Cosmology notes - Inflation

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INTRODUCTION

The standard model of cosmology is a huge success to explain what the universe like. However, the conventional Big Bang theory requires a set of initial conditions to allow the universe to evolve to its current state. Some scientists therefore proposed a phenomenon called inflation to solve current problems. Inflation is a very unfamiliar physical phenomenon. Within a fraction a second the universe grew exponential at an accelerating rate, which in Einstein gravity a negative pressure or equivalently a nearly constant energy density is required. Here we try to describe why this will happen. Note that equations with labels are important.

CURRENT PROBLEM

Some existing problems trigger the discovery of inflation. We will introduce horizon problem and curvature problem here.

Horizon Problem

We define the *comoving horizon* as the maximum distance a light ray can travel bewteen time 0 and time t:

$$\tau = \int_0^t \frac{\mathrm{d}t'}{a(t')} = \int_0^a \mathrm{d}\ln a \frac{1}{aH} \tag{1}$$

The comoving horizon grows monotonically with time. Current CMB observation implies a near homogeneity temperature at each spatial point at the time of last-scattering. However, we could see from Figure 1, at $z \sim 1100$ not all spatial points have overlapping past light cone, namely, they are casually connected before last-scattering. Therefore, it seems impossible that CMB has a homogeneous temperature in Big Bang universe. This is horizon problem. Here we also introduce *comoving Hubble radius* $(aH)^{-1}$ which would indicates the maximum physical distance ($c\mathrm{d}t = \mathrm{d}t = (aH)^{-1}\mathrm{d}a$) of two causal connected spatial point *now* (namely we use $\mathrm{d}t$ here).

Curvature Problem

To consider the problem, we consider and re-express Friedman Equation

$$1 - \Omega(a) = -\frac{k}{aH} \tag{2}$$

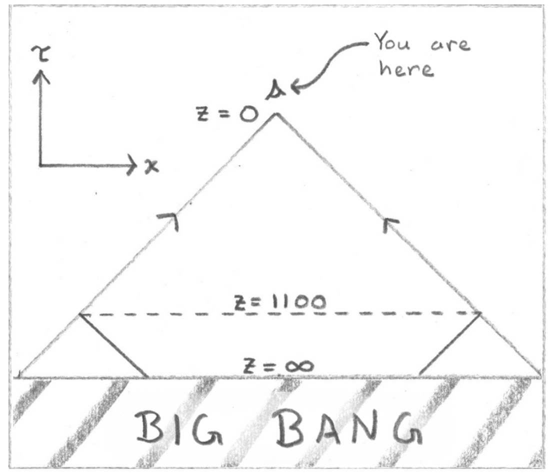

FIG. 1. The comoving diagram of Big Bang universe.

Also from observation we know $\Omega(a) = 1$, indicating that we live in a flat universe now. However, we could obtain

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\ln a} = (1 + 3w)\Omega(\Omega - 1) > 0$$

from Equation 2. Therefore, the Big Bang theory shows us Ω would be less than 1 if we have $\Omega = 1$ today. Why we have $\Omega = 1$ in current universe? This is so-called flatness problem.

INFLATION

An alternative Solution

In order to solve the horizon and flatness problem, some proposed that the universe experienced a period of time when the Hubble radius decreased, or the scalar factor a increased exponentially. Now we simply solve these two problems with the proposal. As for horizon problem, we could see from Figure 2 that those spacelike point at $z \sim 1100$ would have a overlapping past in inflationary universe. Therefore, it is obvious that they have same temperature because they were causal connected before. As for flatness problem, if we have a decreasing Hubble radius during inflation, it would drive the universe toward flatness according to

$$|1 - \Omega(a)| = \frac{1}{(aH)^2}$$

Therefore, an inflationary universe would lead to the homogeneity of CMB as well as the flatness of our universe.

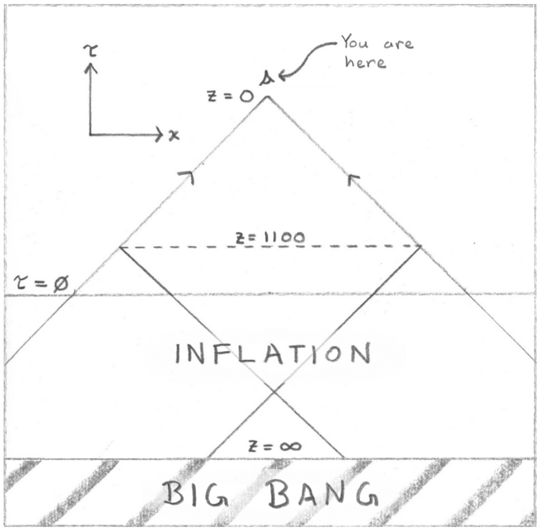

FIG. 2. The comoving diagram of inflationary universe.

Conditions for Inflation

Now we try to derive some basic propeties of inflation. According to the statement above, the Hubble radius would be shrunk, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) < 0 \tag{3}$$

This is a fundamental definition of inflation. We could also obtain from Equation 3

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} > 0$$

Hence, it would be an accelerating expansion during inflation. We introduce a new parameter ε defined by

$$\varepsilon = -\frac{\ddot{H}}{H^2} = -\frac{\mathrm{d}\ln H}{\mathrm{d}\ln a} \tag{4}$$

where $\varepsilon < 1$ during accelarating expansion.

Single-field Inflation

Here we consider a simple model of inflation. Some believe that inflation was caused by a scalar field ϕ. The Lagrangain of ϕ coupled with gravity is

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] = S_{\mathrm{EH}} + S_\phi \tag{5}$$

Here S_{EH} is the sum of the gravitational Einstein-Hilbert action, S_ϕ is the action of a scalar field with canonical kinetic term, and $V(\phi)$ describes the self-interactions of the scalar field.

If we assume a FRW metric for $g_{\mu\nu}$ and a homogeneous field $\phi(t,\mathbf{x}) \equiv \phi(t)$, we could obtain the scalar energy-momentum tensor takes the form of a perfect fluid with

$$\begin{aligned}\rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi)\\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi)\end{aligned} \tag{6}$$

The equation of state is

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 + V(\phi)}{\frac{1}{2}\dot{\phi}^2 - V(\phi)} \tag{7}$$

It indicates that the existence of scalar field ϕ can lead to a negative pressure ($w_\phi < 0$) and accelerated expansion ($w_\phi < -1/3$). The dynamics of ϕ is determined by

$$\begin{aligned}\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0\\ H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\end{aligned} \tag{8}$$

SLOW-ROLL INFLATION

Now we consider the single field inflation under *slow roll* limit: $p_\varphi \simeq \rho_\varphi$ in de Sitter spacetime. In this case the potential of φ is dominant, namely $V(\varphi) \gg \dot{\phi}^2$. Here we will also adopt an additional approximation that the friction term dominant, namely, $|\ddot{\phi}| \ll |3H\dot{\phi}|, |V_{,\phi}|$. The universe expands quasi-exponentially under this circumstance:

$$a(t) \equiv e^{Ht} \tag{9}$$

To characterize the inflation, we will use two slow roll parameters defined as below

$$\begin{aligned}\varepsilon &= -\frac{\dot{H}}{H^2} = -\frac{\mathrm{d}\ln H}{\mathrm{d}N}\\ \eta &= -\frac{\ddot{\phi}}{H\dot{\phi}} = \varepsilon - \frac{1}{2\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}N}\end{aligned} \tag{10}$$

The first parameter ε relates to the evolution of Hubble parameter, and $|\eta| < 1$ ensures the friction dominant condition. The slow roll condition can be also expressed as conditions on the shape of the inflationary potential

$$\begin{aligned}\epsilon_v(\phi) &\equiv \frac{M_{Pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2\\ \eta_v &\equiv M_{Pl}^2\frac{V_{,\phi\phi}}{V} \ll 1\end{aligned} \tag{11}$$

Here we use Planck mass to make these new parameter dimensionless. These two kinds of slow-roll parameters are related as $\varepsilon \approx \epsilon_v$ and $\eta \approx \eta_v - \epsilon_v$. Under slow roll condition we have

$$\begin{aligned}H^2 &\simeq \frac{1}{3}V(\phi)\\ \dot{\phi} &\simeq -\frac{V_{,\phi}}{3H}\end{aligned} \tag{12}$$

Inflation ends when the slow-roll conditions are violated, namely, $\varepsilon(\phi_{\rm end}) \equiv 1$ thus $\epsilon_v(\phi_{\rm end}) \approx 1$. We could define the number of e-fold $\mathrm{d}N \equiv H\mathrm{d}t$ to evaluate the extent of inflation. The number of e-folds before inflation ends is

$$\begin{aligned} N(\phi) &\equiv \ln \frac{a_{end}}{a} \\ &= \int_{\phi_{\rm end}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\varepsilon}} \approx \int_{\phi_{\rm end}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon_v}} \end{aligned} \tag{13}$$

To solve the flatness and horizon problem, the total number of inflationary e-folds would be greater than 60. However, it requires that $N_{\rm CMB} \approx 40 - 60$ from CMB fluctuation. The precise value of N_{CMB} depends on post-inflationary history and reheating of the universe.