# Cosmology notes - Inflation

## INTRODUCTION

The standard model of cosmology is a huge success to explain what the universe like. However, the conventional Big Bang theory requires a set of initial conditions to allow the universe to evolve to its current state. Some scientists therefore proposed a phenomenon called inflation to solve current problems. Inflation is a very unfamiliar physical phenomenon. Within a fraction a second the universe grew exponential at an accelerating rate, which in Einstein gravity a negative pressure or equivalently a nearly constant energy density is required. Here we try to describe why this will happen. Note that equations with labels are important.

## CURRENT PROBLEM

Some existing problems trigger the discovery of inflation. We will introduce horizon problem and curvature problem here.

### Horizon Problem

We define the *comoving horizon* as the maximum distance a light ray can travel bewteen time 0 and time t:

$$\tau = \int_0^t \frac{\mathrm{d}t'}{a(t')} = \int_0^a \mathrm{d}\ln a \frac{1}{aH} \tag{1}$$

The comoving horizon grows monotonically with time. Current CMB observation implies a near homogeneity temperature at each spatial point at the time of last-scattering. However, we could see from Figure 1, at $z \sim 1100$ not all spatial points have overlapping past light cone, namely, they are casually connected before last-scattering. Therefore, it seems impossible that CMB has a homogeneous temperature in Big Bang universe. This is horizon problem. Here we also introduce *comoving Hubble radius* $(aH)^{-1}$ which would indicates the maximum physical distance ($c\mathrm{d}t = \mathrm{d}t = (aH)^{-1}\mathrm{d}a$) of two causal connected spatial point *now* (namely we use $\mathrm{d}t$ here).

### Curvature Problem

To consider the problem, we consider and re-express Friedman Equation

$$1 - \Omega(a) = -\frac{k}{aH} \tag{2}$$

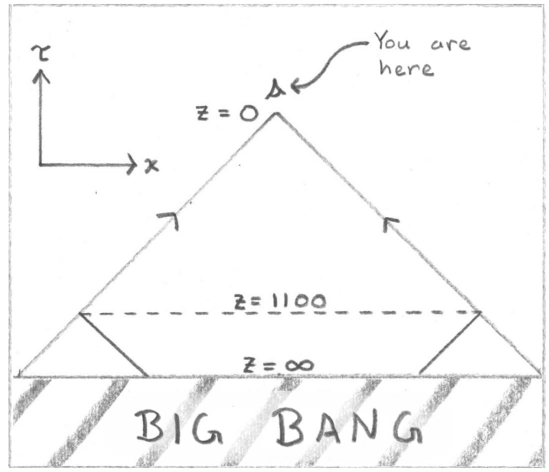


FIG. 1. The comoving diagram of Big Bang universe.

Also from observation we know $\Omega(a) = 1$, indicating that we live in a flat universe now. However, we could obtain

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\ln a} = (1+3w)\Omega(\Omega-1) > 0$$

from Equation 2. Therefore, the Big Bang theory shows us $\Omega$ would be less than 1 if we have $\Omega = 1$ today. Why we have $\Omega = 1$ in current universe? This is so-called flatness problem.

## INFLATION

### An alternative Solution

In order to solve the horizon and flatness problem, some proposed that the universe experienced a period of time when the Hubble radius decreased, or the scalar factor $a$ increased exponentially. Now we simply solve these two problems with the proposal. As for horizon problem, we could see from Figure 2 that those spacelike point at $z \sim 1100$ would have a overlapping past in inflationary universe. Therefore, it is obvious that they have same temperature because they were causal connected before. As for flatness problem, if we have a decreasing Hubble radius during inflation, it would drive the universe toward flatness according to

$$|1 - \Omega(a)| = \frac{1}{(aH)^2}$$

Therefore, an inflationary universe would lead to the homogeneity of CMB as well as the flatness of our universe.

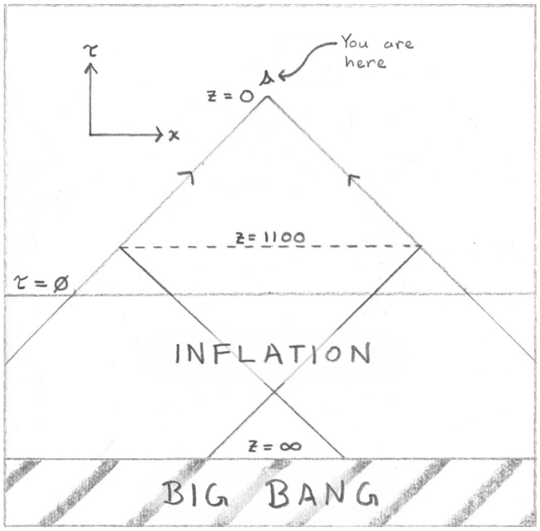


FIG. 2. The comoving diagram of inflationary universe.

### Conditions for Inflation

Now we try to derive some basic propeties of inflation. According to the statement above, the Hubble radius would be shrunk, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) < 0 \tag{3}$$

This is a fundamental definition of inflation. We could also obtain from Equation 3

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} > 0$$

Hence, it would be an accelerating expansion during inflation. We introduce a new parameter $\varepsilon$ defined by

$$\varepsilon = -\frac{\ddot{H}}{H^2} = -\frac{\mathrm{d}\ln H}{\mathrm{d}\ln a} \tag{4}$$

where $\varepsilon < 1$ during accelarating expansion.

### Single-field Inflation

Here we consider a simple model of inflation. Some believe that inflation was caused by a scalar field $\phi$. The Lagrangain of $\phi$ coupled with gravity is

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] = S_{\mathrm{EH}} + S_\phi \tag{5}$$

Here $S_{\mathrm{EH}}$ is the sum of the gravitational Einstein-Hilbert action, $S_\phi$ is the action of a scalar field with canonical kinetic term, and $V(\phi)$ describes the self-interactions of the scalar field.

If we assume a FRW metric for $g_{\mu\nu}$ and a homogeneous field $\phi(t, \mathbf{x}) \equiv \phi(t)$, we could obtain the scalar energy-momentum tensor takes the form of a perfect fluid with

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned} \tag{6}$$

The equation of state is

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 + V(\phi)}{\frac{1}{2}\dot{\phi}^2 - V(\phi)} \tag{7}$$

It indicates that the existence of scalar field $\phi$ can lead to a negative pressure ($w_\phi < 0$) and accelerated expansion ($w_\phi < -1/3$). The dynamics of $\phi$ is determined by

$$\begin{aligned} \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} &= 0 \\ H^2 &= \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \end{aligned} \tag{8}$$

## SLOW-ROLL INFLATION

Now we consider the single field inflation under *slow roll* limit: $p_\varphi \simeq \rho_\varphi$ in de Sitter spacetime. In this case the potential of $\varphi$ is dominant, namely $V(\varphi) \gg \dot{\phi}^2$. Here we will also adopt an additional approximation that the friction term dominant, namely, $|\ddot{\phi}| \ll |3H\dot{\phi}|, |V_{,\phi}|$. The universe expands quasi-exponentially under this circumstance:

$$a(t) \equiv e^{Ht} \tag{9}$$

To characterize the inflation, we will use two slow roll parameters defined as below

$$\begin{aligned} \varepsilon &= -\frac{\dot{H}}{H^2} = -\frac{\mathrm{d}\ln H}{\mathrm{d}N} \\ \eta &= -\frac{\ddot{\phi}}{H\dot{\phi}} = \varepsilon - \frac{1}{2\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}N} \end{aligned} \tag{10}$$

The first parameter $\varepsilon$ relates to the evolution of Hubble parameter, and $|\eta| < 1$ ensures the friction dominant condition. The slow roll condition can be also expressed as conditions on the shape of the inflationary potential

$$\begin{aligned} \epsilon_v(\phi) &\equiv \frac{M_{Pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \\ \eta_v &\equiv M_{Pl}^2\frac{V_{,\phi\phi}}{V} \ll 1 \end{aligned} \tag{11}$$

Here we use Planck mass to make these new parameter dimensionless. These two kinds of slow-roll parameters are related as $\varepsilon \approx \epsilon_v$ and $\eta \approx \eta_v - \epsilon_v$. Under slow roll condition we have

$$\begin{aligned} H^2 &\simeq \frac{1}{3}V(\phi) \\ \dot{\phi} &\simeq -\frac{V_{,\phi}}{3H} \end{aligned} \tag{12}$$

Inflation ends when the slow-roll conditions are violated, namely, $\varepsilon(\phi_{\rm end}) \equiv 1$ thus $\epsilon_v(\phi_{\rm end}) \approx 1$. We could define the number of e-fold $\mathrm{d}N \equiv H\mathrm{d}t$ to evaluate the extent of inflation. The number of e-folds before inflation ends is

$$\begin{aligned} N(\phi) &\equiv \ln \frac{a_{end}}{a} \\ &= \int_{\phi_{\rm end}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\varepsilon}} \approx \int_{\phi_{\rm end}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon_v}} \end{aligned} \tag{13}$$

To solve the flatness and horizon problem, the total number of inflationary e-folds would be greater than 60. However, it requires that $N_{\rm CMB} \approx 40 - 60$ from CMB fluctuation. The precise value of $N_{CMB}$ depends on post-inflationary history and reheating of the universe.