Cosmology notes - Inhomogeneities

INTRODUCTION

In order to study the inhomogeneities in our universe, we should focus on the matter distribution. Here we will solve for the evolution of gravitational potential caused by dark matter. The evolution of cosmological perturbations breaks up into three stages. Early on, all of the modes are outside the horizon ($k\eta \ll 1$) and the potential is constant. At intermediate times, the wavelength fall within the horizon and the universe evolves from radiation domination ($a \ll a_{eq}$) to matter domination ($a \gg a_e q$).At late times, all the modes evolve identically again and remain constant (see Figure 1). If we hope to relate the potential to the primordial potential set up during inflation, we can write

$$\Phi(\vec{k}, a) = \Phi_P(\vec{k}) \times \{\text{Transfer}(k)\} \times \{\text{Growth}(a)\} \quad (1)$$

where Φ_P is primordial gravitational potential. The transfer function is defined as

$$T(k) \equiv \frac{\Phi(k, a_{\text{late}})}{\Phi_{\text{Large-Scale}}(k, a_{\text{late}})} \quad (2)$$

where a_{late} denotes an epoch well after the transfer function regime, while the Large-Scale solution is the primordial Φ decreased by a small amount. From Figure 1 we could see that the largest wavelength perturbations could be regarded as constant. Therefore, we use the potential of large scale as a comparison. The growth function in Equation 1 is defined as

$$D_1(a) \equiv \frac{a\Phi(a)}{\Phi(a_{\text{late}})} \quad (3)$$

with $a > a_{\text{late}}$. The easiest way to probe the potential is to measure the power spectrum of matter distribution. Because the overdensity today to the primordial potential is

$$\delta(\vec{k}, a) = \frac{3}{5}\frac{k^2}{\Omega_m H_0^2}\Phi_P(\vec{k})T(k)D_1(a)$$

we could obtain the power spectrum of matter at late time is

$$P(k, a) = 2\pi^2\delta_H^2 \frac{k^n}{H_0^{n+3}} T^2(k) \left(\frac{D_1(a)}{D_1(a=1)}\right)^2 \quad (4)$$

In general, we use $k^3 P(k)$ to express the power as a dimensionless function.

DARK MATTER OVERDENSITY

Here we only focus on the evolution of dark matter overdensity. Instead of using a full set of 8 Boltzman

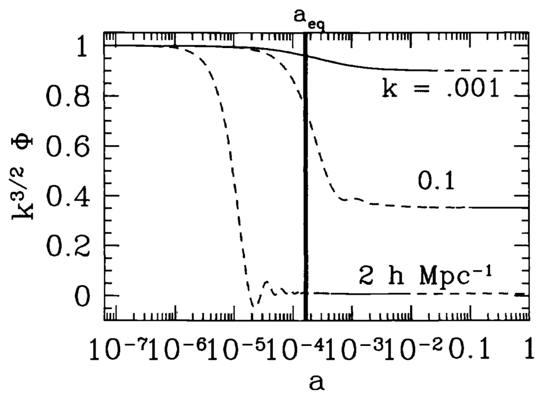

FIG. 1. The linear evolution of the gravitational potential.

equation, we only use five equations.

$$\begin{aligned}
\dot{\Theta}_{r,0} + k\Theta_{r,1} &= -\dot{\Phi} \\
\dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} &= -\frac{k}{3}\Phi \\
\dot{\delta} + ikv &= -3\dot{\Phi} \\
\dot{v} + \frac{\dot{a}}{a}v &= ik\Phi
\end{aligned} \quad (5)$$

Here the subscript $_r$ denotes the zero or first moment of radiation, both photons and neutrinos. Note that it is true in the limit of small baryon density, these two species follow same evolution equations.

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) = 4\pi G a(\rho_{dm}\delta + 4\rho_r\Theta_{r,0}) \quad (6)$$

$$k^2\Phi = 4\pi G a\left[\rho_{dm}\delta + 4\rho_r\Theta_{r,0} + 3\frac{aH}{k}\left(i\rho_{dm}v + 4\rho_r\Theta_{r,1}\right)\right] \quad (7)$$

Here Equation 6 and 7 are equivalent. We use either one together with other 4 equation (with equation number 5) to form a set of 5 equations. The usage of Equation 6 and 7 depends on calculation convenience. The analytic solutions for the dark matter density are difficult. In order to solve them, it is necessary to take some limits to reduce the full set of five equations to two or three. Those solutions correspond to specific limit is valid only for certain time and certain scale. As it is shown in Figure 2, we will analyze four conditions below: super-horizon case, constant Φ case, sub-horizon case and neglect δ case.

Super-Horizon Solution

For modes that are far outside the horizon, $k\eta \ll 1$, and we could drop all terms in the evolution equations

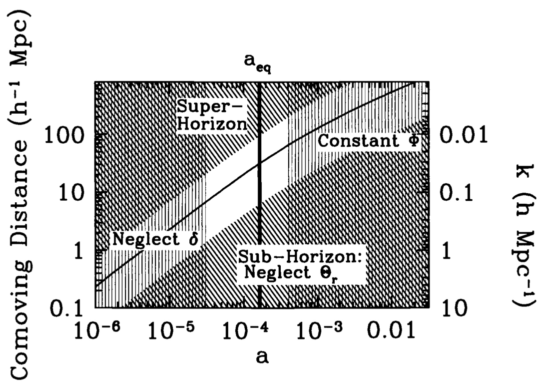

FIG. 2. Physics of the transfer function.

dependent on k. Here we only have

$$\begin{aligned} \dot{\Theta}_{r,0} &= -\dot{\Phi} \\ \dot{\delta} &= -3\dot{\Phi} \\ 3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) &= 4\pi Ga(\rho_{dm}\delta + 4\rho_r\Theta_{r,0}) \end{aligned} \tag{8}$$

The last equation comes from Equation 6. Now we hope to transfer those first order equations to one second order of Φ. The solution for Φ is

$$\Phi = \frac{\Phi(0)}{10}\frac{1}{y^3}(16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16) \tag{9}$$

where $y = a/a_{eq} = \rho_{dm}/\rho$ is introduced as an evolution variable instead of η or a. Here we could find out that, at large y, $\Phi \to (9/10)\Phi(0)$, which indicates the potential on even the largest scales drops by 10% as the universe pass through the epoch of equality.

Through Horizon Crossing

It is worth noting that large-scale potential becomes constant at very late time ($a \geq 10^{-2}$) as the mode crosses the horizon. Radiation is not important in the case. In this case, we have a set of equation

$$\begin{aligned} \dot{\delta} + ikv &= 0 \\ \dot{v} + aHv &= ik\Phi \\ k^2\Phi &= \frac{3}{2}a^2H^2\left[\delta + \frac{3aHiv}{k}\right] \end{aligned} \tag{10}$$

We also need to obtain a second order equation for the solution. Note that we have $\dot{\Phi} = 0$ as initial condition, so the solution must be a constant Φ. After a long algerba we could transform Equation 8 into one second order equation of the form $\alpha\ddot{\Phi} + \beta\dot{\Phi} = 0$, thus it indicates $\Phi =$ const. is a solution of the equation.

Horizon Crossing in Small Scale

When the universe is radiation dominated, the potential is only determined by radiation but not dark matter perturbations. In this case we need to firstly find the solution of $\Theta_{r,0}, \Theta_{r,1}and\Phi$. We have a set of equation

$$\begin{aligned} \dot{\Theta}_{r,0} + k\Theta_{r,1} &= -\dot{\Phi} \\ \dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} &= -\frac{k}{3}\Phi \\ \Phi = \frac{6a^2H^2}{k^2}&\left[\Theta_{r,0} + \frac{3aH}{k}\Theta_{r,1}\right] \end{aligned} \tag{11}$$

The last equation is from Equation 7 after dropping the matter source terms. The solution s a spherical Bessel function. We re-express it in terms of trigonometric functions

$$\Phi = 3\Phi_P(\text{a long trigonometric functions here...})$$

Now we have gravitational potential, and we should determine the evolution of the matter perturbations. Now we should use the last two equations in Equation 5 and thus

$$\ddot{\delta} + \frac{1}{\eta}\dot{\delta} = -3\ddot{\Phi} + k^2\Phi - \frac{3}{\eta}\ddot{\Phi}$$

The solution in this case is

$$\delta(k,\eta) = A\Phi_P\ln(Bk\eta)$$

where A=9.0 and B=0.62 (from Dodelson) or A=9.6 and B=0.44 (from Hu and Sugiyama).

Sub-horizon Evolution

In this case, the radiation pressure causes the gravitational potentials to decay as modes enter the horizon during the radiation era. We have three equations

$$\begin{aligned} \delta' + \frac{ikv}{aHy} &= 3\Phi' \\ v' + \frac{v}{y} &= \frac{ik\Phi}{aHy} \\ k^2\Phi &= \frac{3y}{2(y+1)}a^2H^2\delta \end{aligned} \tag{12}$$

We could simplify Equation 11 as *Meszaros Equation*

$$\delta'' + \frac{2+3y}{2y(y+1)}\delta' - \frac{3}{2y(y+1)}\delta = 0 \tag{13}$$

This equation governs the evolution of sub-horizon CDM perturbations once radiation perturbations have become negligible. The general solution to the *Meszaros Equation* is therefore

$$\delta(k,y) = C_1D_1(y) + C_2D_2(y)$$

for $y \gg y_H$ where y_H is the scale factor when the mode enters the horizon divided by the scale factor at equality. For those modes enter the horizon before equality, we have

$$\begin{aligned} A\Phi_P \ln(By_m/y_H) &= C_1 D_1(y_m) + C_2 D_2(y_m) \\ \frac{A\Phi_P}{y_m} &= C_1 D_1(y_m)' + C_2 D_2'(y_m) \end{aligned} \tag{14}$$

The fitting transfer function (here BBKS) for small-scale is

$$\begin{aligned} T(x \equiv k/k_{eq}) = &\frac{\ln(1+0.171x)}{0.171x}[1+0.284x \\ &+ (1.18x)^2 + (0.399x)^3 + (0.490x)^4]^{-0.25} \end{aligned} \tag{15}$$

Growth Function at late times

At late times all modes have entered the horizon. So the $y \ll 1$ limit for *Meszaros Equation* would apply. The matter overdensity satisfy

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}a^2} + \left(\frac{\mathrm{d}\ln(H)}{\mathrm{d}a} + \frac{3}{a}\right)\frac{\mathrm{d}\delta}{\mathrm{d}a} - \frac{3\Omega_m H_0^2}{2a^5 H^2}\delta = 0$$

Therefore, we could obtain the growth factor

$$D_1(a) = \frac{5\Omega}{2}\frac{H(a)}{H_0}\int_0^a \frac{\mathrm{d}a'}{(a'H(a')/H_0)^3}$$

BEYOND COLD DARK MATTER

In order to obtain an accurate description, we need to focus on three additional components: baryons, massive neutrino and dark energy. If we take baryons into consideration, the power spectrum is supressed on small scales, and there would be small oscillation in transfer function. For the effect from massive neutrino, this kind of neutrino can free-stream out of horizon-scale perturbations at equality, leading to a suppression in power on scale smaller than k_{eq}. When it comes to dark energy, it affects the power spectrum by changing $k_{eq}$and the normalization, and by changing the growth factor at late times.