1 Proposed project

Over the summer I was introduced to Nathan Barker, a former PhD student of Newcastle. Together with my supervisor Andrew Duncan, Nathan described an algorithm they had been writing a paper about. To be specific, NB and AJD have used Higman's solution to the conjugacy problem in the Higman-Thompson groups $G_{n,r}$ to give an algorithm which solves the *power* conjugacy problem. Thus far, I have been working on understanding and implementing that solution.

1.1 History

The story begins with three groups $F < T < V$, which are collectively known as *Thompson's groups.* Their namesake is Richard Thompson, who discovered [10] these groups as part of the search for finitely generated groups with unsolvable word problem. Since then, these groups have been used as counter-examples to a number of other conjectures in group theory, and possess a combination of interesting properties. For instance:

- All three groups are infinite yet finitely presented. Indeed, T and V are rare examples of infinite, finitely presented *simple* groups.
- V contains a copy of every finite group.

See [3] and [9] for the details.

Other researchers are interested in these groups as examples of dynamical systems and self-similar groups. The question of whether F is amenable or not has also been attacked in recent years.

There are multiple ways to think about the elements of these groups, though they are often introduced concretely as groups of functions. The easiest to describe is F. Its elements $f \in F$ are functions $f\colon [0,1] \to [0,1]$ satisfying

1. f is a bijection.
2. f is increasing.
3. f is continuous.
4. f is linear everywhere, except at a finite number of breakpoints of the form $a/2^n$ for $n \in \mathbb{N}$.
5. At points where f is linear, tangents to f have gradients of the form 2^m, for $m \in \mathbb{Z}$.

To obtain the group T, we identify $0 \sim 1$ so that $[0,1]$ is replaced by S^1. This introduces functions for which $f(0) \neq 0$.[1] Next, one obtains V by weakening condition 3 to require only that f is *right*-continuous. This means that f may be discontinous at its breakpoints. See figure 1 for examples.

Diagrams are very useful when working in these groups. Cannon, Floyd and Parry [3] describe tree pair diagrams, which depict elements of F, T and V by using pairs of rooted binary trees. In his thesis [2], Belk introduced 'strand' and 'two-way forest' diagrams to represent and work with elements of F.

[1] We need to relax condition 2 to say that 'f preserves orientation.'

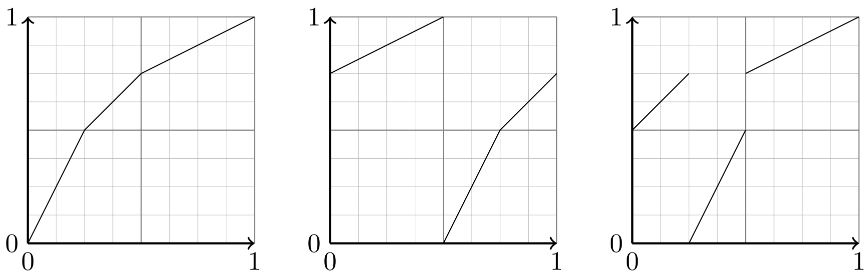

Figure 1: From left to right: example elements of F, T and V.

In a lecture course from July to October 1973 [4], Graham Higman generalised V to an infinite family $\{G_{n,r}\}$ now called the *Higman-Thompson groups*, where V is the special case $V = G_{2,1}$. By describing group elements as automorphisms of a particular algebra, Higman explained how one could solve the conjugacy problem in $G_{n,r}$.

1.2 Implementation

Some tools do exist which which perform computation in these groups. Kogan's `nvTrees` applet [7] allows one to compute in V, and also in multidimensional analogues nV. Belk and Mattucci use strand diagrams in [1] to provide algorithm for solving the conjugacy problem in F, T and V. Hossein has implemented this, giving a polynomial time algorithm: $\mathcal{O}(n)$ in theory with an $\mathcal{O}(n^2)$ implementation [**?**, 5].

To the best of my knowledge, no tool exists which solves the conjugacy problem in the general setting of $G_{n,r}$. I started writing my own Python implementation[2] based on Higman's algorithm in September. As of early December, the program is capable of

- storing group elements $\varphi, \psi \in G_{n,r}$ in memory,
- computing inverses φ^{-1} and products $\varphi\psi$, and
- testing to see if two given elements are conjugate.

See listings 1 and 2 for a brief demonstration.

1.3 Pond-type orbits

As part of testing and debugging my implementation, I found a problem element $\varphi \in G_{2,1}$ which revealed that a lemma of Higman was false [4, lemma 9.6]. This element φ—see figure 2 for an illustration—provided an example of an edge case which we refer to as a *pond-type orbit*. I have worked with AJD to correct the theory and the algorithm so that this edge case is no longer a problem.

[2]Source code and documentation is available online at `https://github.com/DMRobertson/thompsons_v`.

Listing 1: `demonstration.py`

```
from test import setup_script
setup_script(__file__)

"""A quick demonstration of Higman's conjugacy test."""

from thompson.automorphism import Automorphism
from thompson.examples import *

psi = example_5_26_psi
phi = example_5_26_phi

rho = psi.test_conjugate_to(phi)
print('Program found a conjugator:', rho is not None)
print('Rho is genuinely a conjugator:', psi*rho == rho*phi)
print(rho)
```

Listing 2: Output of `demonstration.py`

```
H:\thompsons_v\scripts>demonstration.py
Program found a conjugator: True
Rho is genuinely a conjugator: True
Automorphism: V(2, 1) -> V(2, 1) specified by 3 generators (after expansion
    and reduction).
x1 a1 a1 -> x1 a1
x1 a1 a2 -> x1 a2 a1
x1 a2    -> x1 a2 a2
```

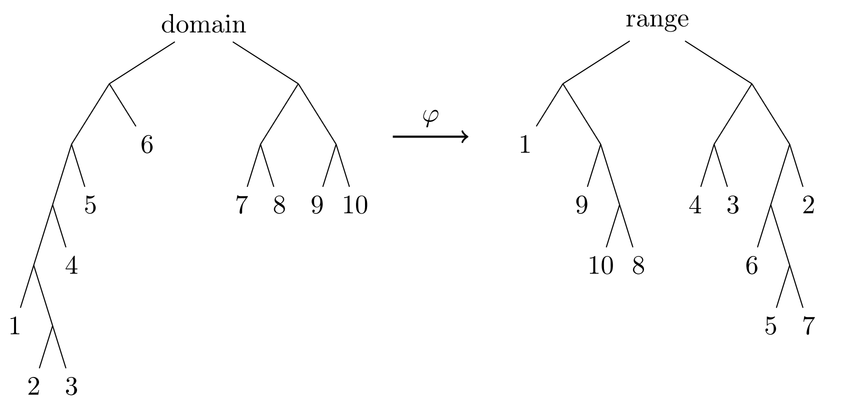

Figure 2: A reduced tree pair diagram illustrating $\varphi \in G_{2,1}$. This was the first example we found of an element with a 'pond'-type orbit.

1.4 Future work

The implementation As of late December, the implementation seems to be quite stable. Randomised testing suggests that the implementation does not produce false positives; work is still ongoing to find evidence that false negatives do not occur. There is still work to be done in implementing the *power* conjugacy test, which was the initial goal of this implementation.

There is much to be done in the realms of complexity analysis. While time has been spent optimising some parts of the program, we are currently lacking a formal complexity analysis of the running time.

The theory We currently have a number of examples of these so-called pond orbits, but our understanding of them is minimal. They are certainly uncommon: a pseudo-random computer search finds an element with a pond orbit approximately 1 time out of 5000. We aim to determine precise conditions for these orbits: when are they formed? It would also be interesting to have a formal estimation of how frequently these pond orbits occur.

The future Underlying this study is the hope that these techniques can be applied to tackle the *simultaneous* conjugacy problem. Namely: for a group G and elements $\psi_1, \ldots \psi_m, \varphi_1, \ldots, \varphi_m \in G$ determine if there exists a single conjugator $\rho \in G$ such that $\psi_i \rho = \rho \varphi_i$ for each $i = 1, \ldots, m$. (Note that the 'ordinary' conjugacy problem the special case $m = 1$.) Mattuci shows in his thesis [8, 6] that this problem is solvable in $G = F$. It is hoped that we can generalise Higman's approach to conjugacy to tackle *simultaneous* conjugacy in $G_{n,r}$.

2 Other details

Conferences This December I attended the *Expanders everywhere!* conference in Neuchâtel, Switzerland. For the rest of this year, I am planning (budget permitting) to attend the thrice-annual meetings of the North British Geometric Group Theory (NBGGT) seminar.

Potential papers The majority of my work thus far is a contribution to AJD's and NB's paper. Looking to the future, I think there is scope for a short paper on the theoretical and practical complexity of my implementation. Any results obtained in the future on the frequency of pond-type orbits may also be paper-worthy.

Courses Any introductory courses on formal complexity analysis would be useful. Other than that, I'm planning to attend MAGIC075, a course on representation theory for groups.

References

[1] J. Belk and F. Matucci. Conjugacy and Dynamics in Thompson's Groups. *ArXiv e-prints*, August 2007. `arXiv:0708.4250 [math.GR]`.

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[5] Nabil Hossain, Robert W McGrail, James Belk, and Francesco Matucci. Deciding conjugacy in Thompson's group F in linear time. In *Symbolic and Numeric Algorithms for Scientific Computing* (SYNASC)*, 2013 15th International Symposium on*, pages 89–96. IEEE, 2013.

[6] Martin Kassabov and Francesco Matucci. The simultaneous conjugacy problem in groups of piecewise linear functions. *Groups, Geometry, and Dynamics*, 6(2):279–315, 2012.

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[9] John Meier. Friends of Thompson's group F. Available from `http://www.cs.lafayette.edu/~meierj/talkspdfs/ThompsonII.pdf`., September 2004.

[10] Richard Thompson. Handwritten notes. Available at `http://www.math.binghamton.edu/matt/thompson/thompson-hd_001.pdf` and `http://www.math.binghamton.edu/matt/thompson/thompson-hd_008.pdf`, 1965.