Extensions of Modules

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1. Algebraic Structures

The rules of ordinary algebra like $x + y = y + x$ or $a(b + c) = ab + ac$ describe more than just the real numbers. In abstract algebra, we study objects which obey different combinations of these rules. In **rings** we can add and multiply; in **vector spaces** we can add vectors and scale them; in **groups** we can only use one operation.

Examples and their elements

Groups		Rings		Vector Spaces	
$\mathcal{D}_3$	symmetries of △	$\mathbb{Z}_n$	$0, 1, \dots, n-1$	$\mathbb{R}$	x
$\mathcal{D}_4$	symmetries of □	$\mathbb{Z}[\sqrt{d}]$	$a + b\sqrt{d}$	$\mathbb{R}^2$	(x, y)
O_2	symmetries of ○	$\mathbb{H}$	$x + yi + zj + wk$	$\mathbb{R}^3$	(x, y, z)
SO_2	rotations of ○	$\mathbb{R}^{2\times 2}$	$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$	$\mathbb{R}^\infty$	$(x_1, x_2, x_3, \dots)$

Structures like these often contain smaller structures of the same type. For instance, the complex numbers $\mathbb{C}$ contain the real numbers $\mathbb{R}$ as a subring, and 3D space contains the x-y plane as a subspace. In certain circumstances, we can fold away these substructures to form a **quotient**.

2. Quotients

Quotients can be created whenever a big structure E consists of copies (**cosets**) of a smaller structure B. We construct the quotient E/B by collapsing each of the copies down to a single point.

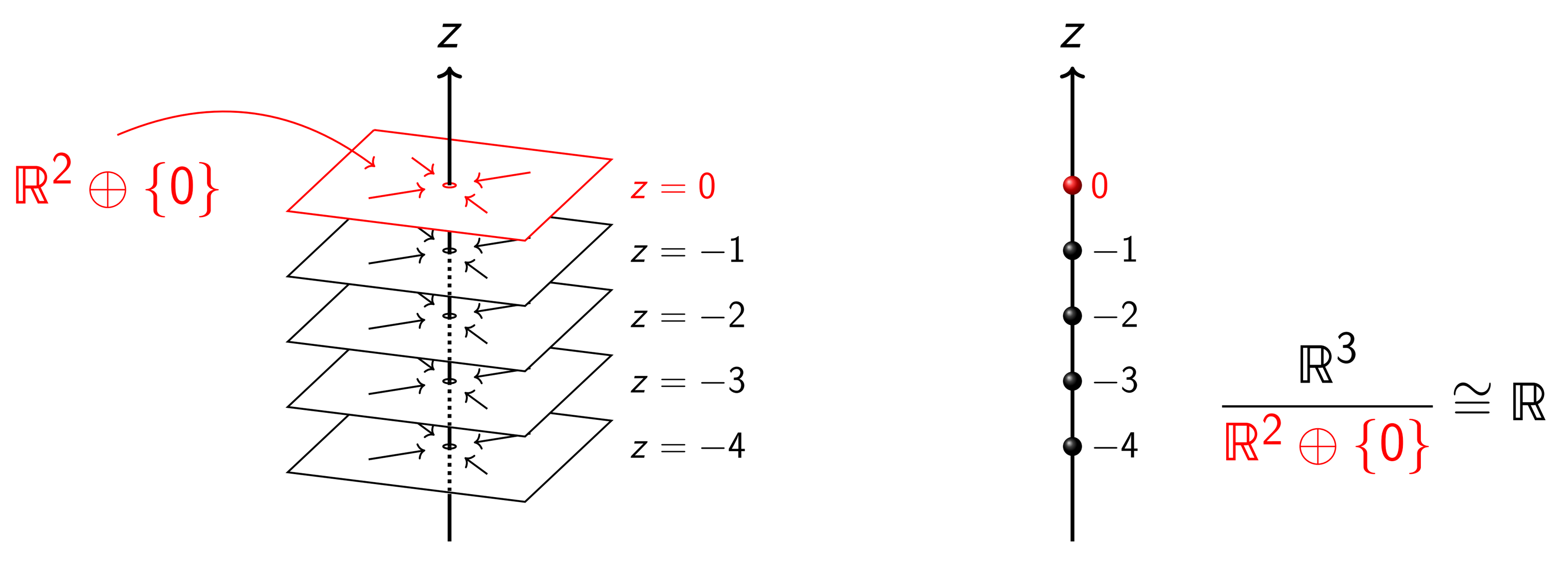


Fig. 1. Factoring out a plane from $\mathbb{R}^3$ gives us a line.

Quotient objects E/B are simplified versions of E, and can be useful when we don't need all of E's detail. Take $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$ (the integers modulo 2) for instance: this simplifies the integers down to 'odd or even'.

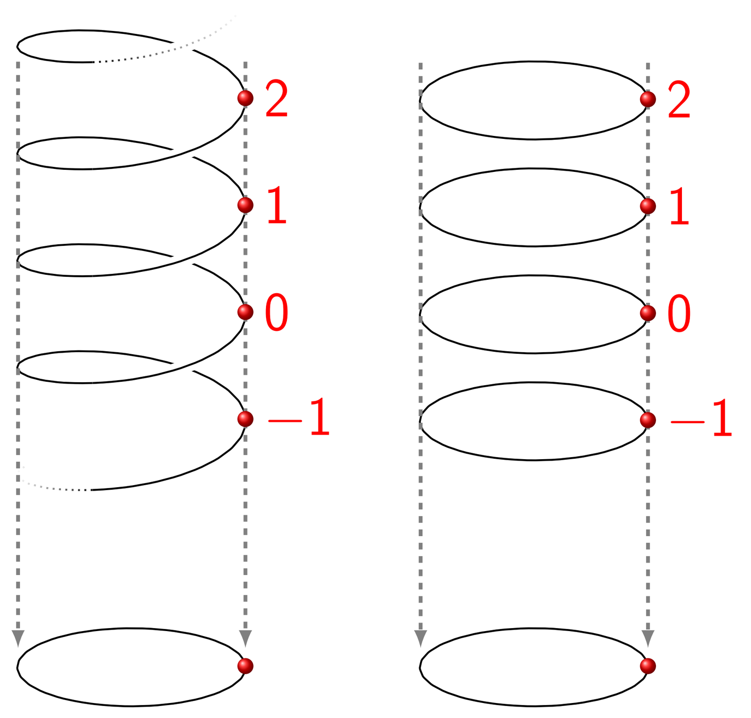
3. Extensions

Let A and B be two algebraic objects (groups, rings, ...). An **extension** of A by B is an object E together with two homomorphisms $\mu\colon B \to E$, $\varepsilon\colon E \to A$ such that

1. μ is injective and ε is surjective;
2. $\operatorname{Im}\mu = \ker\varepsilon$.

These conditions ensure that $E/\mu(B) \cong A$.

Extensions specify a way to form A as a quotient of something by B. They are written as $B \overset{\mu}{\rightarrowtail} E \overset{\varepsilon}{\twoheadrightarrow} A$ to emphasise the maps μ and ε.



Left: The real line bent into a spiral.

$$\mathbb{Z} \rightarrowtail \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$$

Right: Disjoint copies of the circle.

$$\mathbb{Z} \rightarrowtail \mathbb{Z} \oplus (\mathbb{R}/\mathbb{Z}) \twoheadrightarrow \mathbb{R}/\mathbb{Z}$$

Fig. 2. Two extensions of the circle $\mathbb{R}/\mathbb{Z}$ by the integers $\mathbb{Z}$. Both cast a circular shadow.

4. Modules

This project tackles the problem of finding all extensions of A by B where A and B are **Λ-modules**. A module is a generalisation of a vector space where scalars belong to a **ring** Λ instead of a field K. Addition works like normal, but scaling can behave in new ways:

- Scalars may not have an inverse: $x/2$ need not exist, or it might happen that $\lambda x = 0$ for $\lambda \neq 0$ and $x \neq 0$.
- The order of scaling could matter: $(\lambda\mu)x$ may not equal $(\mu\lambda)x$.

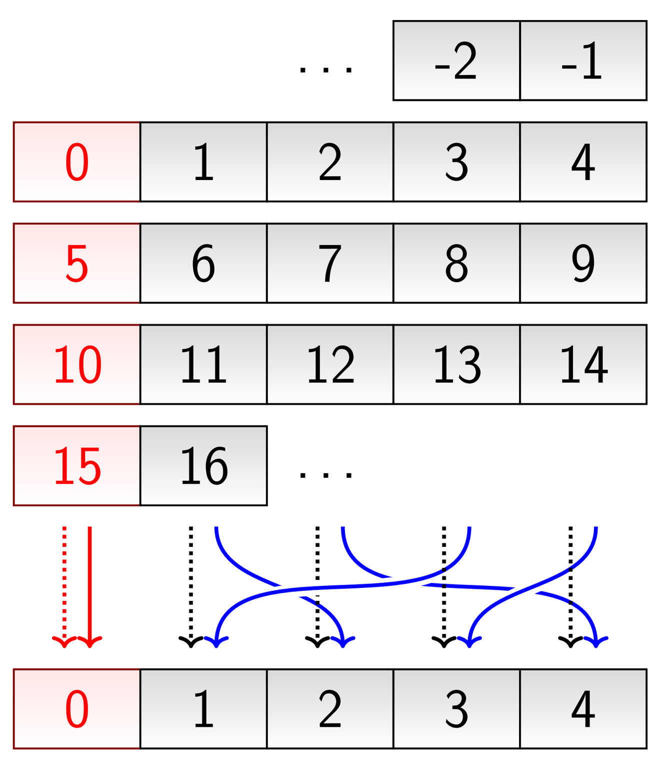
The structure of a module therefore depends heavily upon its scalar ring Λ. Some specific choices give us familiar structures:

- Abelian groups are $\mathbb{Z}$-modules;
- Vector spaces over a field K are K-modules;
- A Ring R can be viewed as an R-module.

5. Equivalence and Ext

To classify all extensions of A by B, we must determine when two extensions are **equivalent**. Informally, this means that one extension's homomorphisms (μ and ε) are compatible with another's.

Imagine we shrink the x-z plane (instead of x-y) in figure 1. The process of forming a quotient would essentially be the same, so the two corresponding extensions would be equivalent. For examples of **inequivalent** extensions, see figures 2 and 3.



The integers folded into themselves to create $\mathbb{Z}_5$ in two ways. In red, the coset $5\mathbb{Z}$ becomes the zero element. This forms two extensions

$$\mathbb{Z} \overset{\times 5}{\rightarrowtail} \mathbb{Z} \underset{\varepsilon}{\overset{\varepsilon'}{\twoheadrightarrow}} \mathbb{Z}_5,$$

where the quotient maps are $\varepsilon(z) = z \bmod 5$ and $\varepsilon'(z) = 2z \bmod 5$.

Fig. 3. Two inequivalent extensions of $\mathbb{Z}_5$ by $\mathbb{Z}$.

Let $E(A, B)$ denote the set of extensions of A by B (up to equivalence). Describing this directly is tricky. However, we can investigate $E(A, B)$ indirectly via the object called $\operatorname{Ext}(A, B)$. The report demonstrates how E and Ext are **naturally equivalent**.

- Equivalent: there is an invertible transformation from E to Ext.
- Natural: transforming before changing A or B gives the same result as transforming after the change.

This means that we can calculate $\operatorname{Ext}(A, B)$ to investigate $E(A, B)$. One immediate consequence of this is that extensions come in **Abelian groups**; this is true simply because $\operatorname{Ext}(A, B)$ is always an Abelian group.

6. Bibliography

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