldots, W_n(t)\}_{t\geq 0}$ be an n-dimensional standard Brownian motion adapted to $\{\mathcal{F}_t\}$.

Let $\{N_i(t)\}_{t\geq 0}$ be independent Poisson processes with intensities $\lambda_i > 0$ for $i = 1, \ldots, n$.

Let $\{J_i^{(k)}\}_{k\geq 1}$ be i.i.d. sequences of jump sizes for each asset i, independent of the Brownian motions and Poisson processes.

2 Jump-Diffusion SDE System

2.1 Primary SDE Formulation

The log-price dynamics follow the multi-dimensional jump-diffusion system:

$$dq_i(t) = \mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t) + \int_{\mathbb{R}} z\tilde{N}_i(dt, dz) \tag{2}$$

where $\tilde{N}_i(dt, dz) = N_i(dt, dz) - \nu_i(dz)dt$ is the compensated Poisson random measure with Lévy measure ν_i.

For Gaussian jumps, this reduces to:

$$dq_i(t) = \mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t) + J_i dN_i(t) \tag{3}$$

where $J_i \sim \mathcal{N}(\mu_{J,i}, \sigma_{J,i}^2)$.

2.2 Matrix Formulation

In vector notation:

$$d\mathbf{q}(t) = \boldsymbol{\mu} dt + \mathbf{\Sigma} d\mathbf{W}(t) + \sum_{i=1}^{n} \mathbf{e}_i J_i dN_i(t) \tag{4}$$

where:

$$\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_n]^T \in \mathbb{R}^n \tag{5}$$
$$\mathbf{\Sigma} = [\sigma_{ij}] \in \mathbb{R}^{n\times n} \tag{6}$$
$$\mathbf{e}_i = i\text{-th unit vector in } \mathbb{R}^n \tag{7}$$

3 Volatility Structure

3.1 Cholesky Parameterization

The volatility matrix must be positive definite: $\mathbf{\Sigma} \succ 0$.

Parameterization via Cholesky decomposition:

$$\mathbf{\Sigma} = \mathbf{L}\mathbf{L}^T \tag{8}$$

where $\mathbf{L} \in \mathbb{R}^{n\times n}$ is lower triangular with $L_{ii} > 0$.

3.2 Covariance Structure

The instantaneous covariance matrix elements:

$$\Sigma_{ij} = \sum_{k=1}^{\min(i,j)} L_{ik} L_{jk} \tag{9}$$

Correlation coefficients:

$$\rho_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}} \tag{10}$$

4 Jump Process Mathematics

4.1 Compound Poisson Process

For each asset i, the jump component is:

$$X_i(t) = \sum_{k=1}^{N_i(t)} J_i^{(k)} \tag{11}$$

where $J_i^{(k)} \sim \mathcal{N}(\mu_{J,i}, \sigma_{J,i}^2)$ are i.i.d. jump sizes.

4.2 Jump Compensation

The compensated jump process:

$$\tilde{X}_i(t) = X_i(t) - \lambda_i \kappa_i t \tag{12}$$

where the compensation factor is:

$$\kappa_i = \mathbb{E}[e^{J_i} - 1] = e^{\mu_{J,i} + \frac{1}{2}\sigma_{J,i}^2} - 1 \tag{13}$$

4.3 Martingale Property

Under the risk-neutral measure $\mathbb{Q}$, the compensated log-price process:

$$\tilde{q}_i(t) = q_i(t) - \int_0^t \left(r - \frac{1}{2}\sum_{j=1}^{n} \sigma_{ij}^2 - \lambda_i \kappa_i \right) ds \tag{14}$$

is a martingale for risk-free rate r.

5 Stochastic Volatility Extension

5.1 Heston Model Integration

Each variance process follows:

$$dv_i(t) = \kappa_i(\theta_i - v_i(t))dt + \xi_i \sqrt{v_i(t)} dZ_i(t) \tag{15}$$

where $Z_i(t)$ are Brownian motions with correlation structure:

$$d\langle W_i, Z_j \rangle_t = \rho_{ij}^{WZ} dt \tag{16}$$

5.2 Feller Condition

For positive variance processes:

$$2\kappa_i \theta_i \geq \xi_i^2, \quad \forall i \tag{17}$$

6 Numerical Integration Schemes

6.1 Euler-Maruyama Discretization

For time step Δt:

$$q_i(t + \Delta t) = q_i(t) + \mu_i \Delta t + \sum_{j=1}^{n} \sigma_{ij} \Delta W_j + \sum_k J_i^{(k)} \Delta N_i^{(k)} \tag{18}$$

where:

$$\Delta W_j \sim \mathcal{N}(0, \Delta t) \tag{19}$$

$$\Delta N_i^{(k)} \sim \text{Bernoulli}(\lambda_i \Delta t) \tag{20}$$

6.2 Higher-Order Schemes

Milstein scheme for improved convergence:

$$q_i(t + \Delta t) = q_i(t) + \mu_i \Delta t + \sum_{j=1}^{n} \sigma_{ij} \Delta W_j \tag{21}$$

$$+ \frac{1}{2} \sum_{j,k=1}^{n} \sigma_{ij} \frac{\partial \sigma_{ik}}{\partial q_j} (\Delta W_j \Delta W_k - \delta_{jk} \Delta t) + \sum_k J_i^{(k)} \Delta N_i^{(k)} \tag{22}$$

7 Likelihood Function

7.1 Discrete-Time Likelihood

For observations $\{\Delta \mathbf{q}_t\}_{t=1}^T$ with $\Delta \mathbf{q}_t = \mathbf{q}_t - \mathbf{q}_{t-1}$:

$$\mathcal{L}(\boldsymbol{\theta}) = \prod_{t=1}^{T} p(\Delta \mathbf{q}_t | \mathbf{q}_{t-1}, \boldsymbol{\theta}) \tag{23}$$

7.2 Gaussian Component

For the diffusion component:

$$p_{diff}(\Delta \mathbf{q}_t) = \frac{1}{(2\pi)^{n/2} |\mathbf{\Sigma} \Delta t|^{1/2}} \exp\left(-\frac{1}{2\Delta t} (\Delta \mathbf{q}_t - \boldsymbol{\mu} \Delta t)^T \mathbf{\Sigma}^{-1} (\Delta \mathbf{q}_t - \boldsymbol{\mu} \Delta t) \right) \tag{24}$$

7.3 Log-Likelihood

$$\ell(\boldsymbol{\theta}) = -\frac{Tn}{2} \log(2\pi) - \frac{T}{2} \log |\mathbf{\Sigma}| - \frac{Tn}{2} \log(\Delta t) \tag{25}$$

$$- \frac{1}{2\Delta t} \sum_{t=1}^{T} (\Delta \mathbf{q}_t - \boldsymbol{\mu} \Delta t)^T \mathbf{\Sigma}^{-1} (\Delta \mathbf{q}_t - \boldsymbol{\mu} \Delta t) \tag{26}$$

8 Bayesian Framework

8.1 Prior Distributions

$$\mu_i \sim \mathcal{N}(0, \tau_\mu^2) \tag{27}$$

$$L_{ii} \sim \text{LogNormal}(m_L, s_L^2) \tag{28}$$

$$L_{ij} \sim \mathcal{N}(0, \tau_L^2), \quad i > j \tag{29}$$

$$\lambda_i \sim \text{Gamma}(\alpha_\lambda, \beta_\lambda) \tag{30}$$

$$\mu_{J,i} \sim \mathcal{N}(0, \tau_J^2) \tag{31}$$

$$\sigma_{J,i} \sim \text{LogNormal}(m_J, s_J^2) \tag{32}$$

8.2 Variational Inference

Approximate posterior:

$$q(\boldsymbol{\theta}) = \prod_k q_k(\theta_k) \tag{33}$$

Evidence Lower BOund (ELBO):

$$\mathcal{L}_{ELBO} = \mathbb{E}_{q(\boldsymbol{\theta})}[\log p(\mathbf{data}|\boldsymbol{\theta})] - \text{KL}[q(\boldsymbol{\theta}) \| p(\boldsymbol{\theta})] \tag{34}$$

8.3 KL Divergence Terms

For Gaussian variational posteriors $q(\theta_k) = \mathcal{N}(\mu_k, \sigma_k^2)$ with Gaussian priors $p(\theta_k) = \mathcal{N}(0, \tau_k^2)$:

$$\text{KL}[q(\theta_k) \| p(\theta_k)] = \frac{1}{2} \left[\frac{\mu_k^2 + \sigma_k^2}{\tau_k^2} - 1 - \log \frac{\sigma_k^2}{\tau_k^2} \right] \tag{35}$$

9 Risk Measures

9.1 Value at Risk

For portfolio weights $\mathbf{w}$ and time horizon T:

$$\mathrm{VaR}_\alpha(T) = -\mathrm{Quantile}_\alpha[\mathbf{w}^T(\mathbf{q}(T) - \mathbf{q}(0))] \tag{36}$$

9.2 Expected Shortfall

$$\mathrm{ES}_\alpha(T) = -\mathbb{E}[\mathbf{w}^T(\mathbf{q}(T) - \mathbf{q}(0)) \mid \mathbf{w}^T(\mathbf{q}(T) - \mathbf{q}(0)) < -\mathrm{VaR}_\alpha(T)] \tag{37}$$

9.3 Quadratic Variation

The quadratic variation process:

$$[\mathbf{q}]_t = \int_0^t \boldsymbol{\Sigma} ds + \sum_{i=1}^n \sum_{k:T_k^{(i)} \le t} \mathbf{e}_i \mathbf{e}_i^T (J_i^{(k)})^2 \tag{38}$$

where $T_k^{(i)}$ are the jump times for asset i.

10 Characteristic Functions

10.1 Characteristic Function of Log-Prices

For $\boldsymbol{\xi} \in \mathbb{R}^n$:

$$\phi_{\mathbf{q}(t)}(\boldsymbol{\xi}) = \mathbb{E}[e^{i\boldsymbol{\xi}^T \mathbf{q}(t)}] \tag{39}$$

$$= \exp\left(i\boldsymbol{\xi}^T \boldsymbol{\mu} t - \frac{1}{2}\boldsymbol{\xi}^T \boldsymbol{\Sigma} \boldsymbol{\xi} t + \sum_{j=1}^n \lambda_j t(\phi_{J_j}(\xi_j) - 1)\right) \tag{40}$$

where $\phi_{J_j}(\xi) = \exp(i\mu_{J,j}\xi - \frac{1}{2}\sigma_{J,j}^2\xi^2)$ for Gaussian jumps.

11 Moment Generating Functions

11.1 Cumulant Generating Function

$$\psi_{\mathbf{q}(t)}(\boldsymbol{\xi}) = \log \mathbb{E}[e^{\boldsymbol{\xi}^T \mathbf{q}(t)}] \tag{41}$$

$$= \boldsymbol{\xi}^T \boldsymbol{\mu} t + \frac{1}{2}\boldsymbol{\xi}^T \boldsymbol{\Sigma} \boldsymbol{\xi} t + \sum_{j=1}^n \lambda_j t(\psi_{J_j}(\xi_j)) \tag{42}$$

where $\psi_{J_j}(\xi) = \mu_{J,j}\xi + \frac{1}{2}\sigma_{J,j}^2\xi^2$ for Gaussian jumps.

11.2 Moment Formulas

First moment:

$$\mathbb{E}[\mathbf{q}(t)] = \boldsymbol{\mu} t + \sum_{j=1}^n \lambda_j \mu_{J,j} t \mathbf{e}_j \tag{43}$$

Second moment matrix:

$$\mathrm{Var}[\mathbf{q}(t)] = \boldsymbol{\Sigma} t + \sum_{j=1}^n \lambda_j t \mathbf{e}_j \mathbf{e}_j^T (\mu_{J,j}^2 + \sigma_{J,j}^2) \tag{44}$$

12 Optimization Theory

12.1 Gradient Computations

For parameter θ_k, the gradient of the log-likelihood:

$$\frac{\partial \ell}{\partial \theta_k} = \sum_{t=1}^{T} \frac{\partial}{\partial \theta_k} \log p(\Delta \mathbf{q}_t | \boldsymbol{\theta}) \tag{45}$$

12.2 Fisher Information Matrix

$$\mathcal{I}_{jk} = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \theta_j \partial \theta_k}\right] \tag{46}$$

12.3 Constraint Manifolds

The parameter space is constrained to:

$$\mathcal{M} = \{(\boldsymbol{\mu}, \mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\mu}_J, \boldsymbol{\sigma}_J) : \mathbf{L} \text{ lower triangular}, L_{ii} > 0, \tag{47}$$

$$\lambda_i > 0, \sigma_{J,i} > 0\} \tag{48}$$

13 Asymptotic Theory

13.1 Consistency

Under regularity conditions, the maximum likelihood estimator $\hat{\boldsymbol{\theta}}_T$ satisfies:

$$\hat{\boldsymbol{\theta}}_T \xrightarrow{p} \boldsymbol{\theta}_0 \quad \text{as } T \to \infty \tag{49}$$

13.2 Asymptotic Normality

$$\sqrt{T}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0) \xrightarrow{d} \mathcal{N}(0, \mathcal{I}^{-1}) \quad \text{as } T \to \infty \tag{50}$$

where $\mathcal{I}$ is the Fisher information matrix.

14 Stochastic Calculus Results

14.1 Itô's Formula

For $f \in C^{2,1}(\mathbb{R}^n \times [0, \infty))$:

$$df(\mathbf{q}(t), t) = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} dq_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial q_i \partial q_j} d\langle q_i, q_j \rangle_t \tag{51}$$

$$+ \sum_{i=1}^{n} \sum_{k} \left[f(\mathbf{q}(T_k^{(i)}-), T_k^{(i)}) - f(\mathbf{q}(T_k^{(i)}-), T_k^{(i)}) - J_i^{(k)} \frac{\partial f}{\partial q_i} \right] \tag{52}$$

14.2 Girsanov's Theorem

Change of measure density:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(-\int_0^t \boldsymbol{\gamma}^T d\mathbf{W}(s) - \frac{1}{2} \int_0^t \|\gamma\|^2 ds \right) \tag{53}$$

$$\times \prod_{i=1}^{n} \prod_{k: T_k^{(i)} \leq t} \frac{h_i(J_i^{(k)})}{g_i(J_i^{(k)})} \tag{54}$$

where g_i and h_i are the jump size densities under $\mathbb{P}$ and $\mathbb{Q}$ respectively.