UAVBook Supplement
Dropping an Object on a Target

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This supplement will explore a simple technique for dropping an object on a target located at a known inertial position. The wind will have a strong effect on the dropped object and so the first step is to estimate the wind.

1 Wind Estimation

Equation (2.9) gives the wind triangle as

$$V_g \begin{pmatrix} \cos\chi\cos\gamma \\ \sin\chi\cos\gamma \\ -\sin\gamma \end{pmatrix} - \begin{pmatrix} w_n \\ w_e \\ w_d \end{pmatrix} = V_a \begin{pmatrix} \cos\psi\cos\gamma_a \\ \sin\psi\cos\gamma_a \\ -\sin\gamma_a \end{pmatrix},$$

where V_g is the ground speed, χ is the course angle, γ is the flight path angle, V_a is the airspeed, ψ is the heading angle, γ_a is the air mass referenced flight path angle, and $(w_n, w_e, w_d)^\top$ is the wind vector given in NED coordinates. The ground speed V_g and the course angle χ can be measured by GPS. The airspeed V_a is measured by the pitot tube, and the heading angle ψ is measured by the magnetometer. Since the air mass referenced flight path angle cannot be measured directly, we will only measure the north and east components of the wind. Accordingly, setting $\gamma = \gamma_a = 0$ we get

$$\begin{pmatrix} w_n \\ w_e \end{pmatrix} = V_g \begin{pmatrix} \cos\chi\cos\gamma \\ \sin\chi\cos\gamma \end{pmatrix} - V_a \begin{pmatrix} \cos\psi\cos\gamma_a \\ \sin\psi\cos\gamma_a \end{pmatrix}$$

$$= \begin{pmatrix} V_g\cos\chi - V_a\cos\psi \\ V_g\sin\chi - V_a\sin\psi \end{pmatrix}. \tag{1}$$

The wind can be estimated by averaging multiple samples of the vector on the right hand side of Equation (1).

2 Equations of Motion

Let $\mathbf{p} \in \mathbb{R}^3$ be the inertial position of the dropped object in NED coordinates, and let $\mathbf{v} \in \mathbb{R}^3$ be the velocity of the object relative to the air mass, also expressed in NED coordinates. Let $\mathbf{w} \in \mathbf{R}^3$ be the wind velocity, or the velocity of the air mass, expressed in NED coordinates, where we assume that $w_d = 0$ as per the previous section. The kinematics of the dropped object are given by

$$\dot{\mathbf{p}}(t) = \mathbf{v}(t) + \mathbf{w}.$$

Solving the differential equation from time t_0 to time t_1 gives

$$\mathbf{p}(t_1) = \mathbf{p}(t_0) + \int_{t_0}^{t_1} \mathbf{v}(\tau)d\tau + (t_1 - t_0)\mathbf{w}.$$

Define

$$\mathbf{z}(t) \triangleq \int_{t_0}^{t} \mathbf{v}(\tau)d\tau, \tag{2}$$

and solve for the initial position to get

$$\mathbf{p}(t_0) = \mathbf{p}(t_1) - \mathbf{z}(t_1) - (t_1 - t_0)\mathbf{w}. \tag{3}$$

Equation (3) gives the position $\mathbf{p}(t_0)$ where the object should be dropped in order for it to be at position $\mathbf{p}(t_1)$ at time $t_1 - t_0$ into the future, given the known wind vector $\mathbf{w}$. In the following paragraphs we will describe how to obtain $\mathbf{z}(t)$ and the time t_1.

The dynamics of the dropped object relative to the air mass are given by Newton's law as

$$\mathsf{m}\dot{\mathbf{v}} = \mathbf{F}, \qquad \mathbf{v}(t_0) = \mathbf{v}_0$$

where m is the mass of the object, and $\mathbf{F}$ is the total force acting on the object, and where $\mathbf{v_0} \in \mathbb{R}^3$ is the initial velocity at the time the object is dropped. If we ignore wind gusts, then there are two forces that act on the object. The first is gravity which acts in the down direction, and the second is aerodynamic drag, which acts opposite to the direction of the velocity vector. Therefore, the total force is given by

$$\mathbf{F} = \mathsf{m}g\hat{\mathbf{k}} - \frac{1}{2}\rho S C_d \|\mathbf{v}\|^2 \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$$= \mathsf{m}g\hat{\mathbf{k}} - \frac{1}{2}\rho S C_d \|\mathbf{v}\|\mathbf{v}, \tag{4}$$

where g is the gravitational force on a unit mass at sea level, $\hat{\mathbf{k}} = (0, 0, 1)^\top$, ρ is the density of air, S is the planform area of the object, and C_d is the coefficient of drag. If we assume that the dropped object is a sphere, then $S = \pi R^2$ where R is its radius. In addition, if we assume relatively slow speeds, then the coefficient of drag is approximately equal to $C_d = 0.5$ (See `https://www.engineeringtoolbox.com/drag-coefficient-d_627.html`). Therefore, the dynamics of the dropped object are given by

$$\dot{\mathbf{v}}(t) = g\hat{\mathbf{k}} - \frac{1}{2\mathsf{m}}\rho S C_d \|\mathbf{v}(t)\|\mathbf{v}(t). \qquad \mathbf{v}(t_0) = \mathbf{v}_0 \tag{5}$$

3   Solving the Ordinary Differential Equation

Suppose that we are given a differential equation

$$\dot{x}(t) \triangleq \frac{dx}{dt}(t) = f(x(t)), \tag{6}$$

where $f(\cdot)$ is an arbitrary function. Since the derivative is defined as

$$\frac{dx}{dt}(t) \triangleq \lim_{\Delta \to 0} \frac{x(t + \Delta) - x(t)}{\Delta},$$

it can be approximated as

$$\frac{dx}{dt}(t) \approx \frac{x(t + T) - x(t)}{T},$$

where $T > 0$ is an small increment of time, for example $T = 0.01$ seconds. Therefore, the differential equation (6) can be approximated as

$$\frac{x(t + T) - x(t)}{T} \approx f(x(t)).$$

Solving for $x(t)$ give

$$x(t + T) \approx x(t) + Tf(x(t)),$$

which is known as the Euler approximation of the differential equation. For example, given the initial condition $x(t_0)$, we can solve for $x(t)$ k-time steps

in the future via the iteration

$$x(t_0 + T) = x(t_0) + Tf(x(t_0))$$
$$x(t_0 + 2T) = x(t_0 + T) + Tf(x(t_0 + T))$$
$$x(t_0 + 3T) = x(t_0 + 2T) + Tf(x(t_0 + 2T))$$
$$\vdots \qquad \vdots$$
$$x(t_0 + kT) = x(t_0 + (k-1)T) + Tf(x(t_0 + (k-1)T)).$$

Therefore, given the initial velocity $\mathbf{v}(t_0)$, the velocity at time t, $\mathbf{v}(t)$, can be found by integrating Equation (5) forward in time using the Euler approximation described above.

4   Computing the Drop Location

The formula for the drop location is given by Equation (3), where $\mathbf{z}(t)$ is given by Equation (2), implying that $\mathbf{z}(t)$ satisfies the differential equation

$$\dot{\mathbf{z}} = \mathbf{v}, \qquad \mathbf{z}(t_0) = 0.$$

Therefore $\mathbf{z}$ and $\mathbf{v}$ can be simultaneously approximated as

$$\mathbf{v}(t + T) = \mathbf{v}(t) + Tf(\mathbf{v}(t)), \qquad \mathbf{v}(t_0) = \mathbf{v}_0$$
$$\mathbf{z}(t + T) = \mathbf{z}(t) + T\mathbf{v}(t), \qquad \mathbf{z}(t_0) = 0.$$

The variable $\mathbf{z}(t)$ can be interpreted as the position of the object at time t if the initial condition is zero, and there is zero wind. Suppose that V is the nominal airspeed of the UAV that is dropping the object. If the UAV drops the object when it is flying due North then the initial velocity of the object is $\mathbf{v}(t_0) = V\hat{\mathbf{i}}$, where $\hat{\mathbf{i}} = (1, 0, 0)^\top$. Careful inspection of Equation (5) shows that in this case the east component of $\mathbf{v}(t)$ will always remain zero, which implies that the east component of $\mathbf{z}(t)$ will also remain zero. The trajectory $\mathbf{z}(t)$ is therefore the nominal trajectory of the object starting at zero, and with initial heading pointing north. If the initial velocity is instead given by

$$\mathbf{v}(t_0) = VR_\psi \hat{\mathbf{i}},$$

where

$$R_\psi \triangleq \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then it is straightforward to see that

$$\mathbf{z}(t) = R_\psi \hat{\mathbf{z}}(t),$$

where $\hat{\mathbf{z}}(t)$ is the nominal solution when $\mathbf{v}(t_0) = V\hat{\mathbf{i}}$. In the following, we will call $\hat{\mathbf{z}}$ the pre-position.

5   Summary of Object Drop Algorithm

Inputs: Given the desired airspeed V and altitude h_0 at the drop location, and given the position of the target $\mathbf{p}_{\text{target}}$.

Step 1: Solve the difference equations

$$\begin{aligned}
\hat{\mathbf{v}}(t + T) &= \hat{\mathbf{v}}(t) + Tf(\hat{\mathbf{v}}(t)), &\hat{\mathbf{v}}(0) &= V\hat{\mathbf{i}} \\
\hat{\mathbf{z}}(t + T) &= \hat{\mathbf{z}}(t) + T\hat{\mathbf{v}}(t), &\hat{\mathbf{z}}(0) &= 0,
\end{aligned}$$

until the down component of $\hat{\mathbf{z}}$ is greater than or equal to h_0. The associated time is stored as T_{drop}, and the associated pre-position is stored as $\hat{\mathbf{z}}_{\text{drop}}$.

Step 2: Estimate the wind vector $\mathbf{w}$ by averaging multiple samples of Equation (1). Estimate the wind direction as

$$\psi_w = \text{atan2}(w_e, w_n).$$

The idea is to drop the object when the course angle is directly into the wind.

Step 3: Compute the drop location from Equation (3) as

$$\mathbf{p}_{\text{drop}} = \mathbf{p}_{\text{target}} - T_{\text{drop}}\mathbf{w} - R_{\psi_w}\hat{\mathbf{z}}_{\text{drop}},$$

and the drop velocity as

$$\mathbf{v}_{\text{drop}} = V R_{\psi_w}\hat{\mathbf{i}}.$$