Cosmology notes - Boltzman Equation

INTRODUCTION

People are interested in the distribution of different component in the universe: photons, baryon matter, dark matter and even neutrinos. These different components also interact with each other, making the analysis of there distribution more complicated. Note that equations with labels are important. In general, we use Boltzman equation:

$$\frac{\mathrm{d}f(x,p,t)}{\mathrm{d}t} = C[f] \tag{1}$$

where f denotes distribution function to describe the out-of-equilibrium phenomena of photons, baryons and dark matter. $C[f]$ represents collision among different components. Usually we will re-express the total time derivative of time in terms of partial deviatives so that we obtain

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\cdot\frac{\partial x^i}{\partial t} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial f}{\partial \hat{p}^i}\cdot\frac{\partial \hat{p}^i}{\partial t} \tag{2}$$

We need to derive specific equations usable in particular cases for our purpose. In the next few sections we will show Boltzman equation for photons, dark matter and baryon matter.

BOLTZMAN EQUATION FOR PHOTONS

Collisionless Boltzman Equation

To obtain a collisionless Boltzman equation, we firstly need to simplify equation 2. Here we use a perturbed metric written as:

$$\mathrm{d}s^2 = (-1-2\Psi)\mathrm{d}x_0^2 + [a^2\delta_{ij}(1+2\Phi)]\mathrm{d}x_i\mathrm{d}x_j \tag{3}$$

Before we proceed, we also need to derive a useful relation. The masslessness of photons implies that

$$P^2 = g_{\mu\nu}P^\mu P^\nu = -(1+2\Psi)(P^0)^2 + p^2 = 0$$

with $P^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ and $p^2 \equiv g_{ij}P^iP^j$. Therefore we have for P^0:

$$P^0 = p(1-\Psi) \tag{4}$$

When it comes to P^i, if $P^i \equiv C\hat{p}^i$ then $p^2 = a^2(1+2\Phi)C^2$, so $C = p(1-\Phi)/a$ and for P^i:

$$P^i = \frac{p\hat{p}^i(1-\Phi)}{a} \tag{5}$$

Now we start to generate Boltzman equation for photons. Firstly, we claim that the last term of equation 2 should be neglected. Since $\partial f/\partial \hat{p}^i$ and $\partial \hat{p}^i/\partial t$ are both first-order term, the last term is the product of two first-order term, namely a second-order term. We can ignore second-order term in general as approximation.

We then turn to $\mathrm{d}x^i/\mathrm{d}t$ in second term of equation 2. It can be re-expressed by recalling $P^0 = \mathrm{d}t/\mathrm{d}\lambda$ and $P^i = \mathrm{d}x^i/\mathrm{d}\lambda$

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{P^i}{P^0}$$

Thus,

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\hat{p}}{a} \tag{6}$$

where we insert equation 4 and ignore Φ and Ψ for first order equation.

Lastly we focus on $\mathrm{d}p/\mathrm{d}t$ of the third term of equation 2. To begin, we should use geodesic equation

$$\frac{\mathrm{d}P^0}{\mathrm{d}\lambda} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta$$

and equation 4, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[p(1-\Psi)] = -\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+\Psi)$$

Now we multiply both sides by $(1+\Psi)$; drop all terms quadratic in Ψ; and re-express the total time derivative of Ψ in terms of partial derivatives so that

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\left\{\frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right\} - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi)$$

We now need to evaluate the product $\Gamma^0_{\alpha\beta}P^\alpha P^\beta/p$. Recall the Christoffel symbol is best written as a sum of derivatives of the metric, we will have first order form after a long algebra with the help of equation 5

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = -H - \frac{\partial \Phi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} \tag{7}$$

Eventually we have Boltzman equation of collisionless photon

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right] = 0 \tag{8}$$

Working through the terms on the right, the first two are familiar from standard hydrodynamics. The third term dictates that photons lose energy in an expanding universe. Finally, the last two encode the effects of under-/overdense regions on the photon distribution function.

Zero- and first- order Boltzman Equation

Now we need to use the expansion of distribution function for further analysis. We could expand distribution function f about its zero-order Bose-Einstein value. Thus we have

$$f(\vec{x}, p, \hat{p}, t) = \left[\exp\left\{\frac{p}{T(t)[1+\Theta(\vec{x}, \hat{p}, t)]} - 1\right\}\right] \tag{9}$$

Since the perturbation Θ is small, so we can expand

$$f \simeq f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta \tag{10}$$

If we insert equation 10 into equation 8, we have zero-order equation

$$\left[-\frac{\mathrm{d}T/\mathrm{d}t}{T} - \frac{\mathrm{d}a/\mathrm{d}t}{a}\right]\frac{\partial f^{(0)}}{\partial p} = 0 \tag{11}$$

and first-order one

$$-p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right] = 0 \tag{12}$$

Collision term for Compton scattering

To fully describe photons behaviors, we need to take Compton scattering into consideration. Compton scatter process of our interest is

$$e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q'}) + \gamma(\vec{p'})$$

We are interested in the change of distribution of photons with momentum $\vec{p}$. Therefore we must sum over all other momenta which affect $f(\vec{p})$. Schematically, the collision term is

$$C[f(\vec{p})] = \sum_{\vec{p}, \vec{q'}, \vec{p'}} |Amplitude|^2 \left\{f_e(\vec{q'})f(\vec{p'}) - f_e(\vec{q})f(\vec{p})\right\}$$

If we assume the amplitude is constant, $|\mathcal{M}| = 8\pi\sigma_T m_e^2$ where σ_T is the Thomson cross-section, we could derive collision term keeping only first order. It is necessary to use

$$\Theta_0(\vec{x}, t) \equiv \frac{1}{4\pi}\int \mathrm{d}\Omega' \Theta(\hat{p}', \vec{x}, t) \tag{13}$$

in order to integrate over solid angle Ω'. This is so-called monopole term and it does not depend on the direction vector. After a long algebra and we finally have

$$C[f(\vec{p})] = -p\frac{\partial f^{(0)}}{\partial p} n_e \sigma_T [\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v_b}] \tag{14}$$

This is the collision term represent the effect of Compton scattering.

Boltzman Equation for Photons in Fourier Form

Now we have Boltzman equation for photons in terms of conformal time

$$\dot{\Theta} + \hat{p}^i\frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \hat{p}^i\frac{\partial \Psi}{\partial x^i} = n_e\sigma_T a[\Theta_0 - \Theta + \hat{p}\cdot\vec{v_b}] \tag{15}$$

If we transfer the equation to Fourier Form, the differential equation is much simpler to solve. In general, an equation of the form $aA(\vec{x}) = bB(\vec{x})$ gets transformed into $a\tilde{A}(\vec{k}) = b\tilde{B}(\vec{k})$. Besides, after Fourier transform we have $\partial/\partial x^i \to k_i (\equiv k^i)$. If we also have

$$\mu \equiv \frac{\vec{k}\cdot\hat{p}}{k}, \tau(\eta) \equiv \int_\eta^{\eta_0} \mathrm{d}\eta' n_e \sigma T a$$

we could obtain Boltzman equation in Fourier form

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\dot{\tilde{\Psi}} = -\dot{\tau}\left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_b\right] \tag{16}$$

BOLTZMAN EQUATION FOR COLD DARK MATTER

We assume cold dark matter particles will not interact with other component, namely it is collisionless. So we do not need to consider the collision term. We first need to derive constraint condition similar with those of photons. We have

$$P^\mu = \left[E(1-\Psi), p\hat{p}^i\frac{1-\Phi}{a}\right] \tag{17}$$

Here we also start from the total time derivative of distribution function f_{dm}

$$\frac{\mathrm{d}f_{\mathrm{dm}}}{\mathrm{d}t} = \frac{\partial f_{\mathrm{dm}}}{\partial t} + \frac{\partial f_{\mathrm{dm}}}{\partial x^i}\cdot\frac{\partial x^i}{\partial t} + \frac{\partial f_{\mathrm{dm}}}{\partial E}\frac{\partial E}{\partial t} + \frac{\partial f_{\mathrm{dm}}}{\partial \hat{p}^i}\cdot\frac{\partial \hat{p}^i}{\partial t}$$

For non-relativistic matter:

$$\frac{\partial f_{\mathrm{dm}}}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f_{\mathrm{dm}}}{\partial x^i} - \frac{\partial f_{\mathrm{dm}}}{\partial E}\left[\frac{\mathrm{d}a/\mathrm{d}t}{a}\frac{p^2}{E} + \frac{p^2}{E}\frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i p}{a}\frac{\partial \Psi}{\partial x^i}\right] = 0 \tag{18}$$

We could use the definition of dark matter density and the velocity to simplify the equation above.

$$n_{dm} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} f_{dm}$$

$$v^i \equiv \frac{1}{n_{dm}}\int \frac{\mathrm{d}^3p}{(2\pi)^3} f_{dm}\frac{p\hat{p}^i}{E}$$

Therefore we have

$$\frac{\partial n_{dm}}{\partial t} + \frac{1}{a}\frac{\partial(n_{dm}v^i)}{\partial x^i} + 3\left[\frac{\mathrm{d}a/\mathrm{d}t}{a} + \frac{\partial \Phi}{\partial t}\right]n_{dm} = 0 \tag{19}$$

- Zero-order equation

$$\frac{\partial n_{dm}^{(0)}}{\partial t} + 3\frac{\mathrm{d}a/\mathrm{d}t}{a} n_{dm}^{(0)} = 0 \tag{20}$$

where $n_{dm}^{(0)}$ is zero-order, homogeneous part of the density.

- Two first-order equation after Fourier transform

$$\dot{\tilde{\delta}} + ik\tilde{v} + 3\dot{\tilde{\Phi}} = 0 \tag{21}$$

$$\dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} + ik\tilde{\Psi} = 0 \tag{22}$$

BOLTZMAN EQUATION FOR BARYON

We finally need a set of Boltzman equation for electrons and protons. These two components are coupled by Coulomb scattering ($e + p \to e + p$). With such a tight coupling force we have

$$\frac{\rho_e - \rho_e^{(0)}}{\rho_e^{(0)}} = \frac{\rho_p - \rho_p^{(0)}}{\rho_p^{(0)}} \equiv \delta_b$$

Similarly the velocities of two species are also a common value:

$$\vec{v}_e = \vec{v}_p \equiv \vec{v}_b$$

We start from the original form of Boltzman equation for electrons and protons

$$\frac{\mathrm{d}f_e(\vec{x}, \vec{q}, t)}{\mathrm{d}t} = c_{ep} + c_{e\gamma} \tag{23}$$

$$\frac{\mathrm{d}f_p(\vec{x}, \vec{Q}, t)}{\mathrm{d}t} = c_{ep} \tag{24}$$

where c_{ep} denotes the interaction between electrons and protons, and $c_{e\gamma}$ for interaction bewteen electrons and photons via Compton scattering. We could derive Boltzman equation for baryon in a similar way with those for cold dark matter, but we still need to take the collision term into consideration. For first-order equation of matter perturbation, two collision terms in equation 23 vanish because the integration of the first term on the righ is symmetry (we only need to consider this term for the equation of matter). Therefore we have

$$\dot{\tilde{\delta}}_b + ik\tilde{v}_b + 3\dot{\tilde{\Phi}} = 0 \tag{25}$$

For the equation of velocity, the collision term would not vanish. Therefore we have

$$\dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b + ik\tilde{\Psi} = \dot{\tau}\frac{4\rho_\gamma}{3\rho_b}\left[3i\tilde{\Theta}_1 + \tilde{v}_b\right] \tag{26}$$

where we have

$$\Theta_1 \equiv i\int_{-1}^{1}\frac{\mathrm{d}\mu}{2}\mu\Theta\mu$$