Exercice 1 : Quality-augmented Hotelling model Consider the Hotelling model in which consumers are uniformly distributed on the [0; 1]-interval and firms A and B are located at the extreme points. Firms produce a product of quality s_i. Consumer $x \in [0; 1]$ obtains utility $u_A = (r - tx)s_A - p_A$ if she buys one unit of product A and $u_B = (r - t(1 - x))s_B - p_B$ if she buys one unit of product B. Each consumer buys either one unit of product A or one unit of product B.

1 Describe the property of the utility function with respect to quality in two or three sentences.
2 Determine the demand for products A and B at given prices and given qualities.
3 Suppose that qualities s_A and s_B are given and that marginal costs of production are zero. Determine the Nash equilibrium in prices under the assumption that qualities are not too asymmetric implying that both firms have a strictly positive market share in equilibrium.
4 Suppose that qualities are symmetric and that the cost of quality $C(s_i)$ is increasing and strictly convex in s_i. How does the equilibrium profit depend on quality?
5 Compare this finding to the standard quality-augmented Hotelling-model in which consumer x obtains utility $u_A = r + sA - tx - pA$ if she buys product A and $u_B = r + s_B - t(1 - x) - pB$ if she buys product B.

Solution :

1) One begins by representing the Hotelling line:

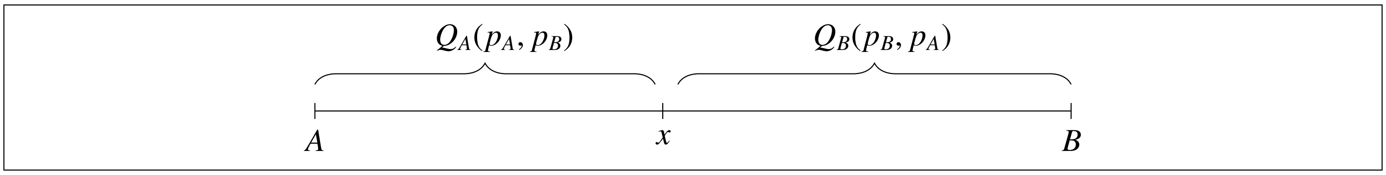

Figure 1 : The Linear City

From the utility functions, we know quality is directly related to reserve price and cost of transport. Indeed, when quality is high enough, consumers will have a higher reservation price and their transport cost will be lower.
It can also be said the higher the gap between reserve price and total transport cost, the higher utility gained from purchasing a great quality product.
In addition, for a x consumer, $\frac{r-tx}{r-t(1-x)}$ might be considered as the degree of sensitivity to quality for the good he buys. Put differently, it is the marginal effect of quality on utility.
Note that the transport cost doesn't necessarily mean a real motion between an a point to a b point. But it rather corresponds to a motion from tastes. That is why the $tx \times s_A$ coefficient is negative.
It simply means consumers enjoy purchasing high-end products.

2) First of all, the indifferent consumer $\tilde{x}$ is the one for whom $u_A = u_B$.
Are equalized the two utility functions:

$$(r - t\tilde{x})s_A - p_A = (r - t(1 - \tilde{x}))s_B - p_B \iff \tilde{x} = \frac{-rs_B + ts_B - p_A + p_B + rs_A}{t(s_B + s_A)} \tag{1}$$

If $0 < \tilde{x} < 1$ (which is a sine qua non condition for resolving the problem), then demands for firm 1 and 2 are respectively:

$$Q_A(p_A, p_B; s_A, s_B) = \tilde{x} \tag{2}$$

And:

$$Q_B(p_A, p_B; s_A, s_B) = 1 - \tilde{x} \tag{3}$$

3) In this question, we assume qualities are asymmetric (whilst preventing the maximal differentiation issue). In order to determine the NE in prices, one has to find both reaction functions. To this end, we begin by computing the firms' profit functions. We have:

$$\pi_i = p_i\,(\frac{-rs_B + ts_B - p_i + p_j + rs_A}{t(s_B + s_A)}) \tag{4}$$

$$\mathscr{P}_{Firm_i}\left|\begin{array}{l} \underset{\{p_i, p_j\}}{\text{Max}}\ \pi_i \\ \\ \quad slc\ p_i; p_j > 0 \end{array}\right. \tag{5}$$

Then, we know that the FOCs give us the two reaction functions.

$$\frac{\partial \pi_i}{\partial p_i} = \frac{1}{2}(p_b + rs_A - rs_B + ts_B) = p_A(p_B) \tag{6}$$

And:

$$\frac{\partial \pi_i}{\partial p_j} = \frac{1}{2}(p_a - rs_A + rs_B + ts_A) = p_B(p_A) \tag{7}$$

SOCs are also verified:

$$\frac{\partial^2 \pi_i}{\partial p_i^2} = -\frac{2}{t(s_A + s_B)} < 0 \tag{8}$$

Since both curves being upward-sloping (thus, goods are strategic complements, i.e $\Pi^i_{ij} > 0$, the NE is the intersection between the two.

We then need to equalize both prices to both reaction functions:

$$\begin{cases} \frac{1}{2}(p_b + rs_A - rs_B + ts_B) &= p_A \\ \frac{1}{2}(p_a - rs_A + rs_B + ts_A) &= p_B \end{cases} \tag{9}$$

Eventually, one can compute firms' prices, by solving the system above.

$$\begin{cases} p_A^*(s_A; s_B) &= \frac{1}{3}(r(s_A - s_B) + t(s_A + 2s_B)) \\ p_B^*(s_A; s_B) &= \frac{1}{3}(r(s_B - s_A) + t(2s_A + s_B)) \end{cases} \tag{10}$$

So, the Nash equilibrium in this quality-augmented Hotelling model is:

$$(p_A^{NE}, p_B^{NE}) = (\frac{1}{3}(r(s_A - s_B) + t(s_A + 2s_B)), \frac{1}{3}(r(s_B - s_A) + t(2s_A + s_B))) \tag{11}$$

The reaction functions can be graphically represented:

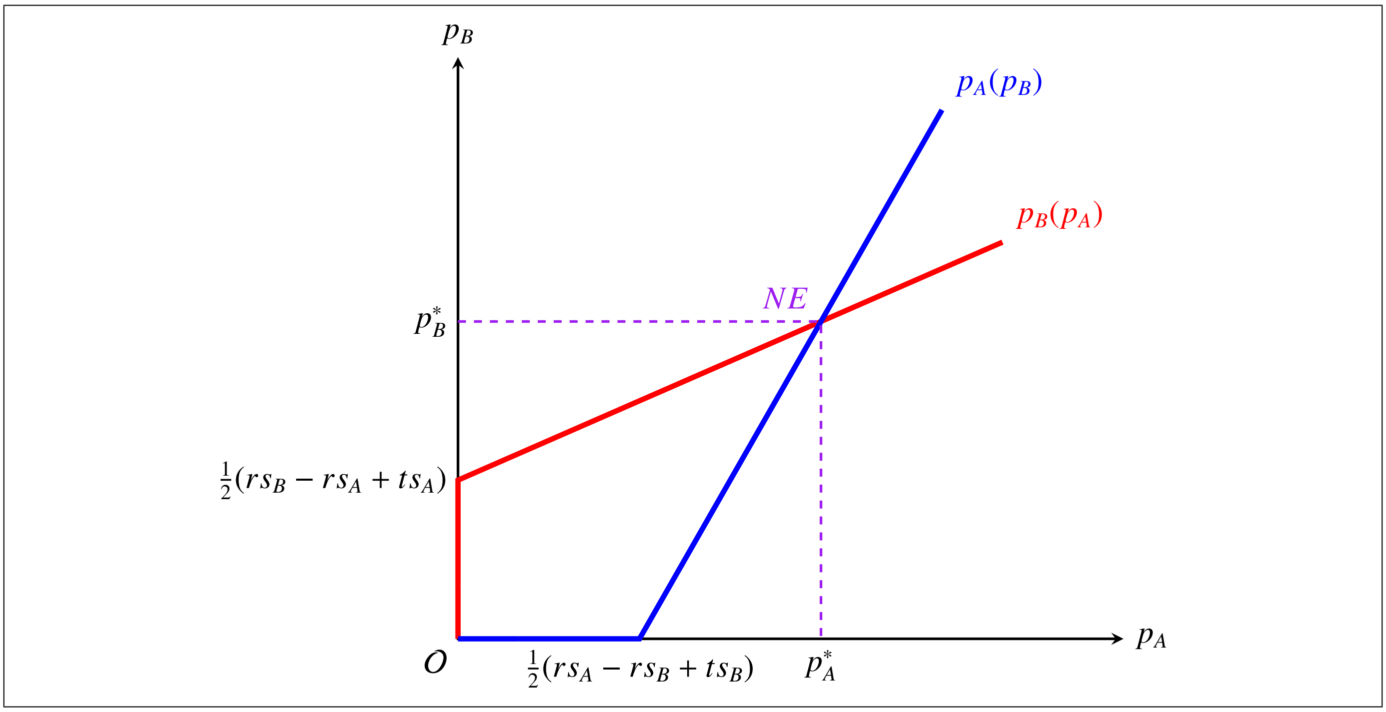

Figure 2 : Pure-strategy Nash Equilibrium

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We can eventually compute firms' equilibrium profit:

$$\begin{cases} \pi_A^{NE} = p_A^{NE}(\tilde{x}) \Leftrightarrow \frac{r(s_A - s_B) + t(s_A + 2s_B))^2}{9t(s_A + s_B)} \\ \pi_B^{NE} = p_B^{NE}(1 - \tilde{x}) \Leftrightarrow \frac{r(s_B - s_A) + t(2s_A + s_B))^2}{9t(s_A + s_B)} \end{cases} \tag{12}$$

One must note the symmetry between both profits. Furthermore, the impact of quality on firms' profits is easily noticeable, as the qualities-difference term appears in both profits.

4) In this case, qualities are symmetric, i.e $s_A = s_B = s_i$.

We also have $C(s_i)$ representing the cost of quality with two assumptions:

$$\frac{\partial C}{\partial s_i} > 0 \tag{13}$$

$$\frac{\partial^2 C}{\partial s_i^2} > 0 \tag{14}$$

Given (4), Firm i's profit can be rewritten as follows, since $p_i = p_j$:

$$\pi_i = p_i(\frac{-rs_B + ts_B - p_i + p_j + rs_A}{t(s_B + s_A)}) - c(s_i) \tag{15}$$

$$\Leftrightarrow \frac{ts_i}{2} - c(s_i) \tag{16}$$

How to get the equilibrium profit ? One has to solve the following equation:

$$\frac{\partial \pi_i}{\partial s_i} = 0 \tag{17}$$

This is nothing else but:

$$\frac{t}{2} - c'(s_i^*) = 0 \tag{18}$$

$$\Leftrightarrow \frac{t}{2} = c'(s_i^*) \tag{19}$$

The latter equation allows us pointing out the effect of quality on profits. Profit function being concave, quality affects profit positively when $t/2 > c'(s_i)$.

Cost of quality has to be lower than the marginal revenue relative to quality. Indeed, the increase in quantity demanded does not cover up quality cost.

5) In our case, if qualities are symmetric, reserve price disappears.

In the standard model with c representing the marginal cost, one can compute firm i's profit as follows:

$$(p_i - c)(1/2 + \frac{(r_i - r_j) - (p_i - p_j)}{2\tau}) \tag{20}$$

Then are computed equilibrium prices:

$$\begin{cases} p_1^* &= c + \tau + \frac{1}{3}(r_1 - r_2) \\ p_2^* &= c + \tau - \frac{1}{3}(r_1 - r_2) \end{cases} \tag{21}$$

In this exercise, marginal cost is equal to 0.

Thus, in the case where qualities are symmetric, they have no effect anymore, since the reserve price disappears.

That is the principal difference between a vertical differentiation model and an asymmetric competition between differentiated products.

Last but not least, if we compute this exercise's marginal utilities with respect to quality, we get:

$$\begin{cases} \dfrac{\partial u_A}{\partial s_A} = r - tx \\ \dfrac{\partial u_B}{\partial s_B} = r - t + tx \end{cases} \tag{22}$$

Whereas in the standard quality-model, marginal utilities with respect to quality are constant and equal to 1. To quote what has been stated before, marginal effect of quality on utility is even more important if consumers have a large reserve price in accordance to their total transport costs.

As a closure, this model could also be augmented with asymmetric information and advertising.

As a matter of fact, one might solve a three-staged game, inwhich the first stage would be a choice of quality for the supplier, the second stage the price determination and the final stage consumers deciding whether or not to purchase the good.