# Cosmology notes - Non-Gaussianity

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## INTRODUCTION

With the help of modern cosmological experiments, we have studied and examined current cosmology theory in a precise extent. The most competitive scenario to explain the evolution of the universe is slow-roll single field inflation with ΛCDM model. The density perturbation under inflation is Gaussian. The proposal of non-gaussianity is motivated by the future work on examination of quantum gravity and extraction of more useful information from CMB. Here we focus on the primordial non-Gaussianity and the detectable bispectrum signal from the scalar primordial perturbation $\zeta$

## IN-IN FORMALISM AND CORRELATION FUNCTION

We will use in-in formalism to calculate the fluctuation of non-gaussianity and the coressponding correlation functions in time dependent background. We are interested in the correlation functions of super-horizon primordial perturbations generated during inflation, or its equivalence, the expectation value of operator $\hat{Q}$

$$\langle Q\rangle \equiv \langle\Omega|Q(t)|\Omega\rangle \tag{1}$$

where $t$ denotes the time at the end of inflation, and $|\Omega\rangle$ is the vacuum state for two interacting field $\phi$ and $\pi$. We consider two a time-dependent background and the perturbation

$$\begin{aligned}\phi_a(\mathbf{x},t) &\equiv \bar{\phi}_a(\mathbf{x},t)+\delta\phi_a(\mathbf{x},t)\\ \pi_a(\mathbf{x},t) &\equiv \bar{\pi}_a(\mathbf{x},t)+\delta\pi_a(\mathbf{x},t)\end{aligned} \tag{2}$$

Note that $Q$ is the product in terms of two field perturbations $\delta\phi$ and $\delta\pi$. The Hamiltonian could be expanded as

$$\begin{aligned}H[\phi(t),\pi(t)] = H[\bar{\phi}(t),\bar{\pi}(t)] + \cdots\\ + \tilde{H}[\delta\phi(t),\delta\pi(t);t]\end{aligned} \tag{3}$$

Here $\tilde{H}$ represents terms of quadratic and higher-orders in perturbation. To have a systematic scheme to do the perturbation theory, we split $\tilde{H}$ into two parts: the quadratic kinematic $H_0$ part and interacting $H_I$ part

$$\begin{aligned}\tilde{H}[\delta\phi(t),\delta\pi(t);t] =& H_0[\delta\phi(t),\delta\pi(t);t]\\ &+ H_I[\delta\phi(t),\delta\pi(t);t]\end{aligned} \tag{4}$$

After a long algebra, we have expectation value from Equation 1

$$\begin{aligned}\langle Q\rangle = \langle\Omega| \left[\bar{T}\exp\left(i\int_{t_0}^{t} H_I(t)\mathrm{d}t\right)\right] Q_I(t)\\ \times \left[T\exp\left(-i\int_{t_0}^{t} H_I(t)\mathrm{d}t\right)\right]|\Omega\rangle\end{aligned} \tag{5}$$

Note that in

$$\begin{aligned}H_I(t) &\equiv H_I\left[\delta\phi^I(t),\delta\pi^I(t);t\right]\\ Q^I(t) &\equiv Q\left[\delta\phi_a^I(\mathbf{x},t),\delta\pi_a^I(\mathbf{x},t)\right]\end{aligned} \tag{6}$$

all the field perturbations are in the interaction picture. It is worth mention that we could also finish the perturbation theory from the Lagrangian formalism. The Lagrangian formalism is equivalent with the Hamiltonian one. To solve the integration, we should series-expand the integrand in powers of $H_I$ to the desired orders, and perform contractions for each term. Finally we could draw the Feynman diagrams which represent the corelation functions, and use them to do the contractions and perform the integration.

## A NO-GO THEOREM

Here we take a simple inflation model for instance to calculate its non-gaussianities. We will find the small non-Gaussianity is difficult to be detected at the end of this section.

At the beginning, we consider this simple inflation model with

- single scalar field inflation
- canonical kinetic term
- always slow-rolls
- in Bunch-Davies vacuum
- in Einstein gravity

The Lagrangian for the single scalar field inflation with canonical kinetic term is

$$S=\int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_P}{2}R-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\phi - V(\phi)\right] \tag{7}$$

Here $V$ is slow-roll potential. We should also follow the general slow-roll parameters $\epsilon = -\dot{H}/H^2$ and $\eta = \dot{\epsilon}/\epsilon H$.

It is convenient to use the ADM formalism to solve the problem.

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right) \tag{8}$$

and thus we could simplify Equation 7. For non-Gaussianity, we only focus on the cubic term of Equation 7, namely

$$\begin{aligned}S_3 = \int \mathrm{d}t\mathrm{d}^3x \Big[& a^3\epsilon^2\zeta\dot{\zeta}^2 + a\epsilon^2\zeta(\partial\zeta)^2 - 2a\epsilon\dot{\zeta}(\partial\zeta)(\partial\chi) \\ &+\frac{a^3\epsilon}{2}\dot{\eta}\zeta^2\dot{\zeta} + \frac{\epsilon}{2a}(\partial\zeta)(\partial\chi)\partial^2\chi \\ &\frac{\epsilon}{4a}(\partial^2\zeta)(\partial\chi)^2 + f(\zeta)\left.\frac{\delta L}{\delta\zeta}\right|_1\Big]\end{aligned} \tag{9}$$

The first line of Equation 9 is proportional to $\epsilon^2$, the second line is proportional to $\epsilon^3$ and thus is negligible. The third line can be absorbed by a field redefinition

$$\begin{aligned}&\zeta \to \zeta_n + f(\zeta_n) \\ &= \zeta_n + \frac{\eta}{4}\zeta^2 + \text{terms with derivatives on } \zeta\end{aligned} \tag{10}$$

All other terms of $f(\zeta)$ vanish outside the horizon.

We will finally focus on the bispectrum for the measurement of non-Gaussianity. According to Equation 5, we expand the exponential to the first order in $H_{I,3}$ to get the leading result

$$\langle\zeta_n^3\rangle = -i\langle 0|\left|\int_{t_0}^{t}\mathrm{d}t\left[\zeta_n(\mathbf{k}_1)\zeta_n(\mathbf{k}_2)\zeta_n(\mathbf{k}_3), H_{I,3}\right]\right||0\rangle \tag{11}$$

After a long algerba, we conclude that the contribution to the bispectrum from the first line of Equation 9 is $f_{NL} = \mathcal{O}(\epsilon)$, and the third line contributes $f_{NL} = \mathcal{O}(\eta)$. The slow-roll parameters are of order $\mathcal{O}(0.01)$, so $f_{NL} \sim \mathcal{O}(0.01)$. Even if we start with Gaussian primordial perturbations, nonlinear effects in CMB evolution will generate $\mathcal{O}(1)$, and a similar number for large scale structures due to the nonlinear gravitational evolution or the galaxy bias. Therefore, it is difficult to distangle galaxy bias and other contaminations and detect such small non-Gaussianity.

## BEYOND NO-GO

In last section we estimate the non-gaussianity of a simple model. It is still necessary to consider other factors to enhance that simple model. In this section we will try to consider some improvement scenario.

### A better Inflation Model

It is a subtle work to construct an explicit and self-consistent inflation model. We should consider various problems listed below

- The $\eta$ problem for slow-roll inflation. In order to have a slow-roll inflation, the mass of the inflation field has to be light enough, $m \ll H$, to maintain a flat potential. However, the natural mass of a light particle in the inflationary background is of order $H$. Either symmetry needs to be imposed or other tuning contributions introduced to solve the problem.

- The h-Problem for DBI inflation. DBI inflation is invented to generate inflation by a different mechanism. It makes use of the warped space in the internal field space. These warped space impose speed limits for the scalar field. A canonical example of warped space is

$$\mathrm{d}s^2 = h(r)^2\left(-\mathrm{d}t^2 + a(t)^2\mathrm{d}\mathbf{x}^2\right) + h(r)^{-2}\mathrm{d}r^2 \tag{12}$$

where $r$ is the extra dimension (or internal space), $h(r) = r/R$ is the warp factor, and $R$ is the length scale of the warped space. In this case, we must have $h \ll HR$ to provide a speed limit that is small enough for inflation. However, one of Einstein equations with metric 12 will have $h < HR$. Therefore, either symmetry, or tuning contributions should be considered to solve the h problem here.

- The field range bound. Some factors will change the bound of the field range $\Delta\phi$. The large field inflation with $V_{large} = 1/2m^2\phi^2$ require the field range to be much larger than $M_P$. In supergravity and string theory, the 4-dimension $M_P$ is obtained by integrating the extra six dimension. So we could obtain $\Delta\phi \leq M_P/(M_{(10)}L)^2$. Due to the dependence of $M_P$ on the volume $V_{(6)}$, increasing the volume only makes the bound tighter.

- The variation of potential. One will encounter problems constructing the large field inflationary potential. Large field potentials that arise from a fundamental theory take the following general form

$$V(\phi) = \sum_{n=0}^{\infty}\lambda_n m_{\text{fund}}^{4-n}\phi^n \tag{13}$$

where $m_{\text{fund}}$ represents typical scales in the theory. For field theory descriptions to hold, $m_{\text{fund}} \ll M_P$. Unless some symmetries are present to forbid an infinite number of terms in Equation 13, or a high degree of fine-tuning is assumed, the shape of potential 13 varies over a scale of $m_{\text{fund}} \ll M_P$. This variation of potential field is too dramatic for the potential to be a successful slow-roll one.

### Shape and running of bispectra

The three-point correlation function is a function of three momenta, $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$. If we assume they are translational and rotational invariance, we are left with $k_1$, $k_2$ and $k_3$, satisfying the triangle inequalities. The triangle information is encoded in $S(k_1, k_2, k_3)$, which is usually used to plot the profiles of bispectra. It is defined as

$$\langle\zeta^3\rangle \equiv S(k_1, k_2, k_3)\frac{1}{(k_1k_2k_3)^2}\tilde{P}_\zeta^2(2\pi)^7\delta^3\left(\sum_{i=1}^{3}\mathbf{k_i}\right) \quad (14)$$

where $\tilde{P}_\zeta$ is the fiducial power spectrum.

The dependence of $S$ on $k_1, k_2, k_3$ is split into two kinds.

- The *shape* of the bispectrum. This refers to the dependence of $S$ on the momenta ratio $k_2/k_1$ and $k_3/k_1$, while fixing the overall momentum scale $K = k_1 + k_2 + k_3$. For example, for bispectra that are approximately scale invariant, the shape is a more important property. The amplitude of bispectra is denoted as $f_{NL}$ when $k_1 = k_2 = k_3$ by matching

$$S(k_1, k_2, k_3) \sim \frac{9}{10}f_{NL} \quad (15)$$

  The $f_{NL}$ will be always used to quantify the level of non-Gaussianities.

- The *running* of the bispectrum. This refers to the dependence of $S$ on the overall momentum scale $K = k_1 + k_2 + k_3$, while fixing the momenta ratio $k_2/k_1$ and $k_3/k_1$. In the running dominant cases, the $f_{NL}$ has a strong scale dependence.

It is useful to quantify the correlations between different non-Gaussian profile. We introduce the inner product of two shapes

$$S{\cdot}S' = \int_{V_k} S(k_1, k_2, k_3)S'(k_1, k_2, k_3)w(k_1, k_2, k_3)\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3 \quad (16)$$

In real data analysis this is performed in the CMB $l$-space. If we choose the weight function

$$w(k_1, k_2, k_3) = \frac{1}{k_1 + k_2 + k_3} \quad (17)$$

the $k$-scaling is close to the $l$-scaling in the data analyses estimator. The estimator involves a triple integral of the bispectrum over three momenta $k_i$. To have a practical computational costs, it is necessary that the integral can be factorized into a multiplication of three integrals, namely a bispectrum in the form $f(k_1)f(k_2)f(k_3)$. This is called *factorizable form.*

## MORE ON SINGLE FIELD INFLATION NON-GAUSSIANITY

In this section we relax some restrictions of the no-go theorem on single field inflation models and study its non-gaussianity.

### Equilateral Shape: Higher Derivative Kinetic Terms

We study large non-gaussianity generated by non-canonical kinetic terms in general single field inflation model. In this case we have

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_P}{2}R + P(X,\phi)\right] \quad (18)$$

where $X \equiv -(1/2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. We will use some quantities such as $c_s^2$, $\Sigma$ and $\lambda$ to characterize some properties of $P$. Note that it is required that the slow-roll parameters should be small during the inflation. To calculate the bispectrum, we look at the cubic action $S_3$. We could decomposed the shape of three-point correlation function into six parts. If we look at the limit $c_s \ll 1$ or $\lambda/\Sigma \gg 1$, the leading order results give two component $S_\lambda$ and $S_c$. This result could be also derived from the scalar field fluctuation while neglecting those terms about gravity. Therefore, we could expands the kinetic terms $P(X, \phi)$ using $\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t)$ to higher derivative order. Both $S_\lambda$ and $S_c$ peak at equilateral limit, and behave as $S \sim k_3/k_1$ in the limit $k_3 \ll k_1 = k_2$, this is *equilateral shape*. *Dirac-Born-Infeld Inflation* is an explicit example for this.

### Other Scenario

There are also some other scenario with a relaxing restriction of single field inflation.

- Sinusoidal running with sharp feature. Another alternative claims the slow-roll parameter can become temporarily large when there are sharp features in inflation potentials or internal field space.

- Resonant running with periodic features. In this case, features may or may not be sharp, but they are periodic. Such feature will induce an oscillatory component to the background evolution, or, to the couplings in the interaction terms. Here the non-gaussianities are generated when modes are sub-horizon.

- Folded shape with a nonstandard vacuum. In this case people will consider a different wave function from the Bunch-Davies vacuum when modes are well within the horizon.