Exercice 1 : Timing Game Use the results of the previous exercise to solve for the Nash equilibria of the endogenous timing game in which firms simultaneously choose whether to play early or to play late. If they both make the same choice (either early or late), the simultaneous Bertrand game follows; if they make different choices, a sequential game follows with the firm having chosen early being the leader. Discuss the economic intuition behind your result.

Solution :

1) To solve this exercise, one must refer to the previous one. From the latter we compute the following results for both firms: profits in a Bertrand duopoly case, profits in a Stackelberg duopoly case, meaning a sequential game, in which firms decide whether to play early or late.
We have determined the results below:

$$\pi_1^B = \frac{2}{225}(5a+2c)^2 \tag{1}$$

$$\pi_2^B = \frac{2}{225}(5a-7c)^2 \tag{2}$$

$$\pi_1^L = \frac{1}{112}(5a+2c)^2 \tag{3}$$

$$\pi_2^F = \frac{1}{1568}(19a-26c)^2 \tag{4}$$

$$\pi_1^F = \frac{1}{1568}(19a+7c)^2 \tag{5}$$

$$\pi_2^L = \frac{1}{112}(5a-7c)^2 \tag{6}$$

For the sake of the model, we make this assumption:

$$a > \frac{7c}{5}$$

First of all, one can write this game using its extensive form.

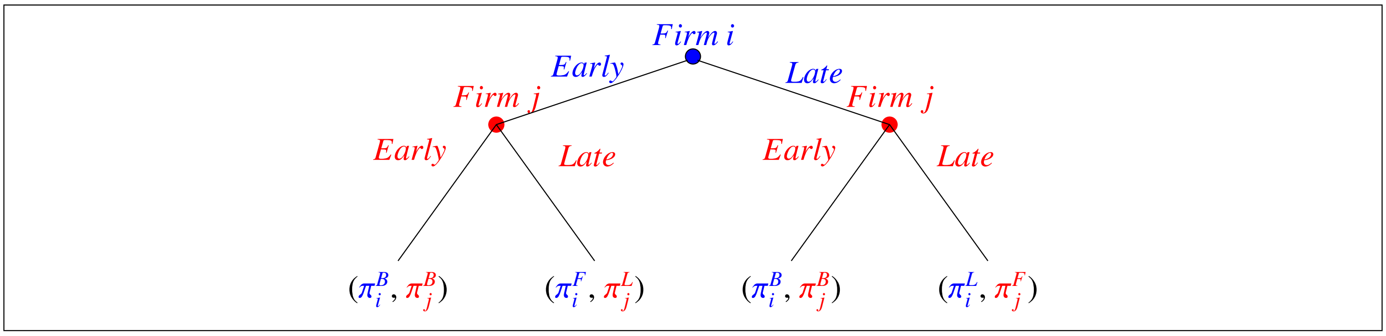

Figure 1 : Game tree

Therefore, we obtain a payoff matrix:

		Firm 2	
		Early	Late
Firm 1	Early	(π_1^B,π_2^B)	(π_1^L,π_2^F)
	Late	(π_1^F,π_2^L)	(π_1^B,π_2^B)

Table 1 : Theoretical payoff matrix

We replace all of our parameters by their respective value in this new payoff matrix:

		Firm 2	
		Early	Late
Firm 1	Early	$\frac{2}{225}(5a+2c)^2$, $\frac{2}{225}(5a-7c)^2$	$\frac{1}{112}(5a+2c)^2$,$\frac{1}{1568}(19a-26c)^2$
	Late	$\frac{1}{1568}(19a+7c)^2$,$\frac{1}{112}(5a-7c)^2$	$\frac{2}{225}(5a+2c)^2$, $\frac{2}{225}(5a-7c)^2$

Table 2 : Actual payoff matrix

Let's begin by looking for the pure-strategy Nash equilibriums. Solving the game backwards, we first focus on the second firm. Looking at the payoff matrix, it is easy to notice the superiority of the Firm 2 profits being a leader under a Stackelberg over the Bertrand case. Indeed:

$$\frac{1}{112}(5a-7c)^2 > \frac{2}{225}(5a-7c)^2$$

Thanks to the previous exercise, we also know that Firm 2 always has a second-mover advantage. Let's recheck this assumption. If so, that means:

$$\frac{1}{1568}(19a-26c)^2 > \frac{1}{112}(5a-7c)^2$$

The latter expression can be rewritten as

$$11a^2 > 10c^2 + 8ac$$

Let's state $a = \frac{7c}{5}$ (it is known a exceeds this, we are just doing that to demonstrate the second mover advantage.) We then obtain, after replacing all a_s by c_s:

$$\frac{539c^2}{25} > \frac{530c^2}{25}$$

This just proves the second mover advantage for the second firm. Basically:

$$\pi_2^F > \pi_2^L > \pi_2^B$$

As regards the first firm, there is an obvious fact. We have:

$$\pi_1^L > \pi_1^B$$

We are also aware of the first-mover advantage for Firm 1. Let's recheck it, just in case.
For a matter of fact, $\pi_1^L > \pi_1^F$ exactly means $7c(2a+c) > 11a^2$. Let's assume :

$$a = \frac{7c}{5} \Leftrightarrow 14\,\frac{7c}{5}c + 7c^2 - 11\,\frac{7c}{5}^2 \Leftrightarrow \frac{126c^2}{25} > 0$$

Hence, one can conclude the first firm has a first-mover advantage. However, the firm has a first-mover advantage only if $c > 0.604a$. Last but not least, let's compare π_1^F with π_1^B.

$$\frac{1}{1568}(19a+7c)^2 > \frac{2}{225}(5a+2c)^2 \Leftrightarrow (5a-7c)(565a+217c) > 0$$

that is to say, assuming a, c being both positive (they are!): $5a > 7c$. This is a condition in our model. Thus, we have $\pi_1^L > \pi_1^F > \pi_1^B$.
What does that mean ? Are witnessed two pure-strategy Nash equilibriums, which are $(Late, Early)$ and $(Early, Late)$.
Is there any mixed-strategy Nash equilibrium ? Let us consider only the cases which are not degenerative. We will see this in a moment. We know from last year lecture that a mixed strategy is an assignment of a probability to each pure strategy. This allows for a player to randomly select a pure strategy. Since probabilities are continuous, there are infinitely many mixed strategies available to a player. Let's just call Firm 1 chooses α the probability corresponding to it playing early whilst Firm 2 is indifferent ($\beta \in (0,1)$) between the two strategies. One then has:

$$\alpha\,\pi_2^B + (1-\alpha)\,\pi_2^L$$

which has to be equalized to:

$$\alpha\,\pi_2^F + (1-\alpha)\,\pi_2^B$$

Resolving the equation we find:

$$\alpha = \frac{\pi_2^B - \pi_2^L}{2\,\pi_2^B - \pi_2^F - \pi_2^L}$$

Let's repeat the operation for Firm 2. This time we are taking β the probability of playing early whilst Firm 1 is indifferent between ($\alpha \in (0,1)$) both strategies. That leads us to the next equation to solve:

$$\beta\,\pi_1^B + (1-\beta)\,\pi_1^L = \beta\,\pi_1^F + (1-\beta)\,\pi_1^B$$

The latter equation gives us the following result:

$$\beta = \frac{\pi_1^B - \pi_1^L}{2\,\pi_1^B - \pi_1^F - \pi_1^L}$$

by perfect symmetry.
Essentially, we have:

$$\begin{cases} \alpha = \frac{\pi_2^B - \pi_2^L}{2\,\pi_2^B - \pi_2^F - \pi_2^L} \\[2em] \beta = \frac{\pi_1^B - \pi_1^L}{2\,\pi_1^B - \pi_1^F - \pi_1^L} \end{cases}$$

We are not done yet, one has still to compute our actual α, β.

$$\begin{cases} \alpha = \frac{\frac{2}{225}(5a-7c)^2 - \frac{1}{112}(5a-7c)^2}{2\,\frac{2}{225}(5a-7c)^2 - \frac{1}{1568}(19a-26c)^2 - \frac{1}{112}(5a-7c)^2} \Leftrightarrow \frac{14\,(5a-7c)^2}{3175a^2-3760ac-878c^2} \\ \\ \beta = \frac{\frac{2}{225}(5a+2c)^2 - \frac{1}{112}(5a+2c)^2}{2\,\frac{2}{225}(5a+2c)^2 - \frac{1}{1568}(19a+7c)^2 - \frac{1}{112}(5a+2c)^2} \Leftrightarrow \frac{14\,(5a+2c)^2}{3175a^2-2590ac-1463c^2} \end{cases}$$

According to the values of a, c, we distinguish 3 cases

- If $5a = 7c \Leftrightarrow \alpha = 0$ and $\beta = 1$. That is exactly our pure-strategy equilibrium.
- If $5a < 7c \Leftrightarrow \alpha, \beta < 0$. The latter is impossible, since probabilities are continuous and positive.
- If $5a > 7c \Leftrightarrow \alpha, \beta > 0$ Once again we have our model condition. This hypothesis allows us to have a mixed-strategy Nash equilibrium.

Based on the 3 Nash equilibriums we found above, there is no point in a Bertrand competition. Let's not forget that a Nash equilibrium is a state such as each player (*i.e* both firms in our case) will gain nothing by changing only its own strategy. Moreover, one knows that in a à la Bertrand competition, firms do not make profit. That is why no Bertrand game is desirable. If firms had symmetric costs, each of them would get higher and higher profits when being follower in the game. In our case, $Firm2$ has got higher costs than $Firm1$, therefore it has a second-mover advantage move. Conversely, $Firm1$ is pretty much a low-cost firm. Hence, it has a first-mover advantage on some conditions, detailed above. Our game only has one subgame perfect equilibrium, which is (Early, Late). The (Late, Early) one only exists theoretically since we know $Firm1$has a first-mover advantage if $c > 0.604a$. As such, she has no interest in playing late, given that agents are rational and want to maximize their profits.