Cosmology notes - Inflation and Perturbation

INTRODUCTION

The standard model of cosmology is a huge success to explain what the universe like. However, there are still something standard model could not answer, such as horizon problem and curvature problem. Some scientists therefore proposed a theory called inflation to solve these problems, but it has not been proved or belied via astronomical observation. Here we focus on perturbations in inflation. Note that equations with labels are important.

AN EXTRA SCALAR FIELD

Equation of Motion

If we consider an arbitrary free scalar field φ (it is different from inflation field ϕ), the Lagrangain is

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$$

Its Euler-Lagrange equation of motion is

$$\frac{1}{\sqrt{-g}} \partial_\nu (g^{\mu\nu} \sqrt{-g} \partial_\mu \varphi) = 0$$

Under conformal coordinates with background FRW metric expressed as $g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$, we have the free scalar equation of motion

$$\ddot{\varphi} + 2\left(\frac{a'}{a}\right)\varphi' - \nabla^2 \varphi = 0 \tag{1}$$

Our main purpose is to solve this equation. We first Fourier expand the field into momentum states φ_k

$$\varphi(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} [\varphi_\mathbf{k}(\tau) b_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} + \varphi^*(\tau) b^*_\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{x}}]$$

Here the coordinate $\mathbf{x}$ are comoving coordinates, and the wavevector $\mathbf{k}$ is a comoving wavevector. The equation of motion for a single mode $\varphi_\mathbf{k}$ is

$$\varphi''_\mathbf{k} + 2\left(\frac{a'}{a}\right)\varphi'_\mathbf{k} + k^2 \varphi_\mathbf{k} = 0$$

It is convenient to introduce a field redefinition

$$u_k \equiv a(\tau)\varphi_\mathbf{k}(\tau)$$

and u_k obeys Klein-Gordon equation to an expanding spacetime

$$u''_k + \left(k^2 - \frac{a''}{a}\right) u_k = 0 \tag{2}$$

Ultraviolet and Infrared limits

Any mode with fixed wavenumber k redshifts with times, so early time corresponds to a physical short wavelength while later time corresponds to a physical infrared wavelength. The solution of equation 2 is different in these two cases.

- Short Wavelength limit with $k \gg a''/a$. In this case, we have an equation for a conformally Minkowski Klein-Gorden field,

$$u''_k + k^2 u_k = 0$$

  with its solution in conformal time and comoving wavenumber

$$u_k(\tau) = \frac{1}{\sqrt{2k}} \left(A_k e^{-ik\tau} + B_k e^{ik\tau}\right) \tag{3}$$

- Long wavelength limit with $k \ll a''/a$. Here we have

$$a'' u_k = a u''_k$$

  with its solution

$$u_k \propto a \Rightarrow \varphi_k = \text{const.} \tag{4}$$

  This is so-called *mode freezing* phenomenon. In this case, field modes φ_k longer than the horizon size asymptote to a constant, *nonzero* amplitude. The amplotude of the field at long wavelength is determined by the integration constants A_k and B_k.

BOUNDARY CONDITION

In order to determine integration constants A_k and B_k in 3, we need to consider the boundary condition in ultraviolet limit, where the field becomes approximately Minkowskian. Therefore, we firstly need to discuss the quantization of Minkowski Space. Firstly, we determine the boundary conditions for the mode function via canonical quantization. We promote the Fourier coefficients to annihilation and creation operators.

$$b_\mathbf{k} \rightarrow \hat{b_\mathbf{k}}, b^*_\mathbf{k} \rightarrow \hat{b^\dagger_\mathbf{k}} \text{ with } \left[\hat{b_\mathbf{k}}, \hat{b^\dagger_\mathbf{k}}\right] \equiv \delta^3(\mathbf{k} - \mathbf{k}')$$

Therefore, the quantum field φ is given by

$$\varphi(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} [\varphi_\mathbf{k}(\tau) b_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} + \text{H.C.}]$$

and the canonical momentum is

$$\Pi(\tau, \mathbf{x}) \equiv \frac{\delta \mathcal{L}}{\delta(\partial_0 \varphi)} = a^2(\tau) \frac{\partial \varphi}{\partial \tau}$$

For ultraviolet mode function 3, the commutation relation

$$[\varphi(\tau,\mathbf{x}),\Pi(\tau,\mathbf{x}')] = i\delta^3(\mathbf{x}-\mathbf{x}')$$

corresponds to

$$u_k\frac{\partial u_k^*}{\partial\tau} - u_k^*\frac{\partial u_k}{\partial\tau} = i$$

which result in a condition on the integration constants

$$|A_k|^2 - |B_k|^2 = 1 \tag{5}$$

Now we need a second boundary condition from vaccum selection, or definition of which state correspond to a zero-particle state for the system, to help us solve the mode equation. Here we use so-called *Bunch-Davies* vacuum

$$u_k(\tau) \propto e^{-ik\tau} \Rightarrow A_k = 1, B_k = 0 \tag{6}$$

since an FRW spacetime is asymptotically Minkowski in the ultraviolet limit.

SOLUTION AND PRIMORDIAL POWER SPECTRUM

With two boundary condition, we can proceed to the solution of equation 2. Firstly we need to re-express the conformal time via equation of state parameter $\epsilon =$ const..

$$\tau = -\left(\frac{1}{aH}\right)\left(\frac{1}{1-\epsilon}\right)$$

Also, we have Friedmann equation

$$\frac{a''}{a} = a^2H^2(2-\epsilon)$$

Therefore, we obtain

$$\tau^2(1-\epsilon)^2u_k'' + \left[(k\tau)^2(1-\epsilon)^2 - (2-\epsilon)\right]u_k = 0 \tag{7}$$

from equation 2. This Bessel equation has solution

$$u_k \propto \sqrt{-k\tau}[J_\nu(-k\tau) \pm iY_\nu(-k\tau)] \tag{8}$$

where $\nu = (3-\epsilon)/(2-2\epsilon)$.

The power spectrum when $k = aH$ is

$$P^{1/2}(k) = \left(\frac{H}{2\pi}\right)_{k=aH} \simeq \left(\sqrt{\frac{2V(\phi(N))}{3\pi m_{PI}^2}}\right)$$

There are two examples of free scalar in inflation. We could constrain the theory of inflation from observation via these two kind of scalar field.

- Gravitational wave mode. The power spectrum in this case is

$$P_T = \frac{16H^2}{\pi m_{PI}^2} \propto k^{n_T} \tag{9}$$

with $n_T = -2\epsilon$

- Perturbations in the density of the universe. This type of perturbation is the dominant component of the CMB anisotrophy and responsible for structure formation. Perturbation in the inflation field $\delta \simeq H/2\pi$ generate density perturbations with power spectrum

$$P_{\mathcal{R}}(k) = \frac{H^2}{\pi m_{PI}^2\epsilon} \propto k^{n_S-1} \tag{10}$$

Here $n_S - 1 \simeq -4\epsilon + 2\eta$ where η is the second slow-roll parameter.

With four observables $P_T, P_{\mathcal{R}}, n_T, n_S$, we also have so-called *consistency condition* for single-field slow-roll inflation

$$r \equiv \frac{P_T}{P_S} = 16\epsilon = -8n_T \tag{11}$$

This relation is testable by accurate measurement of primordial power spectrum, thus giving a strong constrain to the model.